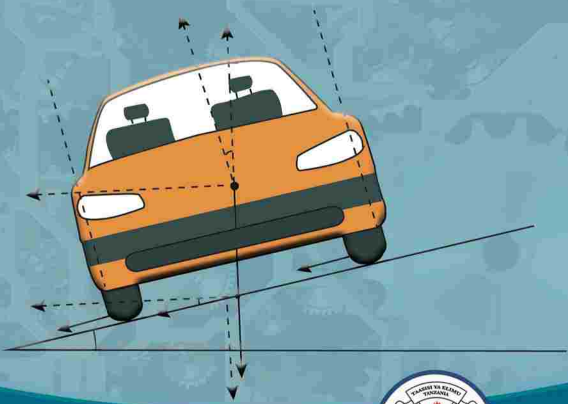
Physics

for Advanced Level Secondary Schools

Student's Book
Form Five



Tanzania Institute of Education

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Physics

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Student's Book

Form Five

THE UNITED REPUBLIC OF TANZANIA
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Preface

This book, *Physics for Advanced Level Secondary Schools,* is written specifically for Form Five students in the United Republic of Tanzania. The book is prepared according to the 2009 Physics Syllabus for Advanced Secondary Education Form V-VI, issued by the Ministry of Education and Vocational Training.

The book is divided into nine chapters, which are: Measurement; Newton's Laws of Motion and projectile motion; Circular motion, simple harmonic motion, and gravitation; Rotation of rigid bodies; Fluid dynamics; Properties of matter; Heat; Vibrations and waves; and Electrostatics. In addition to the content, each chapter contains illustrations, exercises, revision questions, and some practical work. Answers to numerical questions are provided at the end of the book. Learners are encouraged to do all activities and answer all questions so as to enhance their understanding, and promote the acquisition of the intended skills, knowledge, and attitudes.

Tanzania Institute of Education

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Table of contents

Chapter One

Measurement

Introduction

Being an experimental science, Physics needs to relate the theoretical description of nature with experimental observations. Before an explanation of nature can be attempted, accurate observations must be made. The relationship between theory and experimental observation is made through quantitative measurements of various physical quantities. Measurement entails assigning numbers to events or observations. In this chapter, you will learn about the differences between fundamental and derived physical quantities, methods of dimensional analysis, and the relationship between physical quantities. You will also learn about types and sources of errors, ways of determining errors from a graph, and distinguishing between accuracy and precision.

1.1 Physical quantities

Physical quantities can be divided into two types, namely, fundamental and derived quantities. A fundamental physical quantity is not defined in terms of any other quantity whereas the quantities which are defined in terms of other quantities are called derived physical quantities. The measurement of physical quantities involves their comparison with the chosen standard of the same kind of units. The measure of any physical quantity is merely a number and any idea about its magnitude that can be stated in the unit. The standard units for fundamental quantities are called fundamental units. On the other hand, the standard units for derived quantities are called derived units. For example, units for mass, length and time are chosen as fundamental units while units of area, volume, velocity and energy are derived units.

In this section, you will study the interactions existing between the two types of physical quantities. The focus will be on the difference between fundamental and derived physical quantities, the method of dimensional analysis, use of dimensional analysis and the limitations of using the method of dimensional analysis.

1.1.1 Fundamental and derived quantities

Physical quantities which cannot be derived or obtained from any other physical quantities are known as fundamental physical quantities. For example, the fundamental quantity mass can be measured directly using beam balance and hence it does not depend upon other quantities. There are seven fundamental physical quantities. Table 1.1 shows the fundamental quantities with their units.

Table 1.1 Fundamental quantities

Physical quantity	Symbol	SI unit	Symbol
Mass	m	kilogram	kg
Length	l	metre	m
Time	t	second	s
Temperature	T	kelvin	K
Electric current	I	ampere	A
Luminous intensity (brightness)	I	candela	cd
Amount of substance (quantity)	n	mole	mol

Note that, four of the base units namely, the kilogram, ampere, kelvin, and the mole have been redefined in terms of naturally fixed constants, namely the Planck constant (h), the elementary electric charge (e), the Boltzmann constant (k), and the Avogadro constant (N_A) respectively.

On the other hand, derived quantities are defined in terms of fundamental quantities. In order to measure the derived quantity, one must measure the quantities that it depends upon. For example, speed is derived from length and time. Table 1.2 shows examples of derived quantities with their respective units.

Table 1.2 Derived quantities

Derived quantity	Unit of measurement	Unit symbol
Area	square metre	m^2
Volume	cubic metre	m^3
Density	kilogram/cubic metre	kgm^{-3}
Speed	metre/ second	ms^{-1}
Acceleration	meter/second squared	ms^{-2}
Force	newton	N or $kgms^{-2}$
Pressure	newton/ metre squared or pascal	Nm^{-2} or Pa
Potential difference	volts	V

1.1.2 Dimensional analysis

Dimensions are the powers of fundamental physical quantities that represent a certain physical quantity. Dimensions can be represented by square brackets [ ]. Dimensional analysis is the method of establishing a relationship among physical quantities using the three fundamental basic quantities (length, mass and time). Any physically meaningful equation will have the same dimensions on the left and right sides. Therefore, dimensional analysis is important for checking correctness of formula and establishing the relationship among physical quantities. Table 1.3 shows units and dimensions of some common physical quantities.

Table 1.3 Dimensions of some physical quantities

Quantity	Unit	Dimensions
Mass	kg	M
Length	m	L
Time	s	T
Velocity	ms^{-1}	LT^{-1}
Acceleration	ms^{-2}	LT^{-2}
Force	$kgms^{-2}$	MLT^{-2}
Density	kgm^{-3}	ML^{-3}

Example 1.1

Find the dimensional formula for kinetic energy.

Solution

Kinetic energy is given by the expression $\frac{1}{2}mv^2$ where m is the mass and v is the velocity.

Dimensions of kinetic energy $= [\text{mass}] \times [\text{velocity}]^2$,

but $[\text{mass}] = M$ and $[\text{velocity}] = LT^{-1}$.

Since $\frac{1}{2}$ is dimensionless, then,

$$\left[\frac{1}{2}mv^2\right] = ML^2T^{-2}$$

The dimensions of kinetic energy are ML^2T^{-2} where M, L and T are the dimensions of the fundamental quantities mass, length and time, respectively. Therefore, the dimensional formula for kinetic energy is ML^2T^{-2}.

1.1.3 Uses of dimensional analysis

Dimensional analysis is useful in checking correctness of a formula, assigning units of physical quantities and deriving formula.

(a) To check the correctness of formula

Checking correctness of a given equation using dimensional analysis is based on the principle of dimensional homogeneity. The principle works by comparing the dimensions of each term on either side of an equation. It states that, "*An equation is dimensionally correct if the dimensions of the fundamental quantities (mass, length, and time) are the same in each term on either side of the equation*". Only quantities of the same dimensions can be added, subtracted or equated.

Example 1.2

Consider the physical equation $v = u + at$ where v and u are final and initial velocities of a body respectively, a is an acceleration, and t is time. Using methods of dimensional analysis, check whether the equation is dimensionally homogeneous.

Solution

From the principle of dimensional homogeneity, the equation is dimensionally homogeneous if each term on either side of the equation has the same dimensions.

The dimensions of each term are: $[v] = \mathrm{LT}^{-1}$; $[u] = \mathrm{LT}^{-1}$; and $[at] = \mathrm{LT}^{-1}$. Since the dimensions of physical quantities are the same for every term, the equation is dimensionally homogeneous.

(b) To assign units of a physical quantity

The method of dimensional analysis is used to assign units of physical quantities.

For example, the units of the coefficient of viscosity given by $\eta = \dfrac{\pi r^4 \Delta p}{8Ql}$ (where r is the radius of the pipe, Q is the volume flux (volume flow per time), $\dfrac{\Delta p}{l}$ is pressure gradient) can be obtained using dimensional analysis as follows:

$$[\eta] = \frac{(\mathrm{L}^4)(\mathrm{ML}^{-2}\mathrm{T}^{-2})}{\mathrm{L}^3\mathrm{T}^{-1}}; [\eta] = \mathrm{ML}^{-1}\mathrm{T}^{-1}$$ the units of η are $\mathrm{kgm}^{-1}\mathrm{s}^{-1}$ or $\mathrm{Nm}^{-2}\mathrm{s}$.

1.1.4 Relationship between physical quantities

Dimensional analysis can be used to derive an expression of a physical quantity provided the terms upon which the given physical quantity depends are known. This form of dimensional analysis expresses a functional relationship of some variables in the form of an exponential equation. The method involves the following steps:

(i) Identify all the independent variables that are likely or assumed to determine the dependent variable.

(ii) If Q is a variable that depends upon independent variables; $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3, \ldots \mathrm{R}_n$ then $Q \propto \mathrm{R}_1^a \mathrm{R}_2^b \mathrm{R}_3^c \ldots \mathrm{R}_n^m$, where $a, b, c, \ldots, m$ are arbitrary exponent integers.

(iii) Write the above equation in the form $Q = \mathrm{k}\mathrm{R}_1^a \mathrm{R}_2^b \mathrm{R}_3^c \ldots \mathrm{R}_n^m$, k is a dimensionless constant.

(iv) Express each of the quantities in the equation in some base units.

(v) By using dimensional homogeneity, obtain a set of simultaneous equations involving the exponents $a, b, c, \ldots, m$.

(vi) Solve these equations to obtain the value of exponents $a, b, c, \ldots, m$.

(vii) Substitute the values of the exponents in the main equation, and form the non-dimensional parameters by grouping the variables with similar exponents.

This method does not provide the value of a dimensionless constant k. The constant can be determined mathematically or experimentally.

Example 1.3

Consider a small bob hanging freely to a string whose free end is attached to a fixed-support. If the bob is set into periodic oscillation, use dimensional analysis to derive the formula for the period of oscillation of the system.

Solution

The period T of oscillation of the pendulum depends on the length l of the pendulum and the acceleration due to gravity g at the place. With this information the relationship between the physical quantities can be written as

$$T = kl^a g^b \quad (i)$$

where a and b are unknown exponents and k is a dimensionless constant.

Dimensionally, equation (i) can be written as

$$[T] = k[l^a][g^b] \quad (ii)$$

Substituting each physical quantity with its respective base fundamental unit in equation (ii) gives,

$$\mathrm{M^0L^0T^1} = \left(\mathrm{M^0L^aT^0}\right)\left(\mathrm{M^0L^bT^{-2b}}\right) \quad (iii)$$

Comparing LHS and RHS of equation (iii) M: $0 = 0$; L: $0 = a + b$; T: $1 = -2b$;

$$a = \frac{1}{2} \text{ and } b = -\frac{1}{2}$$

Substituting the values on equation (i) gives $T = kl^{\frac{1}{2}}g^{-\frac{1}{2}}$.

The value of k was experimentally found to be 2π. Therefore, the final equation is $T = 2\pi\sqrt{\frac{l}{g}}$.

1.1.5 Limitations of dimensional analysis

Although dimensional analysis is used to check correctness of formulae, derive formulae and assign units of physical quantities, it has some limitations as follows:

(a) If a physical quantity depends on more than three fundamental quantities (M, L and T), then the relation among them cannot be established.

(b) It does not show whether a given physical quantity is a scalar or a vector.

(c) It does not show the value of constants involved in given formula.

(d) It cannot be used for deriving equations containing logarithmic, exponential or trigonometric relations.

(e) It can only verify whether a physical relation is dimensionally correct or not. It cannot show whether the relation is absolutely correct or not. For example, applying this technique $s = ut + 4at^2$ is dimensionally correct whereas the correct relation is $s = ut + \frac{1}{2}at^2$.

Exercise 1.1

1. Use dimensional analysis to check the correctness of the following formula:
   (a) $v^2 = u^2 + 2as$ where u and v are velocities, a is acceleration and s is distance.
   (b) $E = mc^2$ where E is energy, m is mass and c is the velocity of light.
   (c) $T = 2\pi$ where T is period.
2. Write the dimensions of a and b in the relation $p = \frac{b - x^2}{at}$ where p is power, x is distance and t is time.
3. Identify the physical quantity x defined as $x = \frac{IFv^2}{W}$, where I is moment of inertia, F is force, v is velocity, W is work and l is length.

4. Write the dimension of $\frac{a}{b}$ in the relation $F = a\sqrt{x} + bt^2$, where F is force, x is distance and t is time.
5. A jet of water of cross sectional area A and velocity v strikes normally on a stationary flat plate. The mass per unit volume of water is Q. By dimensional analysis, show that an expression for the force F exerted by the jet against the plate is given by kAv^2Q.

1.2 Errors

Measurements of physical quantities are always subjected to some errors. These errors may originate from various sources, mainly from measuring devices, environment, an observer taking the measurements and mathematical computations. Errors are uncertainties in measurements. Therefore, measured values will always deviate from exact values. The difference between the exact value (sometimes taken as a mean value) and the measured value constitutes an error of measurement. The word error should not be confused with mistake which is simply doing something incorrectly or carelessly.

In this section, you will learn types and sources of errors, how to determine errors in measurement, method of estimating errors of derived physical quantities, techniques of determining errors from graph and the differences between accuracy and precision.

1.2.1 Types and sources of errors

Errors of measurement are divided into two types namely systematic and random errors. Systematic errors are caused by instruments. For example, ruler or a beam balance with incorrect scales. These are errors whose cause is known and tend to happen or occur in a systematic pattern. Some specific causes of systematic errors include:

(a) Incorrect design or set up of an instrument which includes construction and calibration.

(b) Incorrect reading or interpretation of the instrument in an experiment.

(c) Limitation of the method used for measurement.

(d) Poor accuracy of formula being used.

Systematic errors can be minimized by proper design and calibration of the measuring instruments.

On the other hand, the causes of random errors are unpredictable and have no systematic pattern. They keep on varying in terms of their magnitude and direction. Causes of random errors include changes in experimental conditions such as pressure, temperature and wind. Also, lack of sensitivity of the instrument and human inaccuracies. Random errors can be minimized by repeating a measurement several times and then finding the arithmetic mean (or average) for all the recorded values.

1.2.2 Determination of errors in measurements

Measured quantities are always subject to errors because of the uncertainties that are involved in the process of measurement.

Hence, it is important to make some analysis and find out the magnitude of those errors and make interpretation. The following are common terms used in errors:

(a) Absolute error

Absolute error is the magnitude of the difference between the true value and the measured value of the quantity. This difference may be positive or negative depending on the circumstances of taking the measurement. Assume a physical quantity to be measured n times and let the measured values be $a_1, a_2, a_3, \ldots, a_n$. The arithmetic mean, a_m of these values becomes

$$a_m = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

hence the absolute error $|\Delta a_n|$in a_n is $|a_n - a_m|$.

(b) Mean absolute error

Since the error may be either positive or negative, it is worth to find the Mean absolute error that is the arithmetic mean of the magnitudes of absolute errors. Mathematically this is expressed as,

$$\Delta a_{\text{mean}} = \frac{|\Delta a_1| + |\Delta a_2| + \ldots + |\Delta a_n|}{n}$$

The final result of measurement can be written as, $a = a_{\text{mean}} \pm \Delta a_{\text{mean}}$.

This implies that value of a is likely to lie between $a + \Delta a_{\text{mean}}$ and $a - \Delta a_{\text{mean}}$.

(c) Relative error or fractional error

The relative error or fractional error is defined as the ratio of the mean absolute error to the mean value of the quantity measured. Relative error $= \dfrac{\Delta a_{\text{mean}}}{a_{\text{mean}}}$.

When the relative error or fractional error is expressed in percentage then percentage error is obtained. Thus,

$$\text{percentage error} = \frac{\Delta a_{\text{mean}}}{a_{\text{mean}}} \times 100\%$$

Relative error gives an indication of how good a measurement is compared to the size of the object being measured. Consider two students measuring objects with a meter stick. One student measures the height of a room and gets a value of (3.125 ± 0.001) m. Another student measures the height of a small cylinder and gets (0.075 ± 0.001) m. Clearly, the overall accuracy of the room height is much better than that of the cylinder. The comparative accuracy of these measurements can be determined by looking at their relative errors as follows:

Relative error in a room height is

$$\frac{0.001\,\text{m}}{3.125\,\text{m}} = 3.2 \times 10^{-4}$$

Relative error in a cylinder height is

$$\frac{0.001\,\text{m}}{0.075\,\text{m}} = 1.3 \times 10^{-2}$$

Clearly, the relative error in the room height is considerably smaller than the relative error in the cylinder height even though the amount of absolute error is the same in each case.

Activity

Determination of errors of mass, length and time of a simple pendulum

Materials: Inextensible string, meter rule, stop watch, retort stand, pendulum bob, and beam balance

Procedure

(a) Measure the mass (m) of the pendulum bob.
(b) Tie the bob on the string and suspend it on the retort stand so that it hangs freely.
(c) Measure the length (l) of the string from the point of suspension to the centre of the bob.
(d) Set the bob to oscillate at small angle.
(e) Measure the time (t) for 10 oscillations.
(f) Repeat step (a) to (e) then record the values of m, l and t as in Table 1.4.

Table 1.4 Measurement of mass, length and time

Measurement	Mass (g)	Length (cm)	Time (s)
First			
Second			
Third			

Question

Write the values of m, l and t including their corresponding errors.

Example 1.4

An object weighs exactly 36.5 grams. When weighed on a faulty scale, it weighs 38 grams.

(a) What is the percentage error in measurement of the faulty scale to the nearest tenth?
(b) If a chick weighs 14 grams on the same defective scale, what is the chick's weight in gram to the nearest tenth?

Solution

(a) Percentage error

$$= \frac{|38\text{ g} - 36.5\text{ g}|}{36.5\text{ g}} \times 100\%$$

which is approximately 4.1%.

(b) Let the true weight of the chick be x, then, $14\,g = x + 0.041x$; $x = 13.4g$.

Therefore, the chick's weight is 13.4 grams.

Example 1.5

The actual length of the playing field is 500 m. A measuring instrument shows the length to be 508 m. Find:

(a) Absolute error in the measured length of the field;
(b) Relative error in the measured length of the field; and
(c) Percentage error in the measured length of the field.

Solution

(a) The absolute error in the length of the field is $|500 - 508|\text{ m} = 8\text{ m}$.
(b) The relative error in the length of the field is $\frac{|500\text{m} - 508\text{m}|}{500\text{m}} = 0.016$.
(c) The percentage error in the length of the field is $\frac{|500\text{m} - 508\text{m}|}{500\text{m}} \times 100\% = 1.6\%$.

1.2.3 Errors of derived physical quantities

In any experimental results, the measured values which are always subjected to various errors can propagate errors when the measured quantities are manipulated.

Such manipulation includes addition, subtraction, multiplication, division and exponents. The propagation of errors in any mathematical computation depends on the formula used to determine the final answer. In this part the errors in sum, difference, product, quotient and exponents will be discussed.

(a) Errors in a sum

Suppose you are given this equation, $x = a + b$. Let Δa be the absolute error in the measurement of a, Δb be the absolute error in the measurement of b and Δx be the absolute error in the value of x,

When a and b are added, such that $x = a + b$

$$x \pm \Delta x = a \pm \Delta a + b \pm \Delta b \qquad (1.1)$$

After expanding equation (1.1), four possible values of Δx are: $\Delta a + \Delta b$, $\Delta a - \Delta b$, $-\Delta a + \Delta b$, and $-\Delta a - \Delta b$. The maximum possible absolute error in x is $\Delta x = \Delta a + \Delta b$. Therefore, the maximum absolute error in the sum of two quantities equals to the sum of the absolute errors in the individual quantities.

Example 1.6

Suppose $a = (20.5 \pm 0.5)\text{cm}$ and $b = (10.0 \pm 0.2)\text{cm}$. Calculate the maximum possible error of $a + b$.

Solution

Let $x = a + b$ then,

$x = 20.5\,\text{cm} + 10.0\,\text{cm} = 30.5\,\text{cm}$ and

$\Delta x = 0.5\,\text{cm} + 0.2\,\text{cm} = \pm 0.7\,\text{cm}$.

Maximum possible error in x is $\pm 0.7\,\text{cm}$.

Therefore, the value of x ranges from 29.8 cm to 31.2 cm.

(b) Errors in a difference

Error in difference can be calculated by following the same procedure used in the sum.

Let $x = a - b$, and Δa be the absolute error in the measurement of a, Δb be the absolute error in the measurement of b and Δx be the absolute error in the measurement of x. Then,

$x + \Delta x = (a \pm \Delta a) - (b \pm \Delta b)$ as in sum, the four possible values of Δx are;

$(\Delta a + \Delta b)$, $(\Delta a - \Delta b)$, $(-\Delta a - \Delta b)$ and $(-\Delta a + \Delta b)$.

The maximum possible error in x is $(\Delta a + \Delta b)$. Hence the value of x can range from, $(a - b) - (\Delta a + \Delta b)$ to $(a - b) + (\Delta a + \Delta b)$

Therefore, the maximum absolute error in the sum is equal to the maximum absolute error in the difference.

(c) Errors in a product

Let $x = ab$, then

$$(x \pm \Delta x) = (a \pm \Delta a)(b \pm \Delta b) \qquad (1.2)$$

By simplifying equation (1.2) and dividing each term by x on both sides gives,

$$\frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \times \frac{\Delta b}{b}$$

Since $\frac{\Delta a}{a}\times\frac{\Delta b}{b}$ is very small, it can be neglected; the possible values of $\frac{\Delta x}{x}$ are: $\frac{\Delta a}{a}+\frac{\Delta b}{b}$, $\frac{\Delta a}{a}-\frac{\Delta b}{b}$, $-\frac{\Delta a}{a}+\frac{\Delta b}{b}$ and

$-\frac{\Delta a}{a}-\frac{\Delta b}{b}$.

But the maximum possible values are $\frac{\Delta a}{a}+\frac{\Delta b}{b}$ and $-\frac{\Delta a}{a}-\frac{\Delta b}{b}$. The maximum value of

$$\frac{\Delta x}{x}=\left(\frac{\Delta a}{a}+\frac{\Delta b}{b}\right)$$

Therefore, maximum fractional error in product of two or more quantities is equal to the sum of fractional errors in the individual quantities.

(d) Error in division

Let $x=\frac{a}{b}$

$$(x\pm\Delta x)=\frac{(a\pm\Delta a)}{(b\pm\Delta b)} \qquad (1.3)$$

simplifing equation (1.3) gives

$$\frac{\Delta x}{x}=\pm\frac{\Delta a}{a}\pm\frac{\Delta b}{b}.$$

Maximum possible value of $\frac{\Delta x}{x}$ are

$\frac{\Delta a}{a}+\frac{\Delta b}{b}$ and $-\frac{\Delta a}{a}-\frac{\Delta b}{b}$, hence,

$$\frac{\Delta x}{x}=\left(\frac{\Delta a}{a}+\frac{\Delta b}{b}\right)$$

Thus, the maximum value of fractional error in division of two quantities is equal to the sum of fractional errors in the individual quantities.

(e) Error in exponents

Let $x=\frac{a^n}{b^m}$, then applying natural logarithms on both sides, $\ln(x)=n\ln a-m\ln b$.

Differentiating both sides gives,

$$\frac{dx}{x}=n\frac{da}{a}-m\frac{db}{b}$$

which can be written in terms of fractional errors as;

$$\frac{\Delta x}{x}=n\frac{\Delta a}{a}-m\frac{\Delta b}{b}$$

Therefore, the maximum value of

$$\frac{\Delta x}{x}=\left(n\frac{\Delta a}{a}+m\frac{\Delta b}{b}\right).$$

This equation can be regarded as a general form for computation of errors in derived physical quantities.

Example 1.7

Calculate percentage error in the determination of $g=4\pi^2\frac{l}{T^2}$ when l and T are measured with $\pm2\%$ and $\pm3\%$ errors respectively.

Solution

Using, $g=4\pi^2\frac{l}{T^2}$, and since $4\pi^2$ is constant, then maximum percentage error in $\frac{\Delta g}{g}$ is $\pm\left(\frac{\Delta l}{l}+\frac{2\Delta T}{T}\right)\times100\%$.

Substituting the values for l and T, the percentage error in g is $\pm8\%$.

Therefore, if the actual value of g is $9.8\,\text{ms}^{-2}$ then approximated value of g may vary from $9.0\,\text{ms}^{-2}$ to $10.6\,\text{ms}^{-2}$.

1.2.4 Errors from a graph

The fact that no individual measurement is accurate often requires experimenters to carry several measurements of a given quantity with the hope that these measurements will cluster about the true value required to be measured. The distribution of these data values is represented graphically by showing a single data point representing the mean value of the data, and error bars to represent the overall distribution of data. Error bars are used on graphs to indicate the error or uncertainty. They look like a cross (Figure 1.1) whose vertical bar gives the error on the ordinate and the horizontal bar gives the error on the abscissa.

For example, the uncertainty associated with a data point $A(x, y) = (3.7\,\text{s} \pm 0.1\text{s},\ 4.0\,\text{m} \pm 0.2\,\text{m})$ on a (x, y) graph is plotted by drawing a cross whose vertical bar goes from 3.8 m to 4.2 m and whose horizontal bar goes from 3.6 s to 3.8 s.

When calculating a gradient from a graph, it is important to determine the magnitude of uncertainty. This is done by drawing two lines of "worst fit" also known as lines of minimum and maximum gradient. These lines are drawn by first constructing a square (or rectangle) around the error bars of the two extreme data points.

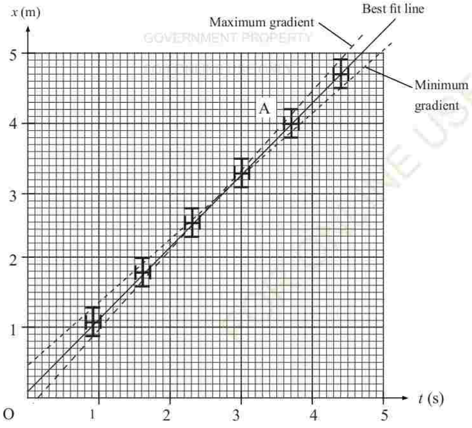

Figure 1.1 Determination of errors from graph

Then the top left corner of the first data point is joined with the bottom right corner of the n^{th} data point and the bottom right corner of the 1^{st} data point is joined with the top left corner of the n^{th} data point (Figure. 1.1).

Suppose the gradient of line of best fit is m, gradient of the worst fit line one is m_1 and gradient of the worst fit line two is m_2. The uncertainty in the gradient of best fit line is then taken as half the difference between errors in the gradients of worst fit lines. Mathematically,

$$\Delta m = \frac{\Delta m_1 + \Delta m_2}{2}, \text{ where } \Delta m_1 = |m_1 - m|$$

and $\Delta m_2 = |m_2 - m|$. This formula applies to a straight line graph and a curve although the gradients of the graph would vary.

Example 1.8

Suppose the slope of the best fit line is 1.0 and slopes of maximum and minimum worst lines are 1.16 and 0.81 respectively. Estimate the value of slope of the graph.

Solution

From the given information,

$$\Delta m_1 = |1.16 - 1.0| \text{ and } \Delta m_2 = |0.81 - 1.0|$$

$$\Delta m = \frac{0.16 + 0.19}{2} \approx 0.18.$$

Therefore, the slope of the graph to the nearest hundredth is 1.0 ± 0.18.

1.2.5 Accuracy and precision

Accuracy and precision have different meanings, although some people use them interchangeably. Precision refers to the closeness of two or more measurements to each other. For example, if you weigh a given substance five times, and you get the same value each time then your measurement is very precise. Precision is determined by the smallest scale division or least counting unit of the measuring instrument. The smaller the least counting unit or the smaller the scale division the greater the precision.

The accuracy is the measure of how close the measured value is to the true value of the quantity. Accuracy in measurements always depends on several factors like personal errors and imperfection in technique or procedure used. Others include instrumental errors and environmental factors like weather changes, wind and temperature. Therefore, accuracy refers to the degree of conformity and correctness of something when compared to a true or absolute value whereas precision refers to a state of strict exactness. Accuracy in measurements can be improved in many ways including the following:

(a) Make measurement with an instrument that has the highest level of precision (smallest possible unit)

(b) Apply correct techniques when using the measuring instrument and when reading the value measured.

(c) Avoid the error due to parallax by taking readings while looking at right angle to the scale of an instrument.

(d) Repeat the same measurement several times to get a good average value.

(e) Take measurement under controlled conditions. If the object you are measuring could change size depending upon weather conditions (expand or shrink), make sure you measure it under the same conditions each time.

Exercise 1.2

1. Explain the basic differences between precision and accuracy.
2. In a certain experiment, the refractive index of a glass was observed to be 1.44, 1.50, 1.48, 1.45, 1.60 and 1.52. From these data calculate:
   (a) Mean absolute error;
   (b) Mean value of the refractive index;
   (c) Fractional error; and
   (d) Percentage error.
3. The relative error in measuring the mass of a certain substance is 5% and in its volume is 2%. What will be the percentage error in the measurement of the density?
4. The initial and the final temperatures of a liquid are found to be (63.5 ± 0.5)°C and (72.6 ± 0.4)°C respectively. Determine the rise in temperature.
5. The period of oscillation of a simple pendulum is $T=2\pi\sqrt{\frac{l}{g}}$. The value of l is 20 cm known to 1 mm accuracy, and time for 100 oscillations of the pendulum is found to be 90 s using a wrist watch of 1 s resolution. What is the accuracy in the determination of g?

Revision exercise 1

1. An experiment shows that the frequency f of a tuning fork depends on the length l of the prongs, density ρ, and the Young's modulus E of the material. Using dimensional analysis derive an expression for the frequency.
2. An explosion that happened in water created a gas bubble within it, and it was found to oscillate with a period of oscillation T. If T is proportional to $p^x\rho^y E^z$ where p is the static pressure, ρ is the density of water and E the total energy of explosion, using methods of dimensions, determine x, y and z.
3. It is suggested that the velocity of water waves in a basin depends on wavelength λ, density of water ρ and the acceleration due to gravity g. Using dimensional analysis check if the dependence of these quantities is correct.
4. Differentiate between an error and a mistake.
5. A certain wire with a length of (125.2 ± 0.1) cm was subjected to an extensional force and caused it to extend to (128.3 ± 0.1) cm. Calculate the elongation of this wire with its error limit.
6. A physical quantity Q is given by the following equation $Q=ka^3b^2c^5d^4$. If the percentage error in each measured value of a, b, c and d is 0.6%, determine the percentage error in Q.

7. The length, breadth and thickness of a glass block as measured by a student were found to be (25.12 ± 0.05)cm, (15.55 ± 0.05)cm and (5.15 ± 0.05)cm. Determine the percentage error of the volume of this glass block.

8. In the ancient years the Earth's daily rotation on its axis was once used to define the standard unit of time. What other types of natural phenomena currently could serve as alternative time standards?

9. (a) You are given a thread and a metre scale. How will you estimate the diameter of the thread?
   (b) A micrometer screw gauge has a pitch of 1.0 mm and 200 divisions on the circular scale. Do you think it is possible to increase the accuracy of the micrometer screw gauge arbitrarily by increasing the number of divisions on the circular scale? Why?
   (c) The mean diameter of a thin brass rod is to be measured by Vernier calipers. Why is a set of 100 measurements of the diameter expected to yield a more reliable estimate than a set of 5 measurements only?

10. What is the unit of volume? A student measured the volume of a cylinder which has the radius r and height h, and wrote the formula of volume as $\pi r^3 h$. Explain whether the student was dimensionally correct or not.

11. A book with many printing errors contains the following four different equations for the displacement y of a particle undergoing a certain periodic motion:
   (a) $y = \sin\left(\frac{2\pi t}{T}\right)$;
   (b) $y = \sin(vt)$;
   (c) $y = \frac{A}{T}\sin\left(\frac{t}{A}\right)$; and
   (d) $y = \left(A\sqrt{2}\right)\left(\sin\left(\frac{2\pi t}{T}\right) + \cos\left(\frac{2\pi t}{T}\right)\right)$,

   where A is maximum displacement of the particle, v is speed of the particle, t is the time and T is the periodic time. Find out the wrong equations on dimensional grounds.

12. Precise measurements of physical quantities are needed in science. For example, to ascertain the speed of an aircraft, one must have an accurate method to find its positions at closely separated instants of time. Think of different examples in modern science where precise measurements of length, time and mass are needed. Also, give a quantitative idea of the precision needed.

Chapter Two

Newton's Laws of Motion and projectile motion

Introduction

In your life, you have experienced many situations in which different objects move. Physicists can describe the motion of such objects. In the process of describing their motion, three questions can be asked: What causes an object to move? What causes an object to stop? And what causes an object to accelerate or decelerate? The answer to these questions is objects move, stop and or acceralate due to the effect of force. In this chapter, you will learn about Newton's Laws of Motion and projectile motion. You will also learn about the application of Newton's Laws of Motion and projectile motion in daily life.

2.1 Newton's Laws of Motion

Motion of bodies was carefully studied and analyzed first by Galileo Galilei and then followed by Sir Isaac Newton. On the basis of his study, Newton articulated three laws of motion: first law, second law and third law of motion. In this section you will learn about force, equilibrant forces on a body, expressions for tension and acceleration of connected bodies, reaction forces, and the principle of the conservation of linear momentum.

2.1.1 Force

The concept of force gives a quantitative description of the interaction between two objects or between an object and its environment. It is defined as a pull or push acting on an object. The result of force is to produce or stop motion of a body. Thus, force can cause a body to move or a moving body to come to a stop. Furthermore, force can cause a body to accelerate or decelerate in a circle with its velocity changing continuously.

Force exists in different types. When a force involves direct contact between two objects we call it contact force. Other types include normal force (a component of force perpendicular to surface), friction force (a component of force parallel to the surface), tension (a force in a cord or rope attached to pulled object) and weight (gravitational attraction the earth exerts on an object).

Being a vector quantity, force is described in terms of magnitude and direction. The SI unit of magnitude of force is newton, abbreviated as N. An object can be affected by several forces simultaneously.

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In this case, the net force (vector sum of all forces) will determine the motion of the object. When there is no force or no net force acting on the body in motion, the body moves with constant velocity and zero acceleration, while the one at rest remains stationary.

The relation between force and motion is described by Newton's three laws of motion. Having discussed some properties of force, we now turn our attention to discuss what makes bodies move the way they do and the way forces affect the motion of a body by using Newton's laws of motion. There are three Newton's laws of motion. The first law says that, when a net force on a body is zero, its motion does not change. The second law relates force to acceleration when the net force is not zero. The third law is a relation between the forces that two interacting bodies exert on each other.

2.1.2 Newton's First Law of Motion

Suppose you are pushing a book along a horizontal table top (i.e. you are applying a horizontal force to it with your hand). When you stop pushing, what happens to the motion of a book? What do you think will keep the book moving? Suppose you are now pushing the book across a smooth surface of a freshly waxed floor, what happens to the motion of a book as you stop pushing it? In the same manner, what happens to the motion of a book moving on a completely frictionless surface when you stop pushing it?

In each of the preceding cases, you may note that after you stop pushing, the book will not continue to move indefinitely; it slows down and stops. So you need to keep pushing (i.e. applying force) for it to continue moving. It may be concluded that, bodies in motion naturally come to rest and that a force is required to sustain motion. Again, let us ask ourselves: what makes the body slow down and stop? The answer is "friction force" which interacts between the lower surface of the body and the surface on which it slides.

In all explained cases, the surfaces exert a frictional force on the book which resists the motion of the book. The only difference in the preceding cases is the magnitude of the frictional force. The slippery (frictionless smooth surface) exerts less friction than the rough surface. These cases show further that, if you could eliminate friction completely, the book would never slow down.

It can therefore be concluded that, when no net force acts on a body, the body either remains at rest or moves with constant velocity in a straight line. This is Newton's first law of motion. The law states that, *"Every body continues in its state of rest or uniform motion in a straight line unless it is acted upon by external force"*.

The tendency of a body to keep moving once it is set in motion or to keep at rest once it is stopped is called Inertia. Thus, Newton's first law of motion is also called the law of inertia.

2.1.3 Newton's Second Law of Motion

The law states that, *"The rate of change of linear momentum is directly proportional to the net externally applied force and the changes take place in the direction of the net force"*.

Consider a body of mass m moving with a velocity v in a straight line. The linear momentum P of the body is defined as the product of mass and its linear velocity such that $P = mv$. From the Newton's second law,

$$\sum F_{ext} = \frac{dP}{dt} \qquad (2.1)$$

where $\sum F_{ext}$ = net external force; P = linear momentum; $\frac{dP}{dt}$ = rate of change of linear momentum.

From the equation $P = mv$; and keeping mass constant; $dP = mdv$ *(2.2)*

Therefore, placing equation (2.2) in equation (2.1) gives $\sum F_{ext} = \frac{mdv}{dt}$, where $\frac{dv}{dt} = a$, the variable a is the acceleration. Therefore,

$$\sum F_{ext} = ma$$

Thus, a net external force acting on an object produces a proportional acceleration. This means acceleration is an effect of a net external force.

Note that, Newton's first law is the special case of Newton's second law. To verify that, let u and v be the initial and final velocities of a moving body whose mass is m and moving in a straight line.

From Newton's second law, net force

$$F = \frac{dP}{dt} = ma$$

Since m ≠ 0, then $a = 0$, that is

$$\frac{v-u}{t} = 0, v-u = 0,$$

therefore, u = v, hence a body is moving with constant velocity or is at rest. This is Newton's first law of motion.

2.1.4 Newton's Third Law of Motion

The law states that, *"To every action (force) there is an equal and opposite reaction"*. Force is a mutual interaction between an object and its environment. For example, when you push an object, the object pushes back at you with an equal force. Suppose that, object A acts on object B; then, force on B due to A, F_{AB} is equal to the force on A due to B, F_{BA} i.e., $F_{AB} = -F_{BA}$, where the negative sign implies that, the two forces are oppositely directed. ***Note that***, these two forces act on different objects and therefore they never cancel each other.

2.1.5 Equilibrant forces on a body

Equilibrant forces are those forces that produce zero acceleration to an object on which they act and therefore establishing equilibrium for that object. One of the simplest cases of a body in equilibrium is a book resting on a table as in Figure 2.1. The forces acting on a book are its weight, W acting downwards, and the normal reaction R_N that the table exerts upward on the book.

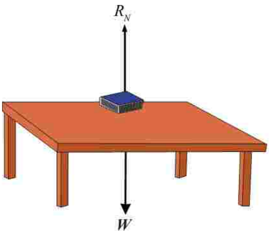


Figure 2.1 Forces on a book resting on a table

Taking the upward direction to be positive and the downward direction to be negative, then the net external force on the book is:

$$\sum F_{ext} = R_N + (-mg) \qquad (2.3)$$

Since there is no net motion vertically, then $\sum F_{ext} = 0$.

Therefore, $0 = R_N - mg$; hence $R_N = mg$.

The analysis on the forces acting on the book shows that the two forces (R_N and mg) add up to give zero, thus, the book is in equilibrium.

Example 2.1

A block of mass, $m = 100\,g$ is placed on a rough inclined plane. The plane makes an angle, $\theta = 30°$ with the horizontal as shown in Figure 2.2. Determine the value of friction force that is required to keep the block at rest.

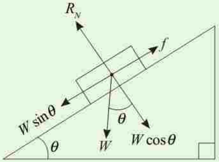


Figure 2.2 Forces acting on a body resting on a rough inclined plane

Solution

The free body diagram (Figure 2.3) for the problem is;

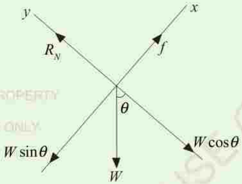


Figure 2.3 Free body diagram for the inclined plane

Net force along the x – axis;

$$\sum F_x = f + (-W \sin\theta)$$

$$ma_x = f - W \sin\theta$$

Since the block is at rest then, $a_x = 0$.

Therefore, $f = W \sin\theta = mg \sin\theta$

$$f = 0.1\,\text{kg} \times 9.8\,\text{ms}^{-2} \times \sin 30° = 0.49\,\text{N}$$

Therefore, friction force which is required to keep the block at rest is $0.49\,\text{N}$.

Note that, since there is no acceleration perpendicular to the plane, the component R_N and $W \cos\theta$ add up to zero.

Example 2.2

A box weighing 8.0 N is supported by two wires with tension T_1 and T_2 (Figure 2.4). Find the tension in each wire.

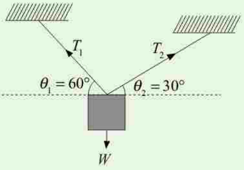


Figure 2.4 Box supported by two wires

Solution

The free body diagram for the problem, Figure 2.5.

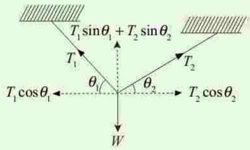


Figure 2.5 Free body diagram for the supported two wires

Net force along the x – axis (horizontal)

$$\sum F_x = T_2\cos\theta_2 + (-T_1\cos\theta_1)$$

$$ma_x = T_2\cos\theta_2 - T_1\cos\theta_1$$

Since the box does not accelerate, $a_x = 0$.

Therefore, $T_1\cos\theta_1 = T_2\cos\theta_2$, but $\theta_1 = 60°$ and $\theta_2 = 30°$

$$T_1 = \frac{\cos 30°}{\cos 60°}T_2 = 1.732T_2 \qquad (i)$$

Net force along the y-axis (vertical)

$$\sum F_y = T_2\sin\theta_2 + T_1\sin\theta_1 + (-W)$$

$$ma_y = T_2\sin\theta_2 + T_1\sin\theta_1 - W$$

Since the box does not accelerate, $a_y = 0$. Therefore,

$$m \times 0 = T_2\sin\theta_2 + T_1\sin\theta_1 - W$$

$$W = T_2\sin\theta_2 + T_1\sin\theta_1$$

$$8 = T_2\sin 30^o + T_1\sin 60^o$$

$$8 = 0.5T_2 + 0.866T_1 \qquad (ii)$$

Substituting equation (i) in equation (ii) gives $8.0 = 0.5T_2 + 0.866 \times 1.73T_2$ and solving for T_2 and T_1;

$$T_2 = \frac{8.0}{0.5 + 1.498} = 4.0\,\text{N} \text{ and } T_1 = 6.9\,\text{N}$$

2.1.6 Motion of connected bodies

Connected bodies can move in vertical direction, horizontal plane and inclined plane.

(a) Connected bodies in vertical motion

Consider two bodies of masses m_1 and m_2 which are connected together using a flexible and massless string. The string is made to pass over a smooth pulley which is fixed to the ceiling so that the two bodies hang freely (Figure 2.6).

(i) If the two masses are equal, the system will be at equilibrium i.e. there will be no motion at all.

(ii) If $m_1 < m_2$, then the system will move in the direction of m_2 with an acceleration a.

(iii) If $m_2 < m_1$, then the system will move in the direction of m_1 with an acceleration a.

Consider case (ii) where $m_1 < m_2$.

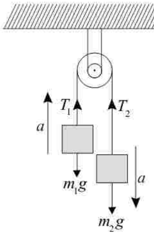


Figure 2.6 Pulley system

The larger weight pulls on the lighter causing the system to accelerate in one direction with an acceleration a. Therefore, the motion of the bodies can be expressed as,

$$T_1 - m_1g = m_1a \quad (2.4)$$

$$m_2g - T_2 = m_2a \quad (2.5)$$

Since the pulley system is smooth, $T_1 = T_2$, then, adding equation (2.4) and (2.5) and rearranging, gives

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g \quad (2.6)$$

Substituting equation (2.6) into equation (2.4) gives

$$T_1 - m_1g = m_1\left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

$$T_1 = m_1g + m_1\left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

$$T_1 = m_1g\left(\frac{m_1 + m_2 + m_2 - m_1}{m_1 + m_2}\right)$$

$$T_1 = \frac{2m_1m_2g}{m_1 + m_2}$$

Also, substituting equation (2.6) into (2.5) gives, $T_2 = \frac{2m_1m_2g}{m_1 + m_2}$. This means $T_1 = T_2 = T$.

Therefore,

$$T = \frac{2m_1m_2g}{m_1 + m_2} \quad (2.7)$$

Example 2.3

Suppose the two bodies in figure 2.6 have the masses m_1 and m_2 of 3 kg and 5 kg respectively. Find the acceleration of each mass and the tension in the string.

Solution

Recall equation, $a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$

$$a = \left(\frac{5\text{kg} - 3\text{kg}}{3\text{kg} + 5\text{kg}}\right) \times 9.8\text{ms}^{-2} = 2.45\text{ms}^{-2}$$

Recall equation, $T = \frac{2m_1m_2g}{m_1 + m_2}$

$$T = \frac{2 \times 3\text{kg} \times 5\text{kg} \times 9.8\text{ms}^{-2}}{3\text{kg} + 5\text{kg}} = 36.75\text{N}$$

Thus, the acceleration of each body is 2.45ms^{-2} and the tension in the string is 36.75 N.

(b) Connected bodies on horizontal plane

Usually one body is pulled horizontally by another, each linked by a tow-bar. This is similar to the pulley but drawn out in a line as in Figure 2.7.

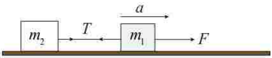


Figure 2.7 *Motion of bodies in horizontal plane*

Assuming no friction,

For m_1: $F - T = m_1 a$ (2.8)

For m_2: $T = m_2 a$ (2.9)

Adding equations (2.8) and (2.9) and rearranging, gives

$$a = \frac{F}{m_1 + m_2}.$$

Substituting value of a into equation (2.9), gives

$$T = \frac{m_2 F}{m_1 + m_2}$$

Consider a case whereby two connected bodies are such that one is resting on a smooth table and the other is hanging through a smooth pulley which is fixed at the edge of the table as in Figure 2.8.

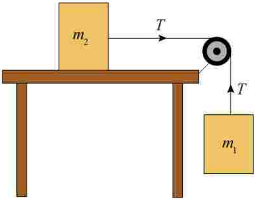


Figure 2.8 Connected bodies

Note that, when a mass of any magnitude m_1 is connected (Figure 2.8), the system will accelerate in the direction of m_1.

The acceleration of the system is

$$a = \frac{m_1}{m_1 + m_2} g$$

and the tension of the string is

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

Suppose the body of mass m_2 is resting on a rough table and the other hanging through a smooth pulley (Figure 2.8). If the motion of the system is in the direction of m_1, then, it follows that,

$$m_1 g - T = m_1 a \quad (2.10)$$

$T - f = m_2 a$, where f is the friction force which always opposes the motion. $T - \mu_k R = m_2 a$ but, $R = m_2 g$

Thus,

$$T - \mu_k m_2 g = m_2 a \quad (2.11)$$

Adding equation (2.10) and (2.11), and rearranging, gives

$$a = \frac{(m_1 - \mu_k m_2) g}{m_1 + m_2} \text{ and } T = \frac{m_1 m_2 (1 + \mu_k) g}{m_1 + m_2}$$

where μ_k is the coefficient of kinetic friction and R is the normal reaction.

Example 2.4

A car with a mass of 600 kg tows a trailer with a mass of 250 kg in a straight line using a rigid tow-bar as shown in Figure 2.9. The resistive force on the car is 200 N and the resistive force on the trailer is 80 N. If the forward thrust produced by the engine of the car is 800 N, find:

(a) The acceleration of the car; and
(b) The tension in the tow-bar.

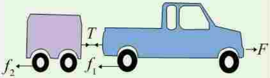


Figure 2.9 A car towing a trailer

Solution

Let m_1 be the mass of a car and m_2 the mass of a trailer. The accelerating force on the car is given by

$$F - T - f_1 = m_1 a \qquad (i)$$

The accelerating force on trailer is given by:

$$T - f_2 = m_2 a \qquad (ii)$$

Adding equation (i) and (ii), and rearranging gives

$$a = \frac{F - f_1 - f_2}{m_1 + m_2}$$

$$a = \frac{800\,\text{N} - 200\,\text{N} - 80\,\text{N}}{600\,\text{kg} + 250\,\text{kg}} = 0.612\,\text{ms}^{-2}$$

Substituting the values of a and f_2 in (ii),

$$T = m_2 a + f_2$$

$$T = 250\,\text{kg} \times 0.612\,\text{ms}^{-2} + 80\,\text{N} = 233\,\text{N}$$

Therefore, acceleration and tension are $a = 0.612\,\text{ms}^{-2}$ and 233 N respectively.

(c) Connected bodies on an inclined plane

Consider two blocks of masses m_1 and m_2 connected by an inextensible string that passes over a smooth pulley as shown in figure 2.10.

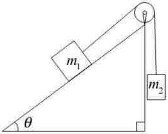


Figure 2.10 Connected bodies on an inclined plane

Suppose $m_2 > m_1$ when the system is released. Mass m_1 will raise up the plane and m_2 will fall vertically downwards. The resultant forces can be shown by a diagram (Figure 2.11).

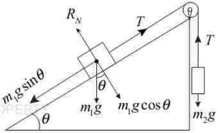


Figure 2.11 Forces acting on connected bodies

Net force along the plane:

$$\sum F_x = T + (-W \sin\theta)$$

$$m_1 a_x = T - m_1 g \sin\theta$$

Since there is no net motion perpendicular to the plane, then

$$R_N = m_1 g \cos\theta$$

Net force on m_2

$$\sum F_y = W_2 - T$$

$$m_2 a_y = m_2 g - T$$

Since the two blocks move as a single system, then they move with the same acceleration. Therefore,

$$a_x = a_y = a$$

Thus, acceleration and tension are:

$$a = \frac{(m_2 - m_1 \sin\theta)g}{m_1 + m_2}$$

and $$T = \frac{m_1 m_2 g(1+\sin\theta)}{m_1 + m_2}$$

Example 2.5

A 10 kg mass on a smooth 30° inclined plane is connected to a 4 kg mass by a light inextensible string passing over a smooth pulley at the top of the plane (Figure 2.12).

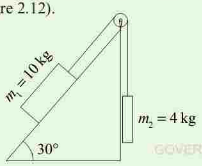


Figure 2.12 Connected masses on an inclined plane

When the bodies are released from rest the 10 kg mass moves down the plane. Find:

(a) The acceleration of the system; and

(b) The tension in the string.

Solution

The free body diagram for m_1 and m_2 is shown in Figure 2.13.

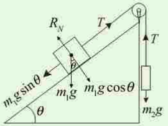


Figure 2.13 Forces acting on connected bodies

Net force parallel to the plane

$$\sum F_x = m_1 g\sin\theta - T$$

$$10\,\text{kg} \times a_x = 10\,\text{kg} \times 9.8\,\text{ms}^{-2} \times \sin 30° - T$$

$$10\text{ kg} \times a_x = 49\text{ kgms}^{-2} - T \qquad (i)$$

Net force on m_2

$$4\,\text{kg} \times a_y = T - 4\,\text{kg} \times 9.8\,\text{ms}^{-2}$$

$$4\text{ kg} \times a_y = T - 39.2\text{ kgms}^{-2} \qquad (ii)$$

But $a_y = a_x = a$ (the acceleration of the system). Solving for a and T; from equation (i) and (ii) gives,

(a) $a = 0.7\,\text{ms}^{-2}$ (b) $T = 42$ N

Example 2.6

Two equal masses connected by a string passing over a frictionless pulley lie on each side of a rough wedge. The wedge faces make angles $\theta_1 = 53°$ and $\theta_2 = 47°$ to the horizontal. Find the coefficient of friction μ for which the masses move at constant velocity.

Solution

Since the masses are equal, the direction of motion will be down the steeper slope. The resultant force on the ascending mass m in the direction of motion is

$$\sum F_1 = mg\sin\theta_1 - T - \mu mg\cos\theta_1 \quad (i)$$

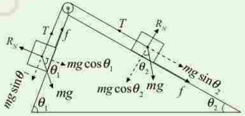


Figure 2.14 Forces acting on connected bodies on inclined rough surfaces

The resultant force on the other mass m is

$$\sum F_2 = T - mg\sin\theta_2 - \mu mg\cos\theta_2 \quad (ii)$$

Motion at constant velocity implies that both forces vanish. Adding equation (i) and (ii), $\sum F_1 = \sum F_2 = 0$, thus,

$$0 = mg(\sin\theta_1 - \sin\theta_2) - \mu mg(\cos\theta_1 + \cos\theta_2).$$

Thus,

$$\mu = \frac{\sin\theta_1 - \sin\theta_2}{\cos\theta_1 + \cos\theta_2}; \text{ and}$$

$$\mu = \frac{\sin 53° - \sin 47°}{\cos 53° + \cos 47°}$$

$$\mu = \frac{0.799 - 0.731}{0.602 + 0.682} = \frac{0.068}{1.284} = 0.05$$

Therefore, the coefficient of friction, μ is 0.05.

(d) Mass ascending or descending in a lift

Consider a person of mass m standing in an accelerating lift. The lift could accelerate upwards (ascending) or downwards (descending). ***Note that,*** there are only two forces acting on the person; the weight downward and the upward reaction of the floor (Figure 2.15).

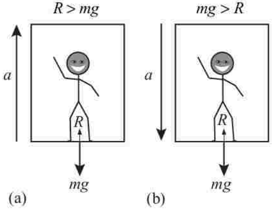


Figure 2.15 Ascending and descending mass in a lift

You are aware of your weight because the ground (or whatever supports us) exerts an upward push on us as a result of the downward push our feet exert on the ground. It is this upward push which makes us feel the force of gravity. When a lift suddenly starts upward the push of the floor on our feet increases and we feel heavier. In fact, we judge our weight from the upward push exerted on us by the floor.

During ascending (Figure 2.15 (a)),

$$R - mg = ma$$

$$R = m(g + a)$$

If our feet are completely unsupported we experience weightlessness. Passengers in a lift that has a continuous downward acceleration equal to g would get no support from the floor since both would be falling with the same acceleration as the lift. There is no upward push on them, and so no sensation of weight is felt. The condition is experienced when we jump off a wall or dive into a swimming pool, as we are then in free fall.

During descending (Figure 2.15 (b)),

$$mg - R = ma; \quad R = m(g - a)$$

When $a = g$, $R = 0$, thus, a person feels weightlessness.

Example 2.7

A person with a mass 100 kg stands in a lift. Find the force exerted by the lift floor on the person when the lift is moving:

(a) Upwards at $3\,\text{ms}^{-2}$; and

(b) Downwards at $4\,\text{ms}^{-2}$.

Solution

(a) Consider the movement upwards,

$$R = mg + ma$$

$R = 100\,\text{kg} \times 9.8\,\text{ms}^{-2} + 100\,\text{kg} \times 3\,\text{ms}^{-2}$,

$R = 1280\,\text{N}$

Therefore, reaction of the floor when the lift ascends is 1280 N.

(b) Consider the movement downwards,

$R = m(g - a)$

$R = 100\,\text{kg} \times \left(9.8\,\text{ms}^{-2} - 4\,\text{ms}^{-2}\right) = 580\,\text{N}$

Therefore, reaction of the floor when the lift descends is 580 N.

2.1.7 Conservation of linear momentum

Consider two objects of masses m_1 and m_2 that are involved in a collision as shown in Figure 2.16.

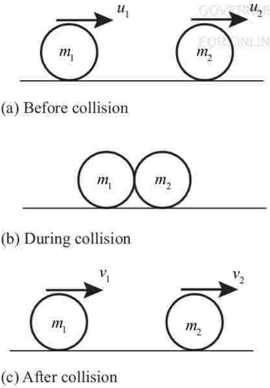


Figure 2.16 Colliding objects

If F_{12} is the force on m_1 due to m_2 and F_{21} is the force on m_2 due to m_1 during the collision and p is linear momentum; then, applying Newton's third law of motion you get

$$F_{12} = -F_{21} \qquad (2.12)$$

Since the two forces act at the same time interval, then, according to Newton's second law of motion, the force acting on mass m_1 is

$$F_{12} = \frac{dp_1}{dt} \qquad (2.13)$$

and the force acting on mass m_2 is

$$F_{21} = -\frac{dp_2}{dt} \qquad (2.14)$$

Substituting equations (2.13) and (2.14) into equation (2.12), you get:

$$dp_1 = -dp_2$$

$$dp_1 + dp_2 = 0$$

$$m_1(v_1 - u_1) + m_2(v_2 - u_2) = 0$$

Rearranging the terms, gives,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad (2.15)$$

Equation (2.15) is called the law of conservation of linear momentum. The law states that, *"Provided that no net external forces act on a system, then, the total momentum of the system before collision is equal to the total momentum of the system after collision"*. The point to be noted is that, individual momenta of various bodies in the system may change, but their total vector sum remains unchanged.

Impulsive forces

A collision is a relatively short lived event whereby two or more objects exert forces on each other. During collision, relatively large forces are exerted on the colliding objects. These forces are called impulsive forces. An impulsive force is not a constant force. It varies from zero (just before collision), increases to maximum (during collision) and decreases to zero (just after collision) (Figure. 2.17).

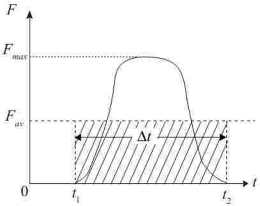


Figure 2.17 Variation of impulsive force with time

Applying Newton's second law of motion to an object involved in collision:

$$F = \frac{dp}{dt}, \text{ hence } Fdt = dp$$

By definition Fdt is the impulse of the force F acting for a duration dt. Thus, impulse of the force is measured by the total change of momentum (dp). Kicking a football and hitting a cricket-ball with a bat are examples of impulse.

When considering systems where mass changes while velocity remains constant, for example liquid emerging from a hosepipe, hovering bird, etc., the following relation can be obtained.

From $\Delta p = \Delta(mv)$, when velocity is constant, $\Delta p = v\Delta m$.

For a small change of momentum and mass, $\Delta p = dp$, $\Delta m = dm$ and $\Delta t = dt$, thus, change in momentum is given as; $dp = vdm$.

Force due to change in mass is obtained as follows;

$$F = \frac{dp}{dt} = v\frac{dm}{dt} \qquad (2.16)$$

But, $dm = \rho Adx$, where A is the area, ρ is the density and dx is the distance. It follows that,

$$F = v\frac{\rho Adx}{dt}$$

Since $\frac{dx}{dt} = v$, then

$$F = \rho Av^2$$

Therefore, the force exerted by the fluid is ρAv^2.

Example 2.8

A ball of mass 2 kg moving horizontally to the right at a speed of 20 m/s strikes a wall and bounces back horizontally at a speed of 20 m/s. If the impact lasted for 0.01 seconds, determine the average force exerted by the ball on the wall during collision.

Solution

$$F = m\frac{dv}{dt} = \frac{m(v-u)}{t}$$

$$F = \frac{2\,\text{kg}(-20\,\text{ms}^{-1} - 20\,\text{ms}^{-1})}{0.01\,\text{s}}$$

$$F = -8000\ \text{kg ms}^{-2}$$

The negative sign indicates a direction in which force F acts. Therefore, the average force exerted by the ball is $8000\,\text{kgms}^{-2}$ or 8000 N.

Example 2.9

Rain falls vertically onto a plane roof, 1.5 m square, which is inclined to the horizontal at an angle of 30°. The rain drops strike the roof with a vertical velocity of $3\ \text{ms}^{-1}$, and a volume of $2.5\times10^{-2}\ \text{m}^3$ of water is collected from the roof in one minute. Assuming that the conditions are steady and that the velocity of the raindrops after impact is zero, calculate:

(a) The vertical force exerted on the roof by the impact of the falling rain; and

(b) The pressure normal to the roof due to the impact of the rain.
(Use density of water is $1000\ \text{kgm}^{-3}$).

Solution

From equation (2.16),

$$F = v\frac{dm}{dt} = v\frac{\rho dV}{dt}$$

where, $\frac{dV}{dt}$ is the rate of change of volume of the water collected from the roof.

$$F = 3\text{ms}^{-1}\times 1000\,\text{kgm}^{-3}\times\frac{2.5\times10^{-2}\,\text{m}^3}{60\text{s}}$$

$$= 1.25\,\text{N}$$

Therefore, the vertical force exerted on the roof is 1.25N .

(b) Pressure normal to the roof, $P = \frac{F_N}{A}$.

Since the roof is inclined at an angle of 30°, the force F_N, normal to the roof is

$$F_N = F\cos\theta$$

$$F_N = 1.25\,\text{N}\times\cos 30°$$

$$= 1.25\,\text{N}\times 0.866 = 1.08\,\text{N}$$

$$P = \frac{1.08\,\text{N}}{1.5\text{m}\times 1.5\text{m}} = 0.48\,\text{Nm}^{-2}$$

Therefore, the pressure normal to the roof is $0.48\,\text{Nm}^{-2}$.

Collisions

When a body in motion interacts with another body (either at rest or in motion), a collision is said to have taken place. When such a collision takes place, velocity of bodies may change. The velocities after the collision can be determined, considering that, during the collision, the law of conservation of linear momentum and energy hold. Collision can be either elastic or inelastic. In this part, you are going to learn collisions that occur in one dimension and two dimensions, and that the colling bodies make contact during a collision.

(a) Elastic collision

Elastic collision is a type of collision in which the total kinetic energy of the colliding bodies is conserved. This means that the total kinetic energy and the momentum of colliding bodies are conserved.

(b) Inelastic collision

Inelastic collision is a type of collision whereby the kinetic energy of the colliding bodies is not conserved but the momentum is conserved. Kinetic energy is not conserved due to the fact that some of the mechanical energy is lost in the collision. Energy is lost in form of heat or is used in deformation of bodies. When it happens that the two colliding bodies stick to each other after collision, it is referred to as perfect inelastic collision. So, they move together with a common velocity after the impact. Hence equation (2.15) becomes $m_1u_1 + m_2u_2 = (m_1 + m_2)v$. where v is the final velocity of the single body after collision. In most cases for this to happen, the objects must stick together and move with common velocity as a single unit. For example, consider a wooden block of mass M swinging from fixed strings of length l. If a bullet of mass m is fired horizontally at velocity u and hits the block, it becomes embedded, then the masses swing in a to and fro motion as shown in Figure 2.18.

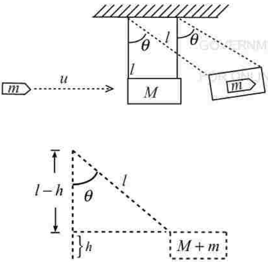


Figure 2.18 Ballistic pendulum

From conservation of linear momentum $mu + M \times 0 = (m + M)v$. Where v is the common velocity of block and bullet after collision. Therefore,

$$v = \frac{mu}{m+M} \qquad (2.17)$$

Also from conservation of energy, total kinetic energy after collision = total potential energy at extreme point

$$\frac{1}{2}(m+M)v^2 = (m+M)gh \qquad (2.18)$$

Substituting equation 2.17 into 2.18 we end up with

$$\left(\frac{mu}{M+m}\right)^2 = 2gh, \text{ then } h = \frac{1}{2g}\left(\frac{mu}{M+m}\right)^2$$

$\cos\theta = 1 - \frac{h}{l}$, substituting the

value of h; $\cos\theta = 1 - \frac{1}{2gl}\left(\frac{mu}{M+m}\right)^2$

$$\theta = \cos^{-1}\left(1 - \frac{1}{2gl}\left(\frac{mu}{M+m}\right)^2\right) \qquad (2.19)$$

The angle θ is the angle of displacement when the bullet hits the block.

Example 2.10

A 10 g bullet is fired with a velocity of $300\,\text{ms}^{-1}$ into a pendulum bob which has a mass of 990 g. How high does the pendulum bob with the bullet embedded swing after the collision?

Solution

From the law of conservation of momentum,

$m_1u_1 + m_2u_2 = (m_1 + m_2)v$

$0.01\,\text{kg} \times 300\,\text{ms}^{-1} = (0.01 + 0.99)\,\text{kg} \times v$

Therefore, $v = 3\ \text{ms}^{-1}$

In conserving mechanical energy at point A and B; $K_A + U_A = K_B + U_B$, where K_A, K_B and U_A, U_B are kinetic and potential energies respectively.

$$\frac{1}{2}(m_1+m_2)v^2+0=0+(m_1+m_2)gh$$

$$h=\frac{1}{2}\frac{v^2}{g},\quad h=\frac{(3\,\text{ms}^{-1})^2}{2\times 9.8\,\text{ms}^{-2}}=0.46\,\text{m}$$

Therefore, the pendulum bob will swing at a height of $0.46\,\text{m}$.

In practice, there is always some loss of kinetic energy during collision. The "lost" energy is converted to other forms of energy such as heat energy, sound energy, light energy, and energy of deformation. The degree of loss of energy during collision can be described by the coefficient of restitution (e). This coefficient depends on the elastic properties and nature of the surfaces of colliding objects. Therefore, it is possible to classify a collision as elastic, inelastic, or perfect inelastic according to the value of e that is associated with it. The coefficient of restitution e is defined as the ratio of the relative velocity of separation to the relative velocity of approach.

$$e=\frac{\text{relative velocity of separation}}{\text{relative velocity of approach}};$$

$$e=\frac{v_2-v_1}{u_1-u_2}$$

From Figure 2.16, u_1-u_2, v_2-v_1 are relative velocities of approach and separation respectively.

When $e=1$; such collision is said to be perfectly elastic. This means the kinetic energy of the system remains constant (conserved). Thus, initial kinetic energy is equal to final kinetic energy

$$\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2 \qquad (2.20)$$

Which can be written as,

$$m_1(u_1+v_1)(u_1-v_1)=m_2(v_2+u_2)(v_2-u_2)$$

Equation (2.15) can be written as,

$$m_1(u_1-v_1)=m_2(v_2-u_2) \qquad (2.21)$$

Placing equation (2.21) into (2.20) gives

$$m_1(u_1+v_1)(u_1-v_1)=(v_2+u_2)m_1(u_1-v_1)$$

Dividing both sides by $m_1(u_1-v_1)$ gives

$$u_1-u_2=v_2-v_1 \qquad (2.22)$$

Equation (2.22) shows that when the kinetic energy of system of colliding objects is conserved, the coefficient of restitution (e) equals to 1. Thus, for perfectly elastic collision in one dimension, the relative velocity of approach before collision is equal to relative velocity of recession (separation) after collision. When $e=0$ the collision is said to be perfectly inelastic collision. Such that, $0=\frac{v_2-v_1}{u_1-u_2}$; this is true only when $v_2=v_1$.

For inelastic collision, velocity of separation is always less than velocity of approach, hence $\frac{v_2-v_1}{u_1-u_2}<1$ or $e<1$. This result shows that, after collision the colliding objects move with a common velocity. In most cases for this to happen, the objects must stick together and move as a single unit.

Example 2.11

A ball of mass $0.1\,\text{kg}$ moving horizontally at a speed of $5\,\text{ms}^{-1}$ collides head on with a ball of $0.3\,\text{kg}$ at rest. Assuming that the collision is perfectly elastic, determine the final velocities of the two balls.

Solution

From the law of conservation of linear momentum $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

But $m_1 = 0.1\text{kg},\ m_2 = 0.3\text{kg},$

$u_1 = 5\text{ms}^{-1},\ u_2 = 0\text{ms}^{-1}$

$$0.1v_1 + 0.3v_2 = 0.5 \qquad (i)$$

For perfect elastic collision, $\frac{v_2 - v_1}{u_1 - u_2} = 1$

$$v_2 - v_1 = 5 \qquad (ii)$$

Solving equation (i) and (ii) gives, $v_1 = -2.5\text{ms}^{-1}$ and $v_2 = 2.5\text{ms}^{-1}$.

It shows that the two balls move with the same velocity in the opposite directions. When collision occurs in two dimensions it is treated by considering horizontal and vertical directions. Consider a particle of mass m_1 colliding elastically with a particle of mass m_2 which initially is at rest. Let u_1 be the initial velocity of mass m_1, move along the x-direction. After the collision, the two particles move with velocities v_1 and v_2 making an angle θ_1 and θ_2 with the x-axis respectively (Figure 2.19).

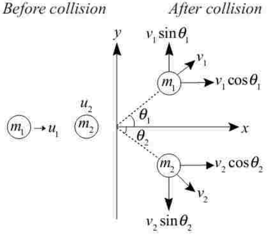


Figure 2.19 Collision system

According to the law of conservation of linear momentum, linear momentum in x-direction is given by

$$m_1u_1 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 \qquad (2.23)$$

For y-direction, since initially the y-component of momentum is zero, then,

$$0 = m_1v_1\sin\theta_1 + m_2(-v_2\sin\theta_2) \qquad (2.24)$$

The law requires that in an elastic collision, the total kinetic energy before collision equals the total kinetic energy after collision.

Thus, noting the initial conditions, $u_2 = 0$.

$$\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad (2.25)$$

The three equations (2.23), (2.24) and (2.25) express the entire contents of the conservation laws. The motion after collision involves four unknowns v_1, v_2, θ_1 and θ_2, while assuming that, the values of m_1, m_2, u_1 and u_2 are known. The four unknown quantities cannot be determined by only three equations. In order to find their values, at least one quantity should be known.

Example 2.12

A small spherical body slides with velocity v and without rolling on a smooth horizontal table and collides with an identical sphere which is initially at rest on the table. After the collision the two spheres slide without rolling away from the point of impact, the velocity of the first sphere being in a direction of 30° to its previous velocity. Assuming that energy is conserved, and that there are no horizontal external forces acting, calculate the speed and direction of travel of the target sphere away from the point of impact.

Solution

From Figure (2.19), considering the horizontal direction,

$$m_1u_1 + m_2u_2 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$$

where $m_1 = m_2, u_2 = 0, u_1 = v$ and $\theta_1 = 30°$

Thus, $v = v_1\cos 30° + v_2\cos\theta_2$

$$\cos\theta_2 = \frac{v - \frac{\sqrt{3}}{2}v_1}{v_2} \qquad (i)$$

Vertical direction,

$$0 = v_1\sin 30° - v_2\sin\theta_2$$

$$\sin\theta_2 = \frac{v_1}{2v_2} \qquad (ii)$$

Squaring (i) and (ii), and adding, you get, $v_2^2 = v_1^2 + v^2 - v_1v\sqrt{3}$ *(iii)*

Since kinetic energy in an elastic collision is conserved, then,

$$\frac{1}{2}m_1v^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$v_2^2 = v^2 - v_1^2 \qquad (iv)$$

Substituting equation (iv) into (iii), and simplifying;

$$v_1 = \frac{\sqrt{3}}{2}v \text{ and } v_2 = \frac{v}{2}$$

The speed of the target sphere is $\frac{v}{2}$.

From equation (ii),

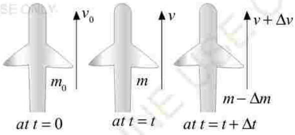
$$\sin\theta_2 = \frac{\frac{\sqrt{3}}{2}v}{2\left(\frac{v}{2}\right)} = \frac{\sqrt{3}}{2},\ \theta_2 = 60°$$

Therefore, the speed and direction of the target sphere are $\frac{v}{2}$ and 60° respectively.

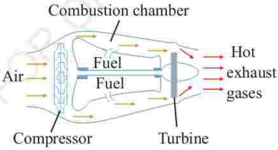
2.1.8 Momentum of systems with varying masses and velocity

There are numerous cases where momentum changes are produced by reaction or explosive forces. An example is a bullet fired from a rifle; initially, the total momentum of the bullet and rifle is zero. From the principle of the conservation of linear momentum, when the bullet is fired the total momentum of bullet and rifle is still zero, since no external force has acted on them. The momentum of the rifle is thus equal and opposite to that of the bullet, resulting into reaction force. This also applies to the systems in which both mass and velocity change, for example in rocket and jet propulsion, sand on conveyor belts and hosepipes.

Consider a system whose mass and velocity change as shown in Figure 2.20(a).



(a) Rocket propulsion



(b) Jet engine

Figure 2.20 Systems of changing mass and velocity

At initial time t, a system of mass m is moving with a velocity v. At a later time $(t+\Delta t)$, the mass is $(m-\Delta m)$, and it moves with a speed $(v+\Delta v)$. The ejected mass Δm moves with a speed u.

As change in mass, velocity and time become very small, then,

$\Delta m = dm,\ \Delta v = dv$ and $\Delta t = dt$

From principle of conservation of linear momentum,

total initial momentum = total final momentum

$$mv = (m-dm)(v+dv)+udm$$

$$mv = mv + mdv - vdm - dmdv + udm$$

But the term $dmdv$ is the product of two small quantities and therefore can be neglected.

$$mv - mv = mdv - vdm + udm$$

$$0 = mdv + (u-v)dm$$

$$mdv = -(u-v)dm$$

The quantity $(u-v)$ gives the relative velocity of the ejected mass with respect to the system denoted as v_{rel}. Thus,

$$dv = -v_{rel}\frac{dm}{m} \qquad (2.26)$$

The above equation can be applied in all systems of changing mass.

Rockets propulsion

One of the interesting illustrations where the principle of conservation of linear momentum is applied is that of rocket propulsion. Here the rocket is propelled as a reaction of the ejected gas produced during the combustion of the fuel. In this case the mass of the rocket goes on changing constantly as a result of ejection of gases formed during combustion of the fuel. The mass of the rocket constantly decreases as the gases get ejected, its acceleration and velocity go on increasing all the time (Figure 2.20(a)).

The velocity of a rocket in outer space (negligible gravitational pull of the earth) can be obtained using equation (2.26).

$$\int_{v_i}^{v_f} dv = \int_{m_i}^{m_f} -v_{rel}\frac{dm}{m},\ \left[v\right]_{v_i}^{v_f} = -v_{rel}\left[\ln m\right]_{m_i}^{m_f}$$

$$v_f = v_i + v_{rel}\ln\left(\frac{m_i}{m_f}\right)$$

The thrust on a rocket is a recoil (reaction) force exerted on the rocket by the exhaust gases. The expression for thrust can be obtained by using equation (2.26).

$$dv = -v_{rel}\frac{dm}{m},\ mdv = -v_{rel}dm$$

Divide by dt throughout the equation to get, $m\frac{dv}{dt} = -v_{rel}\frac{dm}{dt}$ (2.27)

The thrust on a rocket is $-v_{rel}\frac{dm}{dt}$. This is the force that propels the rocket forward.

The negative sign in equations (2.26) and (2.27) shows an increase in speed (positive $\frac{dv}{dt}$) corresponding to a decrease in rocket mass (negative $\frac{dm}{dt}$).

Note that, when the rocket is under the influence of earth's gravity, equation (2.27) is modified to become.

$$m\frac{dv}{dt} = -v_{rel}\frac{dm}{dt} - mg \qquad (2.28)$$

Rearranging equation (2.28),

$$\frac{dv}{dt}=\frac{-v_{rel}}{m}\frac{dm}{dt}-g,\ dv=-v_{rel}\frac{dm}{m}-gdt$$

$$\int_{v_i}^{v_f}dv=-\int_{m_i}^{m_f}v_{rel}\frac{dm}{m}-\int_0^t g\,dt$$

$$v_f=v_i+v_{rel}\ln\left(\frac{m_i}{m_f}\right)-gt \qquad (2.29)$$

Thus final velocity is given by equation (2.29), and acceleration is given by

$$a=\frac{v_{rel}}{t}\ln\left(\frac{m_i}{m_f}\right)-g,$$

This applies where the force of gravitation is experienced.

Jet propulsion

A jet engine uses the surrounding air for its oxygen supply and so is unsuitable for space travel. The compressor draws in air at the front, compresses it, fuel is injected and the mixture burns to produce hot exhaust gases which escape at high speed from the rear end of the engine. These cause forward propulsion and drive the turbine (Figure 2.20 (b)) which in turn rotates the compressor and hence the jet takes off. Suppose air of mass m_a enters the front end of the jet with incoming velocity v_i which then mixes with fuel of mass m_f in the combustion chamber. The mixture of air and fuel burn and the exhaust (burnt) gases will be ejected with velocity v_0 through the rear end of the jet.

The initial linear momentum P_i of incoming air is given by;

$$P_i=m_a v_i$$

The final linear momentum P_f of outgoing burnt gases is given by;

$$P_f=\left(m_a+m_f\right)v_0$$

The change in linear momentum is given by $\quad dP=\left(m_a+m_f\right)v_0-m_a v_i$

The force F exerted by the burnt gases equals to the rate of change of linear momentum $\frac{dP}{dt}$, i.e.,

$$F=\frac{\left(m_a+m_f\right)v_0}{t}-\frac{m_a v_i}{t}$$

$$F=\left(\frac{m_a}{t}+\frac{m_f}{t}\right)v_0-\left(\frac{m_a}{t}\right)v_i$$

where $\frac{m_a}{t}$ and $\frac{m_f}{t}$ are the rates at which air enters the jet and the fuel burns respectively. The force exerted by the burnt gases to the rear end of the jet (from left to right) has the same magnitude as that the rear end of the engine produces from right to left. In turn and with reference to Newton's third law of motion, the rear end of the jet produces the same force but in opposite direction which makes the jet engine to take off.

Example 2.13

A jet aircraft is travelling at $225\ \text{ms}^{-1}$ in a horizontal flight. The engine takes in air at a rate of 85kgs^{-1} and burns fuel at a rate of 3kgs^{-1}. If the exhaust gases are ejected at 650ms^{-1} relative to the aircraft; find the thrust of the jet engine.

Solution

Velocity of incoming air (velocity of

jet aircraft) $v_i = 225\,\text{ms}^{-1}$ and velocity of outgoing burnt gases $v_0 = 650\,\text{ms}^{-1}$. The rate at which air enters the front end of the jet, $\frac{m_a}{t} = 85\,\text{kgs}^{-1}$ and the rate at which fuel burns, $\frac{m_f}{t} = 3\,\text{kgs}^{-1}$.

Then thrust on the jet engine is obtained from;

$$F = \left(\frac{m_a}{t} + \frac{m_f}{t}\right)v_0 - \left(\frac{m_a}{t}\right)v_i$$

$$= 88\,\text{kgs}^{-1} \times 650\,\text{ms}^{-1} - \left(85\,\text{kgs}^{-1}\right) \times 225\,\text{ms}^{-1}$$

$$= 38075\ \text{N}$$

Therefore, thrust of the jet engine is 38075 N.

Example 2.14

A rocket moving in free space has a speed of $3.0 \times 10^3\,\text{ms}^{-1}$ relative to the earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of $5.0 \times 10^3\,\text{ms}^{-1}$ relative to the rocket.

(a) What is the speed of the rocket relative to the earth once the rocket's mass is reduced to one-half its mass before ignition?

(b) What is the thrust on the rocket if it burns fuel at $0.77\ \text{kgs}^{-1}$?

Solution

(a) $v_f - v_i = v_{rel} \ln\left(\frac{m_i}{m_f}\right)$,

$$v_f = v_i + v_{rel} \ln\left(\frac{m_i}{m_f}\right)$$

$$v_f = 3 \times 10^3\ \text{ms}^{-1} + 5.0 \times 10^3\ \text{ms}^{-1} \ln\left[\frac{m_i}{0.5m_i}\right]$$

$$= 6465.7\ \text{ms}^{-1}$$

Therefore, speed of the rocket relative to the earth is $6466\,\text{ms}^{-1}$.

(b) Thrust, $F = v_{rel}\frac{dm}{dt}$,

$$F = 5.0 \times 10^3\,\text{ms}^{-1} \times 0.77\,\text{kgs}^{-1} = 3850\,\text{N}$$

Therefore, the thrust on rocket is 3850 N.

Example 2.15

Two fire fighters must apply a total force of 600 N to a steady hose that is discharging water at 3600 litres/min. Estimate the speed of the water as it exits the nozzle.

Solution

$$F = v_{rel}\frac{dm}{dt},\quad v_{rel} = \frac{F}{\frac{dm}{dt}} = \frac{F}{\frac{\rho dV}{dt}}$$

$$v_{rel} = \frac{600\,\text{N}}{1000\,\text{kgm}^{-3} \times 0.06\,\text{m}^3\text{s}^{-1}} = 10\,\text{ms}^{-1}$$

Therefore, the speed of water is $10\,\text{ms}^{-1}$.

2.1.9 Applications of Newton's Laws of Motion

Newton's laws of motion are widely applied and experienced in daily life. Some of the applications are:

As one turns a corner when travelling in a car, his/her body keeps moving in a straight line while the car turns the corner. It is as if he/she is pushed into the side of the seat. Also, when a moving bus stops suddenly, the passengers feel a jerk in a forward direction. This is because the upper part of the passenger's body tends to remain in continuous motion while the lower part of the body comes at rest suddenly. That is why it is advised to fasten a seat belt when travelling in a car to ensure safety throughout the journey.

When beating a dusty coat with a stick, the coat comes in motion while the dust particles remain in a state of rest and thus get removed.

When you jump off a small rowing boat in water, you will push yourself forward towards the water. The same force you use to push forward will make the boat move backwards.

When you walk on the ground you press the ground backward with your feet as a reaction and the ground gives you an equal and opposite impulse forward which sets you in motion.
If a ball is kicked in air, it will rise in air and eventually fall back to the ground. This is due to air resistance and pull of gravity.

When a rocket propels, it pushes out a burning gas (action) and exhausted gas pushes on the rocket (reaction) with an equal thrust but opposite in direction. The net rate of change in linear momentum produces the forward acceleration of a rocket.

Exercise 2.1

1. Explain why;
   (a) when you push against a wall, you feel like the wall is pushing against you.
   (b) when one end of a horizontal plane is lifted, a certain stage (angle of repose) is reached when an object resting on it begins to slide down the plane.
   (c) if you push on a heavy box which is at rest, you must apply some force to start its motion. However, once the box is sliding, you can apply a smaller force to maintain its motion.
2. Give reasons as to why;
   (a) when a train suddenly starts, the passengers standing in a compartment tend to fall backwards.
   (b) when brakes are applied on the train, passengers inside it tend to fall forward.
   (c) a passenger sitting in the rear of a bus claims that she was injured as the driver slammed on the brakes, causing a suitcase to come flying towards her from the front of the bus. If you were the judge in this case, what disposition would you make? Why?
3. Analyse the motion of a stone dropped in water in terms of its speed and acceleration as it falls. Assume that a resistive force is acting on the stone that increases as the velocity of the stone increases.
4. A mass $m_1 = 1\text{kg}$ lies on a smooth table and is attached by a string

and a frictionless pulley to a mass $m_2 = 0.01\,kg$ hanging from the edge of the table (Figure 2.21). The system is released from rest.

(a) Calculate the distance the mass m_1 moves across the table in the first 10 seconds.

(b) How long will it take for the mass m_1 to travel 1 m from its initial position?

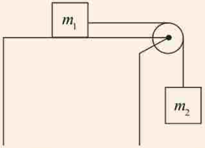


Figure 2.21 Connected masses on a pulley

5. (a) Can a body be in equilibrium when only one force acts on it? Explain your answer.

(b) A person standing in a lift holds a spring balance with a load of 5 kg suspended from it. What would be the reading of the balance when the lift is descending with an acceleration of $3.8\,ms^{-2}$?

6. A book of mass M rests on a long table with a piece of paper of mass $m = 0.1M$ in between. The coefficient of friction between all surfaces is $\mu_s = 0.1$. The paper is pulled with horizontal force P (Figure 2.22).

(a) What is the minimum value of P required to cause any motion?

(b) With what force must the paper be pulled in order to extract it from between the book and the table?

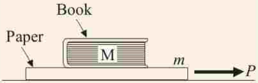


Figure 2.22 A book resting on a table

7. (a) Explain why the load on the rear wheels of a motor car increases when the vehicle is accelerates.

(b) Figure 2.23 shows a painter in a crate which hangs alongside a building. When the painter who weighs 1000 N pulls on the rope, the force that the painter exerts on the floor of the crate is 450 N. If the crate weighs 250 N, find the acceleration of the system.

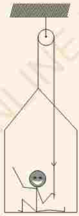


Figure 2.23 A painter in a crate

8. A box of mass 2 kg lies on a rough horizontal floor, coefficient of friction is 0.2. A light string is attached to the box in order to pull it

across the floor. If the tension in the string is T, find the tension that must be exceeded for motion to occur if the string is

(a) horizontal.

(b) $45°$ above the horizontal.

(c) $45°$ below the horizontal.

9. (a) A jet engine on a test bed takes in $20\,\text{kg}$ of air per second at a velocity of $100\,\text{ms}^{-1}$ and burns $0.8\,\text{kg}$ of fuel per second. After compression and heating, the exhaust gases are ejected at velocity of $500\,\text{ms}^{-1}$ relative to the air craft. Calculate the thrust of the engine.

(b) A fire engine pumps water at such a rate that the velocity of water leaving the nozzle inclined at angle $60°$ to the horizontal is $15\,\text{ms}^{-1}$. Calculate the pressure exerted on the wall, assuming the rebound of the water is neglected and $1\,\text{m}^3$ of water has a mass of $1000\,\text{kg}$.

(c) A bullet is fired from a gun with a horizontal velocity of $500\,\text{ms}^{-1}$. The mass of the gun is $4\,\text{kg}$ and the mass of the bullet is $50\,\text{g}$. Find the initial speed of recoil of the gun and the gain in kinetic energy of the system.

10. (a) (i) What are head-on and oblique collisions?

(ii) In perfectly inelastic collisions between two objects, there are events in which all of the original kinetic energy is transformed to forms other than kinetic. Give an example of such an event.

(b) By using your own environment, describe the applications of the principle of conservation of linear momentum in daily life situations.

(c) A ball of mass m moving at $5\,\text{ms}^{-1}$ collides with a ball of mass $2\,\text{kg}$ which is at rest. After collision, the first ball acquired the velocity of $2\,\text{ms}^{-1}$ at an angle of $50°$ relative to its original direction. What is the velocity of the second ball after collision?

2.2 Projectile motion

When an object is thrown in air, its motion is influenced entirely by gravity and air resistance. The motion of the object is a two dimensional motion because as the object moves in air, it covers both horizontal and vertical displacement. The horizontal component does not have any acceleration, hence constant magnitude. The vertical component, however, has acceleration equal to the acceleration due to gravity, but, in opposite direction and hence, its magnitude is different at different points and is directed vertically downwards. Therefore, projectile motion is a two dimensional motion of an object in air which is influenced entirely by gravity and air resistance.

Examples of projectile motion include the motion of a bomb released by a moving plane, a thrown stone, a bullet fired from a gun, a ball thrown in any direction and an athlete doing a high jump. In this section, you will learn the concept of projectile motion, derivation of mathematical relations and applications of projectile motion.

2.2.1 Motions of projectile

Since a projectile moves horizontally as well as vertically, two coordinates are required to specify its position at any time. Let us discuss the salient features of projectile motion in which air resistance is negligible.

Figure 2.24 shows a projectile projected from the origin O with initial velocity u at an angle θ with horizontal (x-axis). The projectile rises to the maximum height (H) at point B and then descends, and finally strikes the ground at point P.

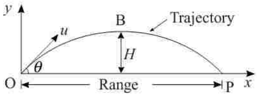


Figure 2.24 Motion of a projectile

The angle θ is called the angle of projection and the horizontal distance OP is called the range of the projectile. Mathematical analysis of the motion will help to define important parameters of the motion. In the analysis, x and y coordinates are treated separately. The x-component of acceleration due to gravity is zero and the y-component is constant and its value is $-g$.

From figure (2.24), vertical and horizontal velocities, and displacements may be obtained using equations of motion.

From first equation of motion,

$$v_y = u_y + a_y t$$

$$v_y = u\sin\theta - gt \qquad (2.30)$$

Also, $v_x = u\cos\theta - 0\times t$

$$v_x = u\cos\theta \qquad (2.31)$$

Therefore, equations (2.30) and (2.31) show the velocities for vertical and horizontal components respectively.

It should be noted that, the horizontal component v_x of velocity is constant throughout the motion because there is no horizontal acceleration (i.e. $a_x = 0$).

Similarly, vertical and horizontal displacements can be obtained using second equation of motion;

$$s_y = u_y t + \frac{1}{2}(a_y)t^2$$

$$s_y = (u\sin\theta)t - \frac{1}{2}gt^2 \qquad (2.32)$$

$$s_x = (u\cos\theta)t + \frac{1}{2}\times 0\times t^2$$

$$s_x = (u\cos\theta)t \qquad (2.33)$$

Therefore, equations (2.32) and (2.33) show the vertical and horizontal displacements respectively.

Likewise, vertical displacement s_y can be obtained from third equation of motion,

$$v_y^2 = u_y^2 + 2a_y s_y$$

$$v_y^2 = u^2\sin^2\theta - 2gs_y$$

$$s_y = \frac{u^2\sin^2\theta - v_y^2}{2g}$$

2.2.2 Parameters of projectile motion

Projectile motion consists of various parameters, which include: trajectory, time of flight, maximum height, time to reach maximum height, range and velocity of projectile at any point.

(a) Trajectory

Trajectory is the path traced by a projectile from the point of projection O, to the point of landing P, (Figure 2.24).

Trajectory equations

From equations (2.33) and (2.32),

$$s_x = (u\cos\theta)t;\ t = \frac{s_x}{u\cos\theta}$$

$$s_y = (u\sin\theta)t - \frac{1}{2}gt^2$$

$$s_y = u\sin\theta\left(\frac{s_x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{s_x}{u\cos\theta}\right)^2$$

$$s_y = s_x\tan\theta - \frac{1}{2}\frac{g{s_x}^2}{u^2\cos^2\theta} \qquad (2.34)$$

Let $s_y = y,\ s_x = x,\ \frac{-g}{2u^2\cos^2\theta} = a,$

$\tan\theta = b$

Equation (2.34) can be reduced to

$y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta}$, i.e., $y = bx + ax^2$.

This is the equation of parabola.

Thus, equation (2.34) is the trajectory equation which is parabolic in nature.

(b) Time of flight

Time of flight is the total time taken by a projectile from the instant when it is projected to the time when it strikes the point in a horizontal plane passing through the point of projection, i.e., from point O to P (Figure 2.24). From the same figure, let B be the point where the vertical component of velocity becomes zero. Hence, at this point equation (2.30),

$$v_y = 0$$

$$0 = u\sin\theta - gt,\ t = \frac{u\sin\theta}{g}$$

This time t is the time taken by a projectile to reach maximum height, i.e. from O to B in (Figure 2.24). The total time taken by a projectile to reach the point in a horizontal plane passing through the point of projection is double of this time t. Hence, the total time of flight is given by

$$T = 2t = \frac{2u\sin\theta}{g} \qquad (2.35)$$

This can alternatively be obtained by considering that the total vertical distance travelled by the particle is zero. Therefore, from equation (2.32),

$$s_y = 0$$

$$0 = (u\sin\theta)T - \frac{1}{2}gT^2,\ T = \frac{2u\sin\theta}{g}$$

(c) Maximum height

Maximum height is the maximum vertical distance attained by the projectile above the point of projection. It is obtained using the relation;

$$v_y^2 = u_y^2 + 2a_y s_y$$

At maximum height (point B)

(Figure 2.24), $v_y = 0$ and $s_y = H$, thus,

$$0 = (u\sin\theta)^2 - 2gH,\ H = \frac{u^2\sin^2\theta}{2g}$$

Note that, maximum height H_{max} is obtained when $\theta = 90°$ and $\sin^2\theta = 1$.

Thus,

$$H_{max} = \frac{u^2}{2g}$$

(d) Horizontal range

Horizontal range R is the horizontal distance covered by the projectile from projection point O to the landing point P (Figure 2.24). From equation (2.33),

$$s_x = (u\cos\theta)t$$

At the point of landing, the time spent in air by a projectile is T and the horizontal distance is R. Therefore,

$$R = (u\cos\theta)T \qquad (2.36)$$

Substituting equation (2.35) in equation (2.36) gives,

$$R = u\cos\theta\left(\frac{2u\sin\theta}{g}\right),\ R = \frac{2u^2\cos\theta\sin\theta}{g}$$

Using the trigonometric identity,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$2\sin\theta\cos\theta = \sin 2\theta$ gives

$$R = \frac{u^2\sin 2\theta}{g} \qquad (2.37)$$

Note that, the maximum horizontal range is obtained when $\sin 2\theta = 1$ or $\theta = 45°$ and therefore,

$$R_{max} = \frac{u^2}{g}$$

Hence, a projectile which is projected making an angle 45° with the horizontal has a maximum range.

(e) Velocity of a projectile at any point

The velocity v of a projectile at any point along the trajectory is obtained by adding the horizontal component and the vertical component of its velocity (Figure 2.25).

$$v^2 = v_x^2 + v_y^2$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

On simplifying gives,

$$v = \sqrt{u^2 + g^2t^2 - 2ugt\sin\theta}$$

The direction of a projectile at any point is given by

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Example 2.16

A ball is thrown with a speed of $17\,\text{ms}^{-1}$ at a projection angle of 58° above the horizontal. Assuming the point of return of the ball is at the same horizontal level as the point of projection, determine;

(a) time of flight.

(b) the range.

(c) maximum height.

(d) time taken to reach maximum height.

Solution

(a) Time of flight, $T = \dfrac{2u\sin\theta}{g}$

$$T = \frac{2\times 17\,\text{ms}^{-1}\times\sin 58^0}{9.8\,\text{ms}^{-2}} = 2.9\,\text{s}$$

(b) Horizontal range, $R = \dfrac{u^2\sin 2\theta}{g}$

$$R = \frac{\left(17\,\text{ms}^{-1}\right)^2 \times \sin\left(2\times 58^o\right)}{9.8\,\text{ms}^{-2}} = 26.5\,\text{m}$$

(c) Maximum height, $H = \dfrac{u^2 \sin^2\theta}{2g}$

$$H = \frac{\left(17\,\text{ms}^{-1}\right)^2 \times \sin^2 58^o}{2\times 9.8\,\text{ms}^{-2}} = 10.6\,\text{m}$$

(d) Time to reach the maximum height,

$$t = \frac{u\sin\theta}{g}$$

Therefore, $t = \dfrac{17\,\text{ms}^{-1} \times \sin 58^o}{9.8\,\text{ms}^{-2}} = 1.5\,\text{s}$

2.2.3 Special cases of projectile motion

So far we have discussed a projectile projected from the ground and the point of striking the ground is on the same plane as the projection point. There are other cases where either projection point or striking point is not on the ground, for example, projectiles fired from a point above the ground and those fired on an inclined plane.

(a) Projectiles fired from a point above the ground

Projectiles fired from a point above the ground may be horizontally or vertically or at an angle θ with the horizontal.

(i) Consider a projectile fired horizontally with velocity u at a height h above the ground (Figure 2.25). The horizontal velocity remains constant throughout the projectile motion. The downward velocity is zero at the time of firing the projectile and keeps on increasing uniformly with time till the projectile hits the ground.

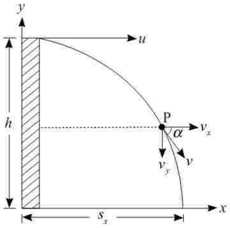


Figure 2.25 Projection above the ground

For horizontal motion ($\theta = 0°$)

$$u_y = u\sin 0° = 0$$

$$u_x = u\cos 0° = u$$

For vertical motion

$v_y = -gt$, negative sign indicates downward velocity.

From equation (2.32), the vertical displacement s_y, from the point of projection to point P (Figure 2.25), is given by

$$s_y = u\sin\left(0°\right)t - \frac{1}{2}gt^2$$

$s_y = -\dfrac{1}{2}gt^2$, negative sign indicates downward displacement.

Time taken by a projectile to hit the ground is obtained from the relation;

$$v_y^2 = u_y^2 + 2a_y s_y$$

$$\left(-gt\right)^2 = \left(u\sin\left(0°\right)\right)^2 - 2gh$$

where $u_y = u\sin(0^\circ)$, $v_y = u\sin(0^\circ) - gt$

and $s_y = h$

$$t = \sqrt{\frac{2h}{g}}$$

Therefore, time taken by a projectile to hit the ground is given by; $t = \sqrt{\frac{2h}{g}}$

Horizontal distance from the point of projection to the point where the projectile hits the ground is given by;

$$s_x = (u\cos\theta)t$$

$$s_x = u\cos(0^\circ) \times \sqrt{\frac{2h}{g}},\ s_x = u\sqrt{\frac{2h}{g}}$$

(ii) Consider a projectile fired from height h above the ground at an angle θ with a velocity u (Figure 2.26).

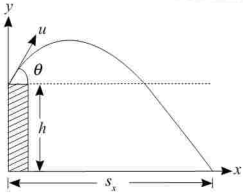


Figure 2.26 Projection at an angle

Motion along the vertical;

$$v_y = u\sin\theta - gt;$$

$$y = h + (u\sin\theta)t - \frac{1}{2}gt^2$$

Motion along horizontal

$$v_x = u\cos\theta;\ \ x = (u\cos\theta)t$$

Time of flight is obtained from the vertical motion equation.

$$0 = h + (u\sin\theta)t - \frac{1}{2}gt^2 \qquad (2.38)$$

Rearranging equation (2.38) and solving gives,

$$t = \frac{u\sin\theta \pm \sqrt{u^2\sin^2\theta + 2hg}}{g}$$

Since $\sqrt{u^2\sin^2\theta + 2hg} > u\sin\theta$, then, time taken by a projectile to reach the ground is given by;

$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2hg}}{g}$$

$$x = (u\cos\theta)t,$$

$$x = u\cos\theta\left(\frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}\right)$$

Example 2.17

A stone is thrown at an angle 45° with the horizontal, from the top of a building 30 m high with an initial velocity of $20\,\text{ms}^{-1}$. Calculate;

(a) time of flight

(b) horizontal distance at which the stone strikes the ground

(c) velocity and direction with which the stone strikes the ground.

Solution

(a) Using equation (2.38),

$$-30\,\text{m} = 20\,\text{ms}^{-1} \times \sin 45^\circ \times t - \frac{1}{2} \times 9.8\,\text{ms}^{-2} \times t^2$$

Solving for t gives $t = 4.3\,\text{s}$ or $t = -1.4\,\text{s}$ but time cannot be negative, hence the time of flight is $t = 4.3\,\text{s}$.

(b) Horizontal distance, $s_x = (u\cos\theta)t$.

$$s_x = 20\,\text{ms}^{-1} \times \cos 45° \times 4.3\,\text{s} = 60.8\,\text{m}$$

(c) Velocity is given by the relation

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v_x = u\cos\theta$$

$$v_x = 20\,\text{ms}^{-1} \times \cos 45° = 14.14\,\text{ms}^{-1}$$

$$v_y = u\sin\theta - gt$$

$$v_y = 20\,\text{ms}^{-1} \times \sin 45° - 9.8\,\text{ms}^{-2} \times 4.3\,\text{s}$$

$$= -28\,\text{ms}^{-1}$$

$$v = \sqrt{\left(14.14\,\text{ms}^{-1}\right)^2 + \left(-28\,\text{ms}^{-1}\right)^2}$$

$$= 31.37\,\text{ms}^{-1}$$

This is the magnitude of the velocity with which the stone strikes the ground.

The direction α of the velocity is calculated using the relation,

$$\tan\alpha = \frac{v_y}{v_x},$$

$$\alpha = \tan^{-1}\left(\frac{-28\,\text{ms}^{-1}}{14.14\,\text{ms}^{-1}}\right) = -63.21°$$

The direction with which the stone strikes the ground is 63.21° below the horizontal.

(b) Projectile on an inclined plane

Projectile motion on an inclined plane is one of the various types of projectile motion. The main distinguishing aspect is that, points of projection and point of striking the ground are not on the same plane.

This type of motion can be discussed in terms of a new pair of coordinates, with x-axis along the incline and y-axis perpendicular to the plane. Figure 2.27 shows a projectile fired initially with velocity u at an angle θ with the horizontal. The inclined plane is at an angle α with the horizontal, therefore the firing angle with the inclined plane is $(\theta - \alpha)$.

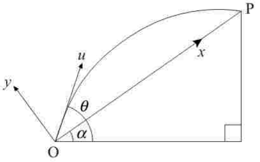


Figure 2.27 Projectile up an inclined plane

Important characterizing aspects of projectile motion up an inclined plane includes:

(i) Coordinate x along the inclined plane and y perpendicular to the inclined plane;

(ii) Angle of projection;

(iii) Range s_x measured along the incline from point O to P;

(iv) Components of initial velocity $u_x = u\cos(\theta - \alpha$ and $u_y = u\sin(\theta - \alpha)$; and

(v) Components of acceleration $a_x = -g\sin\alpha$ and $a_y = -g\cos\alpha$.

When dealing with projectile motion on an inclined plane the common parameters of interest as usual are time of flight, range of flight, maximum range and angle of projection.

Time of flight can be derived from the distance equation $s_y = u_y t + \frac{1}{2}a_y t^2$.

Where $s_y = 0$ at the time of flight.

Thus, $0 = u_y T + \frac{1}{2}a_y T^2$. Substituting expressions for u_y and a_y gives,

$$T = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$$

Consequently, the range of flight is found from the relation, $R = u_x T + \frac{1}{2}a_x T^2$.

Substituting values for u_x, a_x and T,

$$R = u\cos(\theta-\alpha)\left(\frac{2u\sin(\theta-\alpha)}{g\cos\alpha}\right) - \left(\frac{1}{2}g\sin\alpha\right)\left(\frac{2u\sin(\theta-\alpha)}{g\cos\alpha}\right)^2 \quad (2.39)$$

Simplifying equation (2.39),

$$R = \frac{u^2}{g\cos^2\alpha}\left(\sin\left(2\theta-\alpha\right) - \sin\alpha\right)$$

The range is maximum when,

$\sin\left(2\theta-\alpha\right) = 1$, i.e., $2\theta - \alpha = \frac{\pi}{2}$ or

$$\theta = \frac{\pi}{4} - \frac{\alpha}{2}$$

Therefore, maximum range,

$$T = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$$

Example 2.18

A projectile is thrown from the base of an incline of angle $30°$. What should be the angle of projection, as measured from the horizontal direction so that range on the incline is maximum?

Solution

From the relation

$$R = \frac{u^2}{g\cos^2\alpha}\left(\sin\left(2\theta-\alpha\right) - \sin\alpha\right)$$

The range is maximum when

$$\sin\left(2\theta-\alpha\right) = 1 \rightarrow 2\theta = \alpha + 90°$$

$$\theta = \frac{90° + 30°}{2};\ \theta = 60°$$

Therefore, the maximum angle of projection is $60°$.

2.2.4 Applications of projectile motion

Projectile motion is widely applied in every day life. Some applications are as follows.

In football, the amount of force a footballer applies to the ball (how hard the individual kicks) will determine the initial velocity and how fast the ball will travel. The angle at which the footballer kicks the ball determines the height and distance travelled. For example, if the ball is kicked at an angle of $45°$ it will get the maximum range. Projectile motion is very closely associated almost with all types of sports involving jumping or throwing of objects in air.

A soldier who has to target at a particular location must calculate the velocity and angle of throw for the bomb to hit the target. In addition, projectile motion is applied when using fire extinguishers. People who have to extinguish fire at a long distance position the water hose at a certain angle in order to hit the fire.

Projectile motion is also used when food packages are dropped from helicopters or aeroplanes in times of disasters. The distance from which the packages are dropped is important so that these packages may fall in appropriate locations.

Exercise 2.2

1. (a) A bomb is released by a plane flying horizontally and the other one is released by a stationary plane like a helicopter at the same altitude and they reach the ground at the same time. Why?
   (b) A ball is projected horizontally from the top of a building. One second later another ball is projected horizontally from the same point with the same velocity. At what point in the motion will the balls be closest to each other? Will the first ball always be traveling faster than the second ball? What will be the time interval between them when the balls hit the ground? Can the horizontal projection velocity of the second ball be changed so that the balls arrive at the ground at the same time?
2. (a) Does a rocket flight depict projectile motion?
   (b) Two projectiles are thrown with the same magnitude of initial velocity, one at an angle θ with respect to the level ground and the other at angle $(90° - \theta)$. Both projectiles will strike the ground at the same distance from the projection point. Will both projectiles be in the air for the same time interval?
3. (a) What factors determine the span of the jump for one who jumps in a long jump?
   (b) A projectile is fired at an angle of 30° from the horizontal with some initial speed. At what other angle does firing of the projectile results in the same horizontal range if the initial speed is the same in both cases? Neglect air resistance
   (c) The maximum range of a projectile occurs when it is launched at an angle of 45.0° with the horizontal, if air resistance is neglected. If air resistance is not neglected, will the optimum angle be greater or less than 45.0°? Explain.
4. (a) What are the domestic advantages of knowing about projectile motion?
   (b) Draw a free-body diagram for each of the following:
   (i) A projectile in motion in the presence of air resistance;
   (ii) A rocket leaving the launch pad with its engines operating; and
   (iii) An athlete running along a horizontal track.
5. (a) Determine the two possible angles of projection that produce a range of 60 m if the initial velocity of projection is 30 ms^{-1}.
   (b) A man can just throw a stone to a horizontal distance of 75m. With what velocity does he throw it and how long is it in the air?
6. (a) Describe how to throw a projectile so that:

(i) It has zero speed at the top of its trajectory; and

(ii) It has nonzero speed at the top of its trajectory.

(b) A projectile is projected from the foot of an incline of angle 30° with a velocity 30 ms^{-1}. The angle of projection as measured from the horizontal is 60°. Find its speed when the projectile is parallel to the incline.

7. (a) Determine which of the following moving objects obey the equations of projectile motion.

(i) A ball is thrown in an arbitrary direction.

(ii) A jet airplane crosses the sky with its engines thrusting the plane forward.

(iii) A rocket leaves the launch pad

(iv) A rocket moving through the sky after its engines have failed

(v) A stone is thrown under water.

(b) Two projectiles are thrown with the same speed u, but at different angles from the base of an inclined surface of angle "α". The angle of projection with the horizontal is θ for one of the projectiles. If the two projectiles reach the same point on incline, determine the ratio of times of flights for the two projectiles.

8. (a) A ball is held in a person's hand.

(i) Identify all the external forces acting on the ball and the reaction to each.

(ii) If the ball is dropped, what force is exerted on it while it is falling? Identify the reaction force in this case. (Neglect air resistance.)

(b) A projectile is launched with horizontal and vertical velocity components u and v respectively. Show that its trajectory is a parabola and that the maximum height and the range (on level ground) are $H = \frac{v^2}{g}$, $R = \frac{2uv}{g}$ respectively.

9. (a) A body falls freely from rest to the ground a distance h below. In the last one second of its flight it falls a distance $\frac{h}{2}$. Find the value of h.

(b) A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.

10. (a) A projectile is launched at some angle to the horizontal with some initial speed v_i and air resistance is negligible. Is the projectile a freely falling body? What is its acceleration in the vertical direction? What is its acceleration in the horizontal direction?

(b) A ball is projected with a velocity v at an angle θ to the horizontal. It passes through a vertical point y and horizontal point x. If R is the horizontal range, prove that $\tan\theta = \frac{y}{x}\left(\frac{R}{R-x}\right)$.

Revision exercise 2

1. A person is standing in a lift while holding a briefcase. He lets the briefcase go off but it does not fall on the floor of the lift. Describe the motion of the lift.
2. As a rocket is fired from a launching pad, its speed and acceleration increase with time as its engines continue to operate. Explain why this occurs even though the thrust of the engines remains constant.
3. Identify the action-reaction pairs in the following situations;
   (a) a man takes a step
   (b) a snowball hits a person on the back
   (c) a baseball player catches a ball
   (d) a gust of wind strikes a window.

4. (a) A person sitting on the front seat of a stationary car tries to move it by pushing against the dashboard. Will the car move? Explain your answer.
   (b) A truck loaded with sand accelerates along a highway. If the driving force on the truck remains constant, what happens to the truck's acceleration if its trailer leaks sand at a constant rate through a hole at its bottom?
5. (a) What causes a moving body to come to a stop? What causes a body to accelerate or decelerate?
   (b) The driver of a speeding empty truck slams on the brakes and skids to a stop through a distance d.
   (i) If the truck carries a load that doubles its mass, what will be the truck's "skidding distance"?
   (ii) If the initial speed of the truck is halved, what will be the truck's "skidding distance"?
6. (a) Explain how to determine equilibrant forces of a body resting on a horizontal and inclined plane.
   (b) Can action and reaction forces cancel each other? Explain.
   (c) How do you apply Newton's laws of motion in solving various problems in daily life?
7. (a) A mass rests on an inclined plane of angle $\theta = 30°$, the coefficient of static friction is $\mu_s = 0.6$. Draw a diagram showing all the forces acting on the mass and explain their origin. Calculate their values if the mass is $m = 5\text{kg}$ and verify that under these conditions the mass will not slide.
   (b) A mass m is held at rest on an inclined plane, whose slope is α, by means of a horizontal force F (Figure 2.28). If the coefficient of static friction is μ_s, show that the maximum force F_{max} allowed before the body starts to move up the plane is given as $F_{max} = \dfrac{mg(\sin\alpha + \mu_s\cos\alpha)}{\cos\alpha - \mu_s\sin\alpha}$.

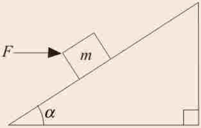


Figure 2.28 A mass rests on an inclined plane

8. A block of mass m is placed on a rough inclined plane at an angle θ with the horizontal. Show that when the block is released, its acceleration is independent of its mass.

9. A particle is projected from a point O and has an initial velocity u at an angle of θ above the horizontal. In the vertical plane of projection it takes a and b axes as the horizontal and vertical axes respectively and O as the origin. Show that the equation of the trajectory when the particle passes through these points (a,b) is $ga^2(1+\tan^2\theta)-2u^2a\tan\theta+2u^2b=0$.

10. An experiment is performed to determine the value of the gravitational acceleration g on earth. Two equal masses M hang at rest from the ends of a string on each side of a frictionless pulley (Figure 2.29). A mass $m = 0.01M$ is placed on the right-hand side. After the heavier side has moved down by $h = 1\text{m}$, the small mass m is removed. The system continues to move for the next 1s, covering a distance of $H = 0.312\,\text{m}$. Find the value of g from these data.

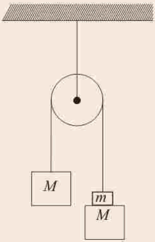


Figure 2.29 Two equal masses hang from the ends of a string

11. A bullet with a mass of $4\,\text{g}$ is horizontally fired at a speed of $600\,\text{ms}^{-1}$ into a ballistic pendulum with a mass of $1\,\text{kg}$ and a thickness of $25\,\text{cm}$. The bullet goes through the pendulum and leaves it with a speed $100\,\text{ms}^{-1}$. Find the magnitude of the constant force that slows down the bullet inside the pendulum and the vertical height through which the pendulum rises.

12. A hose ejects water at a speed of $0.2\,\text{ms}^{-1}$ through a hole of area $0.01\,\text{m}^2$. If the water strikes a wall normally, calculate the force on the wall assuming the velocity of the water normally to the wall is zero after collision.

13. A body is thrown horizontally with a velocity of $5\,\text{ms}^{-1}$ from a tower of $40\,\text{m}$ high. Determine the time of flight and the horizontal distance from the base of a tower to where the body strikes the ground.

14. Two automobiles of equal masses approach an intersection. One vehicle is travelling with velocity 13ms^{-1} towards the east and the other is travelling north with a speed of v. Neither driver sees the other. The vehicles collide at the intersection and stick together, leaving parallel skid marks of an angle of $55°$ north of east. The speed limit for both roads is 56kmh^{-1}, and the driver of the northward moving vehicle claims he was within the speed limit when collision occurred. Is he telling the truth? Give reasons.
15. A 30kg shell at rest, burst and splits into three pieces of equal masses. The first piece flies off vertically with a velocity of 20ms^{-1}, the second piece flies off horizontally with a velocity of 35ms^{-1}. Determine the velocity and direction of the third piece.
16. A space craft's dry mass is 75000kg and the effective exhaust gas velocity of its main engine is 3100ms^{-1}. How much propellant must be carried if the propulsion system is to produce a total velocity of 700ms^{-1}?
17. A large rocket with an exhaust speed of 3000ms^{-1} develops a thrust of $2.4\times10^{7}\text{N}$.
    (a) How much mass is being blasted out of the rocket exhaust per second?
    (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with $v_{rel} = 3.00\text{kms}^{-1}$ and if 90.0% of its initial mass is fuel and oxidizer?
18. (a) A ball is released from a vertical distance of h_1. On striking a level floor, it bounces back to height h_2. Show that the coefficient of restitution between the ball and the floor is given by $e=\sqrt{\frac{h_2}{h_1}}$.
    (b) A 2 kg ball moving horizontally at a speed of 10 ms^{-1} strikes a vertical wall and bounces back with the same speed at an angle of $45°$ above the horizontal. Determine the average force exerted on the wall by the ball if the impact lasted.
    (c) Figure 2.30 shows a fixed pulley carrying a string which has a mass of 4kg attached at one end and a light pulley A attached at the other. Another string passes over a pulley A and carries a mass of 3kg at one end and a mass of 1kg at the other end. Find;
    (i) The acceleration of pulley A
    (ii) The acceleration of the 1kg, 3kg and 4kg masses.
    (iii) The tensions in the strings.

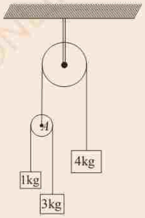


Figure 2.30 Connected masses on a pulley system

19. (a) Explain how you would use a balloon to demonstrate the mechanism responsible for rocket propulsion.
    (b) Can a rocket move forward by pushing the air backward? Explain your answer.
    (c) You are standing perfectly still and then you take a step forward. Before the step your momentum was zero, but afterwards you have some momentum. Is the conservation of momentum violated in this case?
20. If two automobiles collide, they usually do not stick together. Does this mean the collision is elastic? Explain why a head-on collision is likely to be more dangerous than other types of collisions.
21. (a) A body that is thrown upwards to move under the control of gravity only describes parabolic path. What are the quantities that remain constant during the flight?
    (b) A bullet is fired towards the sea from the top of a tall building 98m high built on the beach with the velocity of 49ms^{-1} and at an angle of $30°$ to the horizontal. Determine the distance from the bottom of the building to where the bullet hits the water.
22. Two inclined planes of angles $30°$ and $60°$ are placed touching each other at the base as shown in Figure 2.31. A projectile is projected at right angle with a speed of 103ms^{-1} from point P and hits the other incline at point Q normally. Calculate the time of flight.

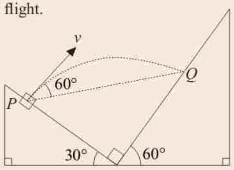


Figure 2.31 Projectile at inclined planes

23. An object is projected so that it just clears two obstacles, each 25 m high, which are situated 160 m from each other; if the time of passing between the obstacles is 2.5 s. Calculate the full range of projection and the initial velocity of the object.
24. (a) A projectile is fired on level ground. Show that for a given range and initial velocity the projection angle has two possible values, which are symmetrically spaced, each side of $45°$.
    (b) A projectile projected from a point on a horizontal plane reaches a greatest height h above the plane and has a horizontal range R. If $R = 2h$, find the angle of projection.
25. A particle is projected vertically upwards with a velocity u, after an interval of time t another particle is projected upwards from the same point and with the same initial velocity. Prove that the particles will meet at the height $\left(\dfrac{4u^2 - g^2t^2}{8g}\right)$.

Chapter Three

Circular motion, simple harmonic motion, and gravitation

Introduction

Circular motion phenomena can be seen in moving objects around us. Examples of these are a speck of dust stuck on a spinning Compact Disk (CD), a stone being whirled around on a string, and a driver on a racing car along a curved road. These objects travel along the perimeter of a circle, repeating their motion over and over. Circular motion takes place on flat and inclined planes, and surprisingly, even though the objects move at a constant speed in a uniform circular path, they still retain acceleration. Likewise, simple harmonic motion is a periodic motion that can be produced using circular motion. One of the simplest forms of periodic motions is uniform circular motion. Simple harmonic motion occurs around us in the form of oscillatory motion. Circular motion is also notable through objects, such as the earth, the moon, and other heavenly objects, moving along their orbits, and held in place through gravitational force. In this chapter, you will learn about uniform circular motion, simple harmonic motion, and gravitation.

3.1 Uniform circular motion

Objects move in circular path in a wide variety of situations such as rotating machine parts, a car rounding a curve with constant radius at constant speed, motion of satellites and so on. So it is important to study this special class of motion in details. Circular motion can be classified into two types namely, uniform and non-uniform motion. Uniform circular motion describes the motion of a body that moves in a circular path at a constant speed. There are two necessary conditions for a body to move in circular motion. The body must be given some initial velocity. The velocity vector is always tangential to the path of the object and perpendicular to the radius of the circular path. Secondly, a force which is always directed at right angle to the velocity vector must act on the body towards the centre. Moreover, in uniform circular motion, the distance (radius) from the axis of rotation remains constant all the time. Through the end of this section, you will be able to explain and apply angular displacement, angular velocity and angular acceleration of uniform circular motion.

3.1.1 Concept of uniform circular motion

In uniform circular motion, the speed of the object remains constant but its direction is constantly changing.

Physics Form V.indd 51 23/07/2020 09:16

For this to happen the force must not act along the direction of motion, but towards the centre (Figure 3.1).

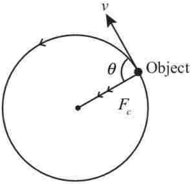


Figure 3.1 Forces in uniform circular motion

Though the force F does not act along v, but it has a component $F\cos\theta$ along vector v. This component will change the speed; consequently for uniform circular motion, $F\cos\theta = 0$. Since $F \neq 0$, then, $\cos\theta = 0$, hence $\theta = 90°$.

Therefore, the force F acting on an object in uniform circular motion is always perpendicular to velocity v. This force is directed towards the centre of the circle, hence is called centripetal force (F_c). The force will produce a centripetal acceleration.

There are some parameters regarding circular motion that need to be well described. Such terms include angular displacement, angular velocity and centripetal acceleration.

(a) Angular displacement

Angular displacement (θ) is the angle turned through by an object moving along a circular path of radius r. It is measured in units of radians (rad). Let the object move from point A to point B in time t and it sweeps out an angle θ about the center O (Figure 3.2).

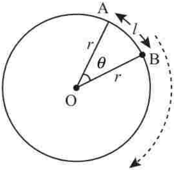


Figure 3.2 A body moving along a circular path

From Figure 3.2, the angular displacement (in radians) is given by

$$\theta = \frac{\text{arc length AB}}{\text{length OB}} = \frac{l}{r} \qquad (3.1)$$

(b) Angular velocity

The angular velocity (ω), of an object moving in circular path is defined as the rate of change of angular displacement of the object.

$$\omega = \frac{\Delta\theta}{\Delta t} \text{ but, } \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \omega \qquad (3.2)$$

where ω is called instantaneous angular velocity; it is measured in radians per second (rad s^{-1}). If the object makes a complete trip around the circle, its angular displacement is 2π radians and the time taken for the trip is called period (T). Therefore, equation (3.2) can be expressed in terms of period T as $\omega = \frac{2\pi}{T}$.

Hence, equation (3.2) can be expressed in terms of equation (3.1) as,

$$\frac{1}{r} \times \frac{dl}{dt} = \omega \text{ or } dl = r\omega dt \qquad (3.3)$$

Consider a body that is moving with a uniform linear velocity (v) on a circular path of radius r having a centre at O as shown in Figure 3.2.

Linear velocity,

$$v = \frac{dl}{dt};\ dl = vdt \qquad (3.4)$$

Comparing equations (3.3) and (3.4); you get

$$v = r\omega \qquad (3.5)$$

Equation (3.5) shows a fundamental relationship between v and ω.

(c) Centripetal acceleration

Centripetal acceleration (a_c) is the acceleration of the body moving around a circle and it is always directed along the radius towards the centre of the circle. This acceleration is also known as radial or normal acceleration.

Consider a body moving on a circular path of radius r such that it passes from point A to B through P with constant speed v (Figure 3.3).

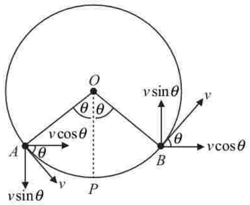


Figure 3.3 Acceleration of a body in circular path

The horizontal acceleration a_x of the body along x-direction is given by,

$$a_x = \frac{v_x}{t} = \frac{v_{Bx} - v_{Ax}}{t}.$$

It then follows that, acceleration along x-direction is zero. i.e.,

$$a_x = \frac{v\cos\theta - v\cos\theta}{t} = 0.$$

The vertical acceleration a_y of the body along y-direction is given by

$$a_y = \frac{v_y}{t} = \frac{v_{By} - v_{Ay}}{t}$$

$$a_y = \frac{v\sin\theta - (-v\sin\theta)}{t}$$

$$a_y = \frac{2v\sin\theta}{t} \qquad (3.6)$$

If t is the time taken by the body to move from A to B; then,

$$v = \frac{arclength\ AB}{t} = \frac{2\theta r}{t}.$$ Therefore,

$$t = \frac{2\theta r}{v} \qquad (3.7)$$

Substituting equation (3.7) into (3.6);

$$a_y = \frac{v^2}{r}\frac{\sin\theta}{\theta} \qquad (3.8)$$

If A and B are considered to be coincident at P, then θ tends to approach zero and that

$\lim_{\Delta\theta\to 0} \frac{\sin\theta}{\theta} = 1$, so that from equation (3.8);

$a_y = \frac{v^2}{r}$ and it is directed along PO acting towards the centre. Hence,

$$a_c = \frac{v^2}{r}.$$

An object of mass, m moving in a circular path of radius r with a constant speed v has a centripetal force F_c whose magnitude is given by $F_c = ma_c$. Hence,

$$F_c = \frac{mv^2}{r}$$

Centripetal force is a net force due to combined effects of inertia from Newton's first law of motion. When a body is moving in a circular path and the centripetal force vanishes, the body would leave the circular path and move with tangential acceleration. Tangential acceleration is the rate of change of linear velocity and lies along the velocity line and is given by $a_t = \frac{dv}{dt}$, where v is the varying velocity (Figure 3.4). When the linear velocity is constant its tangential acceleration is zero.

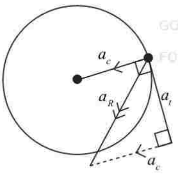


Figure 3.4 Tangential and radial acceleration

Since $\frac{dv}{dt} = r\frac{d\omega}{dt}$, it implies that, $a_t = \alpha r$ where α is the angular acceleration.

The resultant acceleration $a_R = \sqrt{a_t^2 + a_c^2}$

3.1.2 Motion in a horizontal circle

Consider an object of mass m tied to one end of a string and whirled in a circular path on a horizontal plane as shown in Figure 3.5.

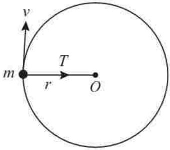


Figure 3.5 Motion in a horizontal circle

There are two forces acting on the object; its weight ($W = mg$) and the tension T. The weight has no component towards the centre O of the circle. Therefore, the only force acting on the object directing it towards the centre is the tension. Hence the tension will provide the necessary centripetal force to keep the object in circular path. It follows that;

$$T = ma_c = \frac{mv^2}{r} \qquad (3.9)$$

A special case of motion in a horizontal circle is that of a conical pendulum. Suppose a small object of mass m is tied to a string AB of length l and then whirled in a horizontal circle of radius r, with B fixed directly above the centre O of the circle as shown in Figure 3.6.

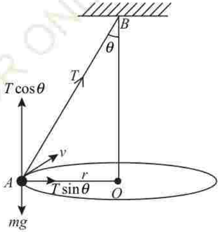


Figure 3.6 Conical pendulum

If the speed is constant, the string turns at a constant angle θ to the vertical. Hence, the only unbalanced force directed towards the centre O of the circle is $T\sin\theta$. The horizontal component of tension provides the centripetal force expressed as

$$T\sin\theta = \frac{mv^2}{r} \qquad (3.10)$$

where T is the tension on the string.

The vertical forces, (mg) and $T\cos\theta$ counter balances each other and produce zero acceleration. That is;

$$T\cos\theta = mg \qquad (3.11)$$

dividing equation (3.10) by (3.11) gives

$$\tan\theta = \frac{v^2}{rg} \text{ but } v^2 = \omega^2 r^2 \text{ thus,}$$

$$\tan\theta = \frac{\omega^2 r}{g}$$

Also,

$\omega = \frac{2\pi}{T}$, implies that,

$$T = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

where T is the period.

From Figure 3.6, $r = l\sin\theta$, thus,

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

which is the period of revolution of conical pendulum.

Example 3.1

A 500 g stone attached to a string is whirled in a horizontal circle at a constant speed of 10.0 ms^{-1}. The length of the string is 1.0 m. Neglecting the effects of gravity, find:

(a) The centripetal acceleration of the stone; and

(b) The centripetal force acting on the stone.

Solution

(a) The centripetal acceleration is given by

$$a_c = \frac{v^2}{r}$$

$$a_c = \frac{(10\,\text{ms}^{-1})^2}{1\,\text{m}} = 100\,\text{ms}^{-2}$$

(b) The centripetal force $F_c = \frac{mv^2}{r}$

$$F_c = 0.5\,\text{kg} \times 100\,\text{ms}^{-2} = 50\,\text{N}$$

Therefore, centripetal acceleration and force acting on the stone are $100\,\text{ms}^{-2}$ and $50\,\text{N}$ respectively.

Example 3.2

A rubber stopper 13 g is attached to a 0.93 m string. The stopper is swung in a horizontal circle, making 10 revolutions in 31.4 seconds. Find the tension in the string.

Solution

Using the relation $T = \frac{mv^2}{r}$ where

$v = \omega r$; $v = (2\pi f)r$

Then, $T = 4\pi^2 f^2 mr$,

$$T = \frac{4\pi^2 \times 10^2}{(31.4\,\text{s})^2} \times 0.013\,\text{kg} \times 0.93\,\text{m} = 0.048\,\text{N}$$

Therefore, tension in the string is 0.048 N.

Physics Form V.indd 55 23/07/2020 09:16

Example 3.3

A small mass of 1kg is attached to the lower end of a string 1m long whose upper end is fixed. The mass is made to rotate in a horizontal circle of radius 0.6 m. If the circular speed of the mass is constant. Find:

(a) Tension F, in the string; and

(b) The period of motion.

Solution

(a) Consider vertical forces.

$$F\cos\theta = mg;\ F = \frac{mg}{\cos\theta}$$

From figure 3.6,

$$\cos\theta = \frac{\sqrt{l^2 - r^2}}{l} = \frac{\sqrt{(1\,\text{m})^2 - (0.6\,\text{m})^2}}{1\,\text{m}}$$

$$= 0.8$$

$$F = \frac{1\,\text{kg} \times 9.8\,\text{ms}^{-2}}{0.8} = 12.25\,\text{N}$$

(b) Using the relation

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}},\ T = 2\pi\sqrt{\frac{1\,\text{m} \times 0.8}{9.8\,\text{ms}^{-2}}} = 1.8\,\text{s}$$

Therefore, tension in the string and period of the motion are 12.25 N and 1.8s respectively.

3.1.3 Motion in a vertical circle

In linear vertical motion, the speed of the body may be constant throughout of the motion. But if a body of mass m is whirled in a vertical circle by a string with constant speed, the tension in the string changes with position of the body along the circular path. On the other hand, the speed as well as the direction of the body may constantly change. Earth's gravitational force is constantly either speeding up the object as it falls or slowing the object down as it rises; thus non-uniform motion.

Consider a block of mass m whirled at the end of the string in a vertical circle of radius r such that it moves from A to B through E and C as shown in Figure 3.7. As the body moves from A to E, it loses potential energy and gains kinetic energy.

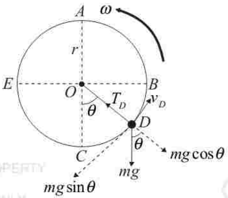


Figure 3.7 Motion in vertical circle

So, from the principle of conservation of mechanical energy; the total energy E_A at A is equal to the total energy E_E at E, that is; $mg(2r) + \frac{1}{2}mv_A^2 = mg(r) + \frac{1}{2}mv_E^2$.

Thus,

$$v_E^2 = 2gr + v_A^2$$

Similarly, the velocity at C and B can be attained using the principle of conservation of mechanical energy.

Suppose the block is at an arbitrary point D. The resultant force F_R on the block is such that; $F_B = T_D - mg\cos\theta$ but $F_R = \frac{mv_D^2}{r}$.

Then,

$$\frac{mv_D^2}{r} = T_D - mg\cos\theta$$

$$T_D = \frac{mv_D^2}{r} + mg\cos\theta \qquad (3.12)$$

Equation (3.12) can be used to obtain tension at any point on the circle.

At point A: $\theta = 180°$, $\cos\theta = -1$, then,

$T_A = \frac{mv_A^2}{r} - mg$ (minimum tension).

At point B: $\theta = 90°$, $\cos\theta = 0$, then,

$T_B = \frac{mv_B^2}{r}$ the weight has no horizontal component towards the centre.

At point C: $\theta = 0°$, $\cos\theta = 1$, then,

$T_C = \frac{mv_C^2}{r} + mg$ hence maximum tension.

In order for an object to successfully complete the circle (loop the loop) in a vertical motion, it must have a minimum (critical) velocity v_c at the top of the circle. This velocity is required to avoid sagging of the string. Therefore, the velocity corresponds to the lowest value of the tension.

Since the lowest possible value of T is 0,

$T_{min} = \frac{mv_c^2}{r} - mg = 0$. Thus, critical velocity v_c is the minimum velocity with which a body passes at the highest position so that it just completes the loop. i.e., $v_c = \sqrt{rg}$.

Example 3.4

A body of $50\,g$ is whirled in a vertical circle of radius $60\,cm$. Determine the maximum and the minimum tensions in the string when its velocity at horizontal position is $8\,ms^{-1}$.

Solution

Maximum tension T_{max} is obtained when the particle is at the bottom of the circle,

$$T_{max} = \frac{mv_b^2}{r} + mg \qquad (i)$$

From energy conservation,

$v_b = \sqrt{v_h^2 + 2gr}$, where v_b and v_h are the velocities at the bottom and horizontal position respectively, and r is the radius of the circle.

$$v_b = \sqrt{\left(8\,ms^{-1}\right)^2 + 2\times 9.8\,ms^{-2}\times 0.6\,m}$$
$$= 8.7\,ms^{-1}$$

Substituting the value of v_b in equation (i),

$$T_{max} = \frac{0.05\,kg\times\left(8.7\,ms^{-1}\right)^2}{0.6\,m} + 0.05\,kg\times 9.8\,ms^{-2}$$
$$= 6.8\,N$$

Minimum tension T_{min} is obtained when the particle is at the top of the circle,

$$T_{min} = \frac{mv_t^2}{r} - mg \qquad (ii)$$

$v_t = \sqrt{v_h^2 - 2gr}$ where v_t is the velocity at the top position

$$v_t = \sqrt{\left(8\,ms^{-1}\right)^2 - 2\times 9.8\,ms^{-2}\times 0.6\,m}$$
$$= 7.2\,ms^{-1}$$

$$T_{\min} = \frac{0.05\text{kg} \times (7.2\text{ms}^{-1})^2}{0.6\text{m}} - 0.05\text{kg} \times 9.8\text{ms}^{-2}$$

$$= 3.8\text{N}$$

Therefore, the maximum and the minimum tensions in the string are 6.8 N and 3.8N respectively.

3.1.4 Vehicles on a level and banked curved road

When a vehicle turns in a circular path, it behaves differently when on flat and banked road. The centripetal and frictional forces acting on the vehicle determine the maximum safety velocity the vehicle can travel. On flat surface, cars must rely only on friction to prevent skidding. During rainy season friction is reduced, hence the turning force becomes smaller. Therefore, banked curves were introduced to prevent skidding. With banked curves, the normal force provides a component of force directing a vehicle towards the centre of the curve. Hence, reducing the vehicle's dependency on friction only to safely navigate a curve.

(a) A car on a level rough curved road

Consider a car making a turn (Figure 3.8.) The portion of the turn can be approximated by an arc of a circle of radius r. If the car makes the turn at a constant speed v; then during that turn, the car goes through a uniform circular motion.

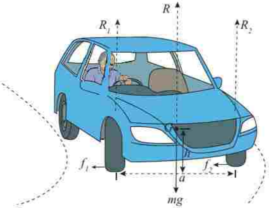


Figure 3.8 A car taking a turn on a rough level road

The necessary centripetal force is provided by the frictional force between the tyres of the car and the road. If R_1 and R_2 are normal reactions of the road of the inner and outer pairs of wheels respectively, then f_1 and f_2 are frictional forces between tyres and the road. Let h be the height of the centre of gravity G, above the ground (road), a is distance between the car wheels and r is radius of the circular path.

Vertical forces;

$$R_1 + R_2 = mg \tag{3.13}$$

Horizontal forces;

$$f_1 + f_2 = \frac{mv^2}{r} \tag{3.14}$$

Taking moments of force about G;

$$f_1 h + f_2 h + R_1\left(\frac{a}{2}\right) = \frac{a}{2}R_2$$

$$(f_1 + f_2)h = \frac{a}{2}(R_2 - R_1) \tag{3.15}$$

Substituting equations (3.13) and (3.14) into (3.15) gives,

$$\frac{2mv^2 h}{ar} = mg - 2R_1$$

$$R_1 = m\left(\frac{g}{2} - \frac{hv^2}{ar}\right) \quad (3.16)$$

Substituting equation (3.15) into (3.13) gives,

$$R_2 = m\left(\frac{g}{2} + \frac{hv^2}{ar}\right) \quad (3.17)$$

The maximum speed v_c at which the vehicle can take the curve without toppling is obtained by setting $R_1 = 0$. Since $m \neq 0$, then $\frac{g}{2} - \frac{hv_c^2}{ar} = 0$

$$v_c = \sqrt{\frac{arg}{2h}} \quad (3.18)$$

Alternatively,

$$f_1 + f_2 = \frac{mv_c^2}{r},$$

$$\mu R_1 + \mu R_2 = \frac{mv_c^2}{r} = \mu mg$$

$$v_c = \sqrt{\mu rg}, \text{ since } f = \mu mg$$

A smooth road offers no friction to the wheels of the vehicle. Therefore, the vehicle taking a corner on such a road will skid outwards away from the road. For this reason, most of the roads are banked at the corners so that the car will not depend on friction only.

(b) A car on a banked rough curved road

In banked road the outer edge is raised to a certain angle θ making a curve above the level of the inner edge. Consider a car moving in a circular curved road above the level of the inner edge as shown in Figure 3.9. Let v_{max} be the maximum speed of the car before it skids.

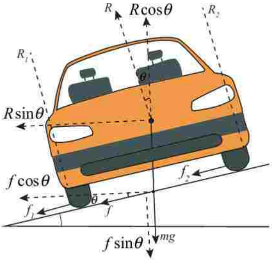


Figure 3.9 A car on a rough banked road

Horizontal forces, $R\sin\theta + f\cos\theta = ma$, where $R = R_1 + R_2$ and $f = f_1 + f_2$. But $f = \mu R$

$$R(\sin\theta + \mu\cos\theta) = \frac{mv_{max}^2}{r} \quad (3.19)$$

Vertical forces,

$$R\cos\theta = f\sin\theta + mg$$

$$R(\cos\theta - \mu\sin\theta) = mg \quad (3.20)$$

dividing equation (3.19) by (3.20) gives,

$$v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}} \quad (3.21)$$

Equation (3.21) gives the expression for the maximum possible speed of the car before it skids to the outside of the curve (up the banking). For the minimum possible speed of the vehicle before skidding, the direction of frictional forces, f_1 and f_2 in Figure 3.9 will be reversed. By similar procedures as above, the final result will be $v_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$.

(c) A car on a banked smooth curved road

A smooth surface offers no friction, but when such a surface is inclined, it can provide the necessary centripetal force for a vehicle to successfully take a corner (Figure 3.10).

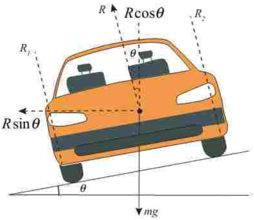


Figure 3.10 A car on banked smooth road

Vertical forces,

$$R\cos\theta = mg \quad (3.22)$$

Horizontal forces,

$$R\sin\theta = \frac{mv^2}{r} \quad (3.23)$$

Dividing equation (3.23) by (3.22) gives, $v_{max} = \sqrt{rg\tan\theta}$. This is the maximum speed a car can take a corner in a smooth banked road without skidding off.

Example 3.5

A car is moving at $30\,\text{kmh}^{-1}$ in a circle of radius $60\,\text{m}$. Find the minimum value of μ_s for the car to make the turn without skidding.

Solution

From the equation $\mu_s = \frac{v^2}{rg}$,

but; $r = 60\,\text{m}$; $g = 9.8\,\text{ms}^{-2}$

$$v = \frac{30\times10^3}{3600}\,\text{ms}^{-1} = 8.3\,\text{ms}^{-1}$$

$$\mu_s = \frac{\left(8.3\,\text{ms}^{-1}\right)^2}{60\,\text{m}\times9.8\,\text{ms}^{-2}} = 0.12$$

Example 3.6

A car whose wheels are 1.4 m apart laterally and whose centre of gravity is 0.5 m above the ground moves round a curve of radius 60 m. Assuming no slipping of the wheels on the road, find the highest speed at which the car can round the curve without overturning.

Solution

Using the equation,

$$v_c = \sqrt{\frac{arg}{2h}}$$

$$v_c = \sqrt{\frac{1.4\,\text{m}\times60\,\text{m}\times9.8\,\text{ms}^{-2}}{2\times0.5\,\text{m}}}$$

$$= 28.69\,\text{ms}^{-1}$$

Therefore, the maximum speed is $28.69\,\text{ms}^{-1}$.

(d) Cyclist on a curved rough level road

A cyclist (such as of motor cycle) taking a corner on a curved rough level road must bend inwards towards the centre of the curved road so as to create the friction force between the tyres and the road (or ground) which is required in order to provide the necessary centripetal force (Figure 3.11).

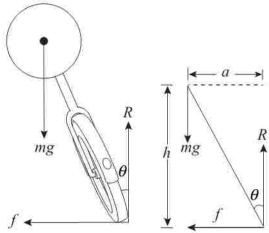


Figure 3.11 Cyclist taking a corner

By principle of moments,

$$\sum clockwise\, moments = \sum anticlockwise\, moments$$

Thus, $fh = Ra$. Where f is the frictional force, R is the normal reaction, a is the distance between R and mg, and h is the height of the centre of gravity.

Hence, $\frac{f}{R} = \frac{a}{h}$; $\frac{mv_c^2}{mgr} = \tan\theta$

and $\mu = \frac{f}{R} = \tan\theta$

Therefore,

$$v_c = \sqrt{rg\tan\theta}$$

This is the maximum speed a cyclist can move along a curved rough level road. However, for a smooth level road $(\tan\theta = 0)$ the cyclist will have velocity of zero and thus skid outward from the road or at rest without bending. It should be noted that the speed limit shown by road symbols are based on these principles.

Example 3.7

A highway road designed for an average speed of $72\,\text{kmh}^{-1}$ has a turn of radius $50\,\text{m}$. To what angle must the road be banked so that cars travelling at $72\,\text{kmh}^{-1}$ may not overturn.

Solution

Using the relation $\tan\theta = \frac{v^2}{rg}$, but

$$v = \frac{72 \times 10^3}{3600}\text{ms}^{-1},\ r = 50\,\text{m},\ g = 9.8\,\text{ms}^{-1}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right),$$

$$\theta = \tan^{-1}\left(\frac{(20\,\text{ms}^{-1})^2}{50\,\text{m} \times 9.8\,\text{ms}^{-2}}\right)$$

$$\theta = \tan^{-1}(0.8163) = 39°$$

Therefore, the road should be banked at 39°.

3.1.5 Applications of circular motion

There are several applications of circular motion in daily life; these include the following:

(a) Rotating fluids

When a liquid in a container is stirred, the centre of the liquid surface forms a hollow. The surface of the liquid will be defined by how the centripetal acceleration changes with radius. A parabolic surface of the liquid may be used in liquid mirror telescopes. The most common liquid used is mercury (or low melting alloys of gallium). In these telescopes, the liquid and its container are rotated at a constant speed around a vertical axis, which causes the surface of the liquid to assume a "paraboloidal" shape regardless of the container's shape.

(b) Centrifugal pump

The main part of a centrifugal pump is the impeller which has a series of curved vanes fitted inside the shroud plates. When a fluid (e.g. water) enters the pump along or near the rotating axis, it is accelerated by the pump impeller. The fluid particles then accelerate radially outward into a volute chamber (casting) from where it exits (Figure 3.12).

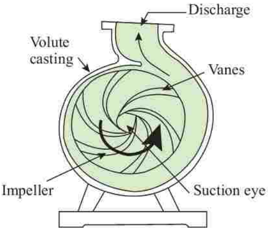


Figure 3.12 Centrifugal pump

(c) Centrifuge

A centrifuge is a device that is used for separating mixtures. It can be used to separate sugar crystals from molasses, cream from milk, bee wax from honey, and constituents of blood and urine samples.

The centrifuge works using the sedimentation principle. The sample of liquid mixture is spanned at relatively high speed, creating a strong centripetal force on the liquid and its content. This force will make denser particle to accelerate outward in the radial direction. When the centrifuge is settled, the heavier (denser) particles settle to the bottom while lighter (less dense) particles rise to the top (Figure 3.13).

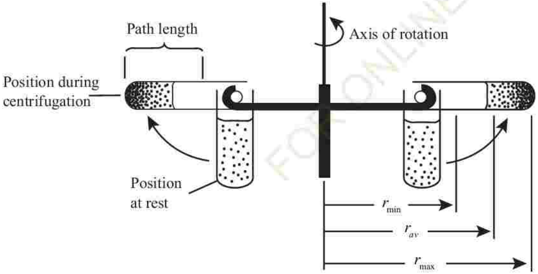


Figure 3.13 Schematic diagram of a centrifuge

Exercise 3.1

1. Give explanation for each of the following observations:
   (a) If there is a net force on a particle in uniform circular motion, why does the particle's linear speed not change?
   (b) As a car rounds a banked circular curve at constant speed, several forces are acting on it: for example, air resistance towards the rear, friction from the pavement in the forward direction, gravity, and the normal force from the tilted road surface. In what direction does the net force point?
2. Imagine you are a driver on an icy road, you approach a curve that has the banking angle calculated for $90\ \text{kph}$. Your passenger suggests you slow down below $90\ \text{kph}$, just to be on the safe side, but you say that you should maintain your speed at $90\ \text{kph}$.
   (a) Who is correct, you or your passenger?
   (b) What would happen if you were to slow down (or speed up)?
   (c) Would your passenger's suggestion be a good one on an unbanked road?
3. Consider a $1\ \text{kg}$ brick being whirled in a vertical circle at the end of $1\ \text{m}$ rope,
   (a) What critical velocity must the brick achieve in order to pass safely through the top of its circular path?
   (b) What would be the critical velocity of the brick if it were to be whirled on the moon where the acceleration due to gravity is one sixth that on the earth?
4. A $500\ \text{g}$ stone attached at the end of a $1\ \text{m}$ long string is whirled in a vertical circle whose centre is $10\ \text{m}$ above the ground. The breaking tension of the string is $100\ \text{N}$. If the string breaks, determine:
   (a) The position of the stone in a circle where the string is most likely to break; and
   (b) The horizontal distance where the stone will strike the ground.
5. A body of mass $8\ \text{kg}$ is moving in a horizontal circle of radius $3\ \text{m}$ with a constant speed of $10\ \text{ms}^{-1}$. Determine:
   (a) The angular velocity; and
   (b) The centripetal force.
6. A car is supposed to move safely around the smooth corner of $200\ \text{m}$ radius with the speed of $60\ \text{kmh}^{-1}$. Find the banking angle for the car to move safely?
7. A mass of $1.5\ \text{kg}$ is attached to a lower end of a string of $2\ \text{m}$ whose upper end is fixed to a rigid support. The mass is then made to move in a horizontal circle of radius $0.8\ \text{m}$. If the circular speed of the mass is constant determine:
   (a) The tension in the string; and
   (b) The period of the motion.
8. A curve in a road has radius of $60\ \text{m}$. The angle of the bank of

the road is $47°$. If the coefficient of static friction between tyres and road is 0.8. Find the maximum and minimum speed a car can move without skidding.

9. A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small block is kept in the bowl and rotates with it without slipping. If the surface of the bowl is smooth and the angle made by the radius through the block with the vertical is θ, show that the angular speed of the bowl is given by, $\sqrt{\frac{g\cos\theta}{R}}$.

10. A simple pendulum is suspended from the ceiling of a car taking a turn of radius 10 m at a speed of 36 kmh^{-1}. Find the angle made by the string of the pendulum with the vertical if this angle does not change during the turn.

11. A ball of mass 0.3kg is tied to one end of a string 0.8m long and rotated in a vertical circle. At what speed of the ball will the tension in the string be zero at the highest point of the circle? What will be the tension at the lowest position?

12. Prove that the velocity v with which a body must be projected at the lowest part of a loop apparatus of radius R in the vertical plane so that it passes at the highest position with minimum velocity, is given by $v = \sqrt{5gR}$.

3.2 Simple harmonic motion

You are most likely familiar with several examples of periodic motion, such as the oscillation of an object on a spring, the motion of a pendulum, children playing on a swing, and the vibrations of a stringed musical instrument. When an object vibrates or oscillates back and forth, over the same path, each oscillation taking the same amount of time, the motion is periodic. Periodic motion is a motion of an object that regularly repeats. In this motion the object returns to a given position after a fixed time interval.

Simple harmonic motion (SHM) is a special case of periodic motion. It occurs in mechanical systems when the force acting on an object is proportional to the displacement of the object relative to the equilibrium position. The force is always directed towards the equilibrium position. Simple harmonic motion forms a basic building block for more complicated periodic motion. It also forms a basis for understanding mechanical waves essential for explaining other phenomena in nature and man-made. For example, when engineers and architects build bridges and tall buildings understanding of mechanical waves plays an important role.

In this section, you will acquire knowledge and skills on simple harmonic motion, explore its characteristics and application in daily life.

3.2.1 The concept of simple harmonic motion

As a model of simple harmonic motion, consider a block of mass m attached to the end of a spring, with the block free to move on horizontal frictionless surface (Figure 3.14).

When the spring is neither compressed nor stretched (Figure 3.14(b)), the block is at equilibrium position, that is $x = 0$. When the block is displaced to a position x (stretched for Figure 3.14(a) and compressed for Figure 3.14(c)), the spring exerts on the block a force F_x, that is proportional to the displacement and is given by Hooke's law, $F_s = -kx$. This force is called a restoring force, because it is directed towards the equilibrium position, hence oppose the displacement from equilibrium.

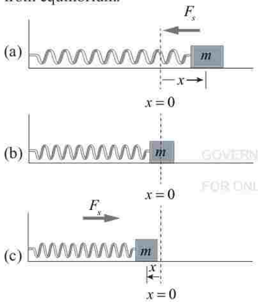


Figure 3.14 A block of mass m attached to a spring

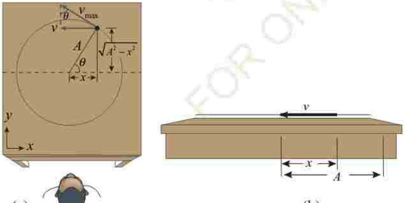
Applying Newton's second law, $\sum F_x = ma_x$ to the motion of the block, the net restoring force in the direction of x will be $-kx = ma_x$, thus $a_x = -\frac{k}{m}x$. This implies the acceleration is proportional to the displacement of the block, and its direction is opposite to the direction of the displacement from equilibrium.

Any system that behaves in this manner is said to exhibit simple harmonic motion. An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement and is always directed towards the equilibrium position.

Sinusoidal representation of SHM

A sinusoidal expression for simple harmonic motion can be derived by comparing the motion to that of an object moving uniformly in a circle. There is nothing actually moving in a circle when a spring oscillates linearly, but it is the mathematical similarity that finds it useful.

Consider a small object of mass m revolving counterclockwise in a circle of radius A, with constant speed v on a table (Figure 3.15).



(a) (b)

Figure 3.15 View of circular motion from (a) above and (b) sideways

As viewed from above, the motion is circular in the x-y plane. But a person who looks at the motion from the edge of the table sees an oscillatory motion back and forth, and this one-dimensional motion corresponds precisely to simple harmonic motion. What the person sees, is the projection of the circular motion onto the x-axis (Figure 3.15(b)). The position of an object undergoing simple harmonic motion as a function of time can be found using the reference circle. From (Figure 3.15(a)), it can be seen that, $\cos\theta = \frac{x}{A}$, so the projection of the object's position on the x-axis is $x = A\cos\theta$.

The object in the reference circle (Figure 3.15) is rotating with uniform angular velocity ω. This can then be written as $\theta = \omega t$ where θ is in radians. Thus $x = A\cos\omega t$. Furthermore, since the angular velocity (specified in radians per second) can be written as $\omega = 2\pi f$, where f is the frequency, then, $x = A\cos(2\pi f)t$. Because the cosine function varies between 1 and –1, x-component equation shows that x varies between A and $-A$. If a pen is attached to a vibrating object as a sheet of paper is moved at a steady rate beneath it, a sinusoidal curve will be drawn that accurately follows the cosine (sinusoidal) function. Figure 3.16 demonstrates how an oscillating mass models sinusoidal wavelike signal.

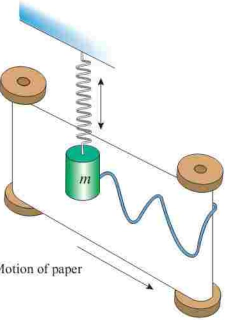


Figure 3.16 A pen attached to an oscillating object of mass m

Example 3.8

Which of the following forces would cause an object to move in simple harmonic motion?

(a) $F = -0.5x^2$ (b) $F = -2.5y$

(c) $F = 9.8x$ (d) $F = -5\theta$

Solution

Both (b) and (d) will give simple harmonic motion because they give force which is proportional to displacement, and minus sign indicates acceleration towards the centre. (a) does not produce SHM since its motion is not proportional to displacement. Similarly, (c) does not produce SHM since its motion is not towards the centre.

3.2.2 Equations of simple harmonic motion

To explore further sinusoidal characteristics of simple harmonic motion, Figure 3.17 will be used to establish its displacement, period, velocity and acceleration. This figure is a combination of ideas presented in Figure 3.15 and 3.16.

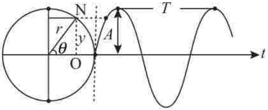


Figure 3.17 Relationship between SHM and circular motion

(a) Displacement

This is the linear distance of the particle from the equilibrium position of the motion i.e., distance ON (Figure 3.17). It is given by, $y = A\sin\theta$, but $\theta = \omega t$

$$y = A\sin\omega t \qquad (3.24)$$

The displacement time curve is shown in Figure 3.18.

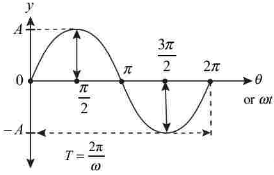


Figure 3.18 Displacement-time curve

(b) Period

The period T, is the time taken by an oscillating object to complete one oscillation or cycle (Figure 3.18).

$$T = \frac{\text{circumference of a cycle}}{\text{speed}}$$

$$T = \frac{2\pi r}{v}$$

Since $v = \omega r$, therefore, $T = \frac{2\pi}{\omega}$.

(c) Linear velocity

Velocity is the rate of change of displacement, that is $v = \frac{dy}{dt}$, but

$$y = A\sin\omega t$$

$\frac{dy}{dt} = A\omega\cos\omega t$. Thus,

$$v = A\omega\cos\omega t \qquad (3.25)$$

Alternatively, squaring both sides of equation (3.24) and (3.25) gives,

$$y^2 = A^2\sin^2\omega t \qquad (3.26)$$

$$v^2 = A^2\omega^2\cos^2\omega t \qquad (3.27)$$

Adding equations (3.26) and (3.27) and solving for v gives,

$$v = \pm\omega\sqrt{A^2 - y^2} \qquad (3.28)$$

The velocity time graph is shown in Figure 3.19.

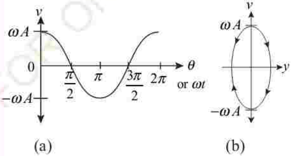


Figure 3.19 (a) Velocity-time curve against time and,
(b) velocity-displacement curve

Example 3.9

A particle executing simple harmonic motion has a period of 4 seconds and an amplitude of 2 cm. Find:

(a) Maximum velocity; and

(b) Velocity at half way of its maximum displacement.

Solution

(a) Using the relation $v = \omega\sqrt{A^2 - y^2}$, maximum velocity is obtained when $y = 0$, hence $v_{max} = \omega A = \frac{2\pi}{T} \times A$,

$$v_{max} = \frac{2\pi}{4\,s} \times 2 \times 10^{-2}\,m = 3.14 \times 10^{-2}\,ms^{-1}$$

(b) The velocity of the particle when $y = \frac{A}{2}$ is given by $v = \frac{2\pi}{T}\sqrt{A^2 - \left(\frac{1}{2}A\right)^2}$

$$v = \frac{2\pi}{4\,s}\sqrt{(0.02\,m)^2 - \left(\frac{1}{2} \times 0.02\,m\right)^2}$$

Hence, $v = 2.72 \times 10^{-2}\,ms^{-1}$.

Therefore, the maximum velocity is $3.14 \times 10^{-2}\,ms^{-1}$ and velocity at a position half way is $2.72 \times 10^{-2}\,ms^{-1}$.

(d) Acceleration

Acceleration is defined as the time rate of change of velocity, i.e., $a = \frac{dv}{dt}$.

Thus, $a = \frac{d}{dt}(A\omega \cos \omega t)$,

$$a = -\omega^2 (A \sin \omega t)$$

$$a = -\omega^2 y \qquad (3.29)$$

If the angular speed ω is constant, then acceleration of a body performing SHM varies linearly with displacement, y (Figure 3.20(b)). Thus SHM can be defined as a to-and-fro motion of an object about an equilibrium position whose acceleration is directly proportional to the displacement.

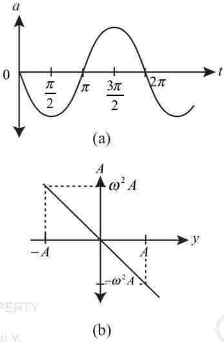


Figure 3.20 Variation of (a) acceleration with time, and (b) acceleration with displacement

Example 3.10

Calculate the period for a particle executing simple harmonic motion with acceleration of $16 cms^{-2}$ at a distance of $4\,cm$ from the equilibrium position.

Solution

Using the relation $a = -\omega^2 y$

$|a| = \left|\left(\frac{2\pi}{T}\right)^2 y\right|$, hence $T = 2\pi\sqrt{\frac{y}{a}}$.

Substituting values of $a = 16\,cms^{-2}$ and $y = 4\,cm$ gives $T = 3.14\,s$

Therefore, the period of oscillation of the particle is 3.14 s.

Example 3.11

The velocity of a particle executing simple harmonic motion is $16\ \text{cms}^{-1}$ at a distance of 8cm and $8\ \text{cms}^{-1}$ at a distance 12cm from mean position. Determine the amplitude of the motion.

Solution

Using the relation $v = \omega\sqrt{A^2 - y^2}$

$$16\,\text{cms}^{-1} = \omega\sqrt{A^2 - (8\,\text{cm})^2} \qquad (i)$$

$$8\,\text{cms}^{-1} = \omega\sqrt{A^2 - (12\,\text{cm})^2} \qquad (ii)$$

Dividing equation (i) by (ii) gives, $A = \pm 13.06$ cm

Hence the amplitude of the system is ± 13.06 cm.

3.2.3 Examples of simple harmonic motion

Simple harmonic motion can be well explained using various examples including: vertical oscillations of a loaded helical spring, oscillations of a liquid in a U-tube, simple pendulum and floating loaded test tube.

(a) Vertical oscillations of a loaded helical spring

Consider a massless helical spring suspended vertically as in Figure 3.21(a). When a mass m is attached on it, the spring stretches to point O (Figure 3.21(b)). Suppose the system is stretched to a further displacement y and then released (Figure 3.21(c)). The resulting vertical oscillation is approximately simple harmonic motion.

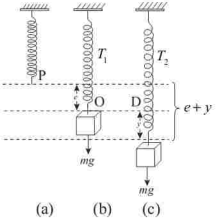


Figure 3.21 Suspended helical spring with mass m

If the system is in equilibrium, then net vertical force is zero. That is $T_1 = mg$, where T_1 is the tension in the spring given by Hooke's law as $T_1 = -ke$, where e is the extension.

Therefore,

$$-ke = mg, \qquad (3.30)$$

where k is spring constant or force constant.

Vertical forces in Figure 3.21 (c) when the mass is displaced downwards by the length y is given by;

$$\sum F_y = T_2 - mg \qquad (3.31)$$

where $T_2 = -k(e + y)$

$\sum F_y$ is called restoring force, since it is directed opposite to displacement y and towards the equilibrium position (point D). It is responsible in bringing the system back to the equilibrium. Since $\sum F_y$ is a net force, it can be written as

$$\sum F_y = ma \qquad (3.32)$$

Equating equation (3.31) and (3.32), gives, $ma = -mg - ke - ky$, but $mg = -ke$

$$a = -\frac{k}{m}y \qquad (3.33)$$

Since k and m are constants, then $a \propto -y$. Hence a loaded helical spring executes simple harmonic motion. Comparing equations (3.29) with (3.33) gives,

$$\omega^2 = \frac{k}{m} \qquad (3.34)$$

The period of oscillation T is obtained from equation (3.34). Since $\omega = \frac{2\pi}{T}$; $T = \frac{2\pi}{\omega}$

Therefore,

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad (3.35)$$

also, $ke = mg$; $\frac{e}{g} = \frac{m}{k}$

Therefore, $$T = 2\pi\sqrt{\frac{e}{g}} \qquad (3.36)$$

In practical situations, usually two or more springs are connected either in series or in parallel to each other.

Example 3.12

A $3.0\,\text{kg}$ ball is attached to a spring of negligible mass and with a spring constant $k = 40\ \text{Nm}^{-1}$. The ball is displaced $0.10\,\text{m}$ from equilibrium and then released. What is the maximum speed of the ball as it undergoes simple harmonic motion?

Solution

Maximum speed occurs at equilibrium, when $y = 0$, that is $v_{max} = \omega A$

But $\omega = \sqrt{\frac{k}{m}}$, $v_{max} = \left(\sqrt{\frac{k}{m}}\right)A$,

$$v_{max} = \sqrt{\frac{40\,\text{Nm}^{-1}}{3\text{kg}}} \times 0.1\text{m} = 0.37\,\text{ms}^{-1}$$

Therefore, the maximum speed is $0.37\,\text{ms}^{-1}$.

Consider two springs of force constants k_1 and k_2 respectively arranged in series and a force F is applied at a free end of lower spring as shown in Figure 3.22.

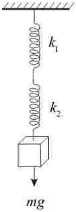


Figure 3.22 Springs with mass m each arranged in series

At equilibrium, F produces different extensions e_1 and e_2 on the springs of k_1 and k_2 respectively. If e_T is the total extension, then $e_T = e_1 + e_2$.

But from Hooke's law,

$$e = -\frac{F}{k} = \frac{F}{k_T} = \frac{F}{k_1} + \frac{F}{k_2}$$

Dividing both sides by F gives

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2}$$

Generally for n springs connected in series

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$

For identical springs $k_1 = k_2 = k$, then,

$$\frac{1}{k_T} = \frac{n}{k} \text{ or } k_T = \frac{k}{n}$$

In parallel connections (Figure 3.23), the same extension is produced in both springs, but the force in the springs is different.

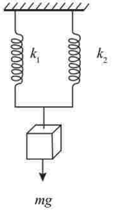


Figure 3.23 Springs connected in parallel

Then, $F = F_1 + F_2$, $k_T e = k_1 e + k_2 e$. For n springs connected in parallel,

$$k_T e = k_1 e + k_2 e + ... + k_n e$$

$$= e\sum_{i=1}^{n} k_i$$

For n identical springs,

$$\sum_{i=1}^{n} k_i = k_T = nk$$

Example 3.13

A car with a mass of 1300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20000 Nm^{-1}. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Solution

To analyze the problem, we first need to consider the effective spring constant of the four springs combined. For a given extension e of the springs, the combined force on the car is the sum of the forces from the individual springs: $F = \sum(-ke) = -\left(\sum k\right)e$, where e has been factored from the sum because it is the same for all four springs.

The effective spring constant for the combined stream $= k_{eff} = \left(\sum k\right)$

Frequency, $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k_{eff}}{m}}$, where m is mass of the car plus people.

$$f = \frac{1}{2\pi}\sqrt{\frac{4 \times 20000\,\text{N/m}}{1460\,\text{kg}}} = 1.18\,\text{Hz}$$

Therefore, the frequency of vibration of the car is, $f = 1.18$ Hz.

(b) Oscillation of liquid in a U-tube

Consider a liquid of density ρ in a U-tube at equilibrium as shown in Figure 3.24(a). If the liquid on one side of the U-tube is depressed by blowing gently down as in Figure 3.24(b) and released, the liquid will oscillate for a short time about the respective initial position O before finally coming to rest.

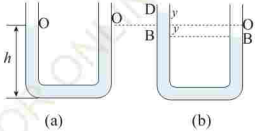


Figure 3.24 Liquid in a U-tube

At some instant, suppose that the level of the liquid on the left side tube is at D, at a height above its original (undisturbed) position O. The level B of the liquid on the other arm is then at a depth y below

Physics Form V.indd 71 23/07/2020 09:17

its original position O. So, the excess pressure on the whole liquid is $2y\rho g$. This excess pressure will exert a force F on the liquid in the tube and restores equilibrium at O;

$F = -2y\rho gA$ where A is the cross-section area of the tube. Since $F = ma$,

$$a = \frac{-2y\rho gA}{m}$$

The mass m of the liquid is given by $m = \rho \times A \times 2h$, thus, $a = \frac{-2\rho Ag}{2\rho Ah}y$.

It follows that,

$$a = -\left(\frac{g}{h}\right)y \qquad (3.37)$$

Note that, acceleration 'a' is directly proportional to displacement 'y' then the oscillations are simple harmonic. Comparing equations (3.37) with (3.29) gives,

$\omega^2 = \frac{g}{h}, \omega = \frac{2\pi}{T}$, which gives,

$$T = 2\pi\sqrt{\frac{h}{g}} \qquad (3.38)$$

where T is the period of oscillation of a liquid in a U-tube.

(c) Oscillations of a simple pendulum

A simple pendulum consists of a thread of length l and a bob of a mass m attached at its end, leaving the free end of a string be attached to a fixed-support as shown in Figure 3.25.

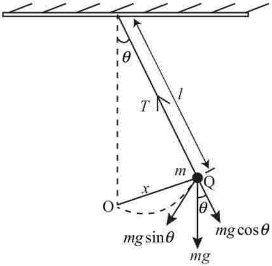


Figure 3.25 The simple pendulum

There are two forces which act on the bob, the weight mg and the tension T in the string.

Resolving the force mg along the tangent and along the thread:

Along the tangent $F = mg\sin\theta$, and along the thread $mg\cos\theta$. Since the force, $F = -mg\sin\theta$ tends to take the bob towards an equilibrium position, then this should be the restoring force. Thus, $a = -g\sin\theta$ where, $F = ma$.

Suppose the bob moving from Q to O accelerates with an acceleration equal to $a = -g\sin\theta$. For strictly SHM, the setting of oscillating body has to be at small angle, whereby, for small angle $\sin\theta \approx \theta$. Thus, $a = -g\theta$. For small oscillations, from arc length, $\frac{x}{l} = \theta$, this results into $a = -\frac{g}{l}x$. Hence from equation (3.29) $a = -\omega^2 x$, then, $\omega^2 = \frac{g}{l}$

but, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$. Therefore, the period of oscillation of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

(d) Floating loaded test tube

Consider a loaded test tube of total mass m floating in a liquid of density ρ (Figure 3.26)

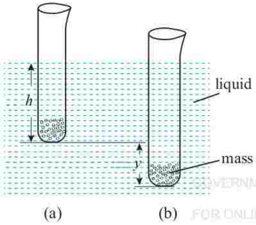


Figure 3.26 Loaded test tube floating in a liquid

When a loaded test tube is placed in the liquid, it is submerged a distance h below water level. When the tube is slightly pushed down and released (Figure 3.26(b)), will oscillate up and down.

The restoring force (F) is equal to excess upthrust (F_{up}) due to height y

where $F = ma$ and $F_{up} = -\rho Ayg$. Hence,

$$ma = -\rho Ayg \qquad (3.39)$$

but,

$$m = \rho Ah \qquad (3.40)$$

where m is the mass of displaced liquid.

Substituting equation (3.40) into (3.39) gives $a = -\frac{g}{h}y$, which implies that, $a \propto -y$.

Hence, it is a simple harmonic motion with a period $T = 2\pi\sqrt{\frac{h}{g}}$.

3.2.4 Energy changes in simple harmonic motion

When a body oscillates in simple harmonic motion, its energy is constantly changing between potential energy and kinetic energy. But the total energy is always constant according to the law of conservation of mechanical energy.

Consider a particle of mass m executing simple harmonic motion with an amplitude A and constant angular velocity ω. If x is its displacement at a time t, then the magnitude of restoring force F is $F = -m\omega^2 x$. Suppose the particle undergoes a further displacement dx, a small amount of work dW is done against the restoring force. This work is equal to the change in potential energy ΔU of the oscillator given by $\Delta U = -Fdx$. Thus, $dW = m\omega^2 x dx$.

The total work done for the displacement x is $\int_0^W dW = \int_0^x m\omega^2 x\, dx$, $W = \frac{1}{2}m\omega^2 x^2$

But, $\omega^2 = \frac{k}{m}$, thus, $W = \frac{1}{2}kx^2$

Therefore, the potential energy

$$U = \frac{1}{2}kx^2 \qquad (3.41)$$

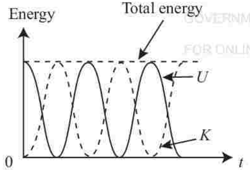
The kinetic energy is given by the relation $K = \frac{1}{2}mv^2$, where v is the velocity when its displacement is x. Then the kinetic energy can be expressed as,

$$K = \frac{1}{2}m\omega^2(A^2 - x^2) \qquad (3.42)$$

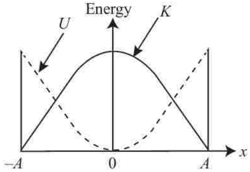
The total energy E_T of the particle at any point is the sum of kinetic energy and potential energy.

$$E_T = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$$
$$= \frac{1}{2}m\omega^2 A^2$$

Therefore, the total energy of a particle executing SHM is always constant. The variations of total energy between potential energy and kinetic energy with displacement or with time are shown in Figure 3.27.



(a)



(b)

Figure 3.27 Energy variation in SHM

When the displacement is zero (at equilibrium position), all the energy is kinetic energy. When displacement is maximum (at the end) all the energy is converted into potential energy.

Example 3.14

A particle of mass 0.25 kg vibrates with a period of 2.0 seconds. If its greatest displacement is 0.4 m, what is its maximum kinetic energy?

Solution

From $K = \frac{1}{2}m\omega^2(A^2 - x^2)$,

$$K_{max} = \frac{1}{2}m\omega^2 A^2 \text{ (at } x = 0\text{)}$$

$$K_{max} = \frac{1}{2} \times 0.25\,\text{kg} \times \left(\frac{2\pi}{2\,\text{s}}\right)^2 \times (0.4\,\text{m})^2$$

$$= 0.197\,\text{J}$$

Therefore, the maximum kinetic energy is 0.197 J.

3.2.5 Applications of simple harmonic motion

Simple harmonic motion plays a role in functioning of different appliances. These include clocks, shock absorbers, musical instruments, gravimeters and seismo-scope.

Clock

A large pendulum clock or vibrating quartz crystal are in periodic motion in order to ensure that indicated time is accurate. This is due to the fact that the oscillator has a constant period because it is in simple harmonic motion, thus it keeps time accurately.

Car shock absorbers

Springs attached to car wheels ensure a smooth car ride in roads with bumps. The absence of shock absorbers will make the car to move up and down when it passes over a bump, thus causing unpleasant condition to passengers. When there are springs in the car the wheel will rise compressing the spring while the car body remain relatively stationary on the compressed spring after passing over the bump. Since the car executes simple harmonic motion, shock absorbers will push the car back to normal place leaving the passengers in pleasant ride.

Musical instruments

In a string instrument, for example violin and guitar, bowing or plucking the string provides the force required to make the spring oscillate and produce sound. The vibration produced in the string causes the air column to execute SHM which will result into producing a regular sound.

Hearing

The ear functions due to SHM phenomena. The sound waves travel through the air and when they arrive at the eardrum, they cause it to vibrate. This signal from the eardrum is sent to the brain for interpretation.

Seismometer

Motion of the ground such as that due to seismic waves from earthquake and volcanic eruption is measured by seismometer. A seismometer consists of a pendulum with stylus at its bottom which is connected to a frame. The pendulum executes simple harmonic motion. During earthquake, the stylus draws a pattern on a paper which describes the ground movement. The pattern represents the strength of the earthquake. Also, gravimeter pendulum executes SHM which will enable measurement of local gravity at a given location.

Example 3.15

A particle oscillating with SHM has a speed of $v = 8.0\,\text{ms}^{-1}$ and an acceleration of $a = 12\,\text{ms}^{-2}$ when it is $3\,\text{m}$ from its equilibrium position. Find:

(a) Amplitude of the motion;
(b) Maximum velocity; and
(c) Maximum acceleration.

Solution

(a) Given $a = \omega^2 y$; $\omega^2 = \dfrac{a}{y}$.

Substituting into $v = \omega\sqrt{A^2 - y^2}$ gives,

$$v = \sqrt{\frac{a}{y}} \times \sqrt{A^2 - y^2}, \text{ solving for } A,$$

$$A = \sqrt{\frac{v^2 y + a y^2}{a}}$$

$$A = \sqrt{\frac{\left(8\,\text{ms}^{-1}\right)^2 \times 3\,\text{m} + 12\,\text{ms}^{-2} \times \left(3\,\text{m}\right)^2}{12\,\text{ms}^{-2}}}$$

$$= 5\,\text{m}$$

Therefore, the amplitude is $5\,\text{m}$.

(b) Maximum velocity,

$$v_{\max} = \omega A = A\sqrt{\frac{a}{y}}$$

$$v_{\max} = 5\,\text{m}\sqrt{\frac{12\,\text{ms}^{-2}}{3\,\text{m}}} = 10\,\text{ms}^{-1}$$

Therefore, the maximum velocity is $10\,\text{ms}^{-1}$.

(c) Maximum acceleration,

$$a_{max} = \omega^2 A = \left(\frac{a}{y}\right) \times A,$$

$$a_{max} = \frac{12\,ms^{-2}}{3\,m} \times 5\,m = 20\,ms^{-2}$$

Therefore, the maximum acceleration is $20\,ms^{-2}$.

Exercise 3.2

1. Think of several examples in everyday life of motion that is at least approximately simple harmonic. In what respects does each deviate from SHM?
2. The analysis of simple harmonic motion neglected the mass of the spring, how would the spring's mass affect the period and frequency of the motion? Explain your reasoning.
3. In any periodic motion, unavoidable friction always causes the amplitude to decrease with time. Does friction also affect the period of SHM? Give a qualitative argument to support your answer. (*Hint:* Does the friction affect the kinetic energy? If so, how does this affect the speed, and therefore the period, of a cycle?)
4. A mass attached by a light spring to the ceiling of an elevator oscillates vertically while the elevator ascends with constant acceleration. Is the period greater than, less than, or the same as when the elevator is at rest? Why?
5. (a) Explain the meaning of the following terms as used in simple harmonic motion:
   (i) Period;
   (ii) Amplitude; and
   (iii) Restoring force.
   (b) How can a uniform motion in a circle be related to a simple harmonic motion?
6. A butcher throws a cut of beef on a spring scale which then oscillates about an equilibrium position with the period $T = 0.5$ seconds. The amplitude of vibrations being $A = 2.0\,cm$ and having displacement of $4.0\,cm$, determine:
   (a) Frequency;
   (b) Maximum acceleration; and
   (c) Maximum velocity.
7. It is found that a load of mass 200 g stretches a spring by 10.0 cm. The same spring is then stretched by an additional 5.0 cm and released. Find:
   (a) Spring constant;
   (b) Period of vibrations and frequency;
   (c) Maximum acceleration; and
   (d) Velocity through equilibrium position.
8. A tray of mass 12 kg is supported by two identical springs (Figure 3.28). When the tray is displaced slightly and released it executes SHM with a period of 1.5 seconds.
   (a) What is the force constant in each spring?
   (b) When a block of mass m is placed above the tray, the period of oscillation changes to 3.0 seconds. What is the value of m?

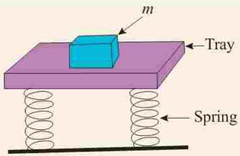


Figure 3.28 SHM of a loaded tray on two identical springs

9. For a particle vibrating with simple harmonic motion the displacement is 12 cm at the instant the velocity is 5 cm/s and the displacement is 5 cm at the instant the velocity is 12 cm/s. Assuming the amplitude is constant, calculate:
   (a) Amplitude;
   (b) Frequency; and
   (c) Period.
10. A wooden cylindrical bar is floating vertically in water 30 cm of its length below the water surface. The bar is slightly dipped and then released to execute vertical oscillation.
   (a) Prove that the oscillations are approximately simple harmonic motion.
   (b) Determine the period of oscillations.

3.3 Gravitation

Gravitation refers to the force of attraction that exists between any two bodies that have mass. It is a universal force affecting the largest and the smallest objects and all forms of matter. Gravitation governs the motion of astronomical bodies. It keeps the moon in orbit around the earth and keeps the earth and the other planets of the solar system in orbit around the sun. On a larger scale, it governs the motion of stars and slows the outward expansion of the entire universe because of the inward attraction of galaxies to other galaxies. An understanding of the law of universal gravitation has allowed scientists to send spacecraft on impressively accurate journeys to other parts of our solar system. Such description of planetary motion provides astronomical data which are important test of the validity of the law.

Typically, the term gravitation refers to the force in general, and the term gravity refers to the earth's gravitational pull. There are several laws governing gravitation and planetary motion. Therefore, in this section, you will learn laws of gravitation specifically the Newton's law of universal gravitation and Kepler's laws of planetary motion. You will also learn how to derive the relationships existing between the laws, determine gravitational potential of a body and applications of the laws.

3.3.1 Kepler's Laws of Planetary Motion

A general study of planetary motion played a big role in the development of Physics. This began by the earliest scientists, basically the Greek astronomers, who attempted to study and explain the movement of the sun and other planets. They assumed the earth was the centre of the universe while the moon, stars and other planets are revolving around it in complex orbits.

In the 15th century Copernicus suggested that the sun was at rest at the centre of universe and so the earth was a planet rotating on its own axis at the same time moving around the sun and other planets had similar motions.

The controversy over these theories stimulated different astronomers to obtain more accurate observational data. Then Tycho Brahe obtained good data on planetary motion. His data were analysed and compiled by Johannes Kepler, who was Brahe's assistant, and found three important regularities with regard to planetary motions and these are known as Kepler's laws of planetary motion.

(a) Kepler's First Law

The law states that *"Planets revolve round the sun in elliptical orbits with the sun as one focus"*. This is known as the law of orbits or ellipses and the phenomenon is shown in Figure 3.29.

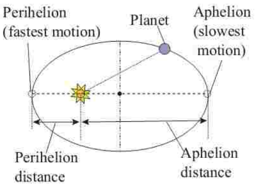


Figure 3.29 Orbit of a planet

With elliptical orbits, a planet is sometimes closer to the sun than it is at other times. The point at which it is closest is called *perihelion*, and the point at which a planet is furthest is called *aphelion*.

(b) Kepler's Second Law

The law states that, *"An imaginary line from the planet to the sun sweeps out equal areas in equal amounts of time"*. Kepler's second law basically says that the planet's speed is not constant. It moves with lowest speed at aphelion and highest at perihelion. The law allows an astronomer to calculate the orbital speed of a planet at any point (Figure. 3.30). This law sometimes is called law of equal areas. The area swept A, B and C in time intervals t are equal.

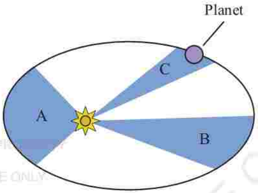


Figure 3.30 Equal areas over equal times

(c) Kepler's Third Law

The law states that, *"The square of the period of revolution of a planet in its orbit is directly proportional to the cube of the average distance from the sun to the planet."* Sometimes this law is known as the law of periods.

$T^2 \propto r^3$, thus, $T^2 = kr^3$, where k is the constant of proportionality.

If T_1 and r_1 is the period and orbital radius of planet A and T_2 and r_2 for planet B, then

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3,$$

T_1 and T_2 are periods of two planets respectively and r_1 and r_2 are average distances of the planets from the sun respectively. This law holds for all space bodies. For example, the moon moving around the earth and all other satellites of the earth.

3.3.2 Newton's Law of Universal Gravitation

Newton pointed out that everybody in the universe attracts every other body. He proposed the law of universal gravitation which states that, *"The force of attraction between two bodies in the universe is directly proportional to the product of their masses and inversely proportional to the square of distance between their centres"*. ***Note that,*** the force of gravitation always acts along the line joining the centres of the two bodies.

Consider two bodies of masses m_1 and m_2 separated by distance r between their centres as in Figure 3.31.

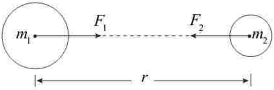


Figure 3.31 Mutual gravitation force

According to Newton's law of gravitation

$$F \propto m_1 m_2 \qquad (3.43)$$

$$F \propto \frac{1}{r^2} \qquad (3.44)$$

Combining equation (3.43) and (3.44) gives,

$$F = k\frac{m_1 m_2}{r^2} \qquad (3.45)$$

where a constant k is called universal gravitational constant,

$$G = 6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}.$$

Re-writing equation (3.45), gives

$$F = G\frac{m_1 m_2}{r^2}. \qquad (3.46)$$

Kepler's Third Law and Newton's Law of Universal Gravitation

Kepler's third law of planetary motion can be derived from Newton's law of universal gravitation. In order to show this relationship, an approximation of circular orbits of planets must be used. Consider a planet of mass m_p revolving around the sun of mass M_s in a circular orbit of radius R as in Figure 3.32.

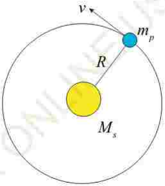


Figure 3.32 *Planetary orbit around the sun*

Since the mass of the sun is much larger than that of a planet, it is correct (for circular orbits) to assume the sun's position at the centre of the circular orbit. Therefore, the centripetal force on the planet is provided by the sun's gravitational pull given by,

$$\frac{m_p v^2}{R} = G\frac{M_s m_p}{R^2}$$

But $v = \omega R$ and $\omega = \frac{2\pi}{T}$, then,

$$T^2 = \frac{4\pi^2 R^3}{GM_s}$$

The term $\frac{4\pi^2}{GM_s}$ is constant, hence

$$T^2 \propto R^3.$$

Example 3.16

Two masses of 800 kg and 600 kg are at a distance of 0.25 m apart. Calculate the magnitude of gravitation force of attraction between them.

Solution

$$F = G\frac{m_1 m_2}{r^2}$$

$$= \frac{6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}\times 800\,\text{kg}\times 600\,\text{kg}}{(0.25\,\text{m})^2}$$

$$F = 5.12\times10^{-4}\,\text{N}$$

Example 3.17

Assuming the orbit of the earth about the sun to be circular (it is actually slightly elliptical) with radius 1.5×10^{11} m, find the mass of the sun. The earth revolves around the sun in 3.12×10^{7} seconds.

Solution

For the earth to revolve around the sun it requires a centripetal force. This centripetal force is provided by the gravitational pull of the sun. Therefore,

centripetal force (F_c) = gravitational force (F_g)

$$\frac{m_e v^2}{r} = G\frac{m_e m_s}{r^2} \rightarrow m_s = \frac{v^2 r}{G},$$

but $v = \omega r$ and $\omega = \frac{2\pi}{T}$, then

$$m_s = \frac{4\pi^2 r^3}{T^2 G}$$

$$m_s = \frac{4\pi^2\times\left(1.5\times10^{11}\,\text{m}\right)^3}{\left(3.12\times10^{7}\,\text{s}\right)^2\times 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}}$$

$$= 2.0\times10^{30}\,\text{kg}$$

Therefore, mass of the sun is about 2.0×10^{30} kg.

Example 3.18

There is a point along the line joining the centres of the Earth and the moon where an object of mass m, does not experience a force of gravity due to the earth and the moon. Determine the position of an object with respect to the earth (Mass of the earth, $m_e = 6.0\times10^{24}$ kg, mass of the moon $m_m = 3.35\times10^{22}$ kg, earth-moon distance $R = 3.8\times10^{8}$ m)

Solution

Let x be the distance from point of zero gravitational force to the centre of the earth as illustrated in Figure 3.33.

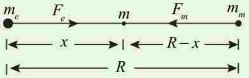


Figure 3.33 *Gravitational force*

The net gravitational force on mass m,

$\sum F = F_m + (-F_e)$, but, $\sum F = 0$,

hence $F_e = F_m$; $\dfrac{Gm_e m}{x^2} = \dfrac{Gm_m m}{(R-x)^2}$

Solving for x gives, $x = (R-x)\left(\dfrac{m_e}{m_m}\right)^{\frac{1}{2}}$

$$x = (3.8\times10^8\,\text{m} - x)\left(\frac{6.0\times10^{24}\,\text{kg}}{3.35\times10^{22}\,\text{kg}}\right)^{\frac{1}{2}}$$

$$= 3.54\times10^8\,\text{m}$$

The position of an object with respect to the earth is, $x = 3.54\times10^8$ m.

3.3.3 Acceleration due to gravity (g)

Consider an object of mass m placed on a uniform spherical earth of mass m_e and radius r_e as shown in Figure 3.34.

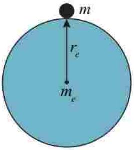


Figure 3.34 A body placed on the surface of the earth

The gravitational force of attraction on mass m is obtained from Newton's law of gravitation $F = G\dfrac{m_e m}{r_e^2}$.

The attractive force which the earth exerts on the object is the weight of that object.

$$F = mg = G\frac{m_e m}{r_e^2}$$

Therefore,

$$g = G\frac{m_e}{r_e^2} \qquad (3.47)$$

The mass of the earth can be calculated from equation 3.47,

$$m_e = \frac{gr_e^2}{G} \qquad (3.48)$$

and the density of the earth

$\rho = \dfrac{m_e}{V_e} = \dfrac{gr_e^2}{G}\left(\dfrac{3}{4\pi r_e^3}\right)$, since the earth is sphere, therefore,

$$\rho = \frac{3g}{4\pi r_e G} \qquad (3.49)$$

Equation (3.47) provides the value of g on the surface of the earth. Experiments show that g varies from place to place on the surface of the earth as well as with altitude.

(a) Variation of g with altitude

Suppose that an object of mass m is at a height h above the surface of the earth (Figure 3.35).

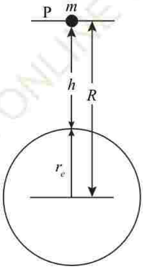


Figure 3.35 Value of g above the earth's surface

Assume the earth to be a uniform sphere of radius r_e and mass m_e, then the value of g at point P is

$$g' = G\frac{m_e}{R^2} \qquad (3.50)$$

Dividing equation (3.50) by (3.47), gives

$$g' = \frac{gr_e^2}{R^2}$$

Therefore, $g' \propto \frac{1}{R^2}$ since gr_e^2 is constant.

If the object is at height close to the surface of the earth, then

$$g' = \frac{gr_e^2}{(r_e+h)^2} = \frac{g}{\left(1+\frac{h}{r_e}\right)^2}, \text{ or}$$

$$g' = g\left(1+\frac{h}{r_e}\right)^{-2}$$

By Binomial expansion and neglecting the higher powers of $\frac{h}{r_e}$,

$$g' = g\left(1-\frac{2h}{r_e}\right) \qquad (3.51)$$

(b) Variation of *g* with depth

Consider a body of mass m placed at a depth d below the surface of the earth as shown in Figure 3.36.

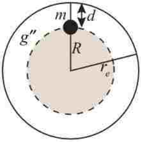


Figure 3.36 Variation of g with depth

The value of acceleration due to gravity on mass m at a depth d is given by,

$$g'' = G\frac{m_e'}{R^2} \qquad (3.52)$$

Where m_e' is the effective mass of the earth that exerts gravitational force on mass m given by,

$$m_e' = \rho\left(\frac{4}{3}\pi R^3\right) \qquad (3.53)$$

but, $m_e = \rho\left(\frac{4}{3}\pi r_e^3\right)$,

hence, $m_e' = \frac{R^3}{r_e^3}m_e$. Substituting into equation (3.52), gives,

$$g'' = \left(\frac{Gm_e}{r_e^2}\right)\frac{R}{r_e}, \text{ but } \frac{Gm_e}{r_e^2} = g.$$

Therefore,

$$g'' = \frac{g}{r_e}R \qquad (3.54)$$

since $\frac{g}{r_e}$ is constant, then $g'' \propto R$, then $R = r_e - d$, and equation (3.54) can be written as,

$$g'' = g\left(1-\frac{d}{r_e}\right) \qquad (3.55)$$

The acceleration due to gravity decreases with increase in depth. If $d = r_e$, then $g = 0$. Therefore, at the centre of the earth an object feels weightless. Equations (3.51) and (3.54) can be presented graphically as shown in Figure 3.37.

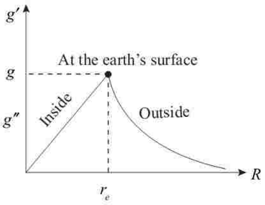


Figure 3.37 Variation of g inside and outside the earth's surface

(c) Variation of *g* with latitude

A latitude of a place is the angle which the line joining the place to the centre of the earth makes with the equatorial plane. As the earth spins on its axis, different places on the surface experience different speed with respect to the earth's axis of rotation. The variation of speed means that there is variation of force of gravity and therefore variation on the acceleration due to gravity. The effect of earth's rotation on the acceleration is shown in Figure 3.38.

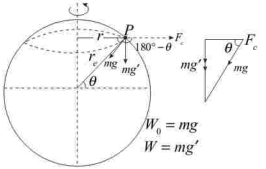


Figure 3.38 Effect of earth's rotation on the value of g

Suppose the weight of a body is W_0, part of this weight is used to provide the centripetal force F_c on the object. Therefore, at point P the measured weight is W (the apparent weight). The relationship between W and W_0 is given as $W^2 = F_c^2 + W_0^2 - 2W_0F_c\cos\theta$.

$$(mg')^2 = \left(m\omega^2 r\right)^2 + (mg)^2 - 2(mg)(m\omega^2 r)\cos\theta \quad (3.56)$$

Substitute $r = r_e\cos\theta$ and simplifying equation (3.56) gives,

$$g' = g\left(1 + \frac{\omega^4 r_e^2\cos^2\theta}{g^2} - \frac{2\omega^2 r_e\cos^2\theta}{g}\right)^{\frac{1}{2}}$$

The quantity $\dfrac{\omega^4 r_e^2\cos^2\theta}{g^2}$ is very small such that, it can be ignored, hence,

$g' = g\left(1 - \dfrac{2\omega^2 r_e\cos^2\theta}{g}\right)^{\frac{1}{2}}$, by expanding and ignoring higher terms,

$$g' = g\left(1 - \frac{\omega^2 r_e\cos^2\theta}{g}\right)$$

$$g' = g - \omega^2 r_e\cos^2\theta \quad (3.57)$$

The value of *g* increases from the equator to the poles. At the equator, $\theta = 0°$; $g' = g - \omega^2 r_e$ and at poles, $\theta = 90°$; $g' = g$.

Example 3.19

A body weighing 72 N on the surface of the earth, moves to a height half the value of the radius of the earth. What is its new weight?

Solution

Using the relation,

$$W = G\frac{m_e m}{r_e^2} \quad (i)$$

$$W' = G\frac{m_e m}{(r_e + h)^2} \quad (ii)$$

dividing equation (ii) by (i) gives

$$W' = \frac{r_e^2}{(r_e + h)^2}W$$

$$W' = \frac{r_e^2}{(r_e + \frac{1}{2}r_e)^2} \times 72\,\text{N} = 32\,\text{N}$$

Hence, the weight at height, h is $32\,\text{N}$.

Example 3.20

At what height from the surface of earth will the value of g be reduced by 36% from the value at the surface? (Radius of earth $r_e = 6400\,\text{km}$).

Solution

From equation $g' = g\left(1 - \frac{2h}{r_e}\right)$,

The change in g, $\frac{g - g'}{g} = \frac{2h}{r_e}$,

$$h = \frac{0.36 \times 6.4 \times 10^6\,\text{m}}{2} = 1.15 \times 10^6\,\text{m}$$

Hence, the value of g will be 36% at a height of $1.15 \times 10^6\,\text{m}$ above the earth's surface.

Example 3.21

What is the acceleration due to gravity on a surface of a planet that has a radius one third that of the earth and the same average density?

Solution

Using $g = G\frac{m_e}{r_e^2}$ and for the planet,

$$g_p = G\frac{m_p}{r_p^2}$$

$$\frac{g_p}{g} = \left(\frac{r_e}{r_p}\right)^2\left(\frac{m_p}{m_e}\right) \quad (i)$$

Since two planets have same density,

$$\frac{m_p}{m_e} = \left(\frac{r_p}{r_e}\right)^3 \quad (ii)$$

substituting (ii) into (i) gives,

$g_p = g\left(\frac{r_p}{r_e}\right)$. But $r_p = \frac{r_e}{3}$, thus,

$$g_p = \frac{1}{3}g$$

Therefore, taking $g = 9.8\,\text{ms}^{-2}$ gives, $g_p = 3.3\,\text{ms}^{-2}$.

3.3.4 Gravitational field and field strength

Gravitational field is defined as the region around a body where another massive object will experience gravitational force of attraction. The concept of field shows that a body due to its mass "modifies" the space around it such that another object (also due to its mass) when brought near the first object, experiences the "modifications" in form of force of gravity. Therefore, the gravitational field of one object will not act as a force of gravity on itself. The gravitational force on a body is exerted by the gravitational field created by other massive bodies. This means that if a very small massive object (called test mass) is placed at a point in space and experiences a gravitational force; then, there is a gravitational field at that point.

Gravitational field strength ($\vec{g}$) at a point is therefore defined as the gravitational force ($\vec{F}$) experienced by unit mass of a test mass at the point, i.e., $\vec{g} = \frac{\vec{F}}{m}$. In SI units, in which the unit of force is 1N, and the unit of mass is 1 kg, the unit of gravitational field strength is one newton per kilogram 1Nkg^{-1} or 1ms^{-2}. Hence, gravitational field strength at a point is equal to the acceleration due to gravity that a unit test mass would experience when placed at that point.

(a) Gravitational potential and potential energy

Gravitational Potential (V_G) at a point in the gravitational field is numerically equal to the amount of work done (W) in bringing a unit mass from infinity to that point. Consider a body of mass m placed outside the earth at point P at a distance r from the centre of the earth. Suppose that the body is moved from point A, a distance r_A to a distance r_B at point B by force of gravity as shown in Figure 3.39.

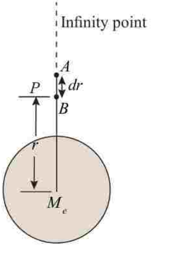


Figure 3.39 Gravitational potential at a point

The total work done per unit mass when a body is moved from A to B by gravitational force is

$$\frac{W_{AB}}{m} = V_B = \int_{r_A}^{r_B} \frac{F_g}{m} dr,$$

substituting $F_g = G\frac{M_e m}{r^2}$ and integrating, gives,

$$V_B = -GM_e\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \qquad (3.58)$$

In calculations of gravitational potential, a reference point is always chosen at infinity. The potential at infinity is taken to be zero. Hence

$$V_B = -G\frac{M_e}{r_B} \qquad (3.59)$$

Therefore, the expression for the potential at a given point is,

$$V_B = -G\frac{M_e}{r} \qquad (3.60)$$

The negative sign in the equation for potential signifies that the object at infinity would fall towards the earth but work is required to move objects from the earth to infinity.

Gravitational potential energy (U) at a point in the gravitational field is numerically equal to the work done in bringing the body from infinity to that point. That is,

$$U_B = W_{\infty B} \qquad (3.61)$$

From equation (3.60) and (3.61), $U_B = W = mV_B$ hence, $U_B = -G\frac{M_e m}{r}$.

Hence, gravitational potential energy is a property of a system of two bodies of masses M_e and m not of a single body. The negative sign shows that objects will have more potential energy as they move away from the earth.

(b) Relationship between gravitational field strength and gravitational potential

The gravitational field strength on the earth's surface is given by $g = G\frac{M_e}{r_e^2}$

The gravitational potential on earth's surface is given by $V = -G\frac{M_e}{r_e}$

dividing the two equations and solving for g gives,

$$g = -\frac{V}{r_e} \quad (3.62)$$

For small changes,

$$g = -\frac{dV}{dr} \quad (3.63)$$

Thus, the gravitational field strength (g) is also called the negative gravitational potential gradient.

Example 3.22

Calculate the gravitational intensity on the surface of Mars assuming it to be a uniform sphere of mass 6.4×10^{22} kg and radius of 3.375×10^{6} m. Use $G = 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$

Solution

$$g = G\frac{M}{r^2}$$

$$g = \frac{6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}\times6.4\times10^{22}\,\text{kg}}{(3.375\times10^{6}\,\text{m})^2}$$

$$= 0.375\,\text{Nkg}^{-1}$$

The gravitational intensity on the surface of Mars is $0.375\,\text{Nkg}^{-1}$.

Example 3.23

Calculate the distance from the earth to the point where the gravitational field due to the earth and the moon cancel out. Given that earth–moon distance is 3.8×10^{8} m and the mass of the earth is 81 times that of the moon.

Solution

Let x be the distance from the earth where the resultant gravitational field strength cancels out. Suppose a unit mass m is put at this point (Figure 3.40).

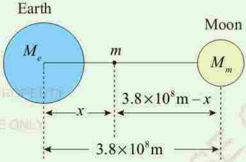


Figure 3.40 Distance between the earth and moon

Gravitational field strength cancel out at $g_e = g_m$

$$G\frac{M_e}{x^2} = G\frac{M_m}{(3.8\times10^8 - x)^2\,\text{m}^2}, \text{ but}$$

$M_e = 81M_m$ solving for x gives,

$$x = 3.42\times10^{8}\,\text{m}$$

Therefore, the gravitational field strength will cancel out at 3.42×10^{8} m away from the earth.

(c) Motion of satellites

A satellite is a body that revolves around another larger body (planet) in space. There are two types of satellites. These are natural satellites and artificial satellites. A natural satellite is a celestial body that revolves around a planet. It is called natural satellite because it is not man-made. For example, the moon is a natural satellite of the planet earth while Titania and Aerial are natural satellites of Uranus.

On the other hand, artificial satellites are man-made satellites that orbit the earth for communication or other purposes. For example, the international space station (ISS), Skylab, sputnik (1 and 2) and Telstar. Artificial satellites orbiting the earth are now quite common and many. They are called earth satellites.

(i) Launching of a satellite

To understand the principle of launching a satellite, consider a ball projected horizontally from a point above the earth's surface (Figure 3.41).

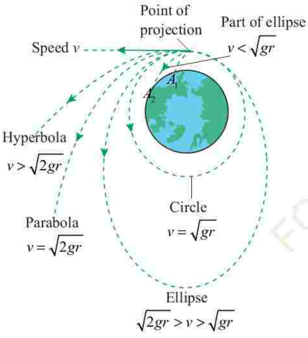


Figure 3.41 Launching of a satellite

If gravity did not act on the ball, then it would follow a straight line path shown by the solid line. But there is gravity so that the ball follows a parabolic path and hits the surface of the earth. If the horizontal velocity is increased the ball will travel a greater horizontal distance before hitting the surface of the earth. As a result, the horizontal range of the ball also increases. Finally, a stage is reached when the horizontal velocity is large enough that the ball's path follows the curvature of the earth. This is the launching velocity which places the ball (or any other body) in a circular orbit around the earth. Further increase in velocity results in the other orbits as shown.

Thus the object in the circular orbit may be regarded as falling but as it falls, its path is concentric with the earth's spherical surface so that the object maintains a fixed distance from the centre of the earth. The velocity required to put the object in its orbit is called the orbital velocity.

(ii) Orbital velocity of a satellite

Orbital velocity is the velocity required to put a satellite into a given circular orbit around the earth. Consider that a satellite of mass m_s is put into a circular orbit around the earth. Suppose m_e and R are the mass of the earth and radius of the orbit respectively as shown in Figure 3.42, where $R = r_e + h$, r_e is radius of the earth.

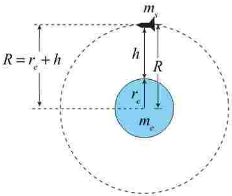


Figure 3.42 Satellite into circular orbit around the earth

The centripetal force on the satellite is provided by the gravitational force of attraction between the satellite and the earth. It then follows that,

$$\frac{m_s v^2}{R} = G\frac{m_e m_s}{R^2}$$

$$v = \sqrt{\frac{Gm_e}{R}} \qquad (3.64)$$

where v is the orbital velocity and R is the orbital radius of satellite.

From equation (3.64) it can be realized that, orbital velocity of a satellite is independent of its mass and decreases as the height increases.

When a satellite revolves close to the earth's surface, the height becomes very small as compared to radius of the earth, such that $r_e + h \approx r_e$. Then the orbital velocity can be approximated to:

$$v = \sqrt{\frac{Gm_e}{r_e}} = \sqrt{gr_e}\text{, since } Gm_e = gr_e^2$$

$$v = \sqrt{9.8\,\text{ms}^{-2} \times 6.4\times10^6\,\text{m}} = 8\times10^3\,\text{ms}^{-1}$$

Thus the orbital velocity of a satellite close to the earth is about $8\times10^3\,\text{ms}^{-1}$.

(iii) Period of a satellite (T)

This is the time taken by a satellite to complete one revolution. Consider a satellite of mass m_s put into a circular orbit of radius R at height h above the earth's surface (Figure 3.42).

The period of the satellite is given by

$$T = \frac{\text{Circumference of the orbit}}{\text{orbital velocity}} = \frac{2\pi R}{v}$$

Substituting $v = \sqrt{\frac{Gm_e}{R}}$ and $R = r_e + h$

gives

$$T = 2\pi\sqrt{\frac{(r_e + h)^3}{Gm_e}} \qquad (3.65)$$

T is the period of a satellite at a distance h from the earth's surface. The period of the satellite depends on the distance from the earth's surface. The greater the height above the earth's surface, the greater the period of revolution.

For Satellite close to the earth, i.e., $r_e >> h$, it follows that, $T = 2\pi\sqrt{\frac{r_e^3}{Gm_e}}$

Hence, the period of revolution of the satellite revolving very close to earth's surface is about 85 minutes.

Example 3.24

A satellite takes 24 hours to revolve on its orbit around the earth. Find the height above the earth at which the satellite should be placed.

Solution

From; $T = 2\pi\sqrt{\frac{(r_e + h)^3}{Gm_e}}$; $h = \sqrt[3]{g\left(\frac{r_e T}{2\pi}\right)^2} - r_e$

$$h = \sqrt[3]{9.8\,\text{ms}^{-2} \times \left(\frac{6.4\times10^{6}\,\text{m}\times 24\times3600\,\text{s}}{2\pi}\right)^{2}}$$

$$-6.4\times10^{6}\,\text{m} = 3.6\times10^{7}\,\text{m}$$

Thus, the satellite should be placed at 3.6×10^{7} m high above the earth's surface so that it revolves with the period of 1 day.

(iv) Energy of satellites

A satellite revolving around the earth has both kinetic energy (*K*) and potential energy (*U*). Thus the total mechanical energy of the satellite is the sum of its kinetic energy and potential energy (Figure 3.39).

Consider a satellite of mass m_s revolving the earth in circular orbit at a height h above the surface of the earth as shown in Figure 3.42.

From $\frac{m_s v^2}{R} = G\frac{m_e m_s}{R^2}$, $m_s v^2 = G\frac{m_e m_s}{R}$,

multiplying by $\frac{1}{2}$ both sides gives,

$$\frac{1}{2}m_s v^2 = G\frac{m_e m_s}{2R}$$

but, $\frac{1}{2}m_s v^2 = K$, and $R = r_e + h$

$$K = G\frac{m_e m_s}{2(r_e+h)} \qquad (3.66)$$

The potential energy U = work done $(m_s V)$ where V is the gravitational potential. But

$V = -\frac{Gm_e}{R}$ and $R = r_e + h$,

Therefore,

$$U = -G\frac{m_e m_s}{r_e+h} \qquad (3.67)$$

The total energy of the satellite is the sum of kinetic energy and potential energy.

$$E_T = K + U,\ E_T = G\frac{m_e m_s}{2(r_e+h)} - G\frac{m_e m_s}{r_e+h}$$

$$E_T = -G\frac{m_e m_s}{2(r_e+h)} \qquad (3.68)$$

Example 3.25

A geostationary satellite orbits the earth at the height of nearly 36000 km from the surface of the earth. What is the potential due to earth's gravity at the site of the satellite? Take radius of the earth $r_e = 6400\,\text{km}$, mass of the earth $m_e = 6\times10^{24}$ kg.

Solution

Consider Figure 3.43 with a geostationary satellite.

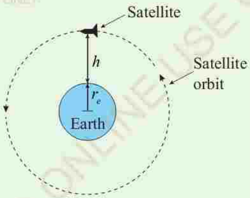


Figure 3.43 A geostationary satellite orbit

The gravitational potential at a height h above the earth's surface is given by;

$V = -G\frac{m_e}{R}$, where $R = r_e + h$,

$$V = -\frac{6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}\times6\times10^{24}\,\text{kg}}{(6.4\times10^{6}\,\text{m} + 3.6\times10^{7}\,\text{m})}$$

$= -9.4 \times 10^6\ \text{Jkg}^{-1}$

Potential due to earth's gravity at the site of the satellite is $-9.4 \times 10^6\ \text{Jkg}^{-1}$.

(v) Parking orbit

The orbit in which a satellite revolving around the earth has period equal to the period of rotation of the earth is called the parking orbit. The satellites placed in parking orbit are called geostationary satellites or synchronous satellites. Since geostationary satellites move with the same period as that of the earth, they appear on the same positon above the surface of the earth all the time as they move. These satellites are coplanar with the equator and move from west to east as the earth; this is the reason why they are called synchronous satellites.

Since the period of revolution is known, we can calculate velocity in the parking orbit. From $v = \sqrt{\frac{Gm_e}{R}}$;

$$R = \frac{vT}{2\pi} \qquad (3.69)$$

also $v = \sqrt{\frac{Gm_e}{R}}$; $R = \frac{Gm_e}{v^2}$ (3.70)

Equating equation (3.69) and (3.70) gives

$$v^3 = \frac{2\pi Gm_e}{T}$$

but $GM_e = gr_e^2$

$$\therefore v = \sqrt[3]{\frac{2\pi gr_e^2}{T}} \qquad (3.71)$$

Substituting the values $r_e = 6.4 \times 10^6\ \text{m}$, $g = 9.8\ \text{ms}^{-2}$, $T = 24 \times 60 \times 60\ \text{s}$ gives velocity of parking orbit $3.08\ \text{kms}^{-1}$.

The height of the parking orbit can be calculated using the relation, $R = \frac{vT}{2\pi}$

Since, $R = r_e + h$, then, $h = \frac{vT}{2\pi} - r_e$

$$h = \frac{(3.08 \times 10^3\ \text{ms}^{-1}) \times (24 \times 3600\ \text{s})}{2\pi} - 6.4 \times 10^6\ \text{m}$$

$$h = 3.6 \times 10^7\ \text{m}$$

Therefore, the parking orbit should be at 3.6×10^7 m high above the earth's surface.

(vi) Escape velocity

Suppose a ball is thrown into air; it rises up to a certain height and then falls back. If it is thrown with a large velocity, it rises to a higher height and falls back again. If a body is projected vertically upwards with sufficient velocity to allow it to move infinitely away from the earth, then, the body never returns and this velocity is called escape velocity. Therefore, escape velocity is defined as the minimum velocity with which a body may be projected such that it escapes from the earth's influence of gravitational force completely.

Suppose the escape velocity of a body is v_E then its kinetic energy at the point of projection is given by $K = \frac{1}{2}mv_E^2$.

The work done to remove the body from the surface of the earth is obtained by conserving mechanical energy of the body at the earth's surface and at infinity.

That is,

$$\frac{1}{2}mv_E^2 + \left(-\frac{Gm_e m}{r_e}\right) = 0$$

$$\frac{1}{2}mv_E^2 = \frac{Gm_e m}{r_e}$$

Therefore,

$$v_E = \sqrt{\frac{2Gm_e}{r_e}} \text{ or } v_E = \sqrt{2gr_e}$$

Substituting the values
$G = 6.67 \times 10^{11}\,\text{Nm}^2\text{kg}^{-2}$, $m_e = 6.0 \times 10^{24}\,\text{kg}$

and $r_e = 6.4 \times 10^6\,\text{m}$ gives, $v_E = 11200\,\text{ms}^{-1}$.

Therefore, in order for the body to escape from the influence of earth's gravitational field, it should be projected with the velocity greater or equal to $11.2\,\text{kms}^{-1}$.

If the body is at a height h from the surface of the earth, then its escape velocity is given by

$$v_E = \sqrt{\frac{2Gm_e}{r_e + h}} \text{ or } v_E = r_e\sqrt{\frac{2g}{r_e + h}}$$

(d) Uses of artificial satellites

When a satellite is to be placed in orbit, it is first carried to a desired height by a rocket. The satellite then turns into the required orbit. The earth's satellites have several uses including; learning about the atmosphere near the earth, weather forecasting and studying radiations from the sun and outer space. Also they are used in receiving and transmitting radio and television signals, and visualization of the actual shape and dimensions of the earth, research and security purposes.

Exercise 3.3

1. A student wrote, "The reason an apple falls downward to meet the earth instead of the earth falling upward to meet the apple is that the earth is much more massive than the apple and therefore exerts a much greater pull than the apple does." Is this explanation correct? If not, what is the correct one?
2. In discussions on satellites by laymen, one often hears questions such as "What keeps the satellite moving in its orbit?" and "What keeps the satellite up?" How do you answer these questions? Are your answers also applicable to the moon?
3. "Astronauts in satellites orbiting around the earth are weightless because the earth's gravity is so weak up there that it is negligible". Is the statement true or false? Explain.
4. What are the differences between gravitation and gravity?
5. An object of mass M is broken into two pieces. What should be their masses if the force of gravitation between them is to be minimum?
6. Explain the following:
   (a) Since the moon is constantly attracted toward the earth by the gravitational interaction, why does it not crash into the earth?
   (b) Which takes more fuel, a voyage from the earth to the moon or from the moon to the earth?
7. Calculate the percentage decrease in weight of a body when taken 32 km below the surface of the earth.

8. Deduce Newton's law of universal gravitation from Kepler's laws of planetary motion.
9. Two masses 800 kg and 600 kg are at a distance 0.25 m apart. Calculate the magnitude of the gravitational field intensity at a point a distance 0.2 m from the 800 kg mass and 0.15 m from the 600 kg mass. Given $G = 6.67 \times 10^{-11}\,\text{Nm}^2\text{kg}^{-2}$
10. What is the change in gravitational potential energy of a body with a mass of 10 kg when taken to a height of r_e from the earth?
11. Gravitational potential at a point 2500 km from the surface of the earth is $-1.5 \times 10^7\,\text{Jkg}^{-1}$. Find the gravitational field strength at this point (Radius of the earth, $r_e = 6400\,\text{km}$).
12. Discuss the following:
    (a) The importance of the artificial satellites and their uses with regard to the planet earth;
    (b) A person sitting in an artificial satellite that is moving around the earth feels weightlessness; and
    (c) Is it possible for a pendulum to vibrate (oscillate) in an artificial satellite?
13. Discuss the importance of parking orbit for our earth.
14. Given that the mass of the moon is $7.5 \times 10^{22}\,\text{kg}$ with a mean radius of $1.75 \times 10^6\,\text{m}$ and the universal gravitational constant, $G = 6.67 \times 10^{-11}\,\text{Nm}^2\text{kg}^{-2}$. Determine the escape velocity from the moon.
15. Assume the radius of the earth to be 6400 km. A body with a mass of 40 kg is moved to a height of 100 km above the surface of the earth.
    (a) Determine the weight of this body at this new position.
    (b) What causes the acceleration due to gravity to vary over the earth's surface?
16. If the acceleration of free fall at the earth's surface is $9.8\,\text{ms}^{-2}$ and the radius of the sarth is 6400 km, calculate the mass of the earth. ($G = 6.67 \times 10^{-11}\,\text{Nm}^2\text{kg}^{-2}$).
17. The gravitational force on a mass of 1 kg at the earth's surface is 10 N. Assuming the Earth is a sphere of radius R , calculate the gravitational force on a satellite of mass 100 kg in a circular orbit of radius 2R.
18. What is the amount of energy needed to launch a satellite of mass 2000 kg from the earth's surface in a parking orbit? (Neglect air resistance).
19. A satellite is revolving in the parking orbit around the earth. If it is suddenly stopped and allowed to fall freely on to the earth, find the speed with which it hits the surface of the earth.
20. If two uniform spheres each with mass M and radius R, touch one another, show that the magnitude of their gravitational force of attraction is $G\left(\frac{M}{2R}\right)^2$.
21. According to Newton's law of universal gravitation, gravitational force is a mutual force. Why then only objects "fall" towards the earth and not the earth "falling" towards other objects in its vicinity?

Revision exercise 3

1. At what point in the motion of a simple pendulum is
   (a) the tension in the string greatest?
   (b) the string tension least?
   (c) the radial acceleration greatest?
   (d) the angular acceleration least?
   (e) the speed greatest?
2. A passenger in a car rounding a sharp curve feels "thrown" toward the outside of the curve.
   (a) What causes this to happen? Is the person really thrown away from the center of the curve?
   (b) Make a free-body diagram of the person.
3. If two planets have the same mass, will they necessarily produce the same gravitational pull on 1.0 kg objects that are
   (a) at their surfaces?
   (b) the same center-to-center distance from both planets (but above their surfaces)? Explain.
4. "If a rock is acted upon by a gravitational force F from the earth when it is at a distance d above the surface of our planet, it will be acted upon by a force $\frac{F}{4}$ if it is raised to $2d$". Is the statement true or false? Explain
5. What is wrong with this statement? "A satellite stays in orbit because the outward centrifugal force just balances the inward centripetal force." Make a free-body diagram of the satellite.
6. A child is sitting 1.50 m from the centre of a highly polished wooden, and rotating disc. The coefficient of static friction between the disc and the child is 0.30. What is the maximum tangential speed that the child can have before slipping off the disc?
7. An object of mass m is resting on top of a hemispherical mound of ice whose radius of curvature is R. The object is given a small push and start sliding down the mound. Show that the object will lose contact with the surface of ice at a vertical height of $\frac{2R}{3}$.
8. The radius r of a rotating room is 4.50 m and the speed v of a child standing against the wall is $12.0\,ms^{-1}$. Find the minimum value of coefficient of static friction required to keep the child pinned against the wall.
9. Determine the angle at which a cyclist should bend to the vertical when he moves a circular path of 64.6 m in circumference for a duration of 10 seconds only.
10. A rod of 20 cm length pivoted on one end is made to rotate in a horizontal plane with a constant angular speed. A ball of mass m is suspended by a string of 20 cm length from the other end of the rod. If the string makes an angle of 30° with the vertical, find the angular speed of the rod.
11. A mass of 0.5 kg is vibrating in a system in which the constant of

the spring used is 100 N/m. The amplitude of vibration is 0.2 m. Determine:
(a) The energy of the system;
(b) The maximum velocity;
(c) The potential energy and kinetic energy when $x = 0.1\text{m}$; and
(d) The maximum acceleration.

12. A simple pendulum has a period of 4.2 seconds. When the pendulum is shortened by 1 m, the period is 3.7 seconds. From the measurements, calculate the acceleration due to gravity and original length of the pendulum.

13. (a) If a displacement of oscillating particle at any time is to be given by an equation $y = a\sin\omega t + b\sin\omega t$. Show that the motion is SHM.
(b) If $a = 3\text{cm}$, $b = 4\text{cm}$ and $\omega = 2\,\text{rad}\,\text{s}^{-1}$. Determine the period, amplitude, maximum velocity and acceleration of the motion in (a).

14. A person of mass 50 kg stands on a platform. The platform oscillates with a frequency of 2 Hz. If the amplitude of oscillations is 0.05 cm, calculate the maximum and the minimum weight of the person recorded by a machine of the platform.

15. Calculate the gravitational field strength and gravitational potential at the surface of the moon given that mass of the moon, $m = 7.34\times10^{22}$ kg, radius of the moon, $R = 1.74\times10^{6}$ m and the gravitational constant $G = 6.67\times10^{-11}\,\text{Nm}^2\text{kg}^{-2}$.

16. A satellite with a mass of 1000 kg moves in a circular orbit with a radius of 7000 km round the earth. Calculate the total energy required to place the satellite in the orbit from the earth's surface, assuming it to be at rest initially. (Take radius of the earth, $r_e = 6.37\times10^{6}$ m and $g = 9.8\,\text{m}\,\text{s}^{-2}$).

17. A rocket is launched vertically from the surface of the earth with an initial velocity v_o. Show that its velocity v at a height h is given by $v^2 = v_o^2 - \dfrac{2gh}{\left(1+\dfrac{h}{R}\right)^2}$ where R is the radius of the earth and g is the acceleration due to gravity.

18. The international space station (ISS) makes 15.65 revolutions per day in its orbit around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

Chapter Four

Rotation of rigid bodies

Introduction

Rotational motion is common phenomenon observed in different moving rigid bodies. Examples of these are the motions of whirled buckets, Digital Video Discs (DVDs), car tyres, circular cutting saws, and ceiling fan blades. Each of these examples involves a body that rotates about an axis that is stationary in some inertial frame of reference. Rotational motion occurs in all scales, from motion of electrons in atoms, to motion of the earth and entire galaxies in the universe. In this chapter, you will learn about the rotation of rigid bodies. It covers the following concepts: angular momentum, centre of mass, moment of inertia, torque, and the kinetic energy of a rotating rigid body.

4.1 Centre of mass

A rigid body is one which does not deform easily under the action of an applied force. A centre of mass of a rigid body or a system of particles refers to a point at which the whole mass of a body or a system of particles is assumed to be concentrated. It can also be defined as the point at which all the mass of the body can be considered to be concentrated when applying an external force. That is, the motion of centre of mass represents the motion of entire body. You can replace the mass of a body by a mass of a single particle placed at the centre of mass of that body. Therefore, the centre of mass of any object is the average position of all the particles of mass that makes up the object. Hence, the centre of mass of regular shape objects (such as rod, disc, cylinder and sphere) is at its geometrical centre of the object. For an irregular object, its centre of mass depends on the shape and mass distribution of its particles.

Consider an irregular object of mass M consisting of a large number of particles each having mass m_1, m_2, m_3, … m_n rotating about an axis OP (Figure 4.1).

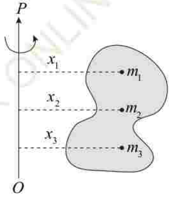


Figure 4.1 Rigid body with axis of rotation OP

Each particle experiences an external force, $F_1 = m_1a, F_2 = m_2a, F_3 = m_3a, \ldots, F_n = m_na$.

The total force experienced by the body is

$$F = m_1 a + m_2 a + m_3 a + ... + m_n a \qquad (4.1)$$

Equation (4.1), can be expressed in terms of the second derivative of displacement as,

$$\sum F = \frac{d^2}{dt^2}\left(m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n\right)$$
$$= \left(\sum_{i=1}^{n} m_i\right) a_{cm} \qquad (4.2)$$

Where a_{cm} is the acceleration of the centre of mass.

Since, $a_{cm} = \frac{d^2 x_{cm}}{dt^2}$ and

$$m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n = \sum_{i=1}^{n} m_i x_i,$$

then equation (4.2) reduces to,

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad (4.3)$$

Similarly,

$$y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \qquad (4.4)$$

For the two dimensional plane figures, the coordinate of the centre of mass is at (x_{cm}, y_{cm}) and its distance from the origin is

$$r_{cm} = \sqrt{x_{cm}^2 + y_{cm}^2} \qquad (4.5)$$

For two connected particles of mass m_1 and m_2 with position vector $\vec{r}_1$ and $\vec{r}_2$ respectively (Figure 4.2), the centre of mass is given by

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad (4.6)$$

where, $\vec{r}_1 = \vec{x}_1 + \vec{y}_1$ and $\vec{r}_2 = \vec{x}_2 + \vec{y}_2$

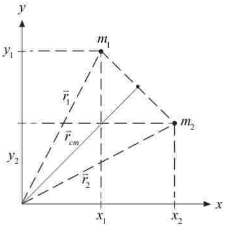


Figure 4.2 Two particle system

The positon of the centre of mass depends upon shape, size and distribution of mass of the body. In addition, the centre of mass of a body may lie within or outside the body. When an external force is applied at the centre of mass, only linear motion is produced (no rotation motion). The motion of the centre of mass is the motion of the whole body.

Example 4.1

Two bodies of masses 0.8 kg and 1.2 kg are located at $(1, -2)$ and $(-3, 4)$ respectively. Find the coordinates of the centre of mass of the system.

Solution

From equation (4.3) and (4.4),

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{0.8\text{ kg} \times 1 + 1.2\text{ kg} \times (-3)}{0.8\text{ kg} + 1.2\text{ kg}} = -1.4$$

and

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$= \frac{0.8\text{ kg} \times (-2) + 1.2\text{ kg} \times 4}{0.8\text{ kg} + 1.2\text{ kg}} = 1.6$$

The coordinates of the centre of mass of the system are $(-1.4, 1.6)$.

Example 4.2

Two bodies of 100 g and 300 g have positon vectors $2\hat{i} + 5\hat{j} + 13\hat{k}$ and $-6\hat{i} + 4\hat{j} - 2\hat{k}$ respectively. Find:

(a) Position vector of the centre of mass; and

(b) Distance of the centre of mass from the origin.

Solution

Form equation (4.5) and (4.6)

(a) $$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{100\text{ g}(2\hat{i} + 5\hat{j} + 13\hat{k}) + 300\text{ g}(-6\hat{i} + 4\hat{j} - 2\hat{k})}{100\text{ g} + 300\text{ g}}$$

$$\vec{r}_{cm} = -4\hat{i} + \frac{17}{4}\hat{j} + \frac{7}{4}\hat{k}$$

(b) $$r_{cm} = \sqrt{x_{cm}^2 + y_{cm}^2 + z_{cm}^2}$$

$$= \sqrt{(-4)^2 + \left(\frac{17}{4}\right)^2 + \left(\frac{7}{4}\right)^2} = 6.09\text{ units}$$

The total mass M of a rigid body can be expressed in terms of the individual mass of the particles as $\sum_{i=1}^{n} m_i$. Therefore, from equation (4.3), the centre of mass of a rigid body can be expressed as,

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M} \quad (4.7)$$

The centre of mass in the *xyz* directions can be expressed as,

$$\vec{x}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{x}_i}{M},\ \vec{y}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{y}_i}{M} \text{ and } \vec{z}_{cm} = \frac{\sum_{i=1}^{n} m_i\vec{z}_i}{M} \quad (4.8)$$

Since the rigid body is a continuous distribution of particles (not discrete particles) equation (4.8) can be expressed as,

$$\vec{x}_{cm} = \frac{1}{M}\int x\,dm,\ \vec{y}_{cm} = \frac{1}{M}\int y\,dm \text{ and}$$

$$\vec{z}_{cm} = \frac{1}{M}\int z\,dm \quad (4.9)$$

In general, the centre of mass of a rigid body of continuous mass M can be expressed as,

$$\vec{r}_{cm} = \frac{1}{M}\int r\,dm \quad (4.10)$$

Example 4.3

Find the centre of mass of uniform rod of length L, along its length.

Solution

Consider a uniform rod with a small element dm of length dx at a distance x from the fixed point.

The mass dm can be expressed in terms of mass density as, $dm = \frac{M}{L}dx$.

Therefore, from equation (4.10), the centre of mass of the rod is,

$$\vec{x}_{cm} = \frac{1}{M}\int_0^L x\,dm = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\left|\frac{x^2}{2}\right|_0^L = \frac{L}{2}$$

Exercise 4.1

1. Explain where the centre of mass of a two particle system lies when one particle is more massive than the other.
2. Does the centre of mass of a rigid body always lie within the body? Give examples to support your answers.
3. What is the difference between centre of mass and centre of gravity?
4. Under what consideration does the centre of mass coincide with the centre of gravity?
5. Prove that the centre of mass of
   (a) two-particle system divides the line joining the particles by the inverse ratios of the masses.
   (b) semi-circular hoop of radius R is given by $\frac{2R}{\pi}$.

4.2 Moment of inertia

Moment of inertia of a rigid body is a measure of how difficult it is to change the state of rotational motion of a rigid body. That is to say, how difficult it is either, to cause a body to rotate when at rest, to stop it when rotating, or to increase or decrease its angular velocity. The moment of inertia of a body depends on the following factors namely; mass, axis of rotation and mass distribution from the axis of rotation of the body. This implies that, a single body will have different values of moment of inertia about different axes of rotation. Hence, the moment of inertia of a body is not unique. This is the reason why it is much easier to rotate a uniform meter rod about its centre (where its moment of inertia is small) than rotating it at one-end (where its moment of inertia is large). The moment of inertia of a rigid body can be deduced from the kinetic energy of the body.

Suppose a rigid body of mass M is rotating with an angular velocity ω about an axis through O perpendicular to the plane of the figure (Figure 4.3).

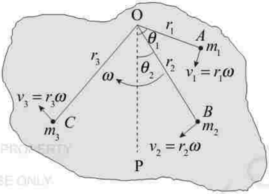


Figure 4.3 Rigid body with several particles

Consider a particle at point A, with mass m_1, which is at a distance r_1 from O. The velocity at A is $v_1 = r_1\omega$, where ω is the same for all particles of the rigid body. The rotational kinetic energy of the particle at A is

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2 \qquad (4.11)$$

The total kinetic energy of the body is the sum of the kinetic energies of all its particles.

$$K.E = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ... + \frac{1}{2}m_nr_n^2\omega^2$$

or

$$K.E = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2 \qquad (4.12)$$

Factoring out the constant values in equation (4.12) the kinetic energy is

$$K.E = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2 \qquad (4.13)$$

The quantity $\sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2$ is called the moment of inertia denoted by I. Therefore, the moment of inertia is the sum of the product of the mass of each particle of a rigid body and the square of its distance from the axis of rotation. That is,

$$I = \sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 \quad (4.14)$$

The SI unit of moment of inertia is kgm^2. Therefore, from equations (4.13) and (4.14), rotational kinetic energy can be expressed in terms of moment of inertia I as,

$$K.E = \frac{1}{2} I \omega^2 \qquad (4.15)$$

Note that, mass is an intrinsic property of an object, whereas moment of inertia depends on the physical arrangement of that mass and the choice of rotation axis. Can you think of a situation in which an object's moment of inertia changes even though its mass does not?

4.2.1 Moment of inertia of a rotating uniform rod

The moment of inertia of a rotating uniform rod (e.g. metre rule) can be determined from its axis of rotation passing either at the centre or at its end.

(a) Moment of inertia of a uniform rod rotating about an axis through its centre

Consider a uniform rod AB of mass M, with length L rotating about an axis PQ passing through the centre of the rod. Then, consider, a small length dx counting for the small mass dm located a distance x from the axis PQ as shown in Figure 4.4.

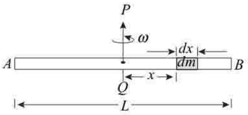


Figure 4.4 Uniform rod with axis of rotation at its centre

The mass of the small element dm is given by

$$dm = \frac{M}{L} dx \qquad (4.16)$$

From equation (4.14), the moment of inertia of the element about an axis PQ passing through the centre of the rod is given as,

$$dI = \frac{M}{L} x^2 dx \qquad (4.17)$$

The total moment of inertia I of the whole rod is the sum of the moment of inertia dI of the small mass elements dm from A to B and can be achieved by integrating equation (4.17) that is,

$$I = \frac{M}{L} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} x^2 \, dx \qquad (4.18)$$

Simplifying equation (4.18), the total moment of inertia I for a uniform rod about an axis through its centre is

$$I = \frac{1}{12}ML^2 \quad (4.19)$$

(b) Moment of inertia of a uniform rod rotating about an axis passing at one end

Consider a uniform rod AB of mass M and length L rotating about an axis PQ passing at one end of the rod. Then, consider, a small length dx counting for the small mass dm located a distance x from the axis PQ (Figure 4.5).

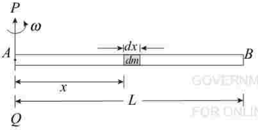


Figure 4.5 Uniform rod with rotation axis at one end

The mass of the small element is $dm = \frac{M}{L}dx$, thus from equation (4.14) the moment of inertia of the element about an axis PQ passing at one end of the rod is, $dI = \frac{M}{L}x^2dx$.

Hence, the total moment of inertia of the rod rotating about an axis PQ passing at one end of the rod is

$$I = \frac{M}{L}\int_0^L x^2\,dx \quad (4.20)$$

Simplifying equation (4.20), the total moment of inertia I for a uniform rod about an axis at one end of the rod is

$$I = \frac{M}{3}L^2 \quad (4.21)$$

4.2.2 Moment of inertia of a ring rotating about an axis through its centre

Consider a uniform ring of mass M and radius r rotating about an axis PQ passing through its centre O, perpendicular to the plane of the ring (Figure 4.6).

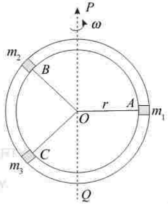


Figure 4.6 Uniform ring with rotation axis through its centre

From equation (4.14) the moment of inertia of small elements A, B, C of mass m_1, m_2, m_3, located a distance r from the centre of the ring is given as

$I_A = m_1r^2$, $I_B = m_2r^2$, and $I_C = m_3r^2$.

Thus, the moment of inertia of the whole ring about an axis PQ through its centre O is given by

$$I = m_1r^2 + m_2r^2 + m_3r^2 + ... + m_nr^2 \quad (4.22)$$

Equation (4.22) can be written as,

$$I = (m_1 + m_2 + m_3 + ... + m_n)r^2 = Mr^2 \quad (4.23)$$

where, $m_1 + m_2 + m_3 + ... + m_n = M$ which is the mass of the uniform ring. Therefore, the total moment of inertia of a uniform ring about an axis through its centre is given as,

$$I = Mr^2 \quad (4.24)$$

4.2.3 Moment of inertia of a disc rotating about an axis through its centre

Consider a uniform disc of mass M and a radius r rotating about an axis PQ passing through its centre O, perpendicular to the plane of the disc. Now consider a small ring element of mass dm and thickness dx at a distance x, rotating about an axis PQ through its centre O, as shown in Figure 4.7.

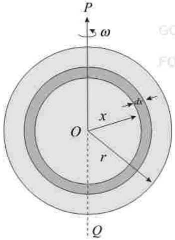


Figure 4.7 Uniform disc with rotation axis through its centre

From equation (4.14), the moment of inertia of the small mass element dm is given as $x^2 dm$. Thus, the total moment of Inertia of the disc about an axis through its centre can be expressed as,

$$I = \int_0^r x^2\, dm \quad (4.25)$$

Since dm is the fraction of the total mass M of the disc, then, the mass and area for both the disc and the ring can be expressed as,

$$\frac{dm}{dA} = \frac{M}{A} \quad (4.26)$$

But, in equation (4.26) the area A of the disc is πr^2 and the area dA of the ring is $2\pi x dx$, then, equation (4.26) can be written as,

$$dm = \frac{M}{A} dA = \frac{2\pi x dx}{\pi r^2} M \quad (4.27)$$

Therefore, from equation (4.25) and (4.27), the total moment of inertia I of the whole disc about an axis through its centre is

$$I = \frac{2M}{r^2} \int_0^r x^3\, dx \quad (4.28)$$

Integrating and simplifying equation (4.28), the total moment of inertia I of the whole disc about an axis through its centre is

$$I = \frac{1}{2} Mr^2 \quad (4.29)$$

Example 4.4

If a flywheel of mass 30 kg and diameter 1m is rotating at 300 revolutions per minute about an axis through its centre, what is the kinetic energy of the flywheel?

Solution

Using equation (4.15) for the kinetic energy ($K.E$) of a rotating rigid body and equation (4.29) for moment of inertia of flywheel (disc), then,

$$K.E = \frac{1}{2}I\omega^2 = \frac{1}{2}\times\left(\frac{1}{2}mr^2\right)\omega^2$$

$$= \frac{1}{2}\times\left(\frac{1}{2}mr^2\right)\times(2\pi f)^2,$$

where $\omega = 2\pi f$

$$K.E = \frac{30\text{ kg}\times(0.5\text{ m})^2\times 4\pi^2\times\left(\frac{300\text{ rev/s}}{60}\right)^2}{4}$$

$$= 1850.6\text{ J}$$

Therefore, the kinetic energy of the flywheel about an axis through its centre is 1850.6 J.

4.2.4 Moment of inertia of a sphere

The moment of inertia of a solid sphere (e.g. ball bearing) and hollow sphere (e.g. football) rotating about an axis through its centre can be derived as follows:

(a) Moment of inertia of a solid sphere rotating about an axis through its centre

Consider a solid sphere of mass M and radius r rotating about an axis PQ passing through its centre O. Now consider an elemental disc of radius a and small mass dm of thickness dy at a distance y from the centre O of the solid sphere (Figure 4.8).

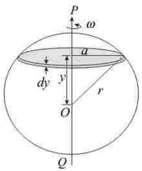


Figure 4.8 Solid sphere with rotation axis through its centre

Then, from equation (4.29), the moment of inertia dI of the disc is,

$$dI = \frac{1}{2}(dm)a^2 \qquad (4.30)$$

Since dm is the fraction of the total mass M of the disc, then, the mass and volume for both the sphere and the disc can be expressed as;

$$\frac{dm}{dV} = \frac{M}{V} \qquad (4.31)$$

The volume V of the sphere is $\frac{4}{3}\pi r^3$ and the volume dV of the disc is $dV = \pi a^2 dy$. Substituting V and dV into equation (4.31), dm can be expressed as,

$$dm = \frac{M}{V}dV = \frac{3Ma^2}{4r^3}dy \qquad (4.32)$$

From Figure 4.8, $a^2 = r^2 - y^2$. Substituting the value of dm and a into equation (4.30), the moment of inertia dI of the disc can be obtained by

$$dI = \frac{3M\left(r^2 - y^2\right)^2}{8r^3}dy \qquad (4.33)$$

Thus, the moment of inertia of the solid sphere about an axis through its centre is obtained by intergrating equation (4.33) as follows:

$$I = \int_{-r}^{r} \frac{3M\left(r^2 - y^2\right)^2}{8r^3} dy \qquad (4.34)$$

Integrating equation (4.34),

$$I = \frac{3M}{8r^3}\left[r^4 y - \frac{2r^2y^3}{3} + \frac{y^5}{5}\right]_{-r}^{r} \qquad (4.35)$$

Simplifying equation (4.35), the total moment of inertia I of the solid sphere about an axis through its centre is

$$I = \frac{2}{5}Mr^2 \qquad (4.36)$$

Example 4.5

The mass of a solid sphere increases by 1%. What will be the percentage increase in the moment of inertia about its axis of symmetry?

Solution

Differentiating the moment of inertia I of a solid sphere about the axis of symmetry i.e. equation (4.36), with respect to M then, $dI = \frac{2}{5}r^2 dM$

The fractional changes in I is the ratio of dI to that of I which gives

$$\frac{dI}{I} = \frac{dM}{M}$$

But the percentage increase in the moment of inertia I about axis of symmetry is

$$\frac{dI}{I}\% = \frac{dM}{M}\% = 1\%$$

Therefore, since the percentage increase in the mass of the sphere is 1%, it follows that, the percentage change in the moment of inertia is also 1%.

(b) Moment of inertia of a hollow thin sphere

Consider a hollow sphere of mass M and radius r rotating about an axis PQ passing through its centre O. Now consider an elemental ring of radius a and small mass dm, of thickness dy, at a distance y from the centre O of the solid sphere (Figure 4.9).

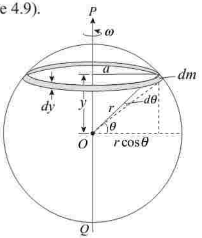


Figure 4.9 Hollow sphere with rotation axis at its centre

Then, from equation (4.24), the moment of inertia, dI of the ring is

$$dI = (dm)a^2 \qquad (4.37)$$

Since dm is the fraction of the total mass M of the ring, then, the mass and area for both the sphere and the ring can be expressed as,

$$\frac{dm}{dA} = \frac{M}{A} \qquad (4.38)$$

From Figure 4.9, the area, A of the sphere is $A = 4\pi r^2$ and the surface area dA of the ring is $dA = 2\pi a dy = 2\pi r a d\theta$ (consider dy to be very small such that $dy = r d\theta$). Substituting A, dA and a into equation (4.38), dm can be expressed as,

$$dm = \frac{M}{2r^2}dy = \frac{Ma}{2r}d\theta \qquad (4.39)$$

Since $a = r\cos\theta$ (Figure 4.9), then, from equation (4.37) and (4.39) the moment of inertia, I of the ring can be expressed as,

$$I = \frac{Mr^2}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \cos\theta \, d\theta$$

$$I = \frac{Mr^2}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\cos\theta \, d\theta \qquad (4.40)$$

Integrating and simplifying equation (4.40), the total moment of inertia, I of the hollow sphere about an axis through its centre is

$$I = \frac{2}{3}Mr^2 \qquad (4.41)$$

Now, suppose the solid and hollow sphere have the same mass and radius, then the ratio of the moment of inertia of solid sphere i.e. equation (4.36) to that of the hollow sphere i.e. equation (4.41) is given as $I_{solid} = \frac{3}{5} I_{hollow}$, that is $I_{solid} < I_{hollow}$.

From the definition of moment of inertia, this implies that, solid sphere rotates much easier than hollow one. The reason for this is, mass distribution in solid sphere is closer to the axis of rotation than in hollow sphere.

4.2.5 Moment of inertia of a cylinder

The moment of inertia of a solid cylinder (e.g. circular iron rod) and hollow cylinder (e.g. hose pipe) rotating about an axis through its centre can be derived as follows:

(a) Moment of inertia of a solid cylinder rotating about an axis through its centre

Consider a solid cylinder of mass, M and radius, r rotating about an axis PQ passing through its centre.

Let the cylinder be divided into small discs of masses $m_1, m_2, ... m_n$ each with radius r (Figure 4.10).

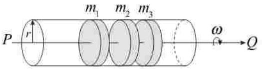


Figure 4.10 Solid circular cylinder with rotation axis at its centre

From equation (4.29), the moment of inertia of the small discs of mass m_1, m_2 and m_3, located a distance r from the centre of the disc, is given by $I_1 = \frac{1}{2}m_1r^2$, $I_2 = \frac{1}{2}m_2r^2$, and $I_3 = \frac{1}{2}m_3r^2$.

Thus, the total moment of inertia of the solid cylinder about an axis PQ through its centre is the sum of moment of inertia of such small discs for the whole solid cylinder given as,

$$I = \frac{1}{2}m_1r^2 + \frac{1}{2}m_2r^2 + \frac{1}{2}m_3r^2 + ... + \frac{1}{2}m_nr^2 \qquad (4.42)$$

Equation (4.42) can be written as,

$$I = \left(m_1 + m_2 + m_3 + ... + m_n\right)\frac{1}{2}r^2 = \frac{1}{2}Mr^2 \qquad (4.43)$$

where, $m_1 + m_2 + m_3 + ... + m_n = M$ which is the mass of the solid cylinder. Therefore, the total moment of inertia of a solid cylinder about an axis through its centre is given as,

$$I = \frac{1}{2}Mr^2 \qquad (4.44)$$

Note that, the formula of the moment of inertia of a solid cylinder about an axis through its centre, equation (4.44), is similar to that of a disc, equation (4.29).

(b) Moment of inertia of a hollow cylinder rotating about an axis through its centre

Consider a hollow cylinder of mass, M and radius, r rotating about an axis PQ passing through its centre. Let the cylinder be divided into small rings of masses $m_1, m_2, ..., m_n$ each with radius r (Figure 4.11).

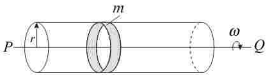


Figure 4.11 Hollow circular cylinder with rotation axis at its centre

From equation (4.24) the moment of inertia of a small rings of mass m_i; located a distance r from the centre of the rings is given by,

$$I_i = m_i r^2$$

Thus, the total moment of inertia of the hollow cylinder about an axis PQ through its centre is the sum of moment of inertia of the small rings for the whole solid cylinder, hence,

$$I = m_1 r^2 + m_2 r^2 + m_3 r^2 + ... + m_n r^2 \quad (4.45)$$

Equation (4.45) can be written as,

$$I = (m_1 + m_2 + m_3 + ... + m_n) r^2 = Mr^2 \quad (4.46)$$

where, $m_1 + m_2 + m_3 + ... + m_n = M$ which is the mass of the hollow cylinder. Therefore, the total moment of inertia of hollow cylinder about an axis through its centre is given as;

$$I = Mr^2 \quad (4.47)$$

Note that, the formula of the moment of inertia of a hollow cylinder about an axis through its centre, equation (4.47) is similar to that of a ring, equation (4.24).

Now, suppose the solid and hollow cylinder have the same mass and radius. Then the ratio of the moment of inertia of solid cylinder (equation (4.44)) to that of the hollow cylinder (equation (4.47)), is given as $I_{solid} = \frac{1}{2} I_{hollow}$, that is, $I_{solid} < I_{hollow}$. From the definition of moment of inertia, this implies, solid cylinder rotates much easier than hollow one. The reason is that, mass distribution in solid cylinder is closer to the axis of rotation than in hollow cylinder.

Exercise 4.2

1. Must a rotating object have a non-zero moment of inertia? Explain.
2. Explain why changing the axis of rotation of an object changes its moment of inertia.
3. Experienced cooks can tell whether an egg is raw or hard boiled by rolling it down a slope (and taking care to catch it at the bottom). How is this possible? Which type of egg should reach the bottom of the slope first?
4. Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

5. Calculate the moment of inertia of a uniform rod of mass 60 g and length 20 cm about an axis perpendicular to its length through
   (a) its centre.
   (b) one end.
6. Find the moment of inertia of a rod 4 cm in diameter and 2 m long, weighing 8 kg about an axis
   (a) perpendicular to the rod and passing through its centre.
   (b) perpendicular to the rod and passing through one end.
   (c) longitudinal axis through the centre of the rod.
7. Four particles of masses 4 kg, 2 kg, 3 kg and 5 kg are fixed at the four corners A, B, C, and D respectively of a square of each side 1 m. Calculate the moment of inertia of the system about
   (a) an axis passing through the point of intersection of the diagonal and perpendicular to the plane of the square.
   (b) the side *AB*.
   (c) the diagonal *BD*.
8. Calculate the moment of inertia of a circular disc of diameter 40 cm, thickness 7 cm and uniform density $9\ \text{gcm}^{-3}$, about a transverse axis through the centre of the disc.
9. Assume the earth is a uniform homogenous sphere of radius 6.37×10^{8} cm and density $5.45\ \text{gcm}^{-3}$. Calculate its moment of inertia about the axis of rotation.

4.3 Axis theorem of rotating bodies

The axes theorems of rotating bodies include the parallel axis theorem and perpendicular axis theorem. These theorems are useful for determining the moment of inertia about axes given that the moment of inertia about other axes are known.

4.3.1 The parallel axis theorem

The moment of inertia of rigid bodies with simple geometry (high symmetry) is relatively easy to calculate provided the rotation axis coincides with an axis of symmetry. The calculation of moment of inertia about an arbitrary axis can be cumbersome. Rigid bodies may have infinitely many moments of inertia because there are infinitely many axes about which they might rotate. Fortunately, the use of an important theorem, called the parallel-axis theorem, often simplifies the calculation.

The parallel-axis theorem states that, *"The moment of inertia I of a rigid body about any axis is given by the sum of the moment of inertia I_C about a parallel axis passing through its centre of mass and the product of its mass M and the square of the perpendicular distance between the two parallel axes d "*i.e, $I = I_c + Md^2$.

To prove the parallel-axis theorem, consider a rigid body of mass M rotating about an axis y located at a distance d from the centre of mass G. Then, consider a particle of mass m_i, at a distance x from a centre of mass G (Figure 4.12).

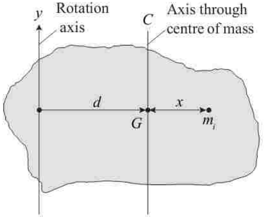


Figure 4.12 Irregular rigid body with rotation axis at its centre

The moment of inertia dI of a particle of mass m_i, a distance x from the axis of rotation is given as,

$$dI = m_i(d+x)^2 \qquad (4.48)$$

Then, the moment of inertia I of the whole rigid body is then given as

$$I = \sum_{i=1}^{n} m_i(d+x)^2 = \sum_{i=1}^{n} m_i(d^2+2xd+x^2)$$

$$I = d^2\sum_{i=1}^{n} m_i + 2d\sum_{i=1}^{n} m_i x + \sum_{i=1}^{n} m_i x^2 \qquad (4.49)$$

From equation (4.49), $\sum_{i=1}^{n} m_i = M$ is the total mass of the rigid body and $\sum_{i=1}^{n} m_i x^2 = I_c$ is the moment of inertia of the rigid body about an axis through its centre of mass G.

$\sum_{i=1}^{n} m_i x = M\bar{x}$, where $\bar{x}$ is the average distance from the centre of mass G to the axis of rotation y, which is zero, since the axis is at the centre. Taking these considerations, the moment of inertia I of a rigid body about any axis is expressed as,

$$I = I_c + Md^2 \qquad (4.50)$$

Applications of the parallel axis theorem

Parallel axis theorem is used to determine moment of inertia of homogeneous rigid bodies with different geometries, for example, cylindrical shell, uniform rod, uniform solid sphere, hollow cylinder, and rectangular plate.

For a uniform rod of mass M and length L, (Figure 4.4) the moment of inertia about an axis through one end is

$$I = I_G + Md^2, \text{ where } d = \frac{L}{2}$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

For a solid sphere of mass M and radius r, the moment of inertia about an axis tangential to the surface is

$$I = I_G + Md^2 = \frac{2}{5}Mr^2 + Mr^2 = \frac{7}{5}Mr^2.$$

For a uniform solid cylinder of mass M and radius r, the moment of inertia about an axis on the surface parallel to its length is

$$I = I_G + Md^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$$

Example 4.6

(i) Consider a uniform ring of mass 200 g and radius 5 cm. Find the moment of inertia of the ring about an axis passing through its edge and perpendicular to the plane of the ring.

(ii) A part of car mechanical linkage (Figure 4.13(b)) has a mass of 3.6 kg.

Its moment of inertia I about an axis through point P, 0.15 m from its centre of mass, is $I = 0.132$ kgm^2. What is the moment of inertia I_C about a parallel axis through the centre of mass?

Solution

(i) Consider a sketch diagram shown in figure 4.13 (a).

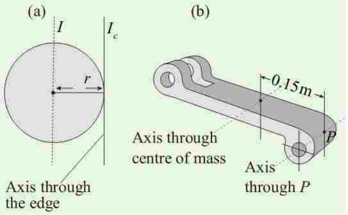


Figure 4.13 (a) Moment of inertia of a ring about its edge, **(b)** car mechanical linkage part

Using the parallel axis theorem, equation (4.50) and the moment of inertia I_c of a ring about its centre equation (4.24), the moment of inertia I of a ring of radius r rotating about its edge can be expressed as;

$$I = I_c + Mr^2 = Mr^2 + Mr^2 = 2Mr^2$$

$$I = 2 \times (200 \times 10^{-3}\,\text{kg}) \times (5 \times 10^{-2}\,\text{m})^2$$
$$= 1 \times 10^{-3}\,\text{kgm}^2$$

where $d = r$ and $I_C = Mr^2$

Therefore, the moment of inertia of the ring rotating about its edge is 1×10^{-3} kgm^2.

(ii) The target variable I_C is obtained by using the parallel-axis theorem equation (4.50). Rearranging the equation,

$$I_C = I - md^2$$

$$md^2 = 3.6\,\text{kg} \times 0.15^2\,\text{m}^2$$
$$= 0.081\,\text{kgm}^2$$

$$I_C = 0.132\,\text{kgm}^2 - 0.081\,\text{kgm}^2$$
$$= 0.051\,\text{kgm}^2$$

4.3.2 The Perpendicular axis theorem

The moment of inertia of a planar object about an axis perpendicular to the plane intersected by two perpendicular axes can be determined using the perpendicular axis theorem.

The theorem states that, *"The moment of inertia of a plane body about an axis perpendicular to its plane is given by the sum of the moments of inertia about any two mutually perpendicular axes in the plane intersecting the first axis"*. i.e., $I_z = I_x + I_y$. Unlike the parallel axis theorem, the perpendicular axis theorem works for planar (two dimensional) bodies only.

Consider a planar body of mass M rotating about z-axis, then, consider a particle P of mass m, at a distance r from a centre of the planar body O (Figure 4.14).

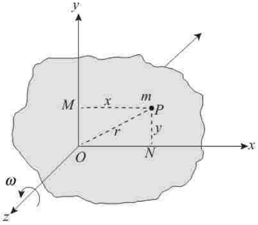


Figure 4.14 Planar rigid body with a rotating axis at its centre

The distance r of the particle m situated at $P(x, y)$ from a point O, can be expressed as, $r^2 = x^2 + y^2$. Then, the moment of inertia of the particle at $P(x, y)$ about z-axis is $mr^2 = m(x^2 + y^2)$.

The moment of inertia of the whole planar body about z-axis is equal to the sum of the moment of all particles of mass m_i at a distance r_i can be expressed as,

$$I_z = \sum_{i=1}^{n} m_i r_i^2 = \sum_{i=1}^{n} m_i\left(x_i^2 + y_i^2\right)$$

$$I_z = \sum_{i=1}^{n} m_i x_i^2 + \sum_{i=1}^{n} m_i y_i^2 \qquad (4.51)$$

where $\sum_{i=1}^{n} m_i x_i^2 = I_y$ is the moment of inertia of the whole body about y-axis and $\sum_{i=1}^{n} m_i y_i^2 = I_x$ is the moment of inertia of the whole planar body about x-axis. Therefore, the moment of inertia about z-axis can be expressed as,

$$I_z = I_x + I_y \qquad (4.52)$$

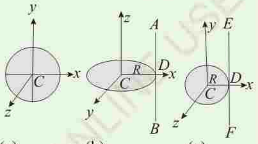
Example 4.7

The moment of inertia of a uniform circular disc of mass M and radius R about an axis passing through its centre and perpendicular to its plane is $\frac{1}{2}MR^2$. Find the moment of inertia of the disc about,

(i) any diameter.
(ii) an axis passing through a point on the edge of the disc and perpendicular to the disc.
(iii) a tangent in the plane of the disc.

Solution

The plane of the disc is the x-y plane as shown in the following figures



(a) (b) (c)

(i) Using perpendicular axis theorem (Figure (a));

$I_z = I_x + I_y$

now, $I_z = I_c = \frac{1}{2}MR^2$ from the symmetry, $I_x = I_y$

$2I_x = \frac{1}{2}MR^2$, thus, $I_x = I_y = \frac{1}{4}MR^2$

(ii) Using parallel axis theorem (Figure (b)), $I = I_c + Md^2$, thus

$$I_{AB} = I_z + M(CD)^2 = I_C + MR^2$$

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

(iii) Using parallel axis theorem (Figure (c))

$$I_{EF} = I_z + MR^2 = \frac{1}{4}MR^2 + MR^2 = \frac{5}{4}MR^2$$

Applications of the perpendicular axis theorem

The perpendicular axis theorem can be used to determine the moment of inertia of a lamina (e.g. a thin planar layer) rotating about an axis perpendicular to the plane. It is also used to determine the moment of inertia of a disc rotating about an axis along its diameter.

(i) Moment of inertia of a lamina rotating about an axis perpendicular to the plane

Consider a lamina of mass M with dimension $x \times y$, rotating about axis OY, passing through its centre O perpendicular to the plane (Figure 4.15). The moment of inertia about this axis can be determined using the perpendicular axis theorem as follows:

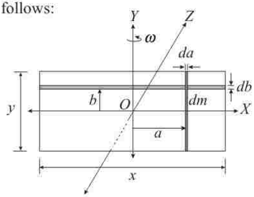


Figure 4.15 Lamina with a rotating axis at its centre

Consider a small mass element dm rotating at a distance a from the axis OY. The moment of inertia dI_y of dm a distance a rotating about axis OY can be expressed as,

$$dI_y = a^2 dm \tag{4.53}$$

Since, dm is the fraction of the total mass M of the lamina, then, the mass and area for both the lamina and the small element dm can be expressed as,

$$\frac{dm}{dA} = \frac{M}{A} \tag{4.54}$$

But in equation (4.54) the area A of the lamina is xy and the area dA of dm is yda, then equation (4.54) can be expressed as,

$$dm = \frac{M}{A}dA = \frac{M}{x}da \tag{4.55}$$

Then, from equation (4.53) the total moment of inertia, I_y of the lamina rotating about axis OY can be expressed as,

$$I_y = 2\frac{M}{x}\int_0^{x/2} a^2\,da = 2\frac{Ma^3}{3x}\Big|_0^{x/2} = \frac{M}{12}x^2 \tag{4.56}$$

The integral part in equation (4.56) is multiplied by 2 to account for the second half of lamina, as the integral is carried for half part of the lamina.

Similarly, the moment of inertia, I_x of the lamina rotating about axis OX can be expressed as,

$$I_x = \frac{M}{12}y^2 \tag{4.57}$$

Using perpendicular axis theorem, the moment of inertia I_z of the lamina rotating about axis OZ can be expressed as,

$$I_x = \frac{M}{12} y^2 \qquad (4.58)$$

(ii) Moment of inertia of a disc rotating about an axis along the diameter

Consider a disc of mass M rotating about $x-$axis or $y-$axis along the diameter of the disc (Figure 4.16). The moment of inertia of the disc about these axes can be determined using the perpendicular axis theorem.

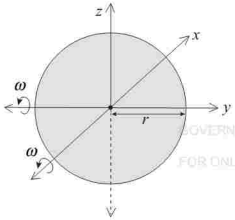


Figure 4.16 A disc with rotating axis along its diameter

Since x-axis and y-axis are along the diameter of the disc, the moment of inertia I_x of the disc about x-axis is equal to moment of inertia I_y of the disc about y-axis. That is,

$$I_x = I_y = I \qquad (4.59)$$

In addition, from equation (4.59), the moment of inertia I_z of the disc of radius r about an axis through its centre and perpendicular to its plane (z-axis) is

$$I_z = \frac{1}{2} Mr^2 \qquad (4.60)$$

Using perpendicular axis theorem i.e. equation (4.52) and (4.59) the moment of the inertia I_z of the disc about an axis through its centre and perpendicular to its plane (z-axis) can be expressed as,

$$I_z = I_x + I_y = 2I_x = 2I_y = 2I \qquad (4.61)$$

Therefore, from equation (4.61) the moment of the inertia I_x or I_y of the disc about an axis along its diameter (x-axis or y-axis) is,

$$I = \frac{I_z}{2} = \frac{1}{4} Mr^2 \qquad (4.62)$$

Exercise 4.3

1. A uniform disc has a mass of 4 kg and a radius of 2 m. Calculate the moment of inertia about an axis perpendicular to its plane,
   (a) through its centre.
   (b) through a point of its circumference.
2. A ring has a radius of 20 cm and a mass of 100 g. Calculate the moment of inertia about an axis,
   (a) perpendicular to its plane through its centre.
   (b) perpendicular to its plane passing through a point on its circumference.
   (c) in its plane passing through the centre.
3. The moment of inertia of a solid sphere of mass 2.5 kg is 4 kgm^2. Find its moment of inertia about a parallel axis at a distance of 0.2 m from its centre.
4. A thin sheet of aluminium of mass 0.025 kg has a length of 0.25 m and

width of 0.1 m. Find its moment of inertia

(a) about an axis perpendicular to the width and passing through the centre of mass m in its plane.

(b) about an axis parallel to the length and passing through the centre of mass m in its plane.

(c) about a perpendicular axis to the plane passing through the centre of mass.

5. Calculate the moment of inertia of thin circular disc of radius 50 cm and mass 2 kg about an axis along the diameter of the disc.

4.4 Radius of gyration of a rotating rigid body

Consider a solid disc of radius r and a total mass M which is uniformly distributed across the area of the disc (Figure 4.17a). The moment of inertia I of the disc about an axis passing through its centre O is $I = \frac{1}{2}Mr^2$.

Suppose the mass M of the disk is shifted to concentrate at a distance k from the axis of rotation (Figure 4.17b), so that, the resulting thin-walled disc has the same moment of inertia as that of the solid disc. The radius k at which the moment of inertia of the solid disc is the same as that of the thin-walled disc is called radius of gyration of the disc.

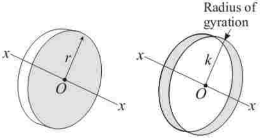


Figure 4.17 (a) A solid disc, and (b) a thin-walled disc

In general, the radius of gyration k of a rigid rotating body is defined as the distance from the axis of rotation to a point where the moment of inertia I of the body remains unchanged if the mass of the body is assumed to be concentrated at that point. That is,

$$k = \sqrt{\frac{I}{M}} \qquad (4.63)$$

If mass M of the body is assumed to be concentrated at a point with a distance k from the axis of rotation, the moment of inertia, by definition is,

$$I = Mk^2 = \sum_{i=1}^{n} mr_i^2 = m\sum_{i=1}^{n} r_i^2 \qquad (4.64)$$

If the body of mass M is made of n particles each of mass m, then $M = mn$ and $m = \frac{M}{n}$. Therefore, from equation (4.64) and (4.63), k can be expressed as,

$$k = \sqrt{\frac{\sum_{i=1}^{n} r_i^2}{n}} = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}} \qquad (4.65)$$

Thus, k represents the root mean square distance and is some kind of an average effective distance of the particles from the axis of rotation. Therefore, the radius of gyration of a body about a particular axis of rotation is equal to the root mean square distance of its particles from the axis of rotation.

Example 4.8

(a) Deduce the radius of gyration k of a rigid body of mass M rotating about various axes as follows:

(i) A uniform rod with axis passing at its centre;

(ii) A uniform rod with axis passing at one-end;

(iii) A uniform ring of radius r with axis passing through its centre perpendicular to its plane;

(iv) A uniform disc of radius r with axis passing through its centre perpendicular to its plane; and

(v) A solid sphere of radius r with axis through its centre.

(b) The radius of gyration of a hollow sphere of mass M and radius R about a certain axis is R. Find the distance of the axis from the centre of the sphere.

Solution

(a) (i)Using equation (4.63) and (4.18), the radius of gyration of a uniform rod with axis of rotation through its centre can be expressed as,

$k = L\sqrt{\frac{1}{12}}$.

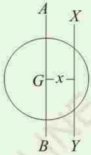
(ii) Using equation (4.63) and (4.21), the radius of gyration of a uniform rod with axis of rotation through one-end can be expressed as, $k = L\sqrt{\frac{1}{3}}$.

(iii)Using equation (4.63) and (4.24), the radius of gyration of a ring with axis of passing through its centre perpendicular to its plane is, $k = r$.

(vi) Using equation (4.63) and (4.29), the radius of gyration of a disc with axis of rotation passing through its centre perpendicular to its plane can be expressed as, $k = r\sqrt{\frac{1}{2}}$.

(v) Using equation (4.63) and (4.36), the radius of gyration of a solid sphere of radius r with axis through its centre can be expressed as, $k = r\sqrt{\frac{2}{5}}$.

(b) Let x be the distance of the axis from the centre of the sphere. From the parallel axis theorem:



$I_{XY} = I_{AB} + Mx^2$;

Thus, $Mk^2 = \frac{2}{3}MR^2 + Mx^2$

Given $k = R$, then

$R^2 = \frac{2}{3}R^2 + x^2$; $x = \frac{R}{\sqrt{3}}$

Therefore, the distance of the axis from the centre of the sphere is $\frac{R}{\sqrt{3}}$.

4.4.1 Compound pendulum

The radius of gyration of a rigid body can be determined by finding the period of rotation about an axis which is at a distance h from the centre of mass of the rigid body. Consider a rigid body suspended from a fixed peg and oscillating about a fixed axis O. Suppose h is the distance OG where G is the centre of mass and θ is the angle made by OG with the vertical at an instant (Figure 4.18). The equilibrium state of the compound pendulum corresponds to the case in which the centre of mass lies vertically below the pivot point. *i.e.* $\theta = 0°$.

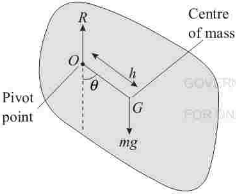


Figure 4.18 Oscillation of rigid body

The torque on the body (compound pendulum) is then $I\alpha$, where α is the angular acceleration given by $\frac{d^2\theta}{dt^2}$, and the opposing torque when θ is small is equal to mgh. Since the perpendicular distance from G to vertical through O is $h\sin\theta$, and $\sin\theta \approx \theta$ when the angle is small, then $I\alpha = -mgh\theta$. Hence,

$$\alpha = \frac{-mgh}{I}\theta.$$

It is clear, by analogy with our previous solutions of SHM equations, that the angular frequency of small amplitude oscillations of a compound pendulum is given by $\omega = \sqrt{\frac{mgh}{I}}$. So the period of oscillation T is given by,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}}$$

Therefore, for a compound pendulum (such as a rod, disc and lamina), the period T is given by

$$T = 2\pi\sqrt{\frac{I}{Mgh}} \qquad (4.66)$$

where, I is the rotational inertia about the axis of suspension and M is the mass of compound pendulum. Using the parallel axis theorem, I is given as,

$$I = M(h^2 + k^2) \qquad (4.67)$$

where k is radius of gyration.

4.4.2 A solid sphere and cylinder

For a torsional motion of a solid sphere or cylinder, the period T can be expressed as,

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad (4.68)$$

where C is the torsional constant of the wire material and I is the moment of inertia of a solid cylinder or sphere about vertical axis through its centre. For a rolling motion of a solid sphere and a cylinder on an inclined plane, the period T can be expressed as,

$$T = 2\pi\sqrt{\frac{r^2 + k^2}{2gh}} \qquad (4.69)$$

where h is the height of the inclined plane, r is the radius of the rolling object and k is the radius of gyration.

Exercise 4.4

1. (a) Define radius of gyration.
   (b) Why is the radius of gyration of a body not unique?
   (c) What is the physical meaning of radius of gyration?
2. Calculate the radius of gyration about a tangent of a hollow sphere of radius 0.5 m parallel to the axis through its centre.
3. A flywheel consists of a solid disc 30 cm in diameter and 2.5 cm thick and two projecting hubs 10 cm in diameter and 7.5 cm long. If the flywheel is made of material with density 8000 kgm^{-3}, find the radius of gyration about the axis of rotation.
4. Small blocks, each of mass m, are clamped at the ends and at the centre of a light rigid rod of length L. Compute the radius of gyration of the system about an axis perpendicular to the rod and passing through a point one-quarter of the length from one end. Neglect the moment of inertia of the rod.
5. What is the radius of gyration of a slender rod of mass 90 g and length 120 cm about an axis perpendicular to its length and passing through 20 cm from one end?

4.5 Torque of a rotating rigid body

When loosening or tightening a bolt or a screw, a twisting force is required to turn the screw about its axis of rotation. The twisting force required to turn the screw is called torque or moment of force. Torque therefore is the force required to rotate an object about an axis of rotation. The effect of torques resembles the translation force which is used to push or pull objects, but torque rotates or twists an object when applied. Thus it can be said that, torque is a turning force or twisting force. In general torque is defined as the product of force applied and the perpendicular distance from the point of force applied. Torque is a vector quantity and its direction is determined by the right hand rule and is perpendicular to both linear force and radius (distance from the axis of rotation). That is,

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (4.70)$$

Its SI unit is newton-metre denoted as Nm. The magnitude of τ is $Fr\sin\theta$ where θ is the angle between $\vec{F}$ and $\vec{r}$.

From equation (4.70), the rotation effects depend on the distance from the point of application of force and the magnitude and direction in which the force is applied called the line of action of the force. This can well be explained when shutting or opening a door.

Consider an open door with hinges at point PCQ (Figure 4.19). It is evident that, it is easy to turn the door from point A with larger distance from the hinge (axis of rotation) than point B which is closer to the hinge. Likewise, turning becomes easier when the line of action of the force is applied perpendicular to the door at A than when applied parallel to the door at that point.

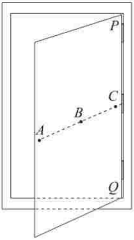


Figure 4.19 Hinged door with points of action of force

4.5.1 Expression for torque of a rotating rigid body

Consider a rigid body rotating about an axis PO with an angular velocity ω, in which, every particle of the body rotates with the same angular velocity (Figure 4.20).

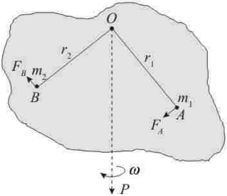


Figure 4.20 Rotation and torque

Let particles A and B with masses m_1 and m_2 rotate with angular velocity, ω. Also, let the line of action of applied force be perpendicular to the position vectors of the particles. Then from Newton's second law of motion, the net force acting on A can be expressed as,

$$F_A = m_1 a_1 = m_1 r_1 \alpha \tag{4.71}$$

where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration which is the rate of change of ω. Thus, using equation (4.70), the moment of force (torque) on particle A can be expressed as,

$$\tau_A = F_A \times r_1 = m_1 r_1^2 \alpha \tag{4.72}$$

Similarly, for particle B, the force can be expressed as,

$$F_B = m_2 a_2 = m_2 r_2 \alpha \tag{4.73}$$

Hence, the moment of force (torque) on particle B is

$$\tau_B = F_B \times r_2 = m_2 r_2^2 \alpha \tag{4.74}$$

Finally, the total moment of the force for the whole body about an axis PO is equal to:

$$\tau = \tau_A + \tau_B + \ldots + \tau_n$$
$$= m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \ldots + m_n r_n^2 \alpha$$

which simplifies to,

$$\tau = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2\right)\alpha \tag{4.75}$$

But $m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2 = I$ is the moment of inertia of the body about axis through O.

From equation (4.75), the moment of force (torque) of the whole body about an axis through O can be expressed as,

$$\tau = I\alpha = I\frac{d\omega}{dt} \tag{4.76}$$

Example 4.9

Masses $m_1 = 0.20$ kg and $m_2 = 0.25$ kg are suspended (Figure 4.21) from a light cord which passes over a wheel of radius 0.15 m and moment of inertia 0.12 kgm^2. Initially, the two masses are held at the same horizontal level. Assuming that the wheel rotates freely about its axis, calculate the speed of each mass and the angular velocity of the wheel when the vertical distance between the masses is 0.3 m.

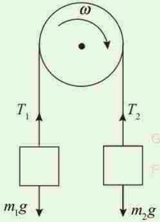


Figure 4.21 Connected masses passing over a wheel

Solution

Since the torque τ is developed on the wheel, then, tension $T_1 \neq T_2$. Thus,

$$T_1 - m_1 g = m_1 a \quad (i)$$

$$m_2 g - T_2 = m_2 a \quad (ii)$$

Adding equation (i) and (ii), it follows that;

$$m_2 g - m_1 g + T_1 - T_2 = m_2 a + m_1 a$$

$$(m_2 - m_1)g - (T_2 - T_1) = (m_2 + m_1)a \quad (iii)$$

but, $\tau = Tr = I\alpha$, where $T = T_2 - T_1$

Since, $\frac{a}{r} = \alpha$, then,

$$T = I\frac{a}{r^2} \quad (iv)$$

Substituting equation (iv) into (iii),

$$(m_2 - m_1)g - I\frac{a}{r^2} = (m_2 + m_1)a$$

$$a = \frac{(m_2 - m_1)g}{\left(\frac{I}{r^2} + m_1 + m_2\right)}$$

$$a = \frac{(0.25 - 0.20)\text{ kg} \times 9.8 \text{ ms}^{-2}}{\left(\frac{0.12 \text{ kgm}^2}{(0.15\text{m})^2}\right) + (0.20 + 0.25)\text{ kg}}$$

$$= 0.085 \text{ ms}^{-2}$$

From the third equation of motion, $v^2 = u^2 + 2as$, where s is distance moved by each mass and $u = 0 \text{ ms}^{-1}$. Therefore,

$$v = \sqrt{2 \times 0.085 \text{ ms}^{-2} \times 0.15 \text{ m}}$$

$$= 0.16 \text{ ms}^{-1}.$$

Therefore, the speed of each mass is 0.16 ms^{-1}.

The Angular velocity ω of the wheel is given by,

$$\omega = \frac{v}{r} = \frac{0.16 \text{ ms}^{-1}}{0.15 \text{ m}}$$

$$= 1.07 \text{ rads}^{-1}$$

Therefore, the angular velocity of the wheel is 1.07 rads^{-1}.

4.5.2 Work done by torque

Consider a force F applied tangentially to a wheel of radius r and allowed to rotate about its centre O through an angle θ (Figure 4.22).

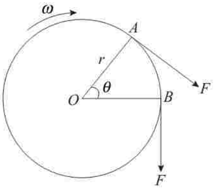


Figure 4.22 A wheel with tangentially applied force

The work done W by the force F on turning the wheel an angle θ (subtended by an arc AB) about an axis O is $W = (\text{Force}) \times (\text{arc distance } AB)$. The arc distance AB is equal to $r\theta$. Therefore W can be expressed as,

$$W = F \times r\theta, \ Fr \times \theta = \tau\theta \qquad (4.77)$$

Example 4.10

Calculate the work done by a torque of 6 Nm if it rotates a wheel through four revolutions.

Solution

Work done,

$W = \tau \times \theta = 6\text{ Nm} \times (4\text{ rev} \times 2\pi)$

$= 150.8\text{ Joules}$

Therefore, the work done by the torque is 150.8 Joules.

4.5.3 Work-energy in rotating objects

Consider a rigid body of moment of inertia I displaced a small angle $d\theta$ from A to B (Figure 4.23).

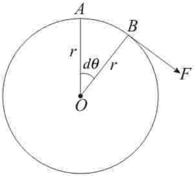


Figure 4.23 Rigid body with a small displacement

The work done dW by a torque τ in turning the object through a small angular displacement $d\theta$ from A to B is given by;

$$dW = \tau d\theta \qquad (4.78)$$

Using equation (4.76), the work done dW in (4.78) can be expressed as,

$$dW = I\left(\frac{d\theta}{dt}\right)d\omega = I\omega\, d\omega \qquad (4.79)$$

The total work done W in turning the wheel from initial ω_i to final angular velocity ω_f can be expressed as,

$$W = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = I\int_{\omega_i}^{\omega_f} \omega\, d\omega$$

$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \qquad (4.80)$$

From equation (4.15), the total work done W in equation (4.80) can be written as,

$$W = \Delta K.E_{rotational} \qquad (4.81)$$

Therefore, equation (4.81) is the rotational work-energy theorem which implies, the work done in rotating a rigid body corresponds to the change in its rotational kinetic energy.

4.5.4 Determination of the moment of inertia of a flywheel

The moment of inertia of a flywheel about a horizontal axle can be determined when one end of string is attached to a mass M and the other end is passed through a hole in the axle of the flywheel, and winding the string round the axle (Figure 4.24).

The mass M reaches reference level (e.g. the ground) when the string is completely unwound from the axle. The number of revolutions n, made by the wheel from when M is released up to it strikes the ground (when the string is released from the axle) is recorded. The flywheel will perform addition rotations after the string is released from the axle. The additional number of revolution n_1 made by the flywheel and the time t taken from when the string is released from the axle until it comes to rest, are also recorded in reference to a mark on the wheel.

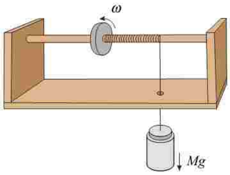


Figure 4.24 Flywheel with axle as its axis of rotation

Now when M is released, the loss in potential energy of M is equal to the gain in kinetic energy of M plus gain in kinetic energy of flywheel and work done against friction. That is,

$$Mgh = \frac{1}{2}Mr^2\omega^2 + \frac{1}{2}I\omega^2 + nf \qquad (4.82)$$

where h is the distance M has fallen, r is the radius of the axle, ω is the angular velocity, I is the moment of inertia of the wheel, and f is the energy per turn expended against friction. Since the energy of rotation of the flywheel when the mass M reaches the ground equals to the work done against friction in n_1 revolutions, then, $f = \frac{1}{2}\frac{I\omega^2}{n_1}$. Then equation (4.82) can be expressed as,

$$Mgh = \frac{1}{2}Mr^2\omega^2 + \frac{1}{2}I\omega^2\left(1 + \frac{n}{n_1}\right) \qquad (4.83)$$

Since the angular velocity of the wheel when M reaches the ground is ω, and the final angular velocity of the wheel is zero after a time t, the average angular velocity is $\frac{\omega}{2} = \frac{2\pi n_1}{t}$. Thus the angular velocity ω is $\omega = \frac{4\pi n_1}{t}$. Using ω and the magnitude of the other quantities in equation (4.83), the moment of inertia I of the flywheel can be determined.

Angle turned through during the n^{th} time

Suppose θ_1 and θ_2 are the angular displacements turned through by a rigid body at times $t_1 = n$ and $t_2 = n - 1$ respectively. From, $\omega = \omega_o + \alpha t$; where

α is the angular acceleration, ω_o and ω are initial and final angular velocity respectively.

$\omega = \frac{d\theta}{dt}$; $d\theta = \omega_o dt + (\alpha t)dt$; integrating within limits,

$\int_{\theta_1}^{\theta_2} d\theta = \omega_o \int_{n-1}^{n} dt + \alpha \int_{n-1}^{n} t\,dt$ results to;

$$\theta_2 - \theta_1 = \omega_o\left(n-(n-1)\right) + \frac{\alpha}{2}\left(n^2 - (n-1)^2\right)$$

$\theta_2 - \theta_1 = \theta_n$, this is the angle turned through during the n^{th} time

$$\theta_n = \omega_o\left(n-n+1\right) + \frac{\alpha}{2}\left(n^2 - n^2 + 2n - 1\right)$$

$$\theta_n = \omega_o + \frac{\alpha}{2}(2n-1) \text{ or}$$

$$\theta_n = \omega_o + \alpha(n - \frac{1}{2})$$

This is the angle turned through by a body during the n^{th} time.

Example 4.11

A flywheel with axle 1.0 cm in diameter is mounted on frictionless bearings and set in motion by applying a steady tension of 2.0 N to a thin thread wound tightly round the axle. The moment of inertia of the system about its axis of rotation is $5.0\times10^{-4}\,\text{kgm}^2$. Calculate:

(a) The angular acceleration of the flywheel when 1.0 m of the thread has been pulled off the axle; and

(b) The constant retarding couple which must be applied to bring the flywheel to rest in one complete turn when tension in the thread having been removed.

Solution

(a) From definition, the torque of the flywheel can be expressed as $\tau = I\alpha = Fr$. Thus, the angular acceleration is,

$$\alpha = \frac{Fr}{I} = \frac{2\text{ N}\times0.5\times10^{-2}\text{ m}}{5\times10^{-4}\text{ kgm}^2}$$

$$= 20\text{ rads}^{-2}$$

Therefore, the angular acceleration of the flywheel is 20 rads^{-2}.

(b) Since the radius of the flywheel is

$$r = \frac{d}{2} = \frac{1\times10^{-2}\text{ m}}{2} = 0.5\times10^{-2}\text{ m}$$

and the length pulled off the axle $s = 1\text{m}$. Using circular motion, the motion of the flywheel when 1.0 m of the thread has been pulled off the axle can be expressed as,

$$\omega_f^2 = \omega_o^2 + 2\alpha\theta = \omega_o^2 + 2\alpha\left(\frac{s}{r}\right)$$

$$\omega_f^2 = 0 + 2\times20\text{ rads}^{-2}\times\frac{1\text{m}}{0.5\times10^{-2}\text{ m}}\text{rad}$$

$$\omega_f = 89.44\text{ rads}^{-1}$$

Now, when the constant retarding couple brings the flywheel to rest in one complete turn, then, the circular motion equation is $0 = \omega_i^2 + 2\alpha'\theta'$ which gives its angular acceleration α', as $\alpha' = \frac{-\omega_i^2}{2\theta'}$.

Therefore, the constant retarding couple which is required to bring the flywheel to rest in one complete turn is

$$\tau' = I\alpha' = -I\frac{\omega_i^2}{2\theta'}$$

$$\tau' = -5\times10^{-4}\text{ kgm}^2\times\frac{\left(89.44\text{rads}^{-1}\right)^2}{2\times2\pi\text{rad}}$$

$$= -0.32\text{ Nm}$$

4.5.5 Kinetic energy of rolling objects

When an object such as a cylinder or ball rolls down a plane, the object is rotating as well as moving down the plane. Therefore, it has both rotational motion and translational motion.

Consider a uniform cylinder of radius r and mass M rolling without slipping down an inclined plane of an angle θ at a height h above the horizontal plane (Figure 4.25).

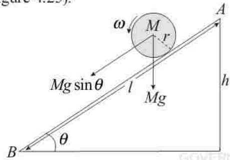


Figure 4.25 Cylinder with axis of rotation along the inclined plane

At any instant, the line of contact AB with the plane is at rest, and therefore, AB can be considered as the axis of rotation at the surface of the cylinder. Then from equation (4.15), kinetic energy of the cylinder about an axis of rotation AB can be expressed as,

$$K.E = \frac{1}{2} I_{AB} \omega^2 \qquad (4.84)$$

Using the parallel axis theorem equation (4.50), the moment of inertia I_{AB} about the surface of the cylinder can be expressed in terms of the moment of inertia I_G about the centre of the cylinder as $I_{AB} = I_G + Mr^2$.

Therefore, equation (4.84) can be expressed as,

$$KE = \frac{1}{2} I_G \omega^2 + \frac{1}{2} Mv^2 \qquad (4.85)$$

where $\frac{1}{2} I_G \omega^2$ is the rational kinetic energy and $\frac{1}{2} Mv^2$ is translational kinetic energy. Therefore, for rolling objects along an incline, the total kinetic energy is the sum of the rotational and translational kinetic energy.

4.5.6 Conservation of mechanical energy in rolling objects

When an object is rolling about an incline (Figure 4.25) its total mechanical energy (E) is the sum of the potential ($P.E = Mgh$) and kinetic energy ($K.E$). Then, E can be expressed as,

$$E = P.E + K.E$$

$$E = Mgh + \left(\frac{1}{2} I_G \omega^2 + \frac{1}{2} Mv^2 \right) \qquad (4.86)$$

The principle of conservation of mechanical energy requires E of the system to remain unchanged (conserved) if no external force (such as friction) is applied to the rolling object. Therefore, the conservation of mechanical energy in a rolling object at any instant can be expressed as,

$$Mgh + \left(\frac{1}{2} I_G \omega^2 + \frac{1}{2} Mv^2 \right) = constant \qquad (4.87)$$

Example 4.12

Determine the velocity at the bottom of a rigid body of mass M rolling from the top of the inclined plane of length l elevated at an angle θ.

Solution

When a rigid body rolls from the top of the incline, its mechanical energy (potential and kinetic) at any instant is conserved. As soon as it begins the rolling, part of its potential energy (U) is converted into kinetic energy $(K.E)$. Its mechanical energy is purely potential at the top and purely kinetic at the bottom of the incline. Then, the conservation of energy for rolling objects equation (4.87), can be expressed as,

$$Mgh + 0 = 0 + \left(\frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2\right) \quad (i)$$

Since the inclined plane is of length l, elevated at an angle θ (Figure 4.25), then from trigonometry, $h = l\sin\theta$, and (i) can be expressed as,

$$Mgl\sin\theta = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 \quad (ii)$$

Using $\omega = \frac{v}{r}$ into (ii) and simplifying, the velocity of the rolling body at the bottom of an incline plane can be expressed as,

$$v = \sqrt{\frac{2gl\sin\theta}{1 + \frac{I}{Mr^2}}} = \sqrt{\frac{2gh}{1 + \frac{I}{Mr^2}}} \quad (iii)$$

which can be expressed as,

$$v = \sqrt{\frac{2gh}{1 + C}} \quad (iv)$$

where $C = \frac{I}{Mr^2}$ is a coefficient (less than or equal to 1) depending on the shape of the body. For example, for a disc $C = \frac{1}{2}$, solid cylinder $C = \frac{1}{2}$ and solid sphere, $C = \frac{2}{5}$. Substituting the value of C in (iv) for different objects, one can compare their respective velocities at the bottom and therefore determine which object will reach the bottom of the incline first.

Example 4.13

Determine the order in which a solid sphere, disc and solid cylinder arrives at the bottom of an incline if all were released at the same time to roll at an inclined plane elevated at an angle θ.

Solution

Using the C for solid sphere, disc and solid cylinder in example 4.12, equation (iv), their respective velocities are

$v_{sphere} = \sqrt{\frac{10}{7}gh}$, $v_{disc} = \sqrt{\frac{4}{3}gh}$ and

$$v_{cylinder} = \sqrt{\frac{4}{3}gh}$$

Therefore, the sphere arrives the bottom first followed by both disc and the solid cylinder as they have same value of C. An alternative approach is to use acceleration where, an object with larger value of linear acceleration a will finish first.

Example 4.14

Consider a solid cylinder of mass M and radius R which is made to roll down a plane without slipping. Find the speed of its centre of mass at the moment when the cylinder reaches the bottom of an inclined plane (Figure 4.26).

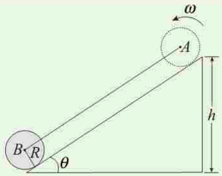


Figure 4.26 A solid cylinder rolling down the plane without slipping

Solution

Using the moment of inertia of a solid cylinder $(I = \frac{1}{2}MR^2)$ about an axis through its centre, $\omega = \frac{v}{R}$ in equation (4.87), the conservation of energy of a rolling object at the bottom can be expressed as,

$$Mgh + 0 = 0 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$$

$$Mgh = \frac{1}{4}Mv^2 + \frac{1}{2}Mv^2 = \frac{3}{4}Mv^2 \qquad (i)$$

Then from (i), the speed v is

$$v = \sqrt{\frac{4}{3}gh} \qquad (ii)$$

Therefore, equation (ii) is the velocity of the centre of mass of a solid cylinder at the bottom of inclined plane.

Example 4.15

A body rolling down an inclined plane has radius R and radius of gyration k.

The body starts moving from the height h and reaches the bottom with velocity v.

Show that, $v = \sqrt{\dfrac{2gh}{1 + \dfrac{k^2}{R^2}}}$

Solution

Consider free body diagram for a body rolling down an inclined plane and has radius R and radius of gyration k starting from height h and reaches the bottom with velocity v (Figure 4.27).

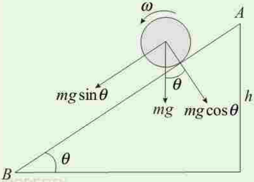


Figure 4.27 A body rolling down an inclined plane

Since the total energy E at the top of incline is purely $P.E$ that is $(K.E = 0)$ and at the bottom of the incline the total energy E is purely $K.E$ that is $(P.E = 0)$. Using the moment of inertia $I = mk^2$ and $\omega = \frac{v}{R}$, the conservation of mechanical energy (4.87) for the rolling body can be expressed as,

$$gh = \frac{1}{2}k^2\left(\frac{v^2}{R^2}\right) + \frac{1}{2}v^2$$

which simplifies to,

$$v = \sqrt{\frac{2gh}{1 + \frac{k^2}{R^2}}}$$

4.5.7 Linear acceleration of rolling objects

Consider a rigid body of radius r and mass M rolling down an inclined plane (Figure 4.28).

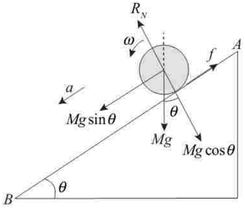


Figure 4.28 Rigid body with axis of rotation along the inclined plane

At any instant, the line of contact AB with the plane is at rest, and therefore, AB can be considered as the axis of rotation at the surface of the rigid body. The component of weight along the incline is $mg\sin\theta$ which provides the torque $(I\alpha)$ on the object about an axis AB. That is,

$$(Mg\sin\theta)r = \tau = I_{AB}\alpha = I_{AB}\frac{a}{r} \qquad (4.88)$$

From equation (4.88), the linear acceleration of the rigid body with axis AB can be expressed as,

$$a = \frac{Mr^2 g\sin\theta}{I_{AB}} \qquad (4.89)$$

From the parallel axes theorem in equation (4.50), then, equation (4.89) can be expressed as,

$$a = \frac{Mr^2 g\sin\theta}{I_G + Mr^2} = \frac{g\sin\theta}{1+\frac{I_G}{Mr^2}} = \frac{g\sin\theta}{1+C} \qquad (4.90)$$

where $C = \frac{I}{Mr^2}$ is a coefficient (less than or equal to 1) depending on the shape of the body.

Example 4.16

Determine the order in which a solid sphere, disc and solid cylinder arrives at the bottom of the inclined if both are released at the same time to roll at an inclined plane elevated at an angle θ.

Solution

Substituting the values of C for solid sphere, disc and solid cylinder in equation (4.90), their respective linear accelerations a are $a_{sphere} = \frac{5}{7}g\sin\theta$,

$a_{disc} = \frac{2}{3}g\sin\theta$ and $a_{cylinder} = \frac{2}{3}g\sin\theta$

Therefore, the sphere arrives the bottom first followed by both disc and the solid cylinder as they have same value of C.

Example 4.17

A sphere and a cylinder having the same mass and radius start from rest at the same point on an inclined plane and they are left to roll down the plane. Determine which one reaches at the bottom first?

Solution

Consider a sphere and cylinder (Figure 4.29a) and its free body diagram (Figure 4.29b)

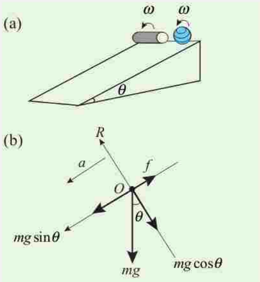


Figure 4.29 (a) Objects on an nclined plane, and (b) free body diagram

Then, the net force F in taking the body down the plane based on Newton's 2[nd] law is,

$$F = mg\sin\theta - f = ma \qquad (i)$$

The moment of force of friction about O is fR which is equal to torque $I\alpha$, about O. Then, the friction force f can be expressed as,

$$f = \frac{I\alpha}{R} \qquad (ii)$$

Using (ii), equation (i) can be expressed as,

$$mg\sin\theta - \frac{I\alpha}{R} = ma \qquad (iii)$$

Using the moment of inertia of a solid sphere in equation (4.36) and that of solid cylinder in equation (4.44), the linear acceleration a for a sphere and a' for a cylinder respectively, $a = \frac{5}{7}g\sin\theta$ and $a' = \frac{2}{3}g\sin\theta$. Since $a > a'$, then the sphere will reach the bottom first.

Example 4.18

A solid cylinder of mass m is placed in a rough inclined plane of inclination θ to the horizontal (Figure 4.30). Show that the minimum frictional force applied for rolling without slipping is $\frac{1}{3}mg\sin\theta$, and the minimum coefficient of friction is $\frac{1}{3}\tan\theta$.

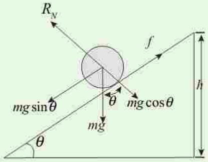


Figure 4.30 Solid cylinder rolling on a rough inclined plane

Solution

For translational motion,

$$f = mg\sin\theta - ma \qquad (i)$$

But for rotational motion,

$$a = \frac{fR^2}{I} \qquad (ii)$$

Since $\alpha = \frac{fR}{I}$ and $\alpha = \frac{a}{R}$ (relationship between liner and rotational acceleration). Using (*ii*) into (*i*),

$$f = mg\sin\theta - m\left(\frac{fR^2}{I}\right)$$

$$= mg\sin\theta - \left(\frac{2mfR^2}{mR^2}\right) = mg\sin\theta - 2f \qquad (iii)$$

since $I = \frac{1}{2}mR^2$ for a solid cylinder.

Therefore, (iii) simplifies to,

$$f = \frac{1}{3} mg \sin\theta \qquad (iv)$$

Similarly, from (Figure 4.30), $f = \mu mg \cos\theta$ and therefore (iv) can be written as,

$$\mu mg \cos\theta = \frac{1}{3} mg \sin\theta$$

Thus, the minimum coefficient of friction is, $\mu = \frac{1}{3}\tan\theta$

Exercise 4.5

1. Is it possible to change the translation kinetic energy of an object without changing its rotational energy? Explain.
2. If you see an object rotating, is there necessarily a net torque acting on it? Why?
3. Can a stationary object have a nonzero angular acceleration? Explain.
4. When tightening a bolt, mechanics sometimes extend the length of a wrench handle by slipping a section of pipe over the handle. Why could this procedure easily damage the bolt?
5. (a) If the forces on an object balance, do the torques necessarily balance? Explain.
   (b) If the torques on an object balance, do the forces necessarily balance? Illustrate your answers with clear examples.
6. A grinding wheel is in the form of a uniform solid disc of radius 7 cm and a mass 2 kg . It starts from rest and accelerates uniformly under the action of a constant torque of 0.6 Nm that the motor exerts on the wheel.
   (a) How long does the wheel take to reach its final operation speed of 1200 rev/min?
   (b) Through how many revolutions does it turn while accelerating?
7. A model airplane with mass 0.750 kg is tethered by a wire so that it flies in a circle 30 m in radius. The airplane engine provides a net thrust of 0.8 N perpendicular to the tethering wire. Find:
   (a) The torque which produces the net thrust about the centre of the circle;
   (b) The angular acceleration of the airplane when it flights at horizontal level; and
   (c) The linear acceleration of the airplane tangent to its flight path.
8. A 15 kg object and a 10 kg object are suspended, joined by a cord that passes over a pulley with a radius of 10 cm and a mass of 3 kg (Figure 4.31). The cord has a negligible mass and does not slip on the pulley. The pulley rotates on its axis without friction. The object starts from rest 3 m apart. Treat the pulley as a uniform disc, and determine the speed of the two objects as they pass each other.

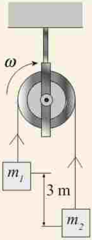


Figure 4.31 Objects suspended on a cord passing over a pulley

9. An object with a weight of 50 N is attached to the free end of a light string wrapped around a wheel of radius 0.25 m and a mass 3 kg. The wheel is a solid disc free to rotate in a vertical plane about the horizontal axis passing through its centre. The suspended object is released 6 m above the floor.
   (a) Determine the tension in the string, the acceleration of the object, and the speed with which the object hits the floor.
   (b) Verify your last answer by using the principle of conservation of energy to find the speed with which the object hits the floor.
10. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl with much larger radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Figure 4.32). Determine the angular speed of the sphere when it reaches the bottom of the bowl.

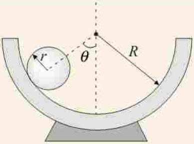


Figure 4.32 Uniform solid sphere inside surface of a hemispherical bowl

11. A solid sphere of mass m and radius r rolls without slipping along the track (Figure 4.33). It starts from rest with the lowest point of the sphere at height h above the bottom of the loop of radius R, much larger than r.
   (a) What is the minimum value of h (in terms of R) such that the sphere completes the loop?
   (b) What are the force components on the sphere at the point P if $h = 3R$?

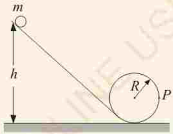


Figure 4.33 Solid sphere rolling without slipping along the track

12. Show that the minimum coefficient of friction for rolling without slipping of a hollow cylinder and solid sphere on an inclined plane are $\frac{1}{2}\tan\theta$ and $\frac{2}{7}\tan\theta$ respectively, where θ is the angle of the inclined plane with the horizontal.

4.6 Angular momentum

When an object is rotating about an axis, its rotational inertia can be characterized using angular momentum. The angular momentum L can be defined as the moment of linear momentum of a body rotating about a fixed axis. It is a vector quantity whose direction is that of the axis of the rotating body and is given a positive sign in the direction in which a right-hand screw would advance if turned in the similar direction. Mathematically, $\vec{L}$ is defined as the product of the linear momentum ($\vec{p} = m\vec{v}$) times the perpendicular distance $\vec{r}$ from the axis of rotation. That is,

$$\vec{L} = \vec{r} \times \vec{p} \qquad (4.91)$$

and its magnitude given as $pr\sin\theta$ where θ is the angle between $\vec{P}$ and $\vec{r}$.

Consider a particle A at a distance r_1 from the axis O of a rotating rigid body that rotates with an angular velocity ω (Figure 4.34).

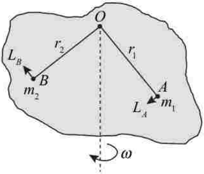


Figure 4.34 Angular momentum and moment of inertia

From equation (4.91), the angular momentum $\vec{L}_A$ of particle A rotating with angular velocity $\vec{\omega} = \frac{\vec{v}_1}{\vec{r}_1}$ about axis O, with $\theta = 0$, can be expressed in magnitude as,

$$L_A = (m_1 v_1) \times r_1 = (m_1 r_1 \omega) \times r_1 = m_1 r_1^2 \omega \quad (4.92)$$

Similarly, for particle B of mass m_2, its angular momentum L_B about axis O is

$$L_B = m_2 r_2^2 \omega \qquad (4.93)$$

In general, the total angular momentum L for n particles about an axis O is,

$$L = (m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2)\omega \qquad (4.94)$$

where, $m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 = I$ is the moment of inertia of the rotating rigid body.

Therefore, the magnitude of angular momentum L of a rigid body rotating with angular velocity ω about an axis O can be expressed as,

$$L = I\omega \qquad (4.95)$$

The vector form of equation (4.95) angular momentum can be written as,

$$\vec{L} = I\vec{\omega} \qquad (4.96)$$

4.6.1 Angular momentum and torque

Using the product rule of differential calculus to equation (4.91), the time rate of change of $\vec{L}$ can be expressed as,

$$\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{v} \times \vec{r}) = m(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt})$$

$$= m(\vec{r} \times \frac{d\vec{v}}{dt}) \qquad (4.97)$$

$$\frac{d\vec{r}}{dt} \times \vec{v} = \vec{v} \times \vec{v} = 0$$

Equation (4.97), is the linear acceleration, and therefore, equation (4.97) can be written as,

$$\vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{\tau} = \frac{d\vec{L}}{dt} \qquad (4.98)$$

Thus equation (4.98) provides the relation between torque τ and the angular momentum L, that is, the torque acting on a body is equal to rate of change of angular momentum of the rigid body. This is analogous to Newton's second law of motion. i.e., $\vec{F} = m\vec{a}$

4.6.2 The principle of conservation of angular momentum

Suppose no external torque $\vec{\tau}_{ext}$ is applied on a rotating rigid body, equation (4.98) can be expressed as,

$$\frac{d\vec{L}}{dt} = 0 \qquad (4.99)$$

This implies, from equation (4.99), the angular momentum, $\vec{L} = I\omega$ is constant and is called the principle of conservation of angular momentum. The principle states that, *"If there is no external torque acting about the axis of the rotation then the angular momentum of a body about that axis of rotation is constant".*

The principle is also expressed as,

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad (4.100)$$

Note that, it is the angular momentum (which is the product of inertia and the angular velocity) that remains constant and not the angular velocity ω. For the angular momentum to remain constant, the moment of Inertia decreases and the angular velocity increases and vice versa. The angular momentum, $\vec{L} = I\vec{\omega}$ is analogous to linear momentum, $\vec{p} = m\vec{v}$, in that, the moment of inertia is replaced with mass, and angular velocity with linear velocity. A summary of the comparison of the dynamic equations for linear and rotational motion is as shown in Table 4.3.

Table 4.3 Comparison of dynamic equations for linear and rotational motion

Linear motion	**Rotational motion**
Mass (linear inertia) m	Moment of Inertia I
Momentum $\vec{p} = m\vec{v}$	Momentum $\vec{L} = I\vec{\omega}$
Newton's second law, $\vec{F} = m\frac{d\vec{v}}{dt}$	Newton's second law $\vec{\tau} = \frac{d\vec{L}}{dt}$
Work $W = Fd$	Work $W = \tau\theta$
Kinetic energy $K.E = \frac{1}{2}mv^2$	Kinetic energy $K.E = \frac{1}{2}I\omega^2$
Power $P = Fv$	Power $P = \tau\omega$
Velocity, $v = u + at$	Angular velocity, $\omega = \omega_o + \alpha t$
Distance, $s = ut + \frac{1}{2}at^2$	Angular displacement, $\theta = \omega_o t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_o^2 + 2\alpha\theta$
$s_n = u + a(n - \frac{1}{2})$	$\theta_n = \omega_o + \alpha(n - \frac{1}{2})$

Example 4.19

If the earth were to suddenly contract to half of its present radius (without any external torque acting on it), by how much would the day be decreased?

Solution

Consider the earth to be a perfect solid sphere of mass M whose radius, angular velocity and moment of inertia of the earth before contraction are R_1, $\omega_1 = \frac{2\pi}{T} = \frac{2\pi}{24}$ and $\frac{2}{5}MR_1^2$ and after contraction are R_2, $\omega_2 = \frac{2\pi}{T}$ and $\frac{2}{5}MR_2^2$ respectively.

During contraction, the angular momentum of the isolated earth is conserved. Therefore,

$$R_1^2\left(\frac{2\pi}{24}\right) = R_2^2\left(\frac{2\pi}{T}\right) \qquad (i)$$

since the earth contracts by a half its radius, using $R_2 = \frac{R_1}{2}$, in (i) it gives $T = 6$ hours. This means a day will last for 6 hours only, and therefore, a day will decrease by 18 hours.

4.6.3 Applications of rotational motion of rigid bodies

Torque has various applications in many common tools used domestically and in industries where it is necessary to turn, tighten or loosen devices. Such tools include spanners and screwdrivers. In general, a longer handle will enable smaller force to accomplish a task. It is easier for example to open a door when the force is applied at a longer distance from the hinge. All these are some of the applications of torque in daily life.

In addition, the basic property of angular momentum is to stabilize objects. For example, if a coin or cycle tyre is placed vertically on a horizontal surface without rolling, and released, it will immediately flip on its side. However, the coin or wheel will sustain its vertical position if rotating. This basic property of angular momentum is gyro-airplane which maintains its position regardless of the change in position in airplane body, hence stabilize airplane position.

Similarly, skaters and divers regulate their rotational motion by just movements of their arms and legs inwardly or outwardly. This is the application of the principle of conservation of angular momentum. When the skater or diver stretch her arms and legs outwards, increases her moment of inertia I, thus, her angular velocity ω is reduced to maintain the initial angular momentum.

Exercise 4.6

1. If two spinning objects have the same angular momentum, will they necessarily have the same rotational kinetic energy? If they have the same rotational kinetic energy, will they necessarily have the same angular momentum? Explain.
2. A circular metal disc of mass 4 kg and diameter 0.4 m makes 10 rev/s about an axis passing through its centre and perpendicular to its plane.

(a) What is the angular momentum about the same axis?

(b) Calculate the magnitude of the torque which will increase the angular momentum by 20% in $10\,\text{seconds}$.

3. A light rigid rod $1\ \text{m}$ in length joins two particles, with masses $4\ \text{kg}$ and $3\ \text{kg}$ at its ends. The combination rotates in the *x-y* plane about a pivot through the centre of the rod. Determine the angular momentum of the system about the origin when the speed of each particle is $5\ \text{m/s}$.

4. A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle. The platform has a mass $M = 100\ \text{kg}$ and a radius $R = 2\,\text{m}$. A student whose mass is $m = 60\ \text{kg}$ walks slowly from the rim of the disk toward its centre. If the angular speed of the system is $2\ \text{rads}^{-1}$ when the student is at the rim, what is the angular speed when he reaches a point $r = 0.5\ \text{m}$ from the centre?

5. A car of mass $900\ \text{kg}$ is moving around a circular path of radius $300\ \text{m}$ with a steady speed of $72\ \text{km/h}$. Calculate its angular momentum.

6. A $2\ \text{kg}$ disc travelling at $3\ \text{ms}^{-1}$ strikes a $1\ \text{kg}$ stick of length $4\ \text{m}$ that is lying flat on nearly frictionless ice (Figure 4.35). Assume that the collision is elastic and that the disc does not deviate from its original line of motion and the moment of inertia of the stick about its centre of mass is $1.33\,\text{kgm}^2$. Find:

(a) The translational speed of the disc;

(b) The translational speed of the stick; and

(c) The angular speed of the stick after the collision.

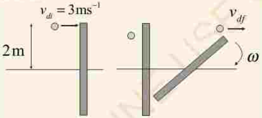


Figure 4.35 A travelling disc striking a stick

Revision exercise 4

1. Explain why the angular velocity of the earth about the sun increases when it comes closer to the sun?
2. Give a physical quantity obtained by a product of moment of inertia and
   (a) angular velocity,
   (b) angular acceleration.
3. How will you determine the direction of a torque? Explain.
4. Why do you use a single finger to turn the door but you use a thumb and finger to open a bottle cork?
5. Suppose you remove two eggs from a refrigerator, one boiled and the other fresh. If you wish to identify the boiled egg without breaking the eggs, you can spin them on the floor and compare their rotational motions. Which egg spins faster? Which rotates more uniformly? Explain.
6. A solid sphere and a solid cylinder, each having the same mass and radius, are released together at the top of an inclined plane and roll without slipping but also with negligible rolling friction. Explain why, despite the fact both must have the same total energy at all times, the sphere will always reach the bottom first.
7. Determine the moment of inertia of a 50 kg thin uniform meter rod rotating about an axis passing through the 25 cm mark perpendicular to its length.
8. The diameter of a ring increase by 2%. What will be the percentage increase in the moment of inertia about the axis of symmetry?
9. Two circular discs of the same mass and thickness are made from metals having different densities. Which disc has the larger rotational inertia about its symmetry axis?
10. What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.
11. Explain briefly why a wheel rolling on a flat horizontal surface cannot be slowed down by static friction?
12. A solid ball, a solid cylinder, and a hollow cylinder roll down a slope. Which one reaches the bottom first? last? Does it matter whether the radii are the same? What about the masses?
13. A wheel of moment of inertia $0.30\ \text{kgm}^2$ mounted on a fixed axle accelerates uniformly from rest to an angular velocity of $60\ \text{rads}^{-1}$ in 12 s. Find:
    (a) The angular acceleration;
    (b) The torque causing the wheel to accelerate; and
    (c) The number of revolutions in this 12 s period.
14. A constant force of 30 N is applied tangentially to a rim of a wheel mounted on a fixed axle and which is initially at rest. The wheel has a moment of inertia of $0.2\ \text{kgm}^2$ and radius of 15 cm.
    (a) What is the torque acting on the wheel?

(b) Find the work done on the wheel in 10 revolutions.

(c) Assuming that no work is done against friction, use energy consideration to find the angular velocity of the wheel after 10 revolutions.

15. A disc and a hoop roll down a slope. They have the same mass and the same radius.

(a) Which one has a greater moment of inertia?

(b) Does one lose more P.E than the other?

(c) Which one acquires the greater speed?

16. An electric motor supplies power of 5×10^{2} W to drive an unloaded flywheel of moment of inertia $2\ \text{kgm}^2$ at a steady speed of 6×10^{2} revolutions per minute. How long will it be before the flywheel comes to rest after the power is switched off, assuming the frictional couple remains constant?

17. An ice dancer is spinning about a vertical axis with his arms extended vertically upwards. Then he allows his arms to fall until they are horizontal.

(a) Will he spin faster or slower when?

(b) Has his kinetic energy been increased or decreased? How do you account for the change?

18. A horizontal disc rotating freely about a vertical axis makes 90 revolutions per minute. A small piece of putty of mass 2.0×10^{-2} kg falls vertically on to the disc and sticks to it at a distance of 5.0×10^{-2} m from the axis. If the number of revolutions per minute is thereby reduced to 80, calculate the moment of inertia of the disc.

19. A sphere of radius r rolls without slipping on a concave surface of large radius of curvature R. Show that the motion of the centre of gravity of the sphere is approximately simple harmonic with a period

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}}$$

where g is the acceleration due to gravity.

20. A thin uniform rod is pivoted about a horizontal axis which passes through a point on the rod 20 cm from the centre of gravity. If the period of oscillation of the rod is 1.58 seconds, find the length of the rod. (Moment of inertia of a uniform rod about an axis through its centre is, $I_c = \frac{ml^2}{12}$, where m is mass and l is length).

21. A cylindrical rocket of diameter 2.0 m develops a spinning motion in space of period 2 seconds about the axis of the cylinder. To eliminate this spin, two jet motors which are attached to the rocket on opposite ends of the diameter are fired until the spinning motion ceases. Each motor turns the rocket in the same direction and provides a constant thrust of 4.0×10^{3} N in a direction

tangential to the surface of the rocket and in the plane perpendicular to its axis. If the moment of inertia of the rocket about its cylindrical axis is $6.0 \times 10^5 \text{ kgm}^2$, find:

(a) Angular acceleration of the spinning of the rocket;

(b) Angular speed;

(c) The time for which the motors are fired; and

(d) The number of revolutions made by the rocket during firing.

22. A flywheel has 8 spokes and a radius of 30 cm. It is mounted on a fixed axle and spinning at 2.5 rev/s. A 24 cm arrow is to be shot parallel to the axle through the wheel without hitting any of the spokes. What minimum speed must the arrow have?

23. A frictionless pulley has the shape of uniform discs of mass 2.5 kg and radius 20 cm. A 1.5 kg stone is attached to a very light wire that is wrapped around the rim of the pulley and the system is released from rest. How far must the stone fall so that the pulley has 4.5 J of kinetic energy?

24. A centrifuge in a medical laboratory rotates at an angular speed of 3600 rev/min. When switched off, it rotates 50 times before coming to rest. Find the constant angular acceleration of the centrifuge.

25. The hub of a washer goes into its cycle, starting from rest and gaining angular speed steadily for 8s. At what time does it turn at 5 rev/s? At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The hub smoothly slows to rest in 12s. Through how many revolutions does the hub turn while it is in motion?

26. A body rotating with uniform angular acceleration covers 24 radians in the 4^{th} second and 36 radians in the 6^{th} second. Calculate:

(a) The angular acceleration and initial angular velocity; and

(b) The angular velocity after 10 seconds.

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Chapter Five

Fluid dynamics

Introduction

Fluids play a vital role in many aspects of everyday life. You drink them, breathe them, and swim in them. They circulate through your body and control the weather. Airplanes fly through them; ships float in them. Fluid dynamics is a study of fluids in motion, which can be described using models that are based on some assumptions and familiar principles and laws such as Bernoulli's Principle, Poisseiulle's Law, and Stokes' Law. The study of fluids provides an understanding of a number of everyday phenomena, such as why an open window and a door together create a draught in a room. In this chapter you will learn about the concept of streamline flow and continuity, viscosity, and turbulent flow.

5.1 Concept of fluid motion

Fluid is any substance that has no fixed shape and can easily flow. Fluid refers to liquids and gases. When looking at the motion of a fluid at different conditions, you can tell the general behaviour of a fluid in motion with respect to varying path it takes. The branch of physics which deals with the study of fluids in motion is called hydrodynamics or fluid dynamics. This section deals with concepts associated with the characteristics of fluid motion.

5.1.1 Compressible and incompressible fluid

Compressibility is an important characteristics of fluids and varies between liquids and gases. A fluid is considered compressible if its density changes with change in its pressure. A gas is considered compressible fluid since it is easy to compress. A liquid on the other hand, is considered incompressible since its density remains the same even if its pressure changes. In general, liquids are called incompressible fluids while gases are called compressible fluids.

(a) Viscous and non-viscous fluid

Viscosity is an intrinsic property of a fluid. It is an internal friction (also called viscous force) exhibited between adjacent fluid layers moving relative to each other. It arises in fluids because the motion of

Physics Form V.indd 135 23/07/2020 09:18

a molecule relative to its neighbours is opposed by the intermolecular forces between them. A fluid with this force is called a viscous fluid while the one without it is termed as a non-viscous fluid. For example, engine oils are viscous fluids while water is a non-viscous fluid.

(b) Steady flow

In a steady flow, velocity, density and pressure at each point in a fluid flow remains constant. Steady flow is also known as streamline flow, orderly flow or uniform flow. For a fluid undergoing steady flow, all particles passing at any given point follow the same path called a streamline. A streamline is a curve whose tangent at any point is along the direction of velocity of the fluid at that point. A special case of steady flow (Figure 5.1) in which the velocities of all particles at given streamlines are the same (though the particles of streamline may move at different speed) is called laminar flow.

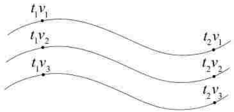


Figure 5.1 Laminar flow

In Figure 5.1 the velocities of particles in the three streams at one particular time t_1 are; v_1, v_2 and v_3, likewise, the same velocities can be observed at any other time t_2. Hence in laminar flow, the velocities of the particles within a stream will remain constant throughout the flow.

(c) Critical velocity and turbulent flow

When the velocity of a steady flow exceeds a particular value, that is, the critical velocity, the motion of particles of the fluid changes from steady to an irregular flow, known as turbulent flow. In a turbulent flow, path and the velocity of particles of fluid change continuously and randomly with time from point to point. The velocity v of a fluid flowing through a pipe depends on coefficient of viscosity η, density ρ of the fluid and radius r of the pipe which can be expressed using methods of dimensional analysis as, $v = \frac{k\eta}{r\rho}$, where k is a constant called Reynold's number R_e. Therefore, v can be expressed as,

$$v = \frac{R_e\eta}{r\rho} \qquad (5.1)$$

When R_e is less than 2000, the fluid flow is laminar or steady, if it is greater than 3000, the flow is turbulent, and between 2000 and 3000, the fluid flow is unstable. For critical velocity v_c the value of R_e is approximately equal to 1100.

5.1.2 The law of mass continuity

The equation of continuity is derived from the principle of conservation of mass which states that, "*Mass of the fluid entering per second at one point is equal to mass of that fluid leaving per second at the other point provided that there are no leaks or sinks of the fluid*". Consider a steady flow of a fluid passing through a tube of cross sectional area A_1 at point P and A_2 at point Q as shown in Figure 5.2 where $\Delta x_1 = v_1\Delta t$ and $\Delta x_2 = v_2\Delta t$.

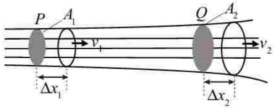


Figure 5.2 Steady flow of a liquid

Let the density and velocity of the liquid at point P is ρ_1 and v_1 respectively. The mass flux past point P is

$$\frac{\Delta m}{\Delta t} = \frac{\Delta(\rho_1 A_1 x_1)}{\Delta t} = \rho_1 A_1 \frac{\Delta x_1}{\Delta t} = \rho_1 A_1 v_1 \text{ and}$$

that at Q is $\frac{\Delta m}{\Delta t} = \rho_2 v_2 A_2$. If there are no leaks along the path of the fluid, the mass flux of the compressible fluid at P is equal to mass flux at Q;

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2 \qquad (5.2)$$

For incompressible fluids, $\rho_1 = \rho_2$, then, equation (5.2) is reduced to

$$A_1 v_1 = A_2 v_2 \qquad (5.3)$$

Equation (5.3) is known as equation of mass continuity.

For incompressible fluids, the product of cross-sectional area A and velocity v of the fluid at any point in a pipe is constant. i.e. $Av = \text{constant}$. This constant is called volume flux or volume flow rate.

Example 5.1

Water flows steadily at the rate of $1000\text{ cm}^3\text{s}^{-1}$ through a horizontal pipe of non-uniform cross-section. Find the velocity of the water at a section where the radius of the pipe is 10 cm.

Solution

From the equation of continuity, the volume of water per second $\frac{V}{t} = Av$

But $\frac{V}{t} = 1000\text{ cm}^3\text{s}^{-1}$, area of cross section of the pipe $A = 100\pi\text{ cm}^2$

Hence, $v = \frac{V}{t} \times \frac{1}{A}$,

$$v = \frac{1000\text{ cm}^3\text{s}^{-1}}{100\pi\text{ cm}^2} = 3.18\text{ cms}^{-1}$$

The velocity of the liquid is 3.18 cms^{-1}.

Exercise 5.1

1. What is meant by the terms:
   (a) Steady flow;
   (b) Turbulent flow;
   (c) Streamline flow; and
   (d) Reynold's number.
2. When a steadily flowing gas flows from a larger-diameter pipe to a smaller-diameter pipe, what happens to;
   (a) its speed,
   (b) its pressure, and
   (c) the spacing between its streamlines?
3. (a) State the law of mass continuity.
   (b) Water enters a cylindrical tube PQ through one end P with a speed v_1 and leaves through the other end Q with speed v_2. If the tube is always completely filled with water, show that the volume of water per second entering the tube is equal to the volume of water per second leaving the tube.

4. The water supply for a city is often provided from reservoirs built on high ground. Water flows from the reservoir, through pipes, and into your home when you turn the tap on your faucet. Why is the water flow more rapid out of a faucet on the first floor of a building than in an apartment on a higher floor?
5. Two water pipes of diameters 1.2 cm and 4 cm are connected in series to a main supply line. Find the ratio of velocities of flow in the two diameters.
6. The flow speed of water through a pipe of cross sectional area $4.0\ \text{cm}^2$ is $5.18\ \text{ms}^{-1}$. The water gradually descends $10\ \text{m}$ as the pipe increase in area to $6.5\ \text{cm}^2$. Find the speed of flow at the lower level.
7. The cylindrical tube of a spray pump has a cross section area of $8.0\,\text{cm}^2$. At its end there are 40 fine holes of diameter $0.1\ \text{mm}$ each. If the liquid flows inside the tube at $1.5\ \text{m}$ per minute, what is the speed of ejection of the liquid through the holes?

5.2 Bernoulli's Principle

Why is it that when you press your thumb over the end of a garden hose so that the opening becomes a small slit, the water comes out at high speed? Is the water under greater pressure when it is inside the hose or when it is out in the air? The relationship between fluid speed, pressure, and elevation was first derived by the Swiss Physicist Daniel Bernoulli. Bernoulli's theorem is derived from the law of conservation of energy. According to the theorem, the total energy (potential energy and kinetic energy) per unit volume remains constant throughout the flow, provided there is no source or sink of the fluid along the length of the pipe. The principle holds for incompressible, irrotational and non-viscous fluid in steady flow. In this section you will learn how to derive Bernoulli's equation and its daily applications.

5.2.1 Derivation of Bernoulli's Equation

Consider a steady flow of irrotational, incompressible and non-viscous fluid flowing in a non-uniform tube (Figure 5.3) from Q_1 at height h_1 to Q_2 at h_2. The cross-sectional areas at Q_1 and Q_2 are A_1 and A_2 respectively, and the corresponding fluid velocities are v_1 and v_2. Since the tube is not horizontal and not uniform, the pressure of the fluid varies among different points of the tube.

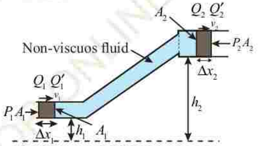


Figure 5.3 Fluid flowing in a non-uniform tube

Let the pressure at Q_1 be P_1 and at Q_2 be P_2. The force at Q_1 is P_1A_1, work done W_1 on the fluid in the region Q_1Q_1' is given by,

$$P_1A_1(v_1\Delta t) = P_1\Delta V$$

and since $\Delta x_1 = v_1\Delta t$ and $A_1\Delta x_1 = \Delta V$; then,

$$W_1 = P_1\Delta V \quad (5.4)$$

According to the equation of continuity, the same volume (V) of fluid will pass through Q_2. The work done (W_2) by the fluid on the right-hand side of the pipe is given by;

$$W_2 = P_2\Delta V \quad (5.5)$$

Therefore, the total work done on the fluid is,

$$\Delta W = W_1 - W_2 = P_1\Delta V - P_2\Delta V$$

$$\Delta W = (P_1 - P_2)\Delta V$$

Let the fluid density be ρ and the mass passing through the pipe as Δm during the time interval Δt.
Hence,

$$\Delta m = \rho A_1 v_1 \Delta t, \quad \Delta m = \rho\Delta V$$

The change in gravitational potential energy $\Delta(P.E)$ can be obtained by,

$$\Delta(P.E.) = mgh_2 - mgh_1 = mg(h_2 - h_1)$$

$$\Delta(P.E.) = \rho g\Delta V(h_2 - h_1)$$

Similarly, the change in kinetic energy is

$$\Delta(K.E.) = \frac{1}{2}\rho\Delta V(v_2^2 - v_1^2)$$

The total change in mechanical energy is

$\Delta U = \Delta(P.E.) + \Delta(K.E.)$, thus,

$$\Delta U = \rho g\Delta V(h_2 - h_1) + \frac{1}{2}\rho\Delta V\left(v_2^2 - v_1^2\right) \quad (5.6)$$

Applying the work-energy theorem in the volume of the fluid, $\Delta W = \Delta U$

$$(p_1 - p_2)\Delta V = \frac{1}{2}\rho\Delta V\left(v_2^1 - v_1^2\right) + \rho g\Delta V(h_2 - h_1) \quad (5.7)$$

Dividing each term by ΔV, and rearranging equation (5.7) gives

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 \quad (5.8)$$

Equation (5.8) is the Bernoulli's equation which can further be written in general form as,

$$P + \frac{1}{2}\rho v^2 + \rho g h = constant \quad (5.9)$$

Thus, Bernoulli's principle states that, "*For a streamline motion of steady, non-rotational, incompressible and non-viscous fluid, the sum of pressure at any point plus the potential energy per unit volume plus the kinetic energy per unit volume is always constant*".

However, for horizontal pipe ($h_1 = h_2$), equation (5.8) can be reduced to

$$P + \frac{1}{2}\rho v^2 = constant \quad (5.10)$$

Example 5.2

Water enters in a house water system through a pipe with 2.0 cm inner diameter at an absolute pressure of 4×10^5 Pa. The pipe leading to the second floor bathroom 5 m high is 1.0 cm inner diameter. If the flow velocity at the inlet pipe is $4\,\text{ms}^{-1}$, find:

(a) The flow velocity; and
(b) Pressure in the bathroom.

Solution

(a) The flow velocity from equation of continuity

$$v_2 = \frac{A_1 v_1}{A_2} \quad \text{(i)}$$

where A_1 and A_2 are areas of large and small pipes respectively, v_1 and v_2 are their corresponding velocities.

But $\frac{A_1}{A_2} = \left(\frac{d_1}{d_2}\right)^2$. Substituting the values $d_2 = 1.0$ cm, $d_1 = 2.0$ cm, $v_1 = 4\,\text{ms}^{-1}$ in (i) gives $v_2 = 16\ \text{ms}^{-1}$

(b) Pressure can be found using Bernoulli's equation;

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) + \rho g(h_1 - h_2)$$

$$P_2 = 4\times10^5\ \text{Pa} + \frac{1}{2}\times1000\ \text{kgm}^{-3}\times \left(\left(4\ \text{ms}^{-1}\right)^2 - \left(16\ \text{ms}^{-1}\right)^2\right) + 1000\ \text{kgm}^{-3}\times9.8\ \text{ms}^{-2}\times\left(0-5\ \text{m}\right)$$

$$P_2 = 2.31\times10^5\ \text{kgms}^{-2} \text{ or } 2.31\times10^5\ \text{Pa}$$

Therefore, the flow velocity and pressure in the bathroom are $16\ \text{ms}^{-1}$ and 2.31×10^5 Pa respectively.

5.2.2 Applications of Bernoulli's Principle

There are several applications of Bernoulli's principle which will be explained in this section. These include flow of a liquid through a wide tank, aerofoil lift, venture meter, atomizer of sprayer and pitot tube.

(a) Fluid flowing from a tank

Consider a tank with some liquid (Figure 5.4). Let point A be at a height h from the bottom and B is at the reference line.

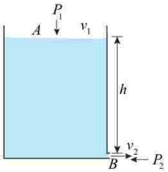


Figure. 5.4 Flowing fluid from a tank

The pressure acting at A and B is atmospheric pressure P. Let v_2 be the velocity of the fluid which is flowing out at B, then,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Assuming the cross-sectional area at A is very large compared to that at B, that is, $v_1 << v_2$, then; $v_2^2 - v_1^2 \approx v_2^2$

Since, $P_1 = P_2 = P$, then, equation (5.8) can be reduced to $\rho g h = \frac{1}{2}\rho v_2^2$. Therefore,

$$v_2 = \sqrt{2gh} \qquad (5.11)$$

This is the velocity of the emerging liquid (velocity of efflux) from a wide tank; it is equal to the velocity of a free fall. In this process all the potential energy is changed into kinetic energy. This process is called Torricelli's theorem which states that, *"The speed of efflux of a liquid from an orifice is the same as the vertical velocity that would be acquired in a free fall."*

Example 5.3

A cylindrical tank with a radius of 1 m rests on a platform 5 m high. Initially the tank is filled with water to a height of 5 m. A plug whose area is $10^{-4}\ \text{m}^2$, is removed from an orifice on the side of the tank at the bottom. Calculate:

(a) Initial speed with which the water flows from the orifice;
(b) Initial speed with which the water will strike the ground;
(c) Initial distance from the tank to the point where water strikes the ground; and
(d) Time taken to empty the tank.

Solution

Consider the water tank in Figure 5.5.

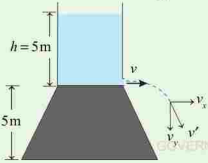


Figure 5.5 Water tank

(a) From Torricelli's theorem, velocity of orifice $v = \sqrt{2gh}$,

$v = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 5 \text{ m}} = 9.9 \text{ ms}^{-1}$

Hence, the initial speed with which the water flows from the orifice is 9.9 ms^{-1}.

(b) The initial speed (say v') can be found from trajectory equation $v' = \sqrt{v_x^2 + v_y^2}$

where v_x is the horizontal velocity and v_y is the vertical velocity.

Hence, $v_x = v$, $v_y = -gt$ and $t = \sqrt{\frac{2h}{g}}$, combining these equations gives $v' = 2\sqrt{gh}$

$v' = 2\sqrt{9.8 \text{ ms}^{-2} \times 5 \text{ m}} = 14 \text{ ms}^{-1}$

Therefore, the initial speed with which the water strikes the ground is 14 ms^{-1}.

(c) The initial horizontal distance from the tank $x_o = vt$

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 5 \text{ m}}{9.8 \text{ ms}^{-2}}} = 1.0 \text{ s}$$

and

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \text{ ms}^{-2} \times 5 \text{m}}$$
$$= 9.9 \text{ ms}^{-1}$$

$$x_o = 9.9 \text{ ms}^{-1} \times 1.0 \text{ s} = 9.9 \text{ m}$$

Therefore, the initial horizontal distance from the tank is 9.9 m.

(d) From Figure 5.5, $v_1 = -\frac{dh}{dt}$ and $v = \sqrt{2gh}$, then, from equation (5.3);

$$-A_1 \frac{dh}{dt} = A_2\sqrt{2gh}$$

The negative sign in the preceding equation indicates that the height of water in the tank is decreasing.

By separable integral,

$$-\frac{A_1}{A_2\sqrt{2g}} \int_h^0 h^{-\frac{1}{2}} dh = \int_0^t dt$$

which simplifies to

$$t = \frac{2A_1}{A_2}\sqrt{\frac{h}{2g}}$$

$$t = \frac{2\pi \text{ m}^2}{10^{-4} \text{ m}^2} \times \sqrt{\frac{5 \text{ m}}{2 \times 9.8 \text{ ms}^{-2}}}$$

$$= 3.17 \times 10^4 \text{ s}$$

Therefore, it will take 3.17×10^4 s to empty the tank.

(b) Aerofoil lift

An aerofoil, just as an aircraft wing, is constructed with the shape, such that, when it moves across air layers, the air flows faster at the top than at the bottom of the wing (Figure 5.6). This creates the pressure difference which makes the up-thrust force much higher to uplift the plane; this is called dynamic lift.

Consider a wing of an airplane in the streamlines of air as shown in Figure 5.6.

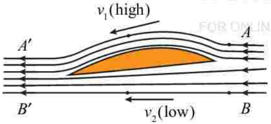


Figure. 5.6 An airplane wing

The time t taken by a volume of particles of air from A to A' and from B to B' is the same, such that,

$v_1 = \frac{AA'}{t}$ and $v_2 = \frac{BB'}{t}$; but $AA' > BB'$, hence $v_1 > v_2$

From Bernoulli's principle, the pressure at B' is greater than that at A', this creates an upthrust force (dynamic lift) to the whole wing, so the whole plane 'floats' in air.

(c) Venturi meter

This is a special instrument which contains a gauge or meter that can be used to measure the speed of flowing liquid like water and oil (Figure 5.7).

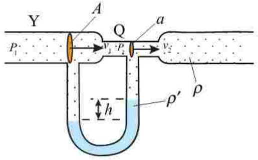


Figure 5.7 Venturi meter

A liquid of density ρ flows in a horizontal pipe from Y to Q, such that, the velocity at Y is v_1 and that at Q is v_2 and, P_1 and P_2 are the pressures at Y and Q respectively. The U-shaped tube containing a liquid of density ρ' is connected with its openings at Y and Q. Since the pipe is horizontal, ($h_1 = h_2$), Bernoulli's equation (5.9) at Y and Q can be simplified to

$$P_1 - P_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \quad (5.12)$$

If A and a are the cross sectional area at Y and Q respectively, then from equation (5.2)

$$v_2 = \frac{v_1 A}{a} \quad (5.13)$$

Substitute equation (5.13) into (5.12),

$$P_1 - P_2 = \frac{1}{2}\rho v_1^2\left(\frac{A^2 - a^2}{a^2}\right) \quad (5.14)$$

Since h in Figure 5.7 is the liquid column height difference between the two arms of the U-shaped tube, then,

$$P_1 - P_2 = gh\rho' \quad (5.15)$$

Equating equations (5.14) and (5.15) and solving for v_1 gives

$$v_1 = a\sqrt{\frac{2gh\rho'}{\rho(A^2 - a^2)}}$$

Knowing the values for densities, cross section areas and height, the speed of flowing liquid past a point Y can be determined.

But, volume flux at Y is, $\frac{V}{t} = Av_1$; thus,

$$\frac{V}{t} = Aa\sqrt{\frac{2gh\rho'}{\rho(A^2 - a^2)}}$$

(d) Atomizer or sprayer

The atomizer or sprayer (Figure 5.8) is a device that is used to spray paint or an insecticide. When the rubber ball of the atomizer is squeezed, air rushes through the narrow neck of the device. In so doing, the pressure in the narrow channel, at A is reduced.

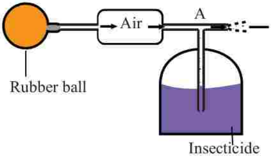


Figure 5.8 Atomizer

Once the pressure at A is reduced, atmospheric pressure pushes the insecticide up the tube towards the narrow channel. The insecticide is then pushed outwards into a fine spray of droplets.

(e) Pitot tube

The pitot tube (flow meter) is a device used to measure the velocity of a moving fluid. It is very often used in airplanes to measure their relative speed. The schematic diagram of a pitot tube is shown in Figure 5.9.

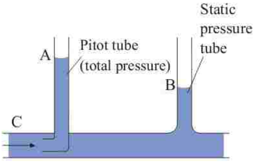


Figure 5.9 Pitot tube

The fluid enters the tube through C and it is immediately brought to stagnation. Hence the pressure P at A, is sometimes called stagnant pressure. Applying Bernoulli principle;

$$P + h\rho g + \frac{1}{2}\rho v^2 = constant$$

The static component is determined from tube B and given as $P + h\rho g$, or P if the flow is horizontal. (i.e. $h = 0$)

The dynamic component is given by $\frac{1}{2}\rho v^2$, hence, the total pressure P_T determined from tube A is given by $P_T = P + \frac{1}{2}\rho v^2$

$$P_T - P = \frac{1}{2}\rho v^2$$

$$v = \sqrt{\frac{2\Delta P}{\rho}} \quad (5.16)$$

where $\Delta P = P_T - P$, which is the pressure difference.

Exercise 5.2

1. The Bernoulli's equation can be written in the form;

$$P+\frac{1}{2}\rho v^2 = Constant$$

   (a) Explain the meaning of each term in the equation.
   (b) State two conditions which must apply for this equation to be true.
   (c) What happens to the internal pressure in a fluid flowing in a horizontal pipe when its speed increases?

2. Explain the following phenomena related to Bernoulli's principle.
   (a) A flag flutter when strong winds blow on it.
   (b) When a fluid flows through a narrow constriction its speed increases.
   (c) Tornadoes often lift the roofs of houses.

3. A fire hose must be able to shoot water to the top of a building which is 28.0 m high when aimed straight up. Water enters this hose at a steady rate of 0.50 m^3s^{-1} and shoots out of a round nozzle. What is the maximum diameter of the nozzle?

4. A manometer connected to a closed tap reads 3.5×10^5 Nm^{-2}. When the valve is opened, the reading of manometer falls to 3.0×10^5 Nm^{-2}, find the velocity of flow of water.

5. A large open tank has two holes in the wall. One is a square hole of side l at a depth y from the top and the other is a circular hole of radius r at a depth $4y$ from the top. When the tank is completely filled with water the quantities of water flowing out per second from both holes are the same. Find the value of the radius.

5.3 Viscosity and turbulent flow

Viscosity characterize the degree of internal friction in fluids. This internal friction, or viscous force, is associated with the resistance experienced by two adjacent layers of fluid moving relative to each other. Viscosity causes part of the kinetic energy of a fluid to be converted to internal energy. Viscosity mechanism is similar to the one by which an object sliding on a rough horizontal surface loses kinetic energy. In previous sections, the discussion focused only on an ideal fluid (steady flow, non-viscous and incompressible fluid). In this section, you will learn about the characteristics of viscous fluid and turbulent flow, specifically, the Newton's law of viscosity, Poiseuille's formula, Stokes' law and applications of viscosity in daily life.

5.3.1 Coefficient of viscosity

Coefficient of viscosity η, of a fluid is a measure of the degree to which the fluid exhibits viscous effects. This effect is described by Newton's law of viscosity which states that, "*The frictional force F between the layers is directly proportional to the area A of the layers and the velocity gradient* $\frac{dv}{dy}$".

Consider a pipe that contains a fluid flowing steadily (Figure 5.10).

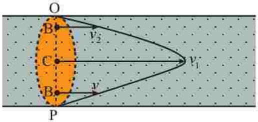


Figure 5.10 Friction between successive layers of a fluid

There are so many cylindrical fluid layers located at different positions within the pipe. Layers have varying speed ranging from zero at the wall of the pipe to the maximum speed at the center of the pipe. Fluid layers between C and B have velocities which are less than that of C but greater than that at P. If A is the surface area of layers in contact, $\frac{v_1 - v_2}{h}$ is the velocity gradient where h is the distance of separation of the two layers with velocities v_1 and v_2. Then according to Newton's law of viscosity

$$F \propto A\left(\frac{v_1 - v_2}{h}\right) \quad (5.17)$$

Introducing the constant of proportionality into equation (5.17), gives,

$$F = \eta A\frac{(v_1 - v_2)}{h} \quad (5.18)$$

where the constant η is called the coefficient of viscosity. Making η the subject of the formula gives

$$\eta = \frac{Fh}{A(v_1 - v_2)} \quad (5.19)$$

Hence, from equation (5.19) the coefficient of viscosity η is defined as the frictional force F per unit area A per unit velocity gradient v_G.

The units of η can be obtained from the method of dimensional analysis as follows;

$$[\eta] = \frac{[F]}{[A]\times[v_G]}$$

But $[F] = MLT^{-2}$, $[A] = L^2$, $[v_G] = T^{-1}$; hence the dimensions of viscosity,

$$\eta = ML^{-1}T^{-1}$$

Therefore, the unit of coefficient of viscosity is $\text{kgm}^{-1}\text{s}^{-1}$ or Nsm^{-2}.

Note that, viscosity of an ideal fluid is zero. The coefficient of viscosity of a liquid decreases with an increase in temperature. But for gases, the coefficient of viscosity, increases with increase in temperature. Viscosity (particularly of oil and grease) is utilized in lubricants for various parts of machines. Viscosity is related to internal friction and hence it affects heat generation in bearings, cylinders and gear sets, therefore, various parts of machines require specific density of lubricants. The knowledge of viscosity is therefore important in measuring and choosing related lubricants for machinery parts. This means that the viscosity of an oil is foremost to be considered when selecting lubricating oil for a specific application.

5.3.2 Poiseuille's Formula

Poiseuille studied the flow of a liquid through a horizontal pipe and found that the volume of liquid flowing out per second $\frac{V}{t}$, depends on the coefficient of viscosity η, the pressure gradient $\frac{P}{l}$, and the radius a of the tube. Thus,

$\frac{V}{t} \propto \eta a \frac{P}{l}$; where P is the pressure difference between the two ends of the pipe of length l.

By method of dimensional analysis,

$$\frac{V}{t} = k\eta^x a^y \left(\frac{P}{l}\right)^z \qquad (5.20)$$

Equating dimensions,

$$M^0L^3T^{-1} = (ML^{-1}T^{-1})^x L^y (ML^{-2}T^{-2})^z$$

Equating powers of like terms,

$M : x + z = 0$

$L : -x + y - 2z = 3$

$L : -x - 2z = -1$

Solving for x, y and z;

$x = -1,\ y = 4, z = 1$

Substituting the values of x, y and z into equation (5.20) gives, $\frac{V}{t} = \frac{kPa^4}{\eta l}$.

The constant of proportionality k was experimentally found to be $\frac{\pi}{8}$. Hence,

$$\frac{V}{t} = \frac{\pi Pa^4}{8\eta l}$$

This is called the Poiseuille's formula.

Through Poiseuille's formula, the coefficient of viscosity of a liquid can be determined. Consider the liquid flowing steadily from a tank of height h through a capillary tube of length l at constant pressure (Figure 5.11).

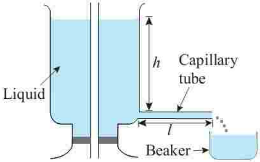


Figure 5.11 Determination of coefficient of viscosity of a liquid

A volume V of the liquid to be collected at time interval t is obtained by applying Poiseuille's formula,

$$\frac{V}{t} = \frac{\pi Pa^4}{8\eta l};\ \text{but}\ \ P = \rho gh$$

where h is the height of the liquid column from the capillary tube to the top of the liquid level and ρ is the density of the liquid.

Hence the coefficient of viscosity can be obtained from the formula

$$\frac{V}{t} = \frac{\pi \rho gha^4}{8\eta l}$$

Example 5.4

Water flows through a horizontal tube with diameter 0.08 m and 4 km length at the rate of 20 litres per second. Calculate the pressure difference required to maintain the flow, given that the coefficient of viscosity of water is $10^{-3}\ \text{Nsm}^{-2}$. Assume only viscous resistance exists.

Solution

From Poiseuille's formula, $\frac{V}{t} = \frac{\pi Pa^4}{8\eta l}$,

where $\frac{V}{t}$ is the volume per second.

The pressure, $P = \frac{V}{t} \times \frac{8\eta l}{\pi a^4}$

$$P = 2\times10^{-2}\ \text{m}^3\text{s}^{-1} \times \frac{8\times10^{-3}\ \text{Nsm}^{-2}\times 4\times10^3\ \text{m}}{\pi\times(0.04\ \text{m})^4}$$

$$P = 7.96\times10^4\ \text{Nm}^{-2}$$

Therefore, pressure difference is $7.96\times10^4\ \text{Nm}^{-2}$.

Example 5.5

A tank of cross-section area A has a viscous liquid to a height h_0 above the base. The liquid is allowed to flow out of the container through a horizontal tube which is narrow and long, connected to the base of the tank. Show that the height h of the remaining liquid at any time t in the tank obeys the equation, $h = h_0 e^{-ct}$.

Solution

Consider Figure 5.12 as a condition for the problem.

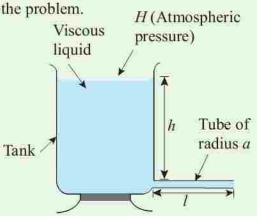


Figure 5.12 Emptying the tank

The liquid flow rate Q_t in the tank is equal to the flow rate Q_b in the tube, i.e $Q_t = Q_b$, then, from Poiseuille's formula;

$$\frac{dh}{dt} = -\frac{\pi\Delta P a^4}{8\eta l A} \qquad (i)$$

$P_1 = h\rho g + H$ and $P_2 = H$, then

$$\Delta P = P_1 - P_2 = h\rho g \qquad (ii)$$

Placing equation (ii) into (i);

$$\frac{dh}{dt} = -\frac{\pi\rho g a^4 h}{8\eta l A};\quad \frac{dh}{h} = -c\,dt$$

where $c = \frac{\pi\rho g a^4}{8\eta l A}$ which is constant.

By integrating, it follows that;

$$\int_{h_0}^{h}\frac{dh}{h} = -c\int_0^t dt$$

$$\ln(h) - \ln(h_0) = -ct \qquad (iii)$$

Equation (iii) can be simplified to, $h = h_0 e^{-ct}$.

5.3.3 Stokes' Law and terminal velocity

When a small solid sphere is dropped into a viscous liquid, the ball will accelerate and eventually reach a point where it moves with a constant velocity known as terminal velocity (Figure 5.13(a)).

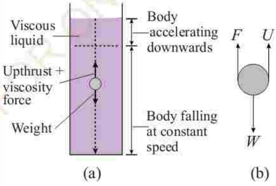


Figure 5.13 Sphere falling through viscous liquid

There are three forces that act upon the ball namely, the weight of the ball, the upthrust and the viscous force (Figure 5.13(b)). At terminal velocity, the net force is zero, since the acceleration is zero, then

$$F + U = W$$

According to Stokes' law, "*The viscous force is proportional to the radius a of the ball, velocity v of the ball, and coefficient of viscosity* η *of the liquid*".

By dimensional analysis,

$$F = ka^x \eta^y v^z \qquad (5.21)$$

$$[F] = [ka^x \eta^y v^z]$$

Equating the dimensions,

$$MLT^{-2} = L^x \left(ML^{-1}T^{-1}\right)^y \left(LT^{-1}\right)^z$$

Equating powers of the like terms gives

$L: \; -y + z + x = 1,$

$T: -y - z = -2, \; M: y = 1$

$x = 1, \; y = 1, \; z = 1$

Substitution of these values into equation (5.21) gives, $F = ka\eta v$

From mathematical analysis the proportional constant, $k = 6\pi$.

Therefore, the viscous force F is expressed as;

$$F = 6\pi a\eta v \qquad (5.22)$$

Equation (5.22) is the expression for Stokes' law. A graph of velocity against time for motion of a ball falling in a viscous fluid and attaining terminal velocity is shown in Figure 5.14.

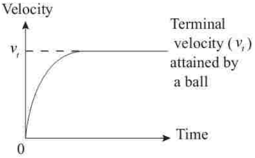


Figure 5.14 Terminal velocity

Suppose ρ is the density of the sphere and σ is the density of a fluid, then, the weight W, of the sphere,

$W = mg$, but $m = \frac{4}{3}\pi\rho a^3$ hence,

$$W = \frac{4}{3}\pi\rho a^3 g$$

Similarly; upthrust U,

$$U = \frac{4}{3}\pi\sigma a^3 g$$

At terminal velocity, $F + U = W$, then

$$6\pi a\eta v_t + \frac{4}{3}\pi\sigma a^3 g = \frac{4}{3}\pi\rho a^3 g$$

$$v_t = \frac{2a^2(\rho - \sigma)g}{9\eta} \qquad (5.23)$$

Thus, equation (5.23) represents the expression of terminal velocity which is a constant velocity attained by a spherical body when falling through a viscous fluid.

Example 5.6

A small oil drop of radius R falls with a terminal velocity of 2.0×10^{-1} ms^{-1} in air. Find the new terminal velocity of the oil drop of half of this radius.

Solution

Terminal velocity of first drop,

$$v_1 = \frac{2(\rho - \sigma)gR^2}{9\eta} \qquad (i)$$

Terminal velocity of the second oil drop,

$$v_2 = \frac{(\rho - \sigma)gR^2}{18\eta} \qquad (ii)$$

Dividing equation (i) by (ii)

$$\frac{v_1}{v_2} = 4;\ v_2 = \frac{v_1}{4}$$

$$v_2 = \frac{0.2\ \text{ms}^{-1}}{4} = 0.05\ \text{ms}^{-1}$$

Therefore, terminal velocity of the second drop is $0.05\ \text{ms}^{-1}$.

Exercise 5.3

1. (a) Define coefficient of viscosity.
   (b) Stokes' law for the viscous force F acting on a sphere of radius a falling with velocity v through a large expense of fluid of coefficient of viscosity η is expressed by the equation $F = 6\pi a\eta v$. State why this equation is true only for sufficiently low velocities.
   (c) Sketch the graph of velocity against time for the motion of the ball falling in a viscous fluid.
2. Explain the following observation as related to fluid dynamics.
   (a) A sphere released in a fluid will fall with diminishing acceleration until it attains a constant terminal velocity.
   (b) Hot water flows more readily than cold water through small leaks in a car radiator.
3. Calculate the velocity of an oil drop of radius 3.0×10^{-5} m falling through the air of coefficient of viscosity $1.8 \times 10^{-5}\ \text{Nm}^{-2}\text{s}$, given that the density of the oil is $8.0 \times 10^{2}\ \text{kgm}^{-3}$, the density of air may be neglected.
4. A square plate with edge length 9.0 cm and mass 488 g is hinged along one side. If air is blown over one surface only, what speed must the air have so as to hold the plate horizontal? The air has a density of $1.212\ \text{kgm}^{-3}$.

Revision exercise 5

1. Give differences between the following terms:
   (a) Compressible and non-compressible fluid;
   (b) Viscous and non-viscous fluid; and
   (c) Steady flow and turbulent flow.
2. A large tank contains water to a depth of 10 m. Water emerges from a small hole on the side of the tank 20 cm below the level of the water surface. Calculate:
   (a) The speed at which the water emerges from the hole; and
   (b) The distance from the base of the tank at which the water strikes the floor on which the tank is standing.

3. (a) Derive Bernoulli's equation for an incompressible non-viscous fluid.
   (b) How is an airplane able to fly upside down?
4. A simple garden syringe used to produce a jet of water consists of a piston with an area of $4.00\ cm^2$ which moves in a horizontal cylinder with a small hole at its end. If the force on the piston is 50 N, calculate the speed at which the water is forced out of the small hole. Assume that the speed of the piston is negligible.
5. Two fast moving rowboats are likely to crush when moving parallel and close to each other. Explain why this is likely to occur.
6. (a) State Poiseuille's formula.
   (b) Using the method of dimensional analysis, derive the Poiseuille's formula.
   (c) Two capillaries of same length and radii in the ratio of 1 : 2 are connected in series and a liquid flows through this system under streamline conditions. If the pressure across the two extreme ends of the combination is 1 m of water, what is the pressure difference across the first capillary?
7. Water flows through a horizontal pipe of varying cross-section at the rate of $10\ m^3s^{-1}$. Find the velocity at a section where the radius of the pipe is 10 cm.
8. Explain why
   (a) a parachute is used while jumping from an aeroplanes.
   (b) a flag flutters when strong winds are blowing on a certain day.
   (c) clouds seem floating in the sky.
   (d) a bigger rain drop falls faster than a smaller one.
9. A flat plate of area $0.1\ m^2$ is placed on a flat surface and is separated from it by a film of oil 10^{-5} m thick whose coefficient of viscosity is $1.5\ Nsm^{-2}$. Calculate the force required to cause the plate to slide on the surface at a constant speed of $1 mms^{-1}$. Assume that the flow is laminar and that the oil adjacent to each surface moves with that surface.
10. Explain why
    (a) it is easier to throw a curve with a tennis ball than a cricket ball.
    (b) airplanes extend wing flaps that increase the area and the angle of attack of the wing during takeoffs and landings.
11. Two drinking glasses having equal weights but different shapes and different cross-sectional areas are filled to the same level with water. According to the expression $P = P_0 + \rho gh$ the pressure is the same at the bottom of both glasses. In view of this, why does one weigh more than the other?
12. Draw sketches of streamline for the following flow systems. Discuss the significant features in each case.

(a) Liquid flowing round a sharp pipe bend.

(b) Air flowing round a moving articulated lorry.

13. Write down the definition of gauge pressure. Sketch a simple liquid U-tube manometer for measuring the pressure difference between two regions of a gas. Give an equation relating the pressure difference to the difference in liquid levels.

14. Briefly explain why
    (a) cars need different oils in hot and cold countries.
    (b) engine runs more freely as it heats up.
    (c) skin lotions are easier to pour in summer than winter.

15. (a) State Bernoulli's principle and explain the Bernoulli's equation for the flow of an ideal fluid in stream line motion. Mention any two applications of Bernoulli's equation.
    (b) Describe different types of flow of fluids. State and explain equation of continuity.

16. (a) Explain why, when the sphere is falling with terminal velocity, $F = W - U$ where F is the Stokes' law force.
    (b) Hence show that the viscosity η can be calculated from the expression

$$\eta = \frac{2r^2 g(\rho - \sigma)}{9v}$$

where v is the velocity of the sphere. Explain why it is important that the experiment is performed using hot oil.

17. Find the rate of flow of water through a pipe 3cm in diameter before turbulent flow occurs. The critical value of Reynold's number is 2000 and viscosity of water is $8.01\times10^{-4}\,\text{Nsm}^{-2}$.

Chapter Six

Properties of matter

Introduction

Everything around us is matter. It can be in a state of solid, liquid, gas or plasma. Matter can be classified depending on its physical and chemical properties. Properties of matter enable us to differentiate materials, and to make appropriate choices of materials for different kinds of tasks. Knowledge of properties of matter is of great value in industrial products. Examples of these products are springs made of steel because steel is highly elastic, and insulating materials used in buildings, or refrigerators. In this chapter, you will learn about some properties namely surface tension, elasticity, and kinetic theory of gases.

6.1 Surface tension

A paper clip can rest on a water surface even though its density is larger than that of water; this is due to surface tension of water. The surface of the liquid behaves like a membrane under tension. Surface tension arises because the molecules of the liquid exert attractive forces on each other. The surface of a liquid behaves like a stretched elastic skin. This is why liquid drops are spherical in the absence of gravitational field. There is zero net force on a molecule inside the volume of the liquid, but a surface molecule is drawn into the volume. Thus the liquid tends to minimize its surface area, just as a stretched membrane does. Due to surface tension insects called pond skaters can walk on water, water rises up in a capillary tube, a needle may be made to float on water and a small piece of soap fixed to the back of a piece of cardboard that is floating on water will cause the cardboard to move over the water surface. Surface tension is the property of liquid by virtual of which its free surface at rest behaves like an elastic skin on a stretched membrane. It has a tendency of contracting so as to occupy a small area as much as possible.

In this section, you will analyse surface tension in terms of the molecular theory and surface energy, then determine the coefficient of surface tension of a liquid and explain the factors that affect surface tension.

6.1.1 Surface tension in terms of the molecular theory

The cohesive force among liquid molecules is responsible for the phenomenon of surface tension. The molecules inside the liquid are attracted equally in all directions by other molecules. In contrast, the molecules on the surface experience an inward pull, because they are partially attracted by other molecules. As a result, a network of force is then formed against the inward pull in order to move molecules to the liquid surface. This results into a large potential energy on surface molecules. In order to attain minimum potential energy and hence stable equilibrium, the free surface of the liquid tends to have the minimum surface area and it behaves like a stretched membrane.

Note that, any equilibrium configuration of an object is one in which the energy is a minimum. This results to liquid taking in a shape such that its surface area is also minimum. This is why a drop of water takes on a spherical shape. Therefore, for a given volume of liquid, a spherical shape is the one that has the smallest surface area.

(a) **Energy of liquid surface**

The fact that a liquid surface is in a state of surface tension can be explained by the intermolecular forces. In the bulk of the liquid, which begins only a few molecular diameters downwards from the surface, a particular molecule such as A (Figure 6.1) is surrounded by an equal number of molecules on all sides.

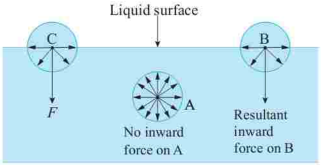


Figure 6.1 Molecular forces in liquid

The average distance apart of the molecules is such that the attractive forces balance the repulsive force. Thus, the average intermolecular forces between A and the surrounding molecules is zero. Consider now a molecule such as C or B on the surface of the liquid (Figure 6.1). There are very few molecules on the vapour side above C or B. If C is displaced very slightly upward, a resultant attractive force F on it, due to the large number of molecules below it, has to be overcome. If all the molecules in the surface were removed to infinity, a definite amount of work would be needed. Consequently, molecules in the surface have potential energy. A molecule in the bulk of the liquid forms bonds with more neighbours than one in the surface. Thus, bonds must be broken, i.e. work must be done to bring the molecule into the surface. Hence, molecules in the surface of the liquid have more potential energy than those in the bulk.

Consider two atoms or molecules exerting forces of attraction on each other as shown in Figure 6.2. If the force F on A moves the molecule a small distance Δr to the right; then, the work done ΔW on A is given by

$$\Delta W = F\Delta r \quad (6.1)$$

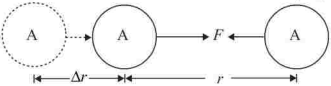


Figure 6.2 Forces of attraction between molecules

Let ΔU be the resulting change in the potential energy, then,

$$\Delta U = -\Delta W \quad (6.2)$$

The negative sign indicates that force is attractive and the potential energy decreases.

Therefore, $\Delta U = -F\Delta r$. Thus, in the limit; $F = -\dfrac{dU}{dr}$

Activity

Demonstration of a floating pin on the surface of water

Requirements: Small dish, pin, water

Procedure

1. Fill the small dish with clean water.
2. Carefully place a pin horizontally on the surface of the water as in Figure 6.3.
3. Observe what happens to the pin.

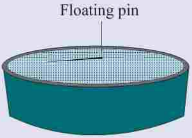


Figure 6.3 Floating pin on the surface of water

4. What causes the observed effect?

(b) Measurement of surface tension

Imagine a tangential line *AB* drawn on the surface of a liquid dividing the liquid into two parts as shown in Figure 6.4.

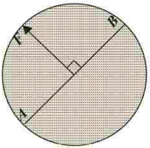


Figure 6.4 Measurement of the surface tension of the liquid

The two formed parts of the surface pull each other with a force directly proportional to the length of the line. It is found that the forces of pull are perpendicular to the line separating the two parts and tangential to the liquid surface.

Let F be the magnitude of the forces exerted on each other by the two parts of the surface across the line segment *AB* of length l. Surface tension is the force per unit length acting in the surface of a liquid perpendicular to one side of a line in the surface.

$\gamma = \dfrac{F}{l}$, where F is the force on either side of line segment *AB* and l is the length of line segment *AB*.

The dimension of surface tension is $\gamma = \frac{MLT^{-2}}{L} = MT^{-2}$ and its SI unit is Nm^{-1}.

Therefore, coefficient of surface tension is the force per unit length of an imaginary line segment drawn in any direction on the free surface of a liquid, the line of action of the force being on the surface and at right angles to the imaginary line segment.

6.1.2 Surface tension in terms of surface energy

Another way of viewing surface tension is in terms of surface energy. A molecule in contact with a neighbour is in a lower state of energy than if it was alone (not in contact with a neighbour). The interior molecules have much more neighbours compared to the surface molecules. Therefore, the surface molecules have higher energy. In order to increase the surface area of certain liquid, work will have to be done against the force of surface tension. This work done is stored in boundary molecules (surface layer) of the liquid in the form of potential energy. Therefore, the liquid surface will have more surface energy due to the increased surface area. The work done against the force of surface tension to increase unit surface area at constant temperature is called surface energy (δ) of the liquid.

i.e. $\delta = \frac{\text{Work done in increasing surface area}}{\text{increase in surface area}}$

Surface energy (also called free surface energy) of the liquid has the same SI units as that for surface tension, that is Nm^{-1} or Jm^{-2}.

Consider stretching a film of liquid on a horizontal frame (Figure 6.5). Since the film has both an upper and lower surface, the force F on AB due to surface tension is given by $F = 2l\gamma$.

If AB is moved a distance x to $A'B'$, then work has to be done against this force. The surface tension is independent of the surface area of the film because as the size of the surface increases more molecules enter it and by so doing maintain the average molecular separation. However, surface tension decreases with increasing temperature, because this decreases the binding energy. Thus, provided AB is moved isothermally to $A'B'$, the force on AB will be constant and work done $= 2x\gamma l$.

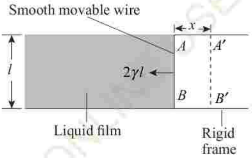


Figure 6.5 Stretching of liquid film in a horizontal frame

The increase in surface area is $2lx$ (upper and lower surface) and therefore the work done per unit area increase (the free surface energy δ) is given by

$$\delta = \frac{2lx\gamma}{2lx};\ \delta = \gamma$$

Therefore, the free surface energy δ is equal to the surface tension γ. Hence surface tension is the work done per unit area in increasing the surface area of a liquid under isothermal condition. Work done in increasing or decreasing the surface area of a liquid is proportional to the change in area, ΔA.

$W = \gamma \Delta A$, where γ is the proportionality constant which is the surface tension.

6.1.3 Coalescing and breaking of liquid drops and bubbles

Understanding surface energy is important in studying behaviours of bubbles and drops. When subjected to various conditions two or more bubbles may merge during contact to form a single large bubble. On the other hand, a large bubble may break down into smaller bubbles. Hence the work done in coalescing and breaking drops and bubbles can be deduced separately.

(a) Coalescence of liquid drops in vacuum

Liquid drops coalesce in the presence of external forces. When drops combine form a single drop (Figure 6.6).

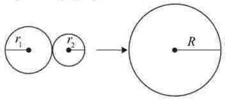


Figure 6.6 Combined liquid drops in vacuum

Consider two liquid drops of radius r_1 and r_2 respectively coalescing in vacuum to form a single drop of radius R under isothermal condition.

By conservation of energy, sum of the surface energies W_1 and W_2 of the two drops equals to the energy W of the formed drop; i.e. $W_1 + W_2 = W$. It follows that, $W = \gamma \times$ change in surface area (A), then,

$4\pi r_1^2\gamma + 4\pi r_2^2\gamma = 4\pi R^2\gamma$, on simplifying gives,

$$R = \sqrt{r_1^2 + r_2^2} \qquad (6.3)$$

For n drops, $R = \sqrt{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}$.

This result for coalescing liquid drops holds also for soap bubbles, except that, the soap bubble has two surfaces in contact with air. i.e., inner and outer surfaces.

Note that, rain drops are a result of coalescence of smaller droplets.

(b) Coalescence of soap bubbles in air

When soap bubbles combine together, a common interface is formed. Consider two soap bubbles of radius r_1, r_2 respectively combining in air to form a common interface of radius R (Figure 6.7).

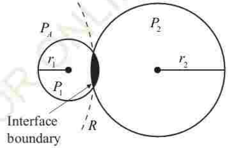


Figure 6.7 Coalescing of soap bubbles in air

Pressure P_1 inside the small bubble is larger than the outside pressure P_A, pressure P_2 inside the large bubble is larger than P_A

and P_1 is larger than P_2.

It then follows that,

$$P_1 - P_A = \frac{4\gamma}{r_1} \text{ and } P_2 - P_A = \frac{4\gamma}{r_2}$$

(refer section 6.1.4)

Excess pressure of the two bubbles,

$$P_1 - P_2 = P; \quad P = \frac{4\gamma}{r_1} - \frac{4\gamma}{r_2}$$

$\frac{4\gamma}{R} = 4\gamma\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$, simplifying this expression gives

$$R = \frac{r_1 r_2}{r_2 - r_1} \qquad (6.4)$$

Therefore, the radius of curvature of the interface is given by $R = \frac{r_1 r_2}{r_2 - r_1}$. This equation holds only under isothermal condition.

(c) Breaking of liquid drops

When a large liquid drop is subjected to external pressure greater than its pressure holding its molecules, it breaks into small droplets. When the drop of radius R breaks into small droplets each of radius r, there is an increase in surface energy but the volume remains constant.

Let the drop break into two small identical-drops. The increase in energy $W = \gamma \times \Delta A$, where ΔA is the increase in surface area.

Original surface area of a large drop is $4\pi R^2$ and that of the two droplets is twice the area of one droplet, i.e. $2 \times 4\pi r^2$.

$$\Delta A = 8\pi r^2 - 4\pi R^2$$

It then follows that,

$$W = \gamma \times \left(8\pi r^2 - 4\pi R^2\right) \qquad (6.5)$$

Generally, if a drop breaks down into n equal droplets, then

$$W = \left(n \times 4\pi r^2 - 4\pi R^2\right)\gamma$$

Since the volume is constant,

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi r_1^3 + \frac{4}{3}\pi r_2^3 + \frac{4}{3}\pi r_3^3 + ... + \frac{4}{3}\pi r_n^3$$

Since the droplets are identical, i.e. $r = r_1 = r_2 = r_3 = r_n$, then,

$$\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3$$

By simplifying, $n = \frac{R^3}{r^3}$, hence

$$W = \gamma \times \left(n \times 4\pi r^2 - 4\pi R^2\right)$$

$$\therefore W = 4\pi\gamma R^2\left(\frac{R}{r} - 1\right) \qquad (6.6)$$

Since $\frac{R}{r} > 1$, there is an increase in energy when a large drop breaks into small ones.

Example 6.1

The surface tension of the soap solution is $0.03\ \text{Nm}^{-1}$. Calculate work required to produce a bubble of radius 0.5 m.

Solution

$$W = \gamma \Delta A$$

Since a soap bubble has two surfaces in contact with air, then

$$\Delta A = 2 \times 4\pi R^2$$

$$\Delta A = 8\pi \times 0.5^2 \text{m}^2 = 2\pi \text{ m}^2$$

$$W = 2\pi \text{ m}^2 \times 0.03 \text{ Nm}^{-1} = 0.19 \text{ J}$$

Example 6.2

Calculate energy in the change of surface area of a soap bubble when its radius decreases from 5 cm to 1 cm. Given that the surface tension of soap solution is $2.0\times10^{-2}\ \text{Nm}^{-1}$.

Solution

$\Delta A = A_f - A_i$, $\Delta A = 4\pi R_f^2 - 4\pi R_i^2$

Substituting the values of R_f and R_i, $\Delta A = 3.02\times10^{-2}\ \text{m}^2$.

Energy (W) in the change of surface area of a soap bubble is given by:

$W = 2\gamma\Delta A$

$$W = 2\times2.0\times10^{-2}\ \text{Nm}^{-1}\times3.02\times10^{-2}\ \text{m}^2$$
$$=1.2\times10^{-3}\ \text{J}$$

Example 6.3

A spherical drop of mercury of radius 2 mm falls to the ground and breaks into 10 small drops of equal size. Calculate the amount of work to be done in the process. Surface tension of mercury, $\gamma = 4.72\times10^{-1}\ \text{Nm}^{-1}$.

Solution

The volume (V_i) of the drop before falling is $\frac{4}{3}\pi R^3$ and volume (V_f) after breaking is $\frac{4}{3}\pi r^3 n$ where n is the number of small drops of mercury.

For the conservation of volume V_i is equal to V_f.

Thus, $\frac{4}{3}\pi R^3 = \left(\frac{4}{3}\pi r^3\right)n$

$$r = \frac{R}{\sqrt[3]{n}} = 9.28\times10^{-4}\ \text{m}$$

Work done (W) in breaking the drop is given by; $W = \gamma\Delta A$, where ΔA is the change in surface area when the drop breaks.

Since, $\Delta A = 4\pi n r^2 - 4\pi R^2$; then by substituting the values of n, r and R; the value of ΔA is $5.80\times10^{-5}\ \text{m}^2$.

Hence,

$$W = 4.72\times10^{-1}\ \text{Nm}^{-1}\times5.80\times10^{-5}\ \text{m}^2$$
$$=2.74\times10^{-5}\ \text{J}$$

6.1.4 Excess pressure inside air bubble or curved liquid surfaces

The force of surface tension is related to the magnitude of the curvature of a liquid surface or a bubble formed in a liquid. Every molecule on the liquid surface experiences a force of surface tension that acts tangentially to the liquid surface at rest. The resultant force normal to the surfaces acts on curved surface of the liquid. For convex surfaces the resultant force is directed inwards the centre of curvature while for concave surfaces the resultant force is directed outwards from the centre of curvature. For the equilibrium of the curved liquid surface there must be an excess pressure force that balances the resultant force due to surface tension. Therefore, for a curved liquid surface in equilibrium, the pressure in its concave side is greater than the pressure on its convex side.

Consider one half of the bubble, A, which is at equilibrium (Figure 6.8). The sum of surface tension force and the external pressure force is equal to internal pressure force on the bubble.

The force on buble A due to the pressure P_1 is given by $P_1 \times \pi r^2$, where πr^2 is the area of the circular face of A and pressure is force per unit area. Similarly, the force on A due to the pressure P_2 is given by $P_2 \times \pi r^2$. Since the circumference of the bubble is $2\pi r$, then the surface tension force acting on the bubble is $2\pi r\gamma$. It follows that,

$$P_1 \pi r^2 + 2\pi r\gamma = P_2 \pi r^2$$

Simplifying gives, $2\gamma = (P_2 - P_1)r$,

therefore, $P_2 - P_1 = \frac{2\gamma}{r}$

Hence, the excess pressure, $P_2 - P_1 = P$ is given by

$$P = \frac{2\gamma}{r} \qquad (6.7)$$

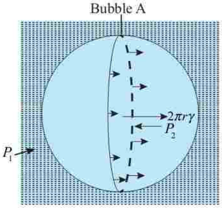


Figure 6.8 Excess pressure in a liquid bubble

Hence, excess pressure for a curved liquid surface is inversely proportional to the radius of the bubble; i.e. the smaller the bubble the greater the excess pressure. This explains why one needs to blow hard to start a balloon growing. Once the balloon has grown less energy is needed to make it expand more.

Excess pressure inside a soap bubble can be calculated by following the same procedure used in a liquid bubble. However, a soap bubble has two liquid surfaces in contact with air, one being inside and the other outside the bubble (Figure 6.9). Hence, the force on one half of the bubble due to surface tension is $2 \times 2\pi r\gamma$.

For the equilibrium of the bubble,

$$P_1 \times \pi r^2 + 4\pi r\gamma = \pi r^2 P_2$$

where P_1 and P_2 are pressure outside and inside the bubble respectively. Excess pressure,

$$P_2 - P_1 = \frac{4\gamma}{r} \qquad (6.8)$$

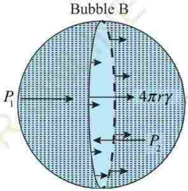


Figure 6.9 Excess pressure insideba soap bubble

This result for excess pressure is related to the result obtained for a bubble formed inside a liquid, equation (6.7).

Example 6.4

Find the excess pressure inside an air bubble of diameter 5 cm in a liquid. The surface tension of the liquid is $25\times10^{-3}\ \text{Nm}^{-1}$.

Solution

Excess pressure, $P=\dfrac{2\gamma}{r}$

$$P=\frac{2\times25\times10^{-3}\ \text{Nm}^{-1}}{2.5\times10^{-2}\ \text{m}}=2\ \text{Nm}^{-2}$$

Therefore, pressure inside the bubble is $2\ \text{Nm}^{-2}$.

Example 6.5

Two spherical soap bubbles of radii 30 mm and 10 mm coalesce so that they have a common interface. If they are made from the same solution and the radii of the bubbles remain the same after joining together, calculate the radius of curvature of their common surface.

Solution

Excess pressure $P=P_1-P_2$, where P_1, P_2 is the pressure inside the bubble with radius of 10 mm and 30 mm respectively.

$$\frac{4\gamma}{R}=\frac{4\gamma}{r_1}-\frac{4\gamma}{r_2}$$

$$\frac{4\gamma}{R}=4\gamma\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$R=\frac{r_1 r_2}{r_2-r_1}$$

$$R=\frac{10\ \text{mm}\times30\ \text{mm}}{30\ \text{mm}-10\ \text{mm}}=15\ \text{mm}$$

Therefore, the radius of curvature of their common surface is 15 mm.

Example 6.6

A soap bubble has a diameter of 4 mm. Calculate the pressure inside it if the atmospheric pressure is 10^5 Pa. Surface tension of soap solution is $2.8\times10^{-2}\text{Nm}^{-1}$.

Solution

$P=P_0+P_\gamma$ where P, P_0 and P_γ are pressures inside the bubble, outside the bubble and due to surface tension respectively.

$$P=P_0+\frac{4\gamma}{r}$$

$$P=10^5\text{Pa}+\frac{4\times2.8\times10^{-2}\ \text{Nm}^{-1}}{2\times10^{-3}\text{m}}$$

$$=1.00056\times10^5\ \text{Pa}$$

Therefore, the pressure inside a soap bubble is 1.00056×10^5 Pa.

6.1.5 Measuring surface tension of a liquid using capillarity method

When a liquid surface is in contact with the surface of a solid, the shape of the liquid surface is usually curved. This effect is caused by the presence of cohesive and adhesive forces. The curvature of the liquid surface is determined by relative strength between cohesive and adhesive forces. If the adhesive force is larger than the cohesive force, the liquid tends to stick to the wall of its container and thus the liquid has a concave meniscus as shown in Figure 6.10(a). On the other hand, if cohesive force is larger than adhesive force, the liquid is pulled away from the wall and the meniscus is convex as seen in Figure 6.10(b). Good examples can be shown by water and mercury for both cases.

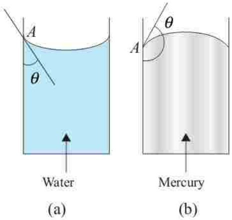


Figure 6.10 Concave and convex meniscus

In Figure 6.10, the angle θ at which the liquid meets the solid surface is called the angle of contact. It is defined as the angle between the solid surface and the tangent to the liquid surface at the point of contact, always measured through the liquid.

The angle of contact is affected by the following factors:

(a) The nature of the liquid and the solid in contact;

(b) The medium that exists above the free surface of the liquid;

(c) Impurities present in the liquid; i.e. adding impurities in the liquid decrease the angle of contact; and

(d) Temperature; contact angle increases with the increase in temperature.

Angle of contact is important in determination of surface tension of a liquid in the phenomenon called capillarity.

Capillarity is the rise or depression of the liquid in a narrow tube immersed in the liquid due to varying intermolecular forces and pressure difference between the upper and lower surfaces.

Suppose γ is the magnitude of the surface tension of the liquid such as water, which rises up a clean capillary tube with an angle of contact zero (Figure 6. 11). The surface tension acts along the boundary of the liquid vertically downwards on the glass. By the law of action and reaction, the glass exerts an equal force in an upward direction on the liquid.

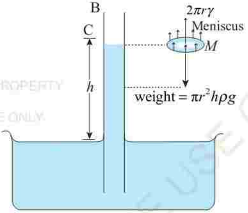


Figure 6.11 Rise in capillary tube

If r is the radius of the capillary tube, the length of liquid in contact with the glass is $2\pi r$. Since surface tension, γ, is the force per unit length acting in the surface of the liquid, then upward force on liquid is given by

$$F_{\gamma} = 2\pi r \times \gamma \qquad (6.9)$$

The upward force on liquid supports the weight of a column of a height h above the outside level of liquid.

The volume V of the liquid column,

$$V = \pi r^2 h$$

The mass m, of the liquid column,

$$m = V \times \rho = \pi r^2 h \rho$$

Thus, the weight W of the liquid is

$$W = \pi r^2 h \rho g$$

From equation (6.9),

upward force F_γ = downward force mg

$$2\pi r \times \gamma = \pi r^2 h \rho g$$

$$\gamma = \frac{rh\rho g}{2} \qquad (6.10)$$

Suppose the angle of contact between the liquid and the tube is θ (Figure 6.12), the magnitude of surface tension (γ) of liquid can be determined by considering vertical component forces.

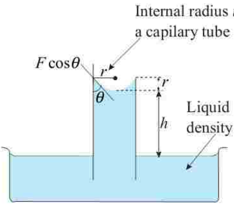


Figure 6.12 Measurement of surface tension using capillary tube

At equilibrium, vertical component force is equal to weight of water in the tube.

$$F\cos\theta = mg, \quad \text{but} \quad m = \rho \times V$$

$$F\cos\theta = \rho V g \qquad (6.11)$$

Where V is the volume of liquid in the tube above the free surface of the liquid given by volume of cylinder of height, h and radius, r plus volume of small cylinder enclosed hemisphere of height, r and radius, r minus volume of hemisphere of radius, r.

$$V = \pi r^2 h + \pi r^2 r - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$V = \pi r^2 \left(h + \frac{r}{3}\right) \qquad (6.12)$$

But $F = 2\gamma \pi r$, substituting (6.12) into (6.11) gives;

$$2\gamma\cos\theta = \rho g r\left(h + \frac{r}{3}\right)$$

$$\gamma = \frac{\rho g r\left(h + \frac{r}{3}\right)}{2\cos\theta}$$

If the tube is very narrow, $\frac{r}{3}$ can be neglected, hence,

$$\gamma = \frac{\rho g r h}{2\cos\theta} \qquad (6.13)$$

The pressure difference for liquid rise in capillary tubes can be described using Figure 6.13. When the capillary tube is placed in water as shown in Figure 6.13(a), initially, water level is the same inside and outside the tube. But the concave meniscus indicates that pressure at M (P_M) is less than that at N (P_N). Hence water in the tube will rise to a height h as shown in Figure 6.13(b).

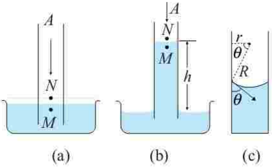


Figure 6.13 Capillary action

Excess pressure is

$$P_N - P_M = \frac{2\gamma}{R}$$

Since the height of the liquid is h and its density is ρ, then

$$P_M = P_A - h\rho g$$

where $P_N = P_A$ which is the atmospheric pressure. Therefore,

$$P_A - \left(P_A - h\rho g\right) = \frac{2\gamma}{R}$$

$$h = \frac{2\gamma}{\rho g R} \qquad (6.14)$$

From Figure 6.13(c), if θ is the angle of contact of the liquid and r is the radius of the capillary tube, R is the radius of curvature of the meniscus, then

$$R = \frac{r}{\cos\theta}$$

substituting R in (6.14) gives

$$h = \frac{2\gamma\cos\theta}{\rho g r} \qquad (6.15)$$

Equation (6.15) shows that the capillary rise h varies inversely with radius r of the tube. Thus the narrower the tube the higher the capillary rise. Capillary rise is applied in many areas including supply of water to tall buildings, absorption of ink by a blotting paper and rise of oil in the wick of a lamp.

Capillary depression on the other hand, occurs when the angle of contact is obtuse ($\theta > 90°$), hence a convex meniscus. Suppose that the depression of the liquid (e.g. mercury) inside a tube of radius r is h (Figure 6.14). The convex meniscus shows that the pressure at M is greater than that at N.

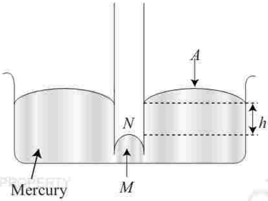


Figure 6.14 Capillary depression

From the general result of excess pressure on curved liquid surfaces,

$$P_M - P_N = \frac{2\gamma\cos\theta}{r}$$

where $P_M = P_A + h\rho g$ and $P_N = P_A$. Hence

$$h = \frac{2\gamma\cos\theta}{r\rho g} \qquad (6.16)$$

Example 6.7

The radius of a capillary tube is 0.25 mm and it is inserted vertically in a liquid whose density is 8×10^2 kgm^{-3} and surface tension is 3×10^{-2} Nm^{-1}. If the angle of contact is 72.5°, determine the height to which liquid will rise in the tube.

Solution

From equation (6.15),

$$h = \frac{2\gamma \cos\theta}{r\rho g}$$

$$= \frac{2 \times 3 \times 10^{-2}\ \text{Nm}^{-1} \times \cos 72.5^\circ}{8 \times 10^{2}\ \text{kgm}^{-3} \times 0.25 \times 10^{-3}\ \text{m} \times 9.8\ \text{ms}^{-2}}$$

$$= 9.2 \times 10^{-3}\ \text{m}$$

Therefore, the liquid in the tube will rise 9.2×10^{-3} m above the normal level.

Example 6.8

Water rises in a capillary tube to a height of 2.0 cm. How high will the water rise to a tube with radius of one third of the former tube?

Solution

In the capillary tube, height h, is inversely proportional to the radius r, i.e. $h \propto \frac{1}{r}$, it then follows that,

$$h_1 r_1 = h_2 r_2$$

$$h_2 = \frac{2\ \text{cm} \times r_1 \times 3}{r_1}, \text{ where } r_2 = \frac{1}{3} \times r_1$$

$$h_2 = 6\ \text{cm}$$

Therefore, water will rise up to 6 cm.

Example 6.9

A glass tube of internal diameter 3 mm is immersed into mercury whose density is 13600 kgm^{-3} and surface tension of 0.45 Nm^{-1}. If the angle of contact of mercury with glass is 135°, calculate the depression of mercury in a glass tube.

Solution

From equation (6.16),

$$h = \frac{2\gamma \cos\theta}{r\rho g}$$

$$= \frac{2 \times 0.45\ \text{Nm}^{-1} \times \cos 135^\circ}{1.5 \times 10^{-3}\ \text{m} \times 13600\ \text{kgm}^{-3} \times 9.8\ \text{ms}^{-2}}$$

$$= -3.18 \times 10^{-3}\ \text{m}$$

Therefore, the depression of mercury in glass tube is 3.18×10^{-3} m.

6.1.6 Factors affecting surface tension

Temperature and impurities affect the surface tension of liquids. Experiments show that the surface tension of liquids (water in particular), decreases with increasing temperature along a fairly smooth curve. The decrease of surface tension with temperature may be due to the greater average separation of the molecules at higher temperature. The force of attraction between molecules is then reduced, which results into reduction of surface energy.

The presence of impurities on the surface or dissolved in a substance directly affects the surface tension of the liquid. Adding impurities to the liquid, reduces or increases the cohesive forces between similar molecules or adhesive forces between different molecules. The surface tension of water, for example, will be increased when highly soluble impurities like salt are added to it, whereas sparingly soluble substances like soap decrease the surface tension of water.

Exercise 6.1

1. Briefly explain the following phenomena as related to surface tension.
   (a) Some insects (bugs) are able to walk on the surface of water.
   (b) A steel needle when placed carefully on water can be made to float. However, when a detergent (such as soap) is added to water, the needle sinks.
   (c) Water wets glass but mercury does not.
   (d) Water moves up a paper towel dipped into it.
2. (a) Explain briefly with the aid of a diagram, what you would expect to happen to a nearly spherical water droplet resting on a clean horizontal surface if a tiny amount of detergent were added to it. How do you account for the change that might occur?
   (b) A soap bubble in vacuum has a radius of 6 cm and another soap bubble has a radius of 8 cm. If the two bubbles coalesce, find the radius of the new bubble formed and state the condition at which this is valid.
3. If the energy required to blow a soap bubble of radius r is E, show that the extra energy needed to double the radius of the bubble is given by, $E = 24\pi\gamma r^2$ where γ is the surface tension of the soap solution.
4. Given that, the excess pressure of one soap bubble is four times the other soap bubble, find the ratio of their volumes.
5. Two soap bubbles of radius r_1 and r_2 such that $r_1 < r_2$, coalesce. Show that the radius of curvature of the common surface in air is $\dfrac{r_1 r_2}{r_2 - r_1}$.
6. A capillary tube is immersed in water of surface tension $7.2\times10^{-2}\ \text{Nm}^{-1}$ and rises to $6.2\ \text{cm}$. By what depth will mercury be depressed if the same capillary is immersed in it? Surface tension of mercury is $0.54\ \text{Nm}^{-1}$, angle of contact of mercury with glass is 140°, and density of mercury is $13600\ \text{kgm}^{-3}$.
7. (a) The surface tension of a soap solution is $0.03\ \text{Nm}^{-1}$. What amount of work is required to produce a bubble of radius $0.5\ \text{cm}$?
   (b) The inside diameters of the two arms of a U–tube are 1.0 mm and 1.5 mm respectively. Now if it is partially filled with water of surface tension of $0.0736\ \text{Nm}^{-1}$ and zero angle of contact, what will be the difference in the level of meniscus between the two arms?
8. Air is forced through a tube of internal diameter of 1.5 mm immersed at a depth of 1.5 cm in a mineral oil having specific gravity of 0.85. Calculate the unit surface energy of the oil if the maximum bubble pressure is $150\ \text{Nm}^{-2}$.
9. The material of a wire has a density of $1.4\ \text{gcm}^{-3}$. If it is not wetted by a liquid of surface tension

4.4×10^{-4} Ncm^{-1}, find the maximum radius of the wire which can float on the surface of the liquid.

10. A square frame of side L is dipped in a liquid. On taking it out, a membrane is formed. If the surface tension of the liquid is γ, find the force acting on the frame.

6.2 Elasticity

Elastic property of materials is important in daily life. You want the wings of an airplane to be able to bend a little, but you would not want them break off. The steel frame of an earthquake-resistant building has to be able to flex, but not too much. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials.

The physical reasons for elastic behaviour can be quite different for different materials. In metals, atomic lattice changes size and shape when forces are applied (energy is added to the system). When forces are removed, the lattice goes back to the original lower energy state. For rubbers and other polymers, elasticity is caused by stretching of polymer chains when forces are applied. So, all substances experience a change in shape or size when exposed to high energy or forces. Elasticity is the tendency or property of a body to return to its original size and shape after it has been deformed. Deformation refers to change of shape or size.

In this section, you will learn the deformation of solids, the concepts of stress and strain, modulus constants and potential energy of deformation.

6.2.1 Elasticity in terms of molecular theory

Unlike fluids (liquids and gases), solids have definite shape and size. At the molecular level, particles of solids tend to maintain their arrangement because they vibrate about a means position and have strong attraction to one another.

When a deforming force is applied to a solid, the intermolecular forces of attraction will resist any change on the equilibrium of the particles. As a result, the particles will be slightly dislocated from their equilibrium position and the solid object as a whole appears to be deformed. If the deforming force is removed, the intermolecular forces restore particles into equilibrium and the solid object regains its form (shape and size).

(a) Stress and strain

Stress is the quantity that is used to describe the applied deforming force F on a body. When an external deforming force is applied to the solid body, an internal restoring force (due to the intermolecular forces of attraction) is developed in the body. This internal restoring force per unit area A of the deformed body is called stress. At equilibrium, the restoring force equals the external applied force. Therefore,

$$stress = \frac{deforming\ force}{cross\text{-}sectional\ area}$$

$$stress = \frac{F}{A} \qquad (6.17)$$

In case the forces applied are along the length of the body (Figure 6.15(a)), the stress is termed as longitudinal stress σ given by,

$$\sigma = \frac{F}{A} \qquad (6.18)$$

In case the body is subjected to a uniform pressure from all sides (Figure 6.15(b)), the stress is termed as hydrostatic (bulk) stress ΔP given by,

$$\Delta P = \frac{F}{A} \qquad (6.19)$$

In fact the hydrostatic stress is the difference between the outside and the inside pressures of the body. In case the forces are acting tangentially (Figure 6.15(c)), the stress is termed a shearing stress. Hence,

$$\textit{shearing stress} = \frac{F}{A}$$

where A is the area to which the force F is acting tangentilly.

The unit of stress is Nm^{-2} or pascal (Pa), and its dimensions are $ML^{-1}T^{-2}$, same as that of pressure.

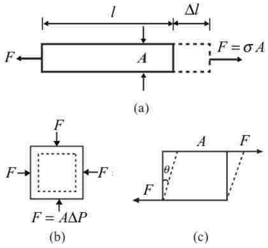


Figure 6.15 Longitudinal, hydrostatic and shearing stress

Strain occurs when the deforming forces act on a body without causing it to move, but bring about a change in its shape and or size. It is defined as the change in the dimension per original dimension of the body. Hence,

$$\textit{strain} = \frac{\textit{change in dimension}}{\textit{original dimension}}$$

The above mentioned stresses yield three types of strains, namely; linear (longitudinal) strain, bulk strain and shear strain. If a body of length l is extended by Δl, then,

$$\textit{linear strain} = \frac{\textit{change in length}}{\textit{original length}}$$

$$\textit{linear strain} = \frac{\Delta l}{l} \qquad (6.20)$$

Similarly, bulk strain occurs when the deforming force produces change in volume of the body. It is measured by the ratio of change of volume of the body to its original volume. Then,

$$\textit{bulk strain} = -\frac{\Delta V}{V} \qquad (6.21)$$

The negative sign signifies that, as the external pressure increases the volume decreases.

When the tangential forces act on a body they change its shape (Figure 6.16). The angle θ through which a line originally perpendicular to the fixed plane is turned, is called shear strain. Shear strain $= \tan\theta$ which is $\frac{AA'}{AF}$. Hence

$$\text{shear strain} = \frac{\Delta L}{L} \qquad (6.22)$$

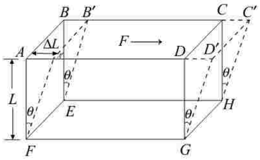


Figure 6.16 Shear strain

Strain being a ratio of two similar quantities is a dimensionless quantity.

6.2.2 Brittle and ductile materials

Robert Hooke experimentally observed that, within the elastic limit (i.e., small strains), the stress is directly proportional to the strain produced in a body. Thus,

$$\frac{stress}{strain} = constant \quad (6.23)$$

This constant is a measure of elasticity of the material and is called the modulus of elasticity. As mentioned earlier strain being dimensionless quantity, modulus of elasticity has the same dimension and the same unit as that of the stress.

To get a clear distinction between brittle and ductile materials let us consider a metallic wire of uniform cross-section area subjected to an increasing load. The stress-strain variation for the wire is shown in Figure 6.17. The portion *OA* of the curve is a sloping straight line. It is the region in which Hooke's law is valid. As can be seen in this region stress is proportional to strain. The point *A* represents the elastic limit.

Within this limit a strain is very small and on removing the applied stress the body regains its original state of zero strain. In other words, it can be said that in this region the body is perfectly elastic. At the moment the elastic limit is exceeded, the strain increases more rapidly than the stress. The region *AB* in Figure 6.17 corresponds to this stage. The extension in this region is partly elastic and partly plastic. This means that if the wire is unloaded in this region, it will not come back to original condition along *OA*. The wire is then said to have acquired a permanent stretch. The point *B* is called the yield point.

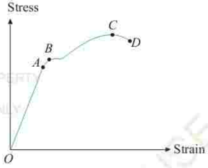


Figure 6.17 Stress-strain curve for an elastic material

Beyond *B* and up to *C*, the strain increases rapidly and in an irregular manner, for small increase in stress or sometimes even without any increase in stress.

Beyond *C* and up to *D*, the extension is plastic. In this region the strain increases steadily with decreasing stress and the cross section of the wire decreases uniformly with the extension. But after *D*, the length of the wire goes on increasing without any addition of a load or even if the load is reduced a little.

The stress corresponding to point *D* is called the ultimate strength or the tensile strength or breaking stress. Eventually, beyond *D* the wire breaks. This point is, therefore, called breaking point.

Those materials for which the portion *BD* of the curve is relatively long are called ductile materials. These materials can undergo large increase in length before breaking and show large plastic range beyond elastic limit. Examples of ductile materials include; copper, silver and iron. The materials, for which the portion *BD* is relatively small and breaks when subjected to a small extension are called brittle materials. Cast iron, glass and ceramic are examples of brittle materials.

6.2.3 Moduli of elasticity

Depending upon the type of stress applied on the body and the corresponding strain, the moduli of elasticity are classified into the following three types: Young's modulus (*Y*), Bulk modulus (*B*) and Shear modulus (*S*).

(a) Young's modulus of elasticity

Young's modulus is the measure of the resistance of a solid to a change in its length when a force is applied perpendicular to a surface. It is given as the ratio of longitudinal stress to the longitudinal strain which is equivalent to the slope of the line in figure 6.17 (within the elastic limit *OA*).

Consider a wire of length *L* and cross-section area *A* fixed at one end to a rigid support as shown in Figure 6.18.

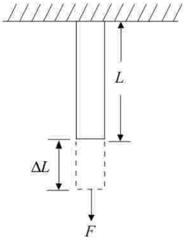


Figure 6.18 A stretched wire by force

If a normal force *F* is applied to the free end, the length of the wire will change by ΔL. Thus,

$$\text{Young's modulus} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$Y = \frac{FL}{A\Delta L} \qquad (6.24)$$

Example 6.10

A wire increases by 10^{-3} of its original length when a stress of $1\times10^{8}\ \text{Nm}^{-2}$ is applied to it. What is the Young's modulus of the material of the wire?

Solution

From equation (6.24),

$$Y = \frac{FL}{A\Delta L}$$

$$= \frac{1\times10^{8}\ \text{Nm}^{-2}}{10^{-3}} = 1\times10^{11}\ \text{Nm}^{-2}$$

Therefore, the Young's modulus is $1\times10^{11}\ \text{Nm}^{-2}$.

(b) The bulk modulus

This refers to the situations in which the volume of a body is changed by the application of an external normal stress. Bulk moduli are possessed by solids, liquids, and gases. The application of a force ΔF (Figure 6.19) which results to change in pressure ΔP, is everywhere normal to the surface of a spherical body that changes its volume by ΔV, but the shape of the body remains unchanged. If the applied pressure is not too large, the compression produced in the body is proportional to the pressure.

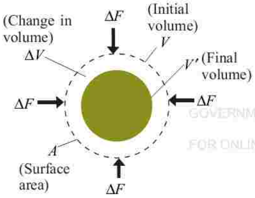


Figure 6.19 Volume subjected to a radial stress

$$Bulk\ modulus = \frac{bulk\ stress}{bulk\ strain}$$

$$B = -\frac{\Delta F V}{A \Delta V}$$

where $\frac{\Delta F}{A} = \Delta P$, $B = -V\frac{\Delta P}{\Delta V}$

When ΔP and ΔV become very small, then, in the limit,

$$B = -V\frac{dP}{dV} \qquad (6.25)$$

The unit of the bulk modulus is Nm^{-2} or Pa. The negative sign in equation (6.25) indicates that the volume decreases with an increase in pressure. The reciprocal of B is called compressibility K of a substance given by $K = \frac{1}{B}$.

A material is therefore easily compressed if it has a small bulk modulus. Gases obviously have much smaller bulk modulus than solids and liquids.

Example 6.11

A solid ball $300\ cm$ in diameter is submerged in a lake of a certain depth such that, the pressure exerted by water is $9.8 \times 10^4\ Nm^{-2}$. Find the change in volume of the ball at this depth. Bulk modulus of the materials of the ball is $10^{12}\ Nm^{-2}$.

Solution

Using the relation

$$B = -V\frac{\Delta P}{\Delta V};\ \Delta V = \frac{-V\Delta P}{B}$$

$$V = \frac{4\pi r^3}{3} = \frac{4\pi \times \left(150 \times 10^{-2}\ m\right)^3}{3}$$

$$V = 14.137\ m^3$$

$$\Delta V = \frac{-\left(14.137\ m^3 \times 9.8 \times 10^4\ Nm^{-2}\right)}{10^{12}\ Nm^{-2}}$$

$$= -1.385 \times 10^{-6}\ m^3$$

Therefore, the change in volume is $-1.385 \times 10^{-6}\ m^3$. The negative sign indicates a decrease in volume of the ball.

Example 6.12

A cube is subjected to a pressure of $5\times10^5\ \text{Nm}^{-2}$. Each side of the cube is shortened by 1% . Find:

(i) The volumetric strain; and

(ii) Bulk modulus of elasticity of the cube.

Solution

Initial volume, $V_i = l^3$, and final volume, $V_f = (l-0.01l)^3 = (0.99l)^3$.

(i) Volumetric strain,

$$\frac{\Delta V}{V} = \frac{l^3-(0.99l)^3}{l^3}$$

$$\frac{\Delta V}{V} = 0.03$$

(ii) Bulk modulus,

$$B = \frac{5\times10^5\ \text{Nm}^{-2}}{0.03}$$

$$B = 1.67\times10^7\ \text{Nm}^{-2}$$

(b) The rigidity or shear modulus

A shear stress is one which changes the shape of a body; the strain which results is called a shear strain. Figure 6.20 shows a solid block *WXYZ* whose lower face is fixed. A force *F* acts on the block tangentially to its upper face. The force provides a shear stress which distorts the block so that its new shape is $WX'Y'Z$.

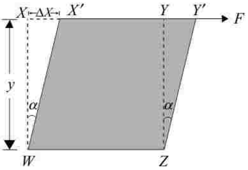


Figure 6.20 Shear forces act on an object

If the tangential stress is $\frac{F}{A}$ and the corresponding shear strain is α radians, then,

$$shear\ modulus = \frac{shear\ stress}{shear\ strain}$$

$$S = \frac{F}{A\alpha} \qquad (6.26)$$

Example 6.13

Aluminium cube having 4 cm in each side is subjected to a tangential force. The top face of the cube is sheared 0.012 cm with respect to the bottom. Find:

(a) The shear strain;

(b) The shear stress; and

(c) The shearing force.

Given that the modulus of rigidity is $2.08\times10^{10}\ \text{Nm}^{-2}$.

Solution

(a) Shear strain, $\alpha = \frac{\Delta L}{L}$

$$\alpha = \frac{0.012\ \text{cm}}{4\ \text{cm}} = 0.003$$

Therefore, the shear strain is 0.003.

(b) shear modulus $(S) = \frac{\text{shear stress}}{\text{shear strain}}$

Shear stress $= S \times$ shear strain

$S = 0.003 \times 2.08 \times 10^{10}\ \text{Nm}^{-2}$

$= 6.24 \times 10^{7}\ \text{Nm}^{-2}$

Therefore, the shear modulus is $6.24 \times 10^{7}\ \text{Nm}^{-2}$.

(c) Shearing force,

$F =$ shear stress $\times$ area of cube face

$F = 6.24 \times 10^{7}\ \text{Nm}^{-2} \times 0.04\,\text{m} \times 0.04\,\text{m}$

$= 9.98 \times 10^{4}\ \text{N}$

Therefore, the shear stress is 9.98×10^{4} N.

6.2.4 Potential energy in deforming a solid body

Consider a wire or any material which extends by an amount x when a force F is applied on it. If the extension is increased by dx, where dx is so small that F can be considered constant; then, the work done is $dW = Fdx$.

The total work done (W) in increasing the extension from 0 to x is equal to elastic potential energy stored in the wire (the strain energy) and is given by;

$$W = \int_0^x F\,dx$$

If the wire obeys Hooke's law, then, $F = kx$, where k is a constant, so that

$$W = \int_0^x kx\,dx$$

$$W = \frac{kx^2}{2} \text{ or } W = \frac{Fx}{2} \qquad (6.27)$$

Similarly, energy stored in a wire per unit volume (U) can be obtained using equation (6.27) as,

$$U = \frac{W}{V} = \frac{Fx}{2AL}$$

$$U = \frac{1}{2} \times \left(\frac{F}{A}\right) \times \left(\frac{x}{L}\right) \qquad (6.28)$$

Therefore,

$$U = \frac{1}{2} \times stress \times strain$$

which is equal to the area under the curve of stress versus strain, up to the elastic limit.

Example 6.14

Calculate the increase in energy of a brass bar of length 0.2 m and cross-sectional area 1 cm^2 when deformed with a force of 49 N along its length. (Young's modulus of brass is $1.0 \times 10^{11}\ \text{N/m}^2$)

Solution

Increase in energy of the bar = Work done in deforming the bar i.e.

$$W = \frac{1}{2}Fx \text{ and } x = \frac{Fl}{YA}, \text{ then,}$$

$$W = \frac{F^2 l}{2AY}$$

$$= \frac{(49\ \text{N})^2 \times 0.2\ \text{m}}{2 \times (1.0 \times 10^{-4}\ \text{m}^2) \times (1.0 \times 10^{11}\ \text{Nm}^{-2})}$$

$$= 2.4 \times 10^{-5}\ \text{J}$$

Therefore, the increase in energy of a brass bar is 2.4×10^{-5} J.

Example 6.15

A 3.0 m length of a rod is suspended horizontally from the ceiling using two vertical wires of equal length tied to its ends. One of the wires is made of steel of cross-sectional area $1\times10^{-3}\,\text{m}^2$ and the other is made of brass of cross-sectional area $4\times10^{-3}\,\text{m}^2$. Find the position along the rod at which a weight may be hung to produce

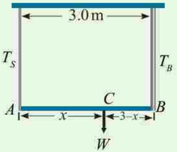
(a) equal stresses in the wires.

(b) equal strains in the wires.

Young's modulus of steel and brass are $2\times10^{11}\,\text{Nm}^{-2}$, and $1\times10^{11}\,\text{Nm}^{-2}$ respectively.

Solution

Let x be the distance from the steel wire at which the weight W is hung. Suppose T_S and T_B are the tensions in the steel and brass wires respectively.



(a) For equal stresses in the wires:

$\sigma_S = \sigma_B$

$$T_S = \frac{T_B}{A_B}\times A_S \qquad (i)$$

Since the system is in equilibrium, take moments about C:

The sum of clockwise moments is equal to the sum of anticlockwise moments.

$$T_S x = T_B \times (3-x) \qquad (ii)$$

Insert equation (i) into (ii)

$$\left(\frac{T_B}{A_B}\times A_S\right) = T_B \times \left(\frac{3-x}{x}\right)$$

$$\frac{A_S}{A_B} = \frac{3-x}{x}$$

$$\frac{1}{4} = \frac{3-x}{x} \qquad (iii)$$

Solving equation (iii), $x = 2.4\,\text{m}$

Therefore, the weight should be placed at a distance of $2.4\,\text{m}$ from the steel wire or $0.6\,\text{m}$ from brass.

(b) For equal strains (S) in the wires:

$S_S = S_B$

$$\text{From } S = \frac{Stress(\sigma)}{Young's\ Modulus(Y)};$$

$$\frac{\sigma_S}{Y_S} = \frac{\sigma_B}{Y_B}$$

$$\text{Since } \sigma_S = \frac{T_S}{A_S} \text{ and } \sigma_B = \frac{T_B}{A_B}$$

$$\frac{T_S}{T_B} = \frac{A_S Y_S}{A_B Y_B}$$

$$= \frac{1\times10^{-3}\,\text{m}^2 \times 2\times10^{11}\,\text{Nm}^{-2}}{4\times10^{-3}\,\text{m}^2 \times 1\times10^{11}\,\text{Nm}^{-2}} = \frac{1}{2}$$

$$\frac{T_S}{T_B} = \frac{3-x}{x} = \frac{1}{2} \qquad (iv)$$

Solving equation (iv); $x = 2\,\text{m}$

Therefore, the weight should hang $2\,\text{m}$ from steel wire or $1\,\text{m}$ from the brass.

6.2.5 Applications of elasticity of materials

The knowledge about elasticity of materials serves a lot of purposes to humans. It is used by engineers in bridge designing to know the maximum load the bridge can withstand without bending or breaking. Also it is used in designing structural details of columns, beams and supports of buildings to avoid bending or breaking due to expansion or contraction.

Winches are used for lifting and moving heavy loads from one place to another. They have a thick metal rope to which the load is attached. In order to lift a load without deforming the rope permanently, it is ensured that the extension should not exceed the elastic limit.

Most parts of structures and machinery are under some kind of stress. In their design, it has to be ensured that applied stress do not exceed the elastic limits of the materials. In railway track structure, the vertical dynamics are significant part of the stress exerted as well as the level of vibration and emitted noise. Thus, elastic materials such as rail pads, under-sleeper pads and under-ballast mats are incorporated to reduce geometrical degradation as well as to decrease noise and vibrations along the track.

Exercise 6.2

1. (a) It is found experimentally that the torque required to twist a hollow cylinder is greater than the torque required to twist a solid cylinder of same length and radius. Explain.
   (b) In the model of a crystalline solid, the particles are assumed to exert both attractive and repulsive forces on each other. Sketch a graph of the potential energy between two particles as a function of the separation of the particles. Explain how the shape of the graph is related to the assumed properties of the particles.
2. (a) Elastic moduli, elastic limit, and strengths of material are all quoted with the same unit, Pascal's. Explain the differences between these three physical quantities.
   (b) Why stresses and strains rather than forces and extensions are generally considered when describing the elastic behavior of solids?
3. (a) Would you expect a rubber band to have a larger or a smaller force constant than that of an iron wire? Explain.
   (b) A steel rod of length 0.6 m and cross-sectional area $2.5 \times 10^{-5}\ \text{m}^2$ at $100\ °\text{C}$ is clamped so that when it cools it is unable to contract. Find the tension in the rod when it has cooled to $20\ °\text{C}$. Young's

modulus of steel is 2.0×10^{11} Pa, linear expansivity of steel is $1.6\times10^{-7}\,°C^{-1}$.

4. (a) If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulus change?

   (b) A wire 2 m long with a cross- sectional area $10^{-6}\,m^2$ is stretched 1 mm by a force of 50 N in the elastic region. Calculate:
   (i) The strain;
   (ii) The Young modulus; and
   (iii) The energy stored in the wire.

5. (a) A large tensile force is needed to increase the length of a steel wire by about 0.1 % but a modest tensile force doubles the length of a rubber band. Explain how the difference in behavior is accounted for by the different molecular structures of steel and rubber.

   (b) Explain why, if a steel wire is formed into a helical spring, the amount of elastic potential energy it can store increases enormously.

   (c) A force of 20 N is applied to the ends of a wire 4 m long, and produces an extension of 0.24 mm. If the diameter of the wire is 2 mm, calculate the stress on the wire, its strain, and the value of Young modulus.

6. A spring is extended by 30 mm when a force of 1.5 N is applied to it. If the spring was un-stretched before applying the mass, calculate the energy stored in the spring when hanging vertically supporting a mass of 0.20 kg. Calculate the loss in potential energy of the mass. Explain why these values differ.

7. Calculate the volume of 1.025×10^{3} kg of sea water at a depth where the pressure is 5.0×10^{7} Pa. Then, calculate the density of sea water at this depth. Bulk modulus of sea water is 2.2×10^{5} Pa and density of surface sea water is $1.025\times10^{3}\,kgm^{-3}$.

8. Two structural beams, beam 1 and beam 2, both have the same cross-sectional area. The tension force required to stretch beam 1 by 1% is four times the force required to stretch beam 2 by 0.5%. Beam 1 has the following properties: $L = 10$ m, $Y = 12\times10^{9}$ Pa, calculate the Young's modulus Y for beam 2.

9. A bone that has the shape of a cylinder has one end fixed to a horizontal surface. If a 35 N force is then applied laterally to the plane of the upper face; determine the lateral displacement, given that the diameter and length of the bone are 1.2 cm and 3 cm respectively. Shear modulus of bone is 80×10^{9} Pa.

10. A mild steel wire of length $2L$ and cross-sectional area A is stretched well within elastic limit, horizontally between two vertical pillars. A mass m is suspended from the mid-point of the wire. Determine the strain produced in the wire.

11. A body of mass, m is hung from the middle point of the steel wire of diameter $0.8\ \text{mm}$ and length $1.2\ \text{m}$ clamped firmly at two points P and Q. The distance between the two points is $1.2\ \text{m}$ in the same horizontal plane such that the middle point sags $1\,\text{cm}$ lower from the original position. Calculate the mass of the body given that Young's modulus of steel is $2\times10^{11}\ \text{Nm}^{-2}$.

6.3 Kinetic theory of gases

Kinetic theory of gases makes use of many assumptions in order to explain the reasons why gases act the way they do. The theory explains the behaviour of gases by considering the motion of the molecules. In the theory, the gas is assumed to consist of a very large number of molecules (one mole is about 6.022×10^{23} molecules) that move about randomly and collide frequently. The pressure of a gas is a force that the gas exerts on the walls of a container. It can be observed that whenever a molecule bounces off a wall it reverses (changes) its direction. The rate of change of the momentum produced is equal to the average force which the gas molecules exert on the wall.

In this section, you will learn to interpret the assumptions of kinetic theory of gases, obtain the expression for pressure of a gas, deduce root mean square speed of gas molecules and establish the relationship existing between kinetic energy and temperature of a gas.

6.3.1 Assumptions of the kinetic theory of gases

The main assumption is that the range of intermolecular forces (both attraction and repulsive) is small compared with the average distance between molecules. The other assumtions are:

(a) Collisions between the molecules and the container are perfectly elastic.

(b) The volume of the gas molecules is negligible compared to the volume of the container in which they occupy.

(c) The time spent in a collision is negligible compared with the time spent by a molecule between collisions.

(d) The intermolecular forces are negligible except during a collision.

(e) Between collisions a molecule moves with uniform velocity in a straight line.

(f) Even in a small volume there is a large number of molecules and that large number of collisions occurs in a small time.

The above assumptions define an ideal gas.

6.3.2 Pressure exerted by gases

Consider an ideal gas enclosed in a cubical container of sides L (Figure 6.21). If a single molecule of a gas of mass m, initially moving towards x-direction has a velocity u_1, then the x-component of momentum is mu_1.

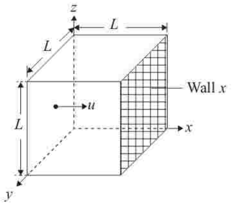


Figure 6.21 Molecules in a cubical container

The molecule will eventually reverse the direction of its momentum by colliding with the wall. Since the collision is elastic, it will rebound with the same speed so that its momentum will now be $-mu_1$. The change in momentum in x-direction is $mu_1 - (-mu_1) = 2mu_1$.

Since the molecule travels a distance $2L$ to-and-fro, the time for such a trip is $\frac{2L}{u_1}$, and therefore, this molecule's rate of change of momentum due to collision with the wall x will be $\frac{mu_1^2}{L}$.

By Newton's second law of motion, the rate of change of momentum is equal to net force, and therefore, $\frac{mu_1^2}{L}$ is the force exerted on the molecule by the wall. Likewise, by Newton's third law of motion, the molecule exerts an equal but oppositely directed force on the wall.

Therefore, force on the wall x is $\frac{mu_1^2}{L}$ and force per unit area is $\frac{mu_1^2}{L^3}$ (since area of the wall $x = L^2$). Therefore, pressure on the wall is $\frac{mu_1^2}{L^3}$.

If there are N molecules in the container and their x-components of velocity are $u_1, u_2, \ldots \ldots u_N$, then, the total pressure, P on wall x will be given by

$$P = \frac{m}{L^3}\left(u_1^2 + u_2^2 + \ldots + u_N^2\right) \qquad (6.29)$$

If $\overline{u^2}$ is the mean value of the squares of all the velocity components in the x-direction, then

$$\overline{u^2} = \frac{u_1^2 + u_2^2 + \ldots + u_N^2}{N}, \text{ implies,}$$

$$N\overline{u^2} = u_1^2 + u_2^2 + \ldots + u_N^2.$$

Thus, from equation (6.29)

$$P = \frac{m}{L^3}\left(N\overline{u^2}\right) \qquad (6.30)$$

For any molecule, $c^2 = u^2 + v^2 + w^2$, where u, v and w are components of velocity along x, y and z respectively, and c^2 is the resultant velocity square. This also holds for the mean square values, therefore, $\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}$.

But since N is large and the molecules move randomly, it follows that, the mean values of u^2, v^2 and w^2 are equal,

i.e. $\overline{u^2} = \overline{v^2} = \overline{w^2}$. Therefore,

$$\overline{c^2} = 3\overline{u^2},\ \overline{u^2} = \frac{\overline{c^2}}{3} \text{ and so } P = \frac{Nm\overline{c^2}}{3L^3}$$

But L^3 = volume V of the gas. Thus,

$$PV = \frac{Nm\overline{c^2}}{3} \qquad (6.31)$$

Since $\rho = \frac{Nm}{V}$ is the density of the gas, Nm is the total mass of the gas, then,

$$P = \frac{1}{3}\rho\overline{c^2} \qquad (6.32)$$

The root mean square speed, v_{rms} of a gas is the square root of $\overline{c^2}$. From equation (6.32),

$$v_{rms} = \sqrt{\frac{3P}{\rho}} \qquad (6.33)$$

For an ideal gas, Charles' law and Boyle's law can be combined to obtain the general relationship between pressure, volume and temperature. Consider a fixed amount of gas of volume V_1 kept in a cylinder at a pressure P_1 and temperature T_1. If it is desired to calculate its volume V_2 at an absolute temperature T_2, you have to increase the pressure to P_2 while keeping the temperature T_1 constant. This is governed by Boyle's law.

$$P_1V_1 = P_2V_2 \qquad (6.34)$$

If it is desired to calculate V_2 at an absolute temperature T_2 and P_1 is kept constant you have to increase the temperature to T_2. This is governed by Charles' law expressed as

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \qquad (6.35)$$

Combining the two laws, that is equation (6.34) and (6.35) yields

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2};\quad \frac{PV}{T} = \text{constant}$$

For a gas with n number of moles,

$$PV = nRT \qquad (6.36)$$

R is called the universal gas constant with a value of 8.31 J mol^{-1}K^{-1}.

Equation (6.36) is called the equation of state because it expresses a relationship between the state variable of the system. Any gas which obeys this equation is called an ideal gas.

6.3.3 Internal energy and kinetic energy of gases

Gas molecules at a particular temperature possess internal energy. This energy depends on the translational kinetic energy and rotational kinetic energy of the molecules. Translational kinetic energy (KE_{tr}) of molecules depends on linear movement of molecules. On the other hand, rotational kinetic energy (KE_{rt}) depends on the number of atoms in a molecule and on the structure of the molecule.These two finally determine the rotational degrees of freedom, f. The degrees of freedom are the number of independent ways the molecules can possess kinetic energy. The total kinetic energy (KE_T) of molecules is the sum of both translational and rotational kinetic energy. For translational kinetic energy of an ideal gas:

$$PV = \frac{Nm\overline{c^2}}{3} \qquad (6.37)$$

Equation (6.37) can also be expressed as;

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right) \qquad (6.38)$$

By comparing equations (6.36) and (6.38) and rearranging,

$$\frac{2}{3}N\left(\frac{1}{2}m\overline{c^2}\right) = nRT,\quad \frac{1}{2}m\overline{c^2} = \frac{3n}{2N}RT$$

Since $\frac{N}{n}$ is the number of molecules per mole, i.e., N_A, the Avogadro number, then,

$$\frac{1}{2}m\overline{c^2} = \frac{3}{2N_A}RT \qquad (6.39)$$

Both R and N_A are universal constants. Therefore, $\frac{R}{N_A}$ is also a universal constant called Boltzmann's constant, $k = 1.38\times10^{-23}\,\text{JK}^{-1}$. Therefore, the average translational kinetic energy of a gas molecule is

$$KE_{tr} = \frac{3}{2}kT \qquad (6.40)$$

Let M be the molar mass of the gas, i.e., $M = N_A m$, then

$\frac{1}{2}N_A m\overline{c^2} = \frac{3}{2}RT$ which results to

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

Similarly, rotational kinetic energy is such that,

$$KE_{rt} = f\times\left(\frac{1}{2}kT\right) \qquad (6.41)$$

From equations (6.40) and (6.41),

$$KE_T = \frac{3}{2}kT + f\left(\frac{1}{2}kT\right) \qquad (6.42)$$

The internal energy of the gas, U is therefore given by

$$U = N\times KE_T$$

For monoatomic gas (with only one atom for each molecule, for example inert gases), $f = 0$

$$\therefore KE_T = \frac{3}{2}kT + 0\times\left(\frac{1}{2}kT\right) = \frac{3}{2}kT$$

and,

$$U = N\times KE_T = \frac{3}{2}\times NkT \text{ or } U = \frac{3}{2}nRT$$

For diatomic gas (with two atoms for each molecule, for example oxygen, hydrogen and nitrogen), $f = 2$,

$$KE_T = \frac{3}{2}kT + 2\times\left(\frac{1}{2}kT\right) = \frac{5}{2}kT \text{ and,}$$

$$U = N\times KE_T = \frac{5}{2}\times NkT \text{ or } U = \frac{5}{2}nRT$$

For polyatomic gas (with more than two atoms for each molecule, for example ozone, carbondioxide etc), $f = 3$ which also results to $KE_T = \frac{5}{2}kT$ and $U = 3NkT$ or $U = 3nRT$.

Example 6.16

What is the root mean square speed of a hydrogen molecule at 27 °C ? ($k = 1.38\times10^{-23}\,\text{JK}^{-1}$).

Solution

From the relation, $v_{rms} = \sqrt{\frac{3kT}{m}}$

$T = 273 + 27 = 300\text{ K}$

Mass of hydrogen $(m) = \frac{2\times10^{-3}\,\text{kg}}{6.023\times10^{23}}$

$m = 3.32\times10^{-27}\,\text{kg}$

$$v_{rms} = \sqrt{\frac{3\times\left(1.38\times10^{-23}\,\text{JK}^{-1}\times300\text{ K}\right)}{3.32\times10^{-27}\,\text{kg}}}$$

$= 1934\,\text{ms}^{-1}$

Example 6.17

Estimate the total number of air molecules in a room of capacity 25.0 m^3 at a temperature of 27 °C and 1 atm pressure, given that $k = 1.38\times10^{-23}\,\text{JK}^{-1}$.

Solution

Let N be the total number of air molecules in a room. Then according to perfect gas equation,

$$PV = NkT$$

$$N = \frac{PV}{kT} = \frac{1.013\times10^{5}\,\text{Nm}^{-2}\times25\,\text{m}^{3}}{1.38\times10^{-23}\,\text{JK}^{-1}\times300\,\text{K}}$$

$$= 6.1\times10^{26}\,\text{molecules}$$

Example 6.18

(a) What is the average translational kinetic energy of an ideal gas molecule at $27\,°\text{C}$? (Given that $k = 1.38\times10^{-23}\,\text{JK}^{-1}$ and $R = 8.314\,\text{Jmol}^{-1}\text{K}^{-1}$)

(b) What is the total random translational kinetic energy of the molecules in 1 mole of this gas at $27\,°\text{C}$?

(c) What is the root mean square speed of oxygen molecules at this temperature?

Solution

Using the relation

(a) $K.E = \frac{3}{2}kT$

$$= \frac{3}{2}\times1.38\times10^{-23}\,\text{JK}^{-1}\times300\,\text{K}$$

$$= 6.21\times10^{-21}\,\text{J}$$

(b) $K.E = \frac{3}{2}nRT$

$$= \frac{3}{2}\times1\,\text{mol}\times8.314\,\text{Jmol}^{-1}\text{K}^{-1}\times300\,\text{K}$$

$$= 3741.3\,\text{J}$$

(c) $v_{rms} = \sqrt{\frac{3kT}{m}}$

Since $n = 1$, then, $m = \frac{M}{N_A}$, where M is molar mass of a compound and m is the mass of an oxygen molecule. Then,

$$m = \frac{32.0\times10^{-3}\,\text{kgmol}^{-1}}{6.023\times10^{23}\,\text{mol}^{-1}}$$

$$= 5.31\times10^{-26}\,\text{kg}$$

$$v_{rms} = \sqrt{\frac{3\times1.38\times10^{-23}\,\text{JK}^{-1}\times300\,\text{K}}{5.31\times10^{-26}\,\text{kg}}}$$

$$= 483.6\,\text{ms}^{-1}$$

Exercise 6.3

1. (a) Briefly explain why the pressure of a gas increases when the volume of a gas is reduced at constant temperature?

   (b) The temperature of an ideal monatomic gas is increased from 25°C to 50°C. Is the average translational kinetic energy of each gas atom doubled? Explain your answer. If your answer is no, what would be the final temperature if the average translational kinetic energy was doubled?

2. Calculate the root mean square speed, v_{rms} of the following system of gas molecules:

Number of Molecules	3	7	5	2	1
Speed (ms^{-1})	2.2	6.1	7.8	4	2.5

3. A sealed vessel has a volume of $1.5\times10^{-3}\,m^3$ and contains oxygen at a pressure of 1.0×10^4 Pa and a temperature of 300 K. Given that the molar gas constant, $R = 8.3\,Jmol^{-1}K^{-1}$, the Avogadro's number, $N_A = 6.023\times10^{23}\,mol^{-1}$, and the molar mass of oxygen $= 32\times10^{-3}\,kgmol^{-1}$. Determine:
   (a) The number of moles of oxygen in the vessel.
   (b) The number of molecules in the vessel.
   (c) The root mean square speed of the molecules in the vessel.
4. By what factors does the mean square speed and the root mean square speed of the molecules of a gas increase when its temperature is doubled?
5. The temperature of a gas is increased in such a way that its volume doubles and its pressure becomes four times the original value. If the root mean square speed of the molecules originally was $250\,ms^{-1}$, what will be its value at the higher temperature?
6. Calculate the root mean square speed of hydrogen molecules and oxygen molecules at the temperature of $0\,°C$. (Relative molecular masses of hydrogen and oxygen are 2 and 32 respectively).

Revision exercise 6

1. It is said that the shape of a liquid is the same as the shape of its container. But, with no container and gravity, what is the natural shape of a drop of water? Why?
2. A small, dry paper clip can rest on the surface of still water. Why can't a heavier paper clip do the same without sinking?
3. Explain the following observations in terms of surface tension.
   (a) A wet tent will let in water if the inside is touched.
   (b) A pond skater can walk on the surface of water but a person cannot.
4. Take a kitchen sieve and immerse it in molten candle wax. Remove it quickly, shaking off excess wax so that the wires of the sieve get a thin coating of wax. Now pour some water into it. Explain what happens.
5. A U-tube has limbs of radii 0.5 cm and 0.5 mm respectively. A liquid of surface tension $7\times10^{-2}\,Nm^{-1}$ is poured into the tube when it is placed vertically. Find the difference in levels of the two limbs. Density of the liquid in the tube is $1000\,kgm^{-3}$ and the angle of contact is zero.
6. The radius of a capillary tube is 0.025 mm. It is held vertically in a liquid whose density is $0.8\times10^3\,kgm^{-3}$, surface tension is

$3.0\times10^{-3}\,\text{Nm}^{-1}$ and the cosine of angle of contact is 0.3. Determine the height to which the liquid will rise in the tube relative to the liquid surface outside. Use acceleration due to gravity, $g = 10\ \text{ms}^{-2}$.

7. Water rises up in a glass capillary up to a height of 9.0 cm while mercury falls down by 3.4 cm in the same capillary. Assume angles of contact for water-glass and mercury glass as $0°$ and $135°$ respectively. Determine the ratio of surface tensions of mercury and water. The density of mercury is $13.6\times10^{3}\,\text{kgm}^{-3}$ and the density of water is $10^{3}\,\text{kgm}^{-3}$.

8. Two soap bubbles in vacuum have a radii 3 cm and 4 cm respectively. If the bubbles are combined to form a single large bubble, calculate the radius of the formed large bubble.

9. Initially a soap bubble in a piston chamber of pressure P_0 has a radius r. If the piston is pulled out until the radius of the soap bubble doubles; show that,

   (a) The new pressure inside the chamber is given by

   $$P = \frac{P_0}{8} - \frac{3\gamma}{2r}$$

   (b) If the piston is compressed until the radius is halved, the new pressure inside the chamber is given by; $P = 8P_0 + \dfrac{24\gamma}{r}$ where P_0 the original air is pressure inside the piston chamber and assume isothermal condition.

10. If olive oil is sprayed onto the surface of a beaker of hot water, it remains as separated droplets on the water surface. As the water cools, the oil forms a continuous thin film on the surface. Suggest a reason for this phenomenon.

11. (a) It is sometimes stated that, by nature of its surface tension, the surface of a liquid behaves as if it was a stretched rubber membrane. To what extent do you think this analogy is justified?

    (b) Explain why the pressure inside a soap bubble is greater than that outside.

12. A soap bubble has diameter of 4 mm. Calculate the pressure inside it if the atmospheric pressure is $10^{5}\,\text{Nm}^{-2}$ and surface tension of soap solution is $2.5\times10^{-2}\,\text{Nm}^{-1}$.

13. Air is introduced through a nozzle into a tank of water to form a stream of bubbles. If the bubbles are intended to have a diameter of 2 mm, calculate how much pressure of the air at the tip of the nozzle must exceed that of the surrounding water. Assume that the value of surface tension between air and water as $72.7\times10^{-3}\,\text{Nm}^{-1}$.

14. Explain the following observation as related to elasticity of a material.

(a) A heavier person compresses a spring mattress more than a lighter person.

(b) Steel nails are rigid and unbending while steel wool is soft and squishy.

15. In designing structures in an earthquake-prone region, how should the natural frequencies of oscillation of a structure relate to typical earthquake frequencies? Should the structures have a large or a small amount of damping?

16. A steel wire, *AB* of length 0.60 m and a cross-sectional area $1.5\times10^{-6}\,\text{m}^2$ is attached at *B* to a copper wire, *BC* of length 0.39 m and cross-sectional area $3.0\times10^{-6}\,\text{m}^3$. The combination is suspended vertically from a fixed point at *A*, and supports a weight of 250 N at *C* (Figure 4.22). Find the extension of each section of the wire (Young's modulus of steel is 2.0×10^{11} Pa. Young's modulus of copper is 1.3×10^{11} Pa).

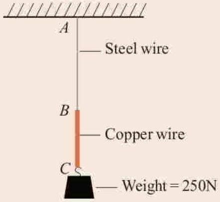


Figure 6.22 Combination of steel and copper wires

17. The Young's modulus of a metal is $8\times10^{9}\,\text{Nm}^{-2}$ and its density is $11\,\text{gcm}^{-3}$. Calculate its density if the metal is subjected to a pressure of $20000\,\text{Ncm}^{-2}$.

18. A copper wire *LM* is fused at one end *M* to an iron wire *MN*. The copper wire has length of 0.9 m and cross-section $0.90\times10^{-6}\,\text{m}^2$. The iron wire has a length of 1.4 m and cross-section $1.30\times10^{-6}\,\text{m}^2$. The compound wire is stretched and its total length increases by 0.01 m. Calculate:

(a) The ratio of extensions of the two wires;

(b) The extension of each wire; and

(c) The tension applied to the compound wire.

Young's modulus for copper and iron are $=1.3\times10^{11}\,\text{Nm}^{-2}$, and $2.1\times10^{11}\,\text{Nm}^{-2}$ respectively.

19. The graph (Figure 6.23) represents stress-strain curves for two different materials, *A* and *B*, where F_A and F_B are respective point at which each material fractures.

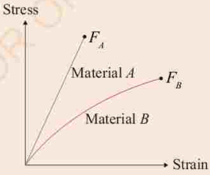


Figure 6.23 Stress-strain curves

State, giving your reasons, which material, A or B,

(a) obeys Hooke's law up to the point of fracture.

(b) is weaker than the other.

(c) has the larger value of Young's modulus.

20. A thin steel wire initially 1.5 m long with a diameter of 0.5 mm is suspended from a rigid support. When a mass of 3.0 kg is attached to the lower end, calculate:

(a) The final extension; and

(b) The energy stored in the wire.

Assume that the material obeys Hooke's law. (Young's modulus for steel $= 2.0 \times 10^{11}\,\text{Nm}^{-2}$)

21. A light rigid bar AB of 0.20 m is suspended horizontally from two vertical wires of steel and brass each of 2.00 m long attached on a ceiling as shown in Figure 6.24.

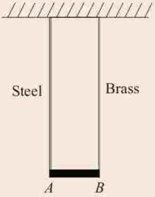


Figure 6.24 A light rigid bar suspended by two wires

If the diameter of the steel wire is 0.60 mm and a mass of 10 kg is suspended from the centre of AB;

(a) What is the tension in each wire?

(b) Calculate the extension of the steel wire and the energy stored in it.

(c) Calculate the diameter of the brass wire.

(d) If the brass wire is replaced by another brass wire of diameter 1.00 mm, where should the mass be suspended so that AB remains horizontal? The young's modulus for steel is 2.0×10^{11} Pa and that of brass is 1.0×10^{11} Pa.

22.(a) Explain what is meant by an ideal gas. What properties are assumed for the model of an ideal gas molecule in deriving the expression $p = \frac{1}{3}\rho\overline{c^2}$ where the symbols have their usual meanings.

(b) How is pressure explained in terms of the kinetic theory of gases? Describe carefully, using diagrams where necessary, the steps in the argument used to obtain an expression for $p = \frac{1}{3}\rho\overline{c^2}$.

23. Show that for a fixed mass of ideal gas at constant temperature, the expression for $p = \frac{1}{3}\rho\overline{c^2}$ can be written as $pV = A$, where A is a constant.

For some real gases, the pressure can be described in terms of the equation $p(A+B)=A$ where B is also a constant for a fixed mass of the gas at a particular temperature. Show that the expression $p(A+B)=A$ implies a pressure less than the value predicted for an ideal gas. Suggest a reason for this in molecular terms.

24. Using the kinetic theory of gases, show that:

    (a) The pressure of an ideal gas is doubled when its volume is halved at constant temperature.

    (b) The pressure of an ideal gas decreases when it expands in a thermally insulated container.

25. A volume of $0.23\ m^3$ contains nitrogen at a pressure of 0.50×10^5 Pa and a temperature of 300 K. Assuming that the gas is ideal, calculate:

    (a) The amount of nitrogen present in moles; and

    (b) The root mean square speed of nitrogen molecules at a temperature of 300 K.

    (Molar mass of nitrogen $= 0.028\ kgmol^{-1}$, molar gas constant $= 8.3\ JK^{-1}mol^{-1}$)

26. Oxygen gas (O_2) has a molar mass of $32\ gmol^{-1}$.

    (a) What is the average translational kinetic energy of an oxygen molecule at a temperature of 300 K?

    (b) What is the average value of the square of its speed?

    (c) What is the root mean square speed?

Chapter Seven

Heat

Introduction

Heat is a form of energy that cannot be seen, though its effects can be felt. Humans get heat from different sources, including the sun, electricity, fire, and gas. On hot days people wear light clothing to improve heat transfer from their bodies to the air, and allow better cooling by evaporation of perspiration. On cold days, they wear heavy clothes, or stay indoors to keep themselves warm. In this chapter, you will learn about thermometric properties of a substance, thermodynamic scale of temperature, thermal conduction, thermal convection, thermal radiation, and the first law of thermodynamics.

7.1 Thermometers

The concept of temperature is rooted in the ideas of 'hotness' or 'coldness' based on the sense of touch. An object that feels hot has higher temperature than when it feels cold. The quantity that indicates how warm or cold an object is relative to some standard is called temperature. Many of the properties of matter that you can measure quantitatively depend on the temperature. The pressure of a gas in a container increases with temperature; a steam boiler may explode if it gets too hot. Temperature is also related to kinetic energies of the molecules of the materials. In this section, you will learn about temperature measurement, determination of the degree of hotness and coldness, and the scales involved in measuring temperature.

7.1.1 Thermometric properties of substance

Thermometry is a branch of science that deals with measurement of temperature. Temperature of a body is the degree or intensity of heat present in a substance or object, such that when two bodies are placed in contact, heat flows from the one at high temperature to the one at low temperature. It is an indicator of the average thermal energy of the molecules. In order to measure temperature, a temperature scale must be established.

Any object which has a physical property that changes in a measurable way as the object gets hotter or colder can be used as the basis of scale of temperature. Such a property is called a thermometric property. Nearly all solids, liquids and gases expand when heated and this expansion is commonly used to specify thermometric properties on which variety of thermometers are based. Therefore, the length of solid bar, volume of liquid or gas can all be used as thermometric properties. Other thermometric properties include properties such as resistity, e.m.f, and resistance of a material. In order to define temperature scale, three distinct conventions are adopted.

(i) A choice of suitable thermometric property.

(ii) A choice of specific functional dependence of temperature on the thermometric property chosen.

(iii) A choice of appropriate number of calibration points to specify uniquely the function chosen.

A thermometer is used to measure temperature. It makes use of a physical property (thermometric property) of a substance which changes continuously with temperature. Table 7.1 shows some thermometric properties of matter used in various thermometers.

Table 7.1 Thermometric properties of matter

Thermometric property	Thermometer
Volume expansion of a gas	Gas thermometer
Volume expansion of a liquid	Laboratory or clinical thermometer
Volume expansion of solid	Bi-metallic strip thermometer
Pressure change in fixed mass of gas	Volume-constant gas thermometer
Change in electromotive force	Thermocouple thermometer
Change in electrical resistance	Resistance thermometer

Thermometers have measurement scales. Scale of temperature is a way to measure temperature quantitatively. There are three temperature scales in use today, namely Celsius, Fahrenheit and kelvin.

7.1.2 Temperature scales

To measure temperature, a temperature scale has to be established as follows:

(i) A substance which is sensitive to temperature is selected, and its properties must be accurate and measurable over wide range of temperature. Its properties must vary in similar way with other physical properties.

(ii) In order to establish a temperature scale, it is necessary to make use of fixed points and assign numbers to them.

(a) The Celsius scale

On the Celsius method of numbering, the lower fixed point is the **ice point**, i.e. the temperature at which pure ice melts at one atmospheric pressure and is assigned the value of 0°C.

On the other hand, the upper fixed point is the **steam point**, i.e. the temperature at which pure water boils at one atmospheric pressure and is assigned the value of 100°C.

Suppose thermometric properties X_{100} and X_0 of the temperature-measuring property are found at the steam and ice points respectively, then, $(X_{100} - X_0)$ gives the fundamental interval of the scale. These properties are proportional to the temperature, i.e. $X \propto \theta$, $X = k\theta$, where k is the proportionality constant.

If X_θ is the value of the property of material at some other temperature θ, then,

$$X_\theta - X_0 = k\theta \qquad (7.1)$$

$$X_{100} - X_0 = 100k \qquad (7.2)$$

Dividing equation(7.1) by (7.2), gives

$$\frac{\theta}{100} = \frac{X_\theta - X_0}{X_{100} - X_0}$$

$$\theta = \left(\frac{X_\theta - X_0}{X_{100} - X_0}\right) \times 100\,°\text{C} \qquad (7.3)$$

Note that, the equation has been defined so that, equal increases in the value of the property represents equal increases of the temperature, i.e. the temperature scale is defined so that, the property varies uniformly or linearly with temperature measured on its own scale.

(b) Fahrenheit scale

On Fahrenheit scale, the boiling point is assigned to 212°F and the freezing point of 32°F. The fundamental interval is then $212°F - 32°F = 180°F$. If you divide by 180, you get 180 equal parts each with 1°F, called one degree Fahrenheit.

Note that, a Celsius temperature T_C, is the number of Celsius degrees above freezing, and the number of Fahrenheit degree above freezing is $\frac{9}{5}$ of this (but freezing on Fahrenheit is at 32°F).

To convert temperature from Celsius to Fahrenheit we multiply the Celsius value by $\frac{9}{5}$ and then add 32°C, i.e.,

$$T_F = \frac{9}{5}T_C + 32 \qquad (7.4)$$

To convert Fahrenheit to Celsius, solve the above equation for T_C to get

$$T_C = \frac{5}{9}(T_F - 32) \qquad (7.5)$$

(c) Thermodynamic scale

Thermodynamic scale of temperature is not an empirical scale of temperature. It is a scale of temperature that is not based on any thermometric property or experimental results. Therefore, it is an absolute scale of temperature (T). Although this scale is theoretical, it is identical with the scale based on pressure variation of an ideal gas at constant volume.

Kelvin suggested that, the standard scale of temperature should be based on ideal or perfect gas or real gases at very low pressure or high temperature, i.e. it should obey Boyle's law.

The two fixed points in the thermodynamic scale of temperature are Absolute zero and Triple point of water.

(i) Absolute zero

Absolute zero is the temperature at which the pressure of an ideal gas becomes zero, and it has the value of 0 K (zero kelvin). The temperature scale which begins at absolute zero is called Kelvin or absolute temperature scale. Absolute zero is equivalent to –273.15°C. The absolute zero is considered to be the lowest possible temperature attained by a substance in which the random motion of the atom and molecules in the substance is at minimum.

(ii) Triple point

Triple point is the point at which vapour, liquid, and solid phases (states) of a substance co-exist in equilibrium. The temperature at the triple point of water is 273.16 K or 0.01°C and the pressure is 4.58 mmHg or 610 Pa. At this point, it is possible to change all of the substance to ice, water, or vapour by making arbitrarily small changes in temperature and pressure (Figure 7.1).

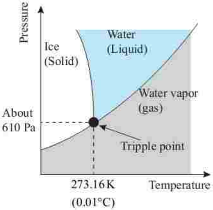


Figure 7.1 Triple point at constant volume

Let X_{tr} be a physical property of a material at its triple point temperature T_{tr}, X_T be the physical property of a material at unknown temperature T (in kelvin). Since the change in temperature is directly proportional to the change in thermometric properties, i.e., $T_{tr} \propto X_{tr}$, then

$$T_{tr} = kX_{tr} \qquad (7.6)$$

where k is the proportionality constant. Similarly,

$$T = kX_T \qquad (7.7)$$

Dividing equation (7.7) by (7.6),

$$T = \left(\frac{X_T}{X_{tr}}\right) \times T_{tr} \qquad (7.8)$$

For water, $T_{tr} = 273.16\,\text{K}$. Thus equation (7.8) can be written as;

$$T = \left(\frac{X_T}{X_{tr}}\right) \times 273.16\,\text{K} \qquad (7.9)$$

The fixed points and their corresponding absolute temperature scales are summarized in Figure 7.2.

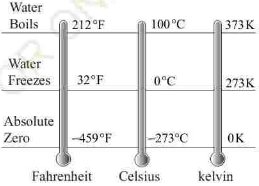


Figure 7.2 Fixed points and absolute temperature scale

7.1.3 Types of thermometers and their uses

To construct a thermometer, you depend on those materials whose properties change uniformly with temperature. For example, the volume of a gas or a liquid increases uniformly with the increase of its temperature, and the length of a solid changes uniformly with increase of its temperature. Similarly, the electric resistance of a wire increases with increase of its temperature. Thermometers are designed to measure the temperature of a body by assigning a numerical value to any given temperature. It uses some measurable properties of matter that change continuously with temperature to measure the unknown temperature. There are many types of thermometers. Some of the common thermometers are liquid-in glass thermometers, gas thermometers, platinum resistance thermometers, thermoelectric thermometers and radiation thermometers (pyrometers).

(a) Liquid-in glass thermometer

A liquid-in glass thermometer is the simplest and most commonly employed type of temperature measurement device. It is one of the oldest thermometers available in the industry. It mainly comprises of:

(i) A bulb which acts as the container holding the liquid whose volume changes with temperature. The bulb also acts as a sensor or gauge which is inserted in the body whose temperature is to be measured (Figure 7.3).

(ii) A stem which is a glass tube containing a tiny capillary tube enlarged at the bottom into a bulb that is partially filled with a "working fluid".

(iii) A temperature scale which is basically preset or imprinted on the stem for displaying temperature readings.

(iv) Point of reference, i.e., a calibration point which is most commonly the ice point.

(v) A working liquid which is generally either mercury or alcohol.

(vi) An inert gas, mainly argon or nitrogen which is filled inside the thermometer above the working liquid to trim down its volatization.

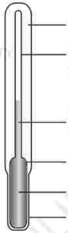


Figure 7.3 Liquid-in glass thermometer

The liquid in glass thermometer utilizes the variation in length of liquid column in a glass with temperature as the thermometric property.

Replacing X in equation (7.3) by length l of liquid column, then,

$$\theta = \left(\frac{l_\theta - l_0}{l_{100} - l_0} \right) \times 100\,°\text{C} \qquad (7.10)$$

where, l_0 = length of liquid column at ice point

l_{100} = length of liquid column at steam point

l_θ = length of liquid column at unknown temperature θ

One of the most common thermometers used in the laboratory and at home make use of the expansion of mercury in a glass tube. Mercury is used because of its good conductivity, uniform expansion for equal amounts of heat gained, opaqueness and luminosity which makes it easier to see. Other thermometric liquids such as alcohols are also used. Each of these thermometric substances have unequal expansion, and therefore, show deviation from a uniform pattern which is an undesirable feature. This shortcoming is taken care of in gas thermometer.

Inaccuracies arise in mercury thermometers from, non-uniformity of the bore of the capillary tube, the gradual change in the zero owing to the bulb shrinking for a number of years after manufacture, and the mercury in the stem not being at the same temperature as that in the bulb.

(b) Gas thermometers

In most accurate work, temperatures are measured by gas thermometer. It is referred as an ideal thermometer because the increase in volume or pressure of a gas with temperature is independent of the nature of the gas. Two types of gas thermometers include: constant-volume gas thermometers and constant-pressure gas thermometers.

(i) Constant-volume gas thermometers

A constant-volume gas thermometer is constructed using a gas, such as hydrogen, helium, nitrogen, oxygen or air at low density as a thermometric substance. It uses the variation of pressure of a gas (kept at a constant volume) with temperature. Experiments show that, there is good agreement in their readings over a wide range of temperatures, especially at low temperatures. Constant-volume gas thermometers are very sensitive, accurate and easily reproducible.

A constant-volume gas thermometer is composed of a bulb filled with a fixed amount of dry air or gas and it is attached to a mercury manometer (Figure 7.4).

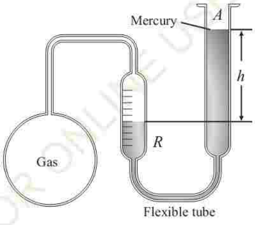


Figure 7.4 Constant-volume gas thermometer

During measurement the glass bulb is placed inside the enclosure whose temperature is to be measured. Keeping the volume of air in the glass bulb constant by raising or lowering the glass tube (to keep the mercury in the left side of the gas tube at the constant reference level R), the pressure of air in glass bulb at ice point $(0°C)$, steam point $(100°C)$ and at the unknown temperature $(\theta°C)$ are determined by recording the corresponding values of height difference h. The pressure, P of the gas is calculated from the relation $P = A + h\rho g$, where A is the atmospheric pressure, and ρ is the density of mercury.

If P_0 and P_{100} are pressures at 0°C and 100°C respectively, then, the temperature of the enclosure can be found by replacing X in equation (7.3) with pressure P;

$$\theta = \left(\frac{P_\theta - P_0}{P_{100} - P_0}\right) \times 100°C \qquad (7.11)$$

(ii) Constant-pressure gas thermometers

These thermometers are based on the thermal expansions of gases at constant pressure. As a thermometric property, it uses the variation of volume of a gas at constant pressure with temperature. If V denotes the volume of a gas at constant pressure, then one can talk of the volume at 0°C, 100°C and θ°C as V_0, V_{100} and V_θ respectively. Replacing X in equation (7.3) with volume V, it follows that,

$$\theta = \left(\frac{V_\theta - V_0}{V_{100} - V_0}\right) \times 100°C \qquad (7.12)$$

(c) Electrical thermometers

Electrical thermometers are classified into two categories, namely, resistance thermometers and thermocouple thermometers.

(i) Resistance thermometers

Resistance thermometers are based on the fact that, resistances of metals are temperature dependent. It is based on the uniform change of electrical resistance with equal rise or fall of temperature, and so resistance R can be used as thermometric property. Resistance thermometers are usually made of platinum because of its high temperature coefficient of resistance and high melting point $(1773\ °C)$. These features make platinum resistance thermometer both sensitive and useful over a wide range of temperature. They are also very accurate over all thermometers "except" gas thermometers, and are stable at high temperature.

The principle of a resistance thermometer

Platinum wire is wound on mica (insulator) and covered with quartz (glass material) and this forms one of the four arms of Wheatstone bridge as shown in Figure 7.5.(b).

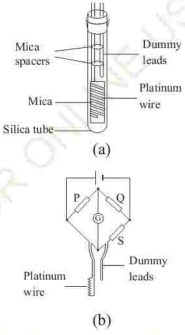


Figure 7.5 Platinum resistance thermometer and its circuit

If R_θ, R_{100}, R_0 are known then θ is calculated as;

$$\theta = \left(\frac{R_\theta - R_0}{R_{100} - R_0}\right) \times 100\ °C \qquad (7.13)$$

When calibrated against constant-volume gas thermometer, the resistance R of platinum is found to vary with Celsius temperature according to equation,

$$R_\theta = R_0(1 + a\theta + b\theta^2) \qquad (7.14)$$

where R_0 is the resistance at 0 °C, R_θ is the resistance at temperature θ, a and b are constants.

Note that, a thermistor is a type of resistance thermometer made from semiconducting materials, and it works on the principle that resistance decreases with increase in temperature.

Example 7.1

A particular resistance thermometer has a resistance of 30Ω at the ice point, 41.580Ω at the steam point and 34.59Ω when immersed in a boiling liquid. A constant volume gas thermometer gives readings of 1.333×10^5Pa, 1.821×10^5Pa and 1.528×10^5Pa at the respective three temperatures. Determine the temperature at which the liquid is boiling:

(a) On the scale of the gas thermometer; and

(b) On the scale of the resistance thermometer

Solution

(a) The Celsius temperature, θ_g according to the gas scale,

$$\theta_g = \left(\frac{P_\theta - P_0}{P_{100} - P_0}\right) \times 100\ °C$$

$$= \left(\frac{1.528\times10^5\text{Pa} - 1.333\times10^5\text{Pa}}{1.821\times10^5\text{Pa} - 1.333\times10^5\text{Pa}}\right) \times 100°C$$

$$= 39.96°C$$

Therefore, temperature on the gas scale is about 39.96°C.

(b) The Celsius temperature, θ_R according to the resistance scale,

$$\theta_R = \left(\frac{R_\theta - R_0}{R_{100} - R_0}\right) \times 100\ °C$$

$$= \left(\frac{34.59\Omega - 30\Omega}{41.58\Omega - 30\Omega}\right) \times 100°C$$

$$= 39.64°C$$

The temperature on the gas scale is about 39.64°C.

(ii) The thermocouple

A thermocouple is a device consisting of two dissimilar metal wires welded together at their ends, forming two electrical junctions. These will set up an electromotive force *(e.m.f)* at the point of contact. A thermocouple works under the Seebeck effect in which thermal *e.m.f* is generated at the contact of the two dissimilar conducting wires. Since the *e.m.f* generated varies continuously as the temperature of the junctions changes, it is used as the thermoelectric property.

In the construction of thermocouple, two junctions are always made. One junction is always maintained at a reference temperature, usually 0°C (hence called 'cold junction') and the other junction (called the hot junction) is connected to the body whose temperature is to be measured (Figure 7.6). The *e.m.f* generated is measured by a high resistance voltmeter in mV. Thermocouple thermometers have small heat capacities because the junctions are small, therefore, they have little effect on the temperature of the body being measured. They can measure rapid fluctuating (changing) temperatures. Also they have a wide range of temperature measurement (–200°C to 1500°C) depending on the type of materials used. However, in thermocouple thermometers, variation of *e.m.f.* with temperature is non-linear. They are also difficult to be recalibrated.

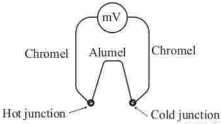


Figure 7.6 Principle of a thermocouple thermometer

The electromotive force (E) generated and the temperature θ between the junctions are related by:

$$E = a\theta + b\theta^2 \qquad (7.15)$$

This relation is parabolic as shown in Figure 7.7, where the values of a and b depend on the materials of the wires used and the temperature difference between the two junctions. As the temperature increases, the *e.m.f.* increases up to the temperature called neutral temperature θ_N, which is independent of the cold junction.

If E_0 and E_{100} are *e.m.f.* at 0°C and 100°C respectively, the temperature to be measured will be found by replacing X in equation (7.3) with *e.m.f.* , E.

$$\theta = \left(\frac{E_\theta - E_0}{E_{100} - E_0}\right) \times 100\ °\text{C} \qquad (7.16)$$

Note that, in practical use several thermocouples are connected in series to form a thermopile so as to give larger e.m.f.

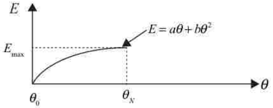


Figure 7.7 Thermocouple e.m.f. as a function of temperature θ

From Figure 7.7: Reference temperature θ_0 which is the temperature of cold junction. Neutral temperature θ_N which is the temperature of hot junction whose *e.m.f.* is maximum.

Inversion temperature θ_i which is the temperature of hot junction when *e.m.f.* is 0 V.

Applying differentiation on $E = a\theta + b\theta^2$, $\frac{dE}{d\theta} = a + 2b\theta$. At $\frac{dE}{d\theta} = 0$, $\theta_N = -\frac{a}{2b}$, where, $\theta = \theta_N$, then,

$$E_{max} = a\theta + b\theta^2 = a\left(-\frac{a}{2b}\right) + b\left(-\frac{a}{2b}\right)^2$$

$$E_{max} = -\frac{a^2}{4b} \qquad (7.17)$$

At E_{max}, $\theta_i = 2\theta_N$ and $\theta_N = \frac{\theta_0 + \theta_i}{2}$.

When the inversion temperature is exceeded, the thermoelectric *e.m.f.* in the thermocouple is reversed. The use of a thermocouple thermometer is restricted in the temperature range between

$0\ °C$ and neutral temperature θ_N. This is because, beyond the neutral temperature θ_N, the thermoelectric *e.m.f.* decreases with increasing temperature.

(d) Pyrometer

A pyrometer is a non-contact type of thermometer used for measuring high temperature using thermal radiation emitted by a hot source. Examples of sources with very high temperatures are ovens and molten metals. Pyrometer consists of an optical component to collect the radiation energy emitted from a surface of an object, a radiation detector that converts radiant energy into an electrical signal, and an indicator to read the measurements.

Other types of thermometer which have similar features as pyrometers include infrared thermometers. They are used to measure relatively low temperature of surfaces that emit like a blackbody.

Example 7.2

In a certain thermocouple, the thermo *e.m.f.* is given by $E = a\theta + \frac{b\theta^2}{2}$, where θ is the temperature of the hot junction. If the cold junction is at $0°C$, $a = 10\mu V\ °C^{-1}$ and $b = -0.05\ \mu V\ °C^{-2}$, calculate:

(a) The neutral temperature, θ_N and the inversion temperature, θ_i; and

(b) The maximum electromotive force, E_{max}.

Solution

(a) $E = a\theta + \frac{b\theta^2}{2}$

Since the neutral temperature is obtained at maximum e.m.f., E_{max}, then, $\frac{dE}{d\theta} = a + b\theta$. At $\frac{dE}{d\theta} = 0$, $\theta_N = -\frac{a}{b}$,

$$\theta_N = -\frac{10\ \mu V\ °C^{-1}}{-0.05\ \mu V\ °C^{-2}} = 200\ °C$$

$$\theta_i = 2 \times \theta_N = 2 \times 200°C$$

$$= 400°C$$

(b) $E_{max} = a\theta_N + \frac{b\theta_N^2}{2}$

$$= 10\ \mu V\ °C^{-1} \times 200\ °C - 0.05\ \mu V\ °C^{-2} \times \frac{(200\ °C)^2}{2}$$

$$E_{max} = 1000\ \mu V$$

Exercise 7.1

1. A faulty thermometer has its fixed points marked 5°C and 95°C. What is the correct temperature in centigrade when this thermometer reads 59 °C ?
2. Explain why
   (a) at least two (2) fixed points are required to define a temperature scale.
   (b) two thermometers using different thermometric properties and calibrated at two fixed points, would not necessarily show the same temperature except at the fixed points.
3. The resistance R_θ of a platinum wire varies with temperature θ according to the equation,

$R_\theta = R_0\left(1+8000b\theta - b\theta^2\right)$ where b is a constant. Calculate the temperature on a platinum scale corresponding to $400°C$ on the gas scale.

4. A liquid-in glass thermometer uses a liquid volume which varies with temperature according to the equation, $V_\theta = V_0\left(1+a\theta+b\theta^2\right)$ where V_θ and V_0 are the volumes of the gas at θ °C and 0 °C respectively, and a and b are constants. If $a = b\times10^3$, what will be the reading of the liquid in glass scale when the actual temperature is $60°C$?

5. A resistance thermometer has a resistance of 28.11Ω at the ice point, 29.10 Ω at the steam point and 28.11Ω at unknown temperature θ. Calculate θ on the scale of this temperature.

6. A particular constant-volume gas thermometer registers a pressure of 1.937×10^4Pa at the triple point of water and 2.618×10^4Pa at the boiling point of a liquid. What is the boiling point of the liquid according to this thermometer?

7. The temperature measurement described in question 6 above was repeated using the same thermometer but with a different quantity of the same gas. The readings on this occasion were 4.0668×10^4Pa at the triple point of water and 5.503×10^4Pa at the boiling point of the liquid.
   (a) What is the boiling point of the liquid according to this measurement?
   (b) Which of the two values is the better approximation to the ideal gas temperature, and why?
   (c) Estimate the ideal gas temperature.

8. A thermocouple thermometer has one of its junctions dipped into steam at $100°C$ while the other junction is dipped into ice at 0 °C. An *e.m.f.* of 1.2 mV is produced. When the junction in ice is removed and placed into an unknown liquid, the thermocouple thermometer produces an e.m.f. of 0.6 mV. What is the temperature of the unknown liquid?

9. The following readings were taken with a simple constant-volume air thermometer. This has a fixed mass of air trapped by a mercury column. What is the room temperature from these readings?

Temperature points		Level of mercury closed limb (mm)	Level of mercury open limb (mm)
1	Bulb in melting ice	136	112
2	Bulb in steam at 1atm	136	390
3	Bulb at room temperature	136	160

10. (a) Two thermometers are constructed in the same way such that, they have equal volume of liquid used and that one has a

spherical bulb and the other has an elongated cylindrical bulb. Which one will respond quickly to temperature changes?

(b) How do you justify that when a body is being heated at melting point, the temperature remains constant?

7.2 Heat transfer

Have you ever wondered why fire walkers do not burn their feet as they step quickly across red hot coal or charcoal? Is it because their feet are wet? What is the physics that explains the phenomenon? You will learn many intriguing applications of heat and heat transfer. Heat flows from an object at a higher temperature to the one with a lower temperature. These objects could be two solids, a solid and liquid or gas, or within a solid, liquid or gas. There are three ways that heat is transferred: conduction (through direct contact), convection (through fluid movement) and radiation (through electromagnetic waves).

7.2.1 Thermal conduction

Conduction is the process in which heat flows from the hotter regions of a material to the colder region without there being any net movement of the material itself.

(a) Conduction of heat in terms of kinetic theory of matter

The Kinetic theory of matter explains heat transfer by conduction, where thermal energy seems to move through a material, warming up cooler areas. Thermal energy can be transferred through conduction from one material to another when they are in direct contact (Figure 7.8).

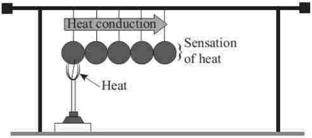


Figure 7.8 Conduction of heat by suspended elastic conducting balls in a row

Heat can also travel along a material as one molecule transfers energy to a neighbouring one. For example, when you put your hand in a container of warm water, your hand will gain heat. This is done through conduction of heat from the water.

Two mechanisms explain how heat is transferred by conduction: Lattice vibration and particle collision. Conduction through solids occurs by a combination of the two mechanisms.

In solids, atoms are bound to each other by a series of bonds. When there is a temperature difference in the solid, the hot side of the solid experiences more vigorous atomic movements. The vibrations are transmitted to the cooler side of the solid. Eventually, they reach equilibrium, where all the atoms are vibrating with the same energy.

Solids, especially metals, have free electrons, which are not bound to any particular atom and thus can freely move about the solid. The electrons on the hot side of the solid move faster than those on the cooler side. As the electrons undergo a series of collisions, the faster electrons give off some of their energy to the slower electrons. Eventually, through a series of random collisions, equilibrium is reached where the electrons are moving at the same average velocity. Conduction through electron collision is more effective than through vibration. This is why metals generally are better heat conductors than ceramic materials which do not have many free electrons. On the other hand, heat is conducted through stationary fluids primarily by molecular collisions, similar to the propagation of sound.

(b) Thermal conductivity

Thermal conductivity of a solid is a measure of the ability of the solid to conduct heat through it. The greater the thermal conductivity of a solid, the greater its ability to conduct heat through it. Consider a slab of materials of cross-section area A and thickness dx subjected to a high temperature θ_1 on one side and lower temperature θ_2 on the other side (Figure 7.9).

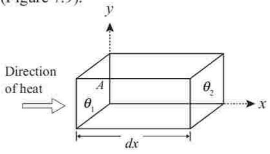


Figure 7.9 A slab

With the aid of Figure 7.9, we can determine the amount of thermal energy dQ conducted through a solid. It is found experimentally that the thermal energy dQ conducted through the slab is;

(i) directly proportional to the area A of the slab; the larger the area, the more thermal energy is transmitted.

(ii) directly proportional to the time dt; the longer the period of time, the more thermal energy is transmitted.

(iii) directly proportional to the temperature difference $(\theta_1 - \theta_2)$ between the faces of the slab; if there is a large temperature difference, a large amount of thermal energy flows.

(iv) inversely proportional to the thickness of the slab, dx; the thicker the slab the greater the distance that thermal energy must pass through. Thus, a thick slab implies a small amount of energy transfer whereas a thin slab implies a larger amount of energy transfer.

The above observations can be expressed as,

$$dQ \propto \frac{A(\theta_1 - \theta_2)}{dx} dt \qquad (7.18)$$

To make equality out of this proportion in equation (7.18), you must introduce a constant of proportionality k. The constant depends on the material that the slab is made of, since it is a known fact that different materials transfer different quantities of thermal energy. Hence,

$$dQ = \frac{-kA(\theta_1 - \theta_2)}{dx} dt \qquad (7.19)$$

Suppose $\theta_1 - \theta_2 = d\theta$, then,

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx} \quad (7.20)$$

where, $\frac{dQ}{dt}$ is the rate of flow of heat from the hotter face to the colder face and is at right angles to the faces (its unit is J/s or Watts, W), $\frac{d\theta}{dx}$ is called the temperature gradient across the section concerned (its unit is Km^{-1}), k is the coefficient of thermal conductivity of the material (its unit is $Js^{-1}m^{-1}K^{-1}$ or $Wm^{-1}K^{-1}$). The coefficient of thermal conductivity of material is the rate of flow of heat per unit area per unit temperature gradient when the heat flow is at right angles to the faces of a thin parallel sided slab of material under steady state conditions. It is a measure of the ability of the material to conduct heat, i.e. the larger the value of k, the faster the heat transfer. The thermal conductivity of some materials is given in Table 7.2.

When heat is flowing in the positive direction of x (Figure 7.9), the temperature gradient is negative, and therefore the presence of negative sign in equation (7.20) makes a positive constant. This is because under steady state condition, the temperature at points within the slab decreases uniformly with distance from hot end to the cold end. It is the existence of the temperature gradient which causes the heat to flow.

Table 7.2 Thermal conductivities of some materials

Material	Thermal conductivity ($Wm^{-1}K^{-1}$)
Silver	420
Copper	380
Aluminium	240
Brass	109
Nickel	87
Iron	80
Lead	35
Mercury	8
Glass (Pyrex)	1.1
Brick	0.6 – 1.0
Rubber	0.2
Air	0.03

Example 7.3

Find the amount of thermal energy that flows per day through a solid oak wall 10.0 cm thick, 3.00 m long, and 2.44 m high, if the temperature of the inside wall is 21.1 °C while the temperature of the outside wall is −6.67 °C. Thermal conductivity of oak is $0.147\ Jm^{-1}\,°C^{-1}$.

Solution

From the relation,

$$Q = \frac{kA(\theta_1 - \theta_2)t}{x}$$

$$Q = \frac{0.147\,Jm^{-1}°C^{-1} \times 3.00\,m \times 2.44\,m \times (21.1°C - (-6.67°C)) \times 24 \times 60 \times 60\,s}{0.1\,m} = 2.58 \times 10^7 J$$

Therefore, thermal energy that flow per day through a solid oak is 2.58×10^7 J.

(i) Heat flow through lagged and unlagged conductors

When a metal bar is heated from one end, heat flow depends on whether the metal bar is lagged or unlagged. If the metal bar is well lagged with a poor conductor of heat such as asbestos and wool, the temperature falls uniformly from the hot end to the cold end of the bar. Suppose a long uniform rod, AB, of length L is thermally insulated so that energy cannot escape by heat from its surface except at the end as shown in Figure 7.10(a), then all heat energy entering one end of the bar eventually leaves the other end.

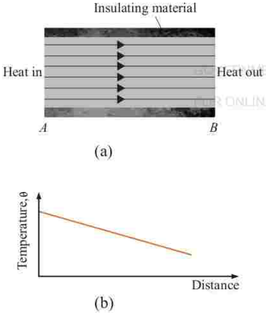


Figure 7.10 Conduction of energy through a uniform, insulated rod of length L

The drop in temperature is linear as shown in Figure 7.10(b). When a steady state has been reached, the temperature at each point along the rod is constant in time. A graph of temperature against length of the bar is shown in Figure 7.10(b).

Since the metal bar is well lagged no heat is lost to the surrounding and a graph of fall of temperature against length of the bar is a straight line.

To a very good approximation, thermal conductivity is independent of temperature. The temperature gradient is the same everywhere along the rod and is

$$\frac{d\theta}{dx} = \frac{|\theta_1 - \theta_2|}{L}$$

Since there is no heat that can escape from the sides of the metal bar, the rate of energy transfer by conduction through the rod is equal, i.e.,

$$\left(\frac{dQ}{dt}\right)_A = \left(\frac{dQ}{dt}\right)_B$$

$$\frac{dQ}{dt} = kA\frac{(\theta_1 - \theta_2)}{L} \qquad (7.21)$$

Equation (7.21) can be written as;

$$\frac{dQ}{dt} = A\frac{(\theta_1 - \theta_2)}{\frac{L}{k}} \qquad (7.22)$$

On the other hand, for unlagged material, heat flows from the hot end to the cold end of the bar but some amount of heat will flow out of the sides of the metal bar to the surrounding by convection and radiation before reaching end point B (Figure 7.11(a)).

When the conditions are steady, the temperature θ measured at points along the length of the bar varies (Figure 7.11(b)).

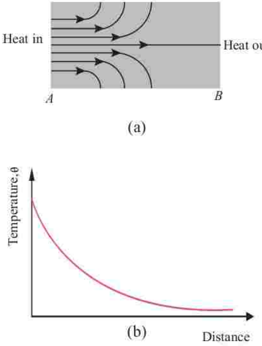


Figure 7.11 Conduction of energy through a uniform, unlagged metal bar of length L

In this case,

$$\left(\frac{dQ}{dt}\right)_A > \left(\frac{dQ}{dt}\right)_B, \text{ therefore,}$$

$$\left(\frac{d\theta}{dx}\right)_A > \left(\frac{d\theta}{dx}\right)_B$$

It follows that, the temperature gradient decreases with distance from the hot end of unlagged uniform bar. The graph (Figure 7.11(b)) shows the steady state temperature distribution of unlagged uniform bar of length L. There is a loss of heat to the surroundings because the metal bar is unlagged and the graph of fall of temperature against length of the bar is not a straight line.

Example 7.4

One face of a copper cube of edge 10 cm is maintained at 100 °C and the opposite face at 0°C. All other surfaces are covered with an insulating material. Find the amount of heat flowing per second through the cube. Thermal conductivity of copper is $385\,\text{Wm}^{-1}\,°\text{C}^{-1}$.

Solution

The heat flows from the hotter face towards the colder face. The amount of heat flowing per second is,

$$\frac{dQ}{dt} = \frac{kA(\theta_1 - \theta_2)}{x}$$

$$\theta_1 - \theta_2 = 100°\text{C} - 0°\text{C} = 100°\text{C}$$

$$\frac{dQ}{dt} = \frac{385\,\text{Wm}^{-1}\,°\text{C}^{-1} \times (0.1\,\text{m})^2 \times 100°\text{C}}{0.1\,\text{m}}$$

$$= 3850\,\text{W}$$

The rate of heat flow through the cube is 3850 W.

(ii) Composite conductors

A composite conductor is one made by joining two or more conductors of different materials joined end to end or side to side. There are two types of composite conductors; conductors in series and conductors in parallel.

Consider a composite conductor made of two different materials each of cross-section area A and, coefficient of thermal conductivities k_1 and k_2 joined end to end as shown in Figure 7.12.

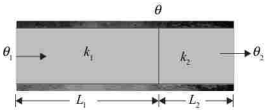


Figure 7.12 Lagged conductors in series

At steady state condition, the heat flowing into one end of the conductor is equal to that flowing out of the other end, given $\theta_1 > \theta > \theta_2$.

$$\frac{Q}{t} = Ak\left(\frac{\theta_1 - \theta_2}{L}\right) \qquad (7.23)$$

For conductor 1,

$$\left(\frac{Q}{t}\right)_1 = Ak_1\left(\frac{\theta_1 - \theta}{L_1}\right) \qquad (7.24)$$

For conductor 2,

$$\left(\frac{Q}{t}\right)_2 = Ak_2\left(\frac{\theta - \theta_2}{L_2}\right) \qquad (7.25)$$

But, $\left(\frac{Q}{t}\right)_1 = \left(\frac{Q}{t}\right)_2 = \frac{Q}{t}$

From (7.24)

$$\theta_1 - \theta = \frac{QL_1}{tk_1A} \qquad (7.26)$$

From (7.25)

$$\theta - \theta_2 = \frac{QL_2}{tk_2A} \qquad (7.27)$$

Adding equations (7.26) and (7.27), and simplifying;

$$\frac{Q}{t} = \frac{A(\theta_1 - \theta_2)}{\frac{L_1}{k_1} + \frac{L_2}{k_2}} \qquad (7.28)$$

In general, for any number of conductors, the total rate of heat flow is given as,

$$\frac{Q}{t} = \frac{A(\theta_1 - \theta_2)}{\sum_i^n \frac{L_i}{k_i}} \qquad (7.29)$$

where $i = 1, 2, 3, \ldots, n$

Suppose $L_1 = L_2 = L$, equation (7.28) becomes

$$\frac{Q}{t} = \frac{Ak_1k_2(\theta_1 - \theta_2)}{L(k_1 + k_2)} \qquad (7.30)$$

Comparing and rearranging equations (7.23) and (7.30), effective conductivity k becomes

$$k = \frac{k_1k_2}{k_1 + k_2} \qquad (7.31)$$

In general, for any number of conductors in series,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots + \frac{1}{k_n} \text{ or } \frac{1}{k} = \sum_{i=1}^{n} \frac{1}{k_i}$$

When dissimilar conductors are joined side to side (in parallel), the left ends of both conductors are kept at the same temperature θ_1 and the right ends of the conductors are kept at the same temperature θ_2. The temperature difference is maintained between the end of each conductor and there is no temperature difference at the junctions.

The rate of flow of heat $\left(\frac{Q}{t}\right)$ through each conductor is different but the rate of flow of heat through the composite conductor is the sum of the rate of heat flow through each conductor.

Consider two dissimilar conductors joined in parallel as shown in Figure 7.13.

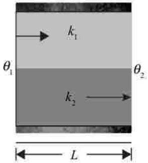


Figure 7.13 Lagged conductors in parallel arrangement

If $\frac{Q}{t}$ is the rate of flow of heat through the composite conductor, then,

$$\frac{Q}{t} = \left(\frac{Q}{t}\right)_1 + \left(\frac{Q}{t}\right)_2$$

$$\frac{Q}{t} = k_1A_1\left(\frac{\theta_1 - \theta_2}{L_1}\right) + k_2A_2\left(\frac{\theta_1 - \theta_2}{L_2}\right)$$

$$\frac{Q}{t} = \left(\frac{k_1A_1}{L_1} + \frac{k_2A_2}{L_2}\right)(\theta_1 - \theta_2) \qquad (7.32)$$

Suppose, $A_1 = A_2 = A$ and $L_1 = L_2 = L$, then,

$$\frac{Q}{t} = A(k_1 + k_2)\left(\frac{\theta_1 - \theta_2}{L}\right) \qquad (7.33)$$

Comparing equations (7.23) and (7.33), where k is the effective thermal conductivity of conductors in parallel, for any number of parallel conductors,

$$k = k_1 + k_2 + k_3 + \ldots + k_n$$

$$k = \sum_{i=1}^{n} k_i \qquad (7.34)$$

Example 7.5

Two perfectly lagged bars x and y are arranged in series and parallel. When the bars are in series the hot end of x is maintained at 90°C and the cold end of y is maintained at 30°C. When the bars are in parallel the hot end of each is maintained at 90°C and the cold end of each is maintained at 30°C. Calculate the ratio of the total rate of flow of heat in parallel arrangement to that in series arrangement. The length of each bar is L and cross section area A. (Thermal conductivity of x is $400\,\text{Wm}^{-1}\text{K}^{-1}$ and that of y is $200\,\text{Wm}^{-1}\text{K}^{-1}$).

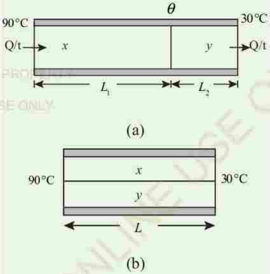


Figure 7.14 Bars in series and parallel arrangement

Solution

(a) Heat flow through bar x,

$$\left(\frac{Q}{t}\right)_x = A \times k_x \times \frac{(90°\text{C} - \theta)}{L_1}$$

$$\left(\frac{Q}{t}\right)_x = 400\,\text{Wm}^{-1}\text{K}^{-1} \times A \times \frac{(90°\text{C} - \theta)}{L_1} \qquad (i)$$

For bar y,

$$\left(\frac{Q}{t}\right)_y = 200\,\text{Wm}^{-1}\text{K}^{-1} \times A \times \frac{(\theta - 30°\text{C})}{L_2} \quad (ii)$$

Since the bars are lagged, heat flow is constant. Thus, $\left(\frac{dQ}{dt}\right)_x = \left(\frac{dQ}{dt}\right)_y$

Equating equations (i) and (ii), and solving $\theta = 70°\text{C}$.

Therefore, the rate of heat flow in series is given by:

$$\left(\frac{Q}{t}\right)_s = 400\,\text{Wm}^{-1}\text{K}^{-1} \times \frac{A}{L} \times (90°\text{C} - 70°\text{C})$$

$$\left(\frac{Q}{t}\right)_s = 8000\,\text{Wm}^{-1} \times \frac{A}{L} \quad (iii)$$

For parallel: $\left(\frac{Q}{t}\right)_p = \left(\frac{Q}{t}\right)_x + \left(\frac{Q}{t}\right)_y$

$$\left(\frac{Q}{t}\right)_p = 400\,\text{Wm}^{-1}\text{K}^{-1} \times \frac{A}{L} \times (90°\text{C} - 30°\text{C}) + 200\,\text{Wm}^{-1}\text{K}^{-1} \times \frac{A}{L} \times (90°\text{C} - 30°\text{C})$$

$$\left(\frac{Q}{t}\right)_p = 24000\,\text{Wm}^{-1} \times \frac{A}{L} + 12000\,\text{Wm}^{-1} \times \frac{A}{L}$$

$$\left(\frac{Q}{t}\right)_p = 36000\,\text{Wm}^{-1} \times \frac{A}{L} \quad (iv)$$

The ratio of rate of heat flow in the lagged parallel to that arranged in series is 9:2.

Example 7.6

Two slabs of lengths L_1 and L_2 and thermal conductivities k_1 and k_2 respectively, are in thermal contact with each other. The temperature of their outer surfaces are θ_1 and θ_2, and $\theta_2 > \theta_1$. Determine:

(a) The temperature θ at the interface; and

(b) The rate of energy transfer by conduction through an area A of the slabs in the steady state condition.

Solution

(a) Suppose the interface temperature is θ, for which $\theta_1 < \theta < \theta_2$. The rate at which energy is transferred through area A of slab 1 is:

$$\frac{dQ}{dt} = k_1 A\left(\frac{\theta - \theta_1}{L_1}\right) \quad (i)$$

The rate at which energy is transferred through the same area A of slab 2 is:

$$\frac{dQ}{dt} = k_2 A\left(\frac{\theta_2 - \theta}{L_2}\right) \quad (ii)$$

Since the two slabs are in a steady state condition, their rates of energy transfer are the same. i.e.

$$k_1 A\left(\frac{\theta - \theta_1}{L_1}\right) = k_2 A\left(\frac{\theta_2 - \theta}{L_2}\right)$$

Therefore,

$$\theta = \left(\frac{k_1 L_2 \theta_1 + k_2 L_1 \theta_2}{k_1 L_2 + k_2 L_1}\right) \quad (iii)$$

(b) Substituting (iii) into either (i) or (ii);

$$\frac{dQ}{dt} = k_1 k_2 A\left(\frac{\theta_2 - \theta_1}{L_1 k_2 + L_2 k_1}\right)$$

Example 7.7

Assuming k is a constant, show that the radial rate of flow of heat in a material between two concentric spheres (Figure 7.15) is given by

$H = \dfrac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1}$ where r_1 and r_2 are the radii of the inner and outer spheres respectively, and T_1 and T_2 are their corresponding temperatures and $T_1 > T_2$.

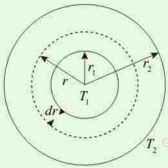


Figure 7.15 Concentric spheres

Solution

The rate of heat flow $H = -kA\dfrac{dT}{dr}$, but $A = 4\pi r^2$.

$$H\frac{dr}{r^2} = -4k\pi dT \qquad (i)$$

$$\int_{r_1}^{r_2} \frac{1}{r^2}dr = \frac{-4\pi k}{H}\int_{T_1}^{T_2} dT$$

$$-\frac{1}{r}\bigg|_{r_1}^{r_2} = \frac{-4\pi k}{H}\left|T\right|_{T_1}^{T_2} \qquad (ii)$$

Simplifying (ii),

$$H = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{(r_2 - r_1)}$$

Thermal resistance (*R*-Value of insulation)

Thermal resistance of a material is the opposition of the material to the flow of heat through it. The thermal resistance R of a slab of a material with area A is defined such that the heat current through the slab is:

$$\frac{dQ}{dt} = \frac{A(\theta_1 - \theta_2)}{R} \qquad (7.35)$$

Thermal resistance R is closely related to the thermal transmittance (U-value) of material as:

$$R = \frac{1}{U}$$

Comparing equation (7.35) with (7.21),

$$R = \frac{L}{k}$$

Since R is measured in m^2KW^{-1}, $U = \dfrac{k}{L}$ is then measured in $Wm^{-2}K^{-1}$.

Therefore, U value is the rate of transfer of heat through a structure (single or composite) per unit temperature difference per unit cross sectional area. The lower the thermal conductivity of the material of which a slab is made, the higher the R-value of the slab.

Example 7.8

A room has a $4\,m \times 4\,m \times 10\,cm$ concrete roof ($k = 1.26\,Wm^{-1}°C^{-1}$). At some instant, the temperature outside is $46°C$ and inside is $32°C$.

(a) Neglecting convection, calculate the amount of heat flowing per second into the room through the roof.

(b) If bricks $(k = 0.65\,\text{Wm}^{-1}\,°\text{C}^{-1})$ of thickness 7.5 cm are laid down on the roof, calculate the new rate of heat flow under the same temperature conditions.

Solution

(a) Thermal resistance of the roof is given by;

$$R_1 = \frac{L_1}{k_1}$$

$$R_1 = \frac{0.10\,\text{m}}{1.26\,\text{Wm}^{-1}\,°\text{C}^{-1}}$$

$$= 7.94 \times 10^{-2}\,\text{m}^2\text{KW}^{-1}$$

Rate of heat flow through the roof,

$$\frac{Q}{t} = \frac{A(\theta_1 - \theta_2)}{R_1}$$

$$\frac{Q}{t} = \frac{16\,\text{m}^2(46\,°\text{C} - 32\,°\text{C})}{7.94 \times 10^{-2}\,\text{m}^2\text{KW}^{-1}} = 2821.2\,\text{W}$$

(b) Thermal resistance of the bricks is given by;

$$R_2 = \frac{L_2}{k_2}$$

$$R_2 = \frac{0.075\,\text{m}}{0.65\,\text{Wm}^{-1}\,°\text{C}^{-1}}$$

$$= 1.15 \times 10^{-1}\,\text{m}^2\text{KW}^{-1}$$

The equivalent thermal resistance of the roof now is

$$R = R_1 + R_2$$

$$R = (79.4 + 115) \times 10^{-3}\,\text{m}^2\text{KW}^{-1}$$

$$= 194.4 \times 10^{-3}\,\text{m}^2\text{KW}^{-1}$$

Therefore, the rate of flow of heat through the roof is;

$$\frac{Q}{t} = \frac{A(\theta_1 - \theta_2)}{R},$$

$$\frac{Q}{t} = \frac{16\,\text{m}^2 \times (46 - 32)\,°\text{C}}{194.4 \times 10^{-3}\,\text{m}^2\text{KW}^{-1}} = 1152\,\text{W}$$

The thermal resistance R acts to impede the flow of thermal energy through the material. The larger the value of R, the smaller the quantity of thermal energy conducted through the roof. For the compound roof wall, the total thermal resistance to thermal energy flow is

$$R = R_1 + R_2 + R_3 + ... + R_n$$

Determination of thermal conductivity by Searle's apparatus

The Searle's apparatus is used for determining thermal conductivity of good conductors of heat (Figure 7.16). The holes at X and Y contain oil to ensure good thermal contact between the thermometers and the bar.

The heater is switched on and water is passed through the copper coil at a constant rate. If the bar is assumed to be perfectly lagged, then it is at steady state (i.e. all four thermometers give steady readings). The rate of flow of heat between X and Y is given by

$$\frac{dQ}{dt} = kA\frac{(\theta_1 - \theta_2)}{x} \qquad (7.36)$$

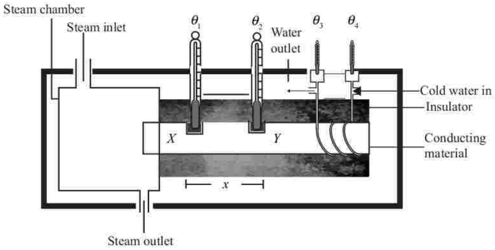


Figure 7.16 Searle's apparatus

Since the bar is assumed to be perfectly lagged and none of the heat is being used to increase temperature (steady state), all the heat which flows along the bar is being used to increase the temperature of the water. If m is the mass of water flowing per unit time and c is the specific heat capacity of water, then the heat required to raise the temperature of the water is given as,

$$\frac{dQ}{dt} = mc(\theta_3 - \theta_4) \quad (7.37)$$

Equating equation (7.36) and (7.37),

$$kA\frac{(\theta_1 - \theta_2)}{x} = mc(\theta_3 - \theta_4) \quad (7.38)$$

The value of k can be determined from equation (7.38).

When the latent heat of water is given, k also can be obtained i.e.,

$$kA\frac{(\theta_1 - \theta_2)}{x} = mL,$$

where L is the latent heat of vaporization of water.

Determination of thermal conductivity by Lee's disc

Lee's disc is used for determining thermal conductivity of poor conductors of heat (Figure 7.17).

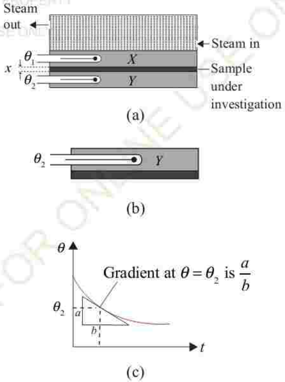


Figure 7.17 (a) Lee's disc, (b) cooling disk, and (c) cooling curves

The sample (e.g. cardboard) is in the form of a thin disc and is sandwiched between the thin base *X* of a steam chest and a thin brass slab *Y*. Steam is passed through the chest and apparatus is left to reach steady state. The sample is thin and therefore to good approximation, no heat is lost from its sides. It follows that, at steady state

$$\frac{dQ}{dt} = kA\frac{(\theta_1 - \theta_2)}{x}$$

where x is the thickness of the poor conductor, A is cross sectional area and k is thermal conductivity of the sample.

Plotting the graph of temperature θ against time t, the rate of temperature change along the disc Y is equal to the gradient of the graph (Figure 7.17c).

The conditions under which Y is losing heat are the same as those at steady state, and therefore,

$$kA\frac{(\theta_1 - \theta_2)}{x} = mc\frac{d\theta}{dt} = mc\frac{a}{b} \qquad (7.39)$$

Thus k can be determined from equation (7.39).

Example 7.9

One end of a copper rod 2 m long and having 1 cm radius is maintained at 250 °C. When a steady state is reached, the rate of heat flow across any cross-section is $2.1\,\text{Js}^{-1}$. What is the temperature of the other end? (Thermal conductivity of copper = $380\,\text{Js}^{-1}\text{m}^{-1}\,°\text{C}^{-1}$).

Solution

$A = \pi r^2$,

$A = 3.14 \times (0.01\,\text{m})^2 = 3.14 \times 10^{-4}\,\text{m}^2$

$$\frac{Q}{t} = kA\frac{(\theta_1 - \theta_2)}{x}, \quad \theta_1 - \theta_2 = \left(\frac{Q}{t}\right) \times \frac{x}{kA}$$

$$\theta_1 - \theta_2 = \frac{2.1\,\text{Js}^{-1} \times 2\text{m}}{380\,\text{Js}^{-1}\text{m}^{-1}\,°\text{C}^{-1} \times 3.14 \times 10^{-4}\text{m}^2}$$

$$= 35.20\,°\text{C}$$

Since $\theta_1 = 250\,°\text{C}$, then it follows that,

$250\,°\text{C} - \theta_2 = 35.20\,°\text{C}, \quad \theta_2 = 214.8\,°\text{C}$

Therefore, the temperature of the other end is $214.8\,°\text{C}$.

Example 7.10

A brass boiler has a base area of $0.15\,\text{m}^2$ and thickness of 1 cm. It boils water at the rate of 6 kilogram per minute when placed on a gas stove. What is the temperature of the part of the flame in contact with the boiler? ($k_{brass} = 109\,\text{Js}^{-1}\text{m}^{-1}\,°\text{C}^{-1}$, heat of vapourization of water is $2256 \times 10^3\,\text{Jkg}^{-1}$).

Solution

The heat gained by the boiler is utilized in vapourizing water,

$$Q = mL \qquad (i)$$

also,

$$\frac{Q}{t} = kA\frac{(\theta_1 - \theta_2)}{x} \qquad (ii)$$

Equating equations (i) and (ii) and rearranging the terms,

$$\theta_1 - \theta_2 = \frac{mLx}{tkA}$$

$\theta_2 = 100\,°\text{C}$ (temperature of steam)

$$\theta_1 = \frac{6\,kg \times 2256 \times 10^3 Jkg^{-1} \times 1 \times 10^{-2} m}{60\,s \times 109\,Js^{-1}m^{-1}°C^{-1} \times 0.15\,m^2} + 100\,°C$$

$$= 237.98\,°C$$

Therefore, the temperature of the part of the flame in contact with the boiler is 237.98°C.

(c) Applications of thermal conduction

Heat transfer is involved in numerous domestic and industrial technologies. The technology is used to solve many problems in thermal mechanics. Different materials have different coefficients of thermal conductivity. This fact has many practical applications in domestic and industrial activities.

Thermal conductivity of air is very low. Clothes effectively shield your body against the loss of heat. Trapped air under clothing acts as excellent insulator and so helps to check the transfer of heat. A few layers of cloth containing air spaces would be a better insulator than one with heavy layer.

All metals particularly copper and silver have large thermal conductivity. Cooking utensils are made of metals as they conduct heat easily. As copper has large thermal conductivity than steel, some steel utensils have their bottom made of copper. Filaments of electric appliances are also made of good conductors of heat.

Insulators have smaller thermal conductivity, for example; air, wood, leather, felt or cotton wool, feather, paper cardboard, asbestos, cork and plastic. In fact, an insulator blocks the transfer of heat. They can be used for insulating refrigerators and houses. The insulator in a refrigerator keeps the heat out of the refrigerator. Different materials (bricks, glass, mud) are used for insulation in the walls and roof of a house. This insulation keeps rooms warm during winter, and cool during summer. Cooking utensils have their handles made of insulators (bakelite, wood). Feathers keep birds warm by not allowing heat energy to flow out of their bodies.

Exercise 7.2.1

1. Calculate the rate of loss of heat through a window of thickness 8 mm and area of 2 m^2 if the temperature difference between the two sides is 20°C. Take thermal conductivity of glass to be 1 $Wm^{-1}K^{-1}$.
2. A 10 cm long brass bar is joined to a copper bar of equal length and diameter so as to form a compound bar with a cross-sectional area of 6.0 cm^2. The junction has negligible thermal resistance and the bar is well lagged. The free end of the brass bar is maintained at 100°C, and the far end of the compound bar is kept at 20°C. Calculate the heat flow per second along the bar and also the temperature at the junction. (Assume k for copper = 400 $Wm^{-1}K^{-1}$ and brass = 10^9 $Wm^{-1}K^{-1}$).
3. Heat is flowing along a uniform, lagged metal bar and the temperature is 80°C at 8 cm from the hot end and 50°C at 20 cm from this end. At what distance from the hot end is the temperature 60°C?

4. One end of a well lagged copper rod is placed in a steam chest and a 0.6 kg mass of copper is attached to the other end of the rod with an area of $2\,\text{cm}^2$. When steam at 100°C is passed into the chest and a steady state is reached, the temperature of the mass of copper rises by 4°C per minute. If the temperature of the surrounding is 15°C, calculate the length of the rod. (Specific heat capacity of copper $= 400\ \text{Jkg}^{-1}\text{K}^{-1}$, thermal conductivity of copper $= 360\ \text{Wm}^{-1}\text{K}^{-1}$).

5. Ice is forming on the surface of a pond. When it is 4.6 cm thick, the temperature of the surface of the ice in contact with the air is 260 K while the surface in contact with the water is at temperature 273 K.
   (a) Calculate the rate of loss of heat per unit area from the water.
   (b) Determine the rate at which the thickness of the ice is increasing. (Thermal conductivity of ice is $2.3\ \text{Wm}^{-1}\text{K}^{-1}$, density of water is $1000\ \text{kgm}^{-3}$, specific latent heat of fusion of ice is $3.25\times10^{5}\ \text{Jkg}^{-1}$).

6. Briefly explain why
   (a) animals in the forest find shelter from cold in holes in the snow.
   (b) warm air rises up but the atmosphere is cooler with increasing altitude.

7. (a) Show that the radial heat flow across the coaxial cylinder is given by;

$$H = 2\pi kL\frac{(T_2 - T_1)}{\ln\left(\frac{r_2}{r_1}\right)},$$

   where k is thermal conductivity, L is length of the cylinder, r_1 and r_2 are radii of inner and outer parts of the cylinder respectively.
   (b) If a copper hot-water cylinder of length 1.0 m and radius 0.20 m of material has thermal conductivity $0.40\ \text{Wm}^{-1}\text{K}^{-1}$, estimate temperature of the outer surface of the lagging, assuming heat loss is through the sides only, if heat has to be supplied at a rate of 0.25 kW to maintain the water at a steady temperature of 60°C.

7.2.2 Thermal convection

In the process of thermal conduction, atoms transfer their energy by colliding with their neighbours. In thermal convection, heat is transferred by actual bulk motion of the medium. The movement of the materials in thermal convection is due to the difference in densities of the hotter and colder parts. You can see currents of water moving about in a flask by placing potassium permanganate crystals at the bottom of the flask containing liquid (Figure 7.18).

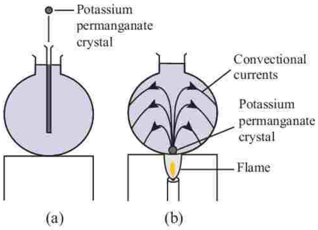


Figure 7.18 Convectional currents of water

As the flask is heated, the liquid expands and its density decreases. The heated molecules thus, move up to the surface of the liquid while the cooler ones move to the bottom to take the place left by warmer molecules. The established circulating current is called convectional current while the process of movement of the molecules is called thermal convection.

If the fluid is circulated by a blower or pump the process is called forced convection, while, if the flow causes differences in density due to thermal expansion it is called natural convection e.g. land breeze and sea breeze.

(a) Factors affecting thermal convection

Thermal convection is affected by a number of factors including excess temperature, medium in which convection takes place, surface area of the body and the volume of cooling body.

Excess temperature occurs when the difference in a body temperature and the temperature of the surrounding create the necessary gradient for convective currents to flow.

Medium in which thermal convection takes place is another factor which affects the rate of thermal convection. For example, the time required to raise temperature of grease by 10°C is not the same as time required to raise the same amount of distilled water through 10°C. The medium determines how fast thermal convection will take place.

Exposed surface area of a body also affects thermal convection. The larger the exposed area of a cooling body the higher the rate of cooling. Similarly, the volume of cooling body affects thermal convection.

(b) Laws of thermal convection

There are two laws; Dulong-Petit (five-fourth power) law and Newton's law of cooling.

(i) Dulong Petit or five-fourth power law

"Under the condition of natural convection, the rate of heat lost by a body is directly proportional to the five-fourth power of the excess temperature over the surrounding provided that the excess temperature is not less than 50°C*."*

$$\frac{dQ}{dt} \propto \left(\theta - \theta_s\right)^{\frac{5}{4}} \qquad (7.40)$$

where θ and θ_s are the temperatures of the body and surrounding respectively.

(ii) Newton's Law of Cooling

"Under the condition of forced convection, the rate of heat lost by a body is directly proportional to the excess temperature over the surrounding".

$$\frac{dQ}{dt} \propto (\theta - \theta_s)$$

$$\frac{dQ}{dt} = k(\theta - \theta_s) \quad (7.41)$$

The Newton's law of cooling is applicable under the following conditions:

(i) Forced convection (for all excess temperatures)

(ii) The excess temperature less than 30°C under natural convection

When the body loses heat Q, its temperature θ falls; if m is its mass, and c is its specific heat capacity, the rate of heat loss is given by $\frac{dQ}{dt} = -mc\frac{d\theta}{dt}$; then its rate of fall of temperature is given by

$$\frac{d\theta}{dt} = -\frac{k}{mc}(\theta - \theta_s) \quad (7.42)$$

For a given object; k, m and c are constants: hence,

$$\frac{k}{mc} = \lambda \quad (7.43)$$

where λ is a constant which represents the nature of the surface and the heat capacity contents.

Substitute equation (7.43) into (7.42) to obtain

$$\frac{d\theta}{dt} = -\lambda(\theta - \theta_s) \quad (7.44)$$

Equation (7.44) is an alternative statement of the Newton's law of cooling, which can now be stated as, *"The rate of fall of temperature (cooling) of an object is proportional to the excess temperature over the surrounding".*

Verification of Newton's law of cooling can be done by plotting the cooling curve of temperature θ versus time t obtained from cooling hot water (Figure 7.19).

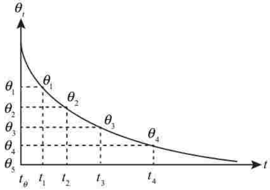


Figure 7.19 Cooling curve

If θ_s is the surrounding (room) temperature, then excess temperature of water is $(\theta - \theta_s)$. Several points (about six) are chosen on the cooling curve and tangents are drawn at these points. The gradient of tangent represents the rate of cooling of the liquid at a particular temperature θ. Then plotting these rates (gradient of tangent) against the excess temperature $(\theta - \theta_s)$ gives a straight line through the origin (Figure 7. 20).

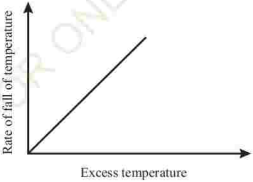


Figure 7.20 Rate of cooling versus excess temperature

A straight line passing through the origin of the graph of rate of temperature fall versus the excess temperature verifies that the given liquid obeys Newton's law of cooling. The instantaneous value of temperature of a cooling object is obtained by integrating equation (7.44), and becomes;

$$\ln(\theta - \theta_s) = -\lambda t + c \qquad (7.45)$$

At $t = 0$, and $\theta = \theta_i$,

$$\ln(\theta_i - \theta_s) = c \qquad (7.46)$$

Substituting equation (7.46) in (7.45),

$$\ln(\theta - \theta_s) = -\lambda t + \ln(\theta_i - \theta_s)$$

Then,

$$\ln\left(\frac{\theta - \theta_s}{\theta_i - \theta_s}\right) = -\lambda t, \; \theta = \theta_s + (\theta_i - \theta_s)e^{-\lambda t} \qquad (7.47)$$

From equation (7.47), it is clearly seen that, when time $t = 0$, $\theta = \theta_i$ and when time $t \to \infty$, $\theta = \theta_s$. Therefore, when an object is cooling, its temperature will never fall below the surrounding temperature.

(c) Applications of thermal convection

In cold countries a fireplace is set at the corner of a room. The air near the fire gets heated, becomes less dense and rises up spreading into the room. Thus the room is kept warm. The colder air moves towards the fire and gets heated in turn.

The mechanism of heating a liquid by a heater is entirely based on convection. The liquid molecules in immediate contact with the heater are heated up and acquire sufficient energy to rise upward. The cool liquid molecules at the top being denser move down to take their place. This cool liquid is in turn heated and moves upward. In this way, convection currents are set up in the liquid which transfer heat to different parts of the container.

Due to currents flowing in the winding of the transformer, enormous heat is produced. Thus a transformer is always kept in a tank containing oil. The warm oil comes in contact with the cooler tank, gives heat to it and descends to the bottom and the process is repeated.

A ventilator or exhaust fan in a room helps to remove warm air due to respiration from a room. The fresh air from outside blows into the room, this is all due to convection current set up in the room.

Kitchen rooms are provided with a chimney through which hot air in the room goes out and fresh air containing oxygen enters through the windows and doors into the kitchen to support the burning of the fuel.

Example 7.11

A body cools in 7 minutes from 60°C to 40°C. What will be its temperature after the next 7 minutes? The temperature of the surrounding is 10°C. Assume Newton's law of cooling holds throughout the process.

Solution

Using the relation, $\ln\left(\frac{\theta - \theta_s}{\theta_i - \theta_s}\right) = -\lambda t$,

when a body cools for the first 7 minutes,

$$\ln\left(\frac{40°C - 10}{60°C - 10°C}°C\right) = -7\,\text{min} \times \lambda$$

$$\ln\left(\frac{3}{5}\right) = -7\,\text{min} \times \lambda \qquad (i)$$

When a body cools for the next 7 minutes,

$$\ln\left(\frac{\theta - 10°C}{40°C - 10°C}\right) = -7\,\text{min} \times \lambda,$$

$$\ln\left(\frac{\theta - 10°C}{30°C}\right) = -7\,\text{min} \times \lambda \qquad (ii)$$

Solving equation (i) and (ii), $\theta = 28°C$

Therefore, the temperature of the body after the next 7 minutes is 28°C.

Example 7.12

A body at 80°C cools to 64°C in 5 minutes and to 52°C in the next 5 minutes. What will be its temperature after another 5 minutes?

Solution

Using; $\ln\left(\frac{\theta - \theta_s}{\theta_i - \theta_s}\right) = -\lambda t$

When a body cools for the first 5 minutes,

$$\ln\left(\frac{64°C - \theta_s}{80°C - \theta_s}\right) = -5\,\text{min} \times \lambda \qquad (i)$$

When a body cools for the next 5 minutes,

$$\ln\left(\frac{52°C - \theta_s}{64°C - \theta_s}\right) = -5\,\text{min} \times \lambda \qquad (ii)$$

Solving from equations (i) and (ii),

$\theta_s = 16°C$

When a body cools for the other 5 minutes,

$$\ln\left(\frac{\theta - 16°C}{52°C - 16°C}\right) = -5\,\text{min} \times \lambda \qquad (iii)$$

From equation (i) and (iii), $\theta = 43°C$.

Therefore, the temperature of the body after another 5 minutes is 43°C.

Example 7.13

A body cools from 80°C to 50°C in 5 minutes. Calculate the time it takes to cool from 60°C to 30°C. The temperature of the surrounding is 20°C.

Solution

Using $\ln\left(\frac{\theta - \theta_s}{\theta_i - \theta_s}\right) = -\lambda t$

When a body cools for the first 5 minutes,

$$\ln\left(\frac{50°C - 20°C}{80°C - 20°C}\right) = -5\,\text{min} \times \lambda$$

$$\ln\left(\frac{3}{6}\right) = -5\,\text{min} \times \lambda,\ \lambda = 0.1386\,\text{min}^{-1}$$

When a body cools for the next time t,

$$\ln\left(\frac{30°C - 20°C}{60°C - 20°C}\right) = -0.1386\,\text{min}^{-1} \times t,$$

$t = 10$ min

Therefore, time taken by the body to cool from 60°C to 30°C is 10 minutes.

Exercise 7.2.2

1. Using Newton's law of cooling,
   (a) show that the temperature of a cooling body at any time *t* obeys the mathematical expression, $\theta = \theta_s\left(1 - e^{-\lambda t}\right) + \theta_i e^{-\lambda t}$ where, θ_s = surrounding temperature, θ_i = initial temperature and λ = constant.
   (b) Using the expression in (a), obtain an expression for the time taken for the body's temperature to become half its value.
2. A liquid takes 5 minutes to cool from 80 °C to 50 °C. How much time will the liquid take to cool from 60 °C to 30 °C? The surrounding temperature is 20 °C.
3. A body initially at 80 °C cools to 64 °C in 5 minutes and to 52 °C in 10 minutes. Determine:
   (a) The surrounding temperature; and
   (b) The temperature after 15 minutes.
4. A body in a room of constant temperature of 18 °C cools from 70 °C to 57 °C in 5 minutes. Assuming Newton's law of cooling to hold all the time, find:
   (a) The temperature of the body after a further time of 5 minutes.
   (b) The time required for the temperature to fall from 57 °C to 34 °C.
5. Wind blows over a hot liquid placed in a beaker in the laboratory whose average room temperature is 27 °C. The liquid is cooling at the rate of 15 °C min^{-1} when it is at a temperature of 87 °C. Calculate the cooling rate when it is at a temperature of 57 °C.
6. Global warming is causing temperatures to rise above expected levels. Traditional building practices and materials are failing to cope up with these changes; as a result, houses are very warm during the hot seasons and moderately cold during cold season. Consider local building practices and building materials and propose modifications in design and construction of houses to cope with climate change.
7. If you have two spoons of the same size, one silver and one stainless steel, there is a quick test to tell which is which. Hold the end of a spoon in each hand, then lower them both into a cup of very hot water. One spoon will feel hot first. Is that the silver spoon or the stainless steel spoon? Explain.

7.2.3 Thermal radiation

All objects above 0 K emit thermal radiation from their surfaces. A portion of this radiant energy may be seen if the surface is at high enough temperature. Also at much lower temperatures, a surface still emits energy although it is little to be detected. Thermal radiation is the radiant energy emitted by a body solely on account of its temperature. This section deals with the process of heat transfer, spectra of thermal radiation emitted by blackbody, laws of blackbody radiation, applications of blackbody radiation in daily life and Prevost's theory of heat exchange.

(a) Heat transfer by radiation

The sun gives us warmth by means of radiant energy which reaches on the earth after passing through the empty space and atmosphere. Heat transfer is not possible by conduction or convection through empty space and atmosphere. Therefore, the way heat is transferred in the empty space is called radiation. It is a process in which heat transfer takes place by electromagnetic waves. These waves travel with the speed of light, $3\times10^{8}\,\mathrm{ms}^{-1}$ and requires no medium for passage. For example, one can feel radiation coming from a warm stove or any burner (Figure 7.21). The heat energy received from a glowing electric lamp or fire is also due to radiation.

The process of thermal radiations can be understood only if one is familiar with the production of electromagnetic waves. According to electromagnetic theory, whenever charged particles are accelerated or decelerated, they create a disturbance in space and carry energy. This disturbance which carries energy is known as an electromagnetic wave. The transfer of energy in these waves is due to the oscillating electric and magnetic fields which change with time.

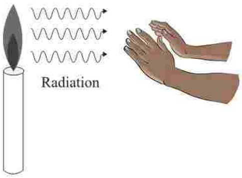


Figure 7.21 Thermal radiation

Electromagnetic spectrum classifies radiation according to wavelengths of the radiation. Main types of radiation are (from short to long wavelengths): gamma rays, X-rays, ultraviolet (UV), visible light, infrared (IR), microwaves, and radio waves. Radiation with shorter wavelengths is more energetic and contains more heat. X-rays, having wavelengths $\sim10^{-9}$m, are very energetic and can be harmful to humans, while visible light with wavelengths $\sim10^{-7}$m contain less energy and therefore have little effect on life. The visible part of the radiation spectrum ranges from violet to red radiation.

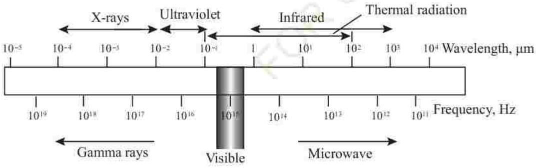


Figure 7.22 Electromagnetic waves spectrum

(b) Blackbody radiation

Every object emits radiation of all wavelengths, though the intensity of different wavelengths may vary considerably. We are familiar with the glowing of an iron block, e.g. a heated rod. When electric current is made to pass through it, it first becomes dull red, then reddish yellow and finally white hot. It means iron emits light as it becomes hot when enough radiation is emitted as visible light for your eyes to respond and at the same time giving off other radiation of different wavelengths. It is important to note that radiation emitted from a body have the following features associated with it:

(i) Dark surfaces are the best emitters of radiant energy.

(ii) The rate of emissions of radiation of a body increases rapidly as its surface temperature increases. In fact, it is proportional to T^4, where T is the surface absolute temperature.

(iii) The predominant wavelength in radiation emission becomes shorter as the temperature of the body increases. A body that glows red is not as hot as the one which glows bluishwhite. Also, this is the reason why the colour of a body changes from light red to dull red, yellow and finally white as it becomes hotter.

The intensity of electromagnetic radiation emitted by a body varies with wavelength at different temperatures (Figure 7.23).

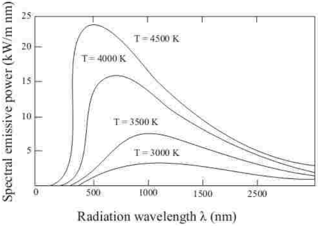


Figure 7.23 Variations of intensity of electromagnetic radiations with wavelength at different temperature

The total emitted radiation is proportional to the area under each curve, and increases with increasing temperature while the corresponding peak wavelength decreases. In fact, all bodies emit heat radiation irrespective of their temperatures. A body at a higher temperature loses these radiation while a body at a lower temperature gains this radiation. But bodies at equal temperatures gain or lose radiation equally.

This is explained by Stefan-Boltzmann law which states that, *"The amount of electromagnetic radiation emitted per unit time from a unit area of a body at absolute temperature in kelvin is directly proportional to the fourth power of absolute temperature of the emitting surface"*. i.e. for the body which is a perfect radiator, $E = \sigma T^4$, where E is the energy radiated per unit area per unit time.

That is;

$$P = AE \text{ and } P = \sigma A T^4$$

where P is the energy radiated by the body per unit time and σ is the Stefan's constant whose value is $5.67 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$.

Suppose that a body's surface at absolute temperature T is at higher temperature than the surroundings at the temperature T_s. The amount of radiation emitted by the body per unit time is

$$P_s = \sigma A T_s^4$$

Therefore, the net loss of energy by the body per unit time is given as;

$$P_{net} = P - P_s \text{ or } P_{net} = \sigma A T^4 - \sigma A T_s^4$$

$$P_{net} = \sigma A\left(T^4 - T_s^4\right) \qquad (7.48)$$

For a body with surface emissivity ε, equation (7.48) can be written as

$$P_{net} = \varepsilon\sigma A\left(T^4 - T_s^4\right) \qquad (7.49)$$

Surface emissivity ε has the value between 0 and 1, and depends on the composition/ nature of surface of the body. A perfect blackbody has high emissivity which is 1 and radiates the maximum. On the other hand, (reflecting) shiny surfaces have very low emissivity which is close to 0 and radiates poorly. An object that radiates energy well also absorbs well and an object that radiates poorly also absorbs poorly. Thus, if the object is hotter than the surroundings, it will lose thermal radiation, and if the body is at a lower temperature than the surroundings, it will gain thermal radiation from the surroundings.

Another law that governs the blackbody radiation is the Wien's displacement law. The wavelength λ_{max} at which the maximum amount of energy is radiated decreases with increase in temperature such that

$$\lambda_{max} T = constant \qquad (7.50)$$

where T is the absolute surface temperature of the blackbody in kelvin. Equation (7.50) is known as Wien's displacement law. The value of the constant is experimentally found to be 2.9×10^{-3} in SI units. Thus Wien's displacement law may be stated as, *"The product of the wavelength, λ_{max} at which maximum amount of energy is radiated and the absolute temperature (T) of the emitting surface is always constant".*

This law can well illustrate the well-known observation that when iron is heated, it first becomes light-red, then dark-red, then yellow and finally it becomes white. The temperature in equation (7.50) must be in kelvin so that a temperature of absolute zero corresponds to no radiation emission. Note also that every object whose temperature is above 0 K including you, emits thermal radiation but the radiation is in the infrared portion of the spectrum, which your eyes are not capable of detecting.

Example 7.14

The temperature of a furnace is 2324 °C and the intensity in its radiation spectrum is maximum nearly at 1200 Å. Calculate the surface temperature of the star that emits radiation of wavelength of nearly 4800 Å.

Solution

According to Wien's displacement law, $\lambda_m T = \text{constant}$.

$$\lambda_1 T_1 = \lambda_2 T_2$$

$$T_2 = \left(\frac{1200\,\text{Å}}{4800\,\text{Å}}\right) \times 2597\text{ K} = 649.25\text{ K}$$

Therefore, surface temperature of the star that emits radiation of wavelength of nearly 4800 Å is 649.25 K.

Example 7.15

A piece of metal loses 255 J of heat per second by radiation when its temperature is 1200 K and the temperature of the surroundings is 300 K. What will be the rate of loss of heat when the temperature of the metal is 600 K?

Solution

The net loss of energy per second by the metal at T_1, (1200 K),

$$P_1 = \varepsilon\sigma A\left(T_1^4 - T_0^4\right) \quad (i)$$

The net loss of energy per second by the metal at T_2, (600 K),

$$P_2 = \varepsilon\sigma A\left(T_2^4 - T_0^4\right) \quad (ii)$$

Dividing equation (i) by (ii) and rearranging terms gives

$$P_2 = \frac{P_1\left(T_2^4 - T_0^4\right)}{\left(T_1^4 - T_0^4\right)}$$

$$= \frac{255\text{Js}^{-1} \times \left(600^4 - 300^4\right)\text{K}^4}{\left(1200^4 - 300^4\right)\text{K}^4} = 15\text{Js}^{-1}$$

Therefore, the rate of loss of heat when the temperature of the metal is 600 K is 15Js^{-1}.

Example 7.16

What is the total rate of radiation of energy from a human body with surface area 1.20 m^2 and surface temperature 30 °C? If the surroundings are at a temperature of 20 °C, what is the net rate of radiative heat loss from the body? The emissivity of the human body is very close to unity, irrespective of skin pigmentation.

Solution

Taking $\varepsilon = 1$, $\sigma = 5.67 \times 10^{-8}\text{ Wm}^{-2}\text{K}^{-4}$ the body radiates at rate;

$$P = \varepsilon\sigma AT^4$$

$$= 1.2\text{m}^2 \times 5.67 \times 10^{-8}\text{ Wm}^{-2}\text{K}^{-4} \times (303\text{K})^4$$

$$= 574\text{ W}$$

The total rate of radiation of energy from a human body is 574 W.

This loss of heat is partly offset by absorption of radiation which depends on the temperature of the surroundings. The net radiative energy transferred is

$$P_{net} = \varepsilon\sigma A\left(T^4 - T_0^4\right)$$

$$= 1.2\text{m}^2 \times 5.67 \times 10^{-8}\text{ Wm}^{-2}\text{K}^{-4} \times \left(303^4 - 293^4\right)\text{K}^4$$

$$= 72\text{ W}$$

(c) Solar constant

Solar constant ϕ_s, is the energy from the sun arriving perpendicularly at the top surface of the earth's atmosphere per unit area per unit time. In order to determine ϕ_s; it is assumed that no part of the energy from the sun is absorbed by layers between the earth and the sun.

$$\phi_s = \frac{P_s}{A}$$

where P_s = Power radiated by the sun as a blackbody,

A = area on which the radiant energy passes

$$\phi_s = \frac{\sigma A T_s^4}{A} = \frac{4\pi R_s^2 T_s^4}{4\pi D^2} \rightarrow \phi_s = \left(\frac{R_s}{D}\right)^2 T_s^4$$

where D is the mean distance from the earth to the sun, R_s is the radius of the sun and T_s is the surface temperature of the sun.

Note that, from knowledge of the solar constant, the surface temperature of the earth T_e can be obtained when it is in radiative equilibrium. i.e., power received by the earth (P_e) is equal to the power radiated by the earth as a blackbody (P_r). Then,

$$P_e = \frac{A_e}{A} \times P_s$$

where A_e = Area of the earth receiving the radiant energy

$$P_e = \frac{\pi R_e^2}{4\pi D^2} \times 4\pi R_s^2 \sigma T_s^4$$

$$P_e = \frac{\pi R_e^2}{D^2} \times R_s^2 \sigma T_s^4 \quad \text{also,}$$

$$P_r = 4\pi R_e^2 \sigma T_e^4$$

Since P_e and P_r are equal at radiative equilibrium; then,

$$4\pi R_e^2 \sigma T_e^4 = \frac{\pi R_e^2}{D^2} \times R_s^2 \sigma T_s^4$$

$$T_e^4 = \left(\frac{R_s}{2D}\right)^2 T_s^4$$

Therefore,

$$T_e = \left(\frac{R_s}{2D}\right)^{\frac{1}{2}} T_s$$

Substituting the values of R_s, D and T_s the effective temperature of the earth's surface can be calculated.

(d) Prevost's theory of heat exchange

The Prevost's theory of heat exchange tells you that, when the temperature of a body is constant, the body loses heat by radiation and gains it by absorption at equal rates. Hence, there is no net radiation and the body and surroundings are in equilibrium.

It was put forward by Prevost, that;

(i) A body radiates heat at a rate depending on its temperature and nature of the surface.

(ii) A body absorbs heat at a rate which depends on its temperature, surface area and surrounding temperature.

(e) Applications of thermal radiation

The loss of radiant energy can be minimized by making a surface of low emissivity. For example, in a thermos flask, a double walled glass bottle with a silver coating on the inner walls reduces heat transfer by radiation because the coating has a low emissivity.

Thus, the three processes of heat transfer are minimum in a thermos flask. So, a flask keeps hot things hot and cold things cold for a fairly long time.

Radiation and convection are the major mechanisms of heat transfer in the atmosphere, the sun and the solar system. The climatic changes are also affected by these processes of heat transfer. Our planet constantly absorbs radiation coming from the sun. In thermal equilibrium, the rate at which our planet absorbs solar radiation must be equal to the rate at which it emits radiation into space. Similarly a premature baby in an incubator can be cooled dangerously by radiation if the walls of the incubator happened to be cold, even when the air in the incubator is warm.

Exercise 7.2.3

1. (a) Show that for radiation, as for conduction and convection, the heat transfer depends on the temperature difference between two bodies.
   (b) Why do floor tiles feel colder than wooden floor even though both are at the same in temperature?
   (c) Why is a blanket able to protect ice from melting?
2. (a) Why does a good absorber of radiant energy appear black?
   (b) The car's radiator is made of steel and is filled with water. You are asked to fill the radiator to the very top with cold water, then the driver drives off without remembering to replace the radiator cap. As the water and the steel radiator heat up, will the level of water drop or rise and overflow? Explain.
3. Two spheres made of the same material have radii 2.0 cm and 3.0 cm and their temperatures are 627 °C and 527 °C respectively. If they are blackbodies, find the ratio of
   (a) the rate at which they are losing heat,
   (b) the rate at which their temperatures are falling,

   when they are placed in room temperature of 290 K.
4. The tungsten filament of an electric lamp has a length of 0.5 m and a diameter of 6×10^{-5} m. The power rating of the lamp is 60 W. Assuming the radiation from the filament is equivalent to 80% that of a perfect blackbody radiator at the same temperature, estimate the steady temperature of the filament given that Stefan-Boltzmann constant $= 5.67 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$.
5. The total external surface area of a dog's body is 0.8 m^2 and the body temperature is 37 °C. At what rate is it losing heat by radiation when it is in a room whose temperature is 17 °C ? Assume that the dog's body behaves as a blackbody given that Stefan-Boltzmann constant is $5.67 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$.

6. The energy arriving per unit area on the Earth's surface per second from the sun is $1.34\times10^{3}\ \text{Wm}^{-2}$.
   The average distance from the Earth to the Sun is 215 times the length of the sun's radius. Given that, both the Earth and the sun are blackbodies, estimate the temperature of the sun. Stefan Boltzmann constant is $5.67\times10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$.

7.3 First law of thermodynamics

Thermodynamics is the name given to the processes in which energy is transferred as heat and work. In previous chapters you learnt that work is done when energy is transferred from one object to another by mechanical means. Also, in section 7.2 you saw that heat is a transfer of energy from one object to another at a lower temperature. Thus, heat is much like work. To distinguish them, heat is defined as a *transfer of energy due to a difference in temperature*, whereas *work* is *a transfer of energy that is not due to a temperature difference*. In discussing thermodynamics, we often refer to particular systems. A system is any object or set of objects that we wish to consider. Everything else in the universe is referred to as its "environment" or the "surroundings." In this section, you will examine the first law of thermodynamics.

The law states that, *"Energy can be converted from one form to another with the interactions of heat, work and internal energy, but it cannot be created no destroyed under any circumstances".* Mathematically, this is represented as $dQ = dU + dW$, where dQ is the heat exchange between a system and its surroundings, dU is the change in internal energy of the system, and dW is the work done by or on the system.

7.3.1 Thermodynamics processes

Before discussing thermodynamic process, let us define thermodynamic state of a system. A system has certain properties such as temperature, pressure and volume whose instanteneous values define the state of the system. For example, in a thermos flask there are 250 ml of water at 50°C and this is the state of the system. If the values of the properties are changed (e.g. adding 50 ml of water at 25°C), the state of the system also changes. Thermodynamic process is a process in which there are changes in the state of a thermodynamic system. An example of thermodynamic process is the car engine where heat is generated by the chemical reaction of oxygen and vaporized gasoline in the engine cylinder. The heated gas pushes on the pistons within the cylinder, doing mechanical work that is used to propel the car.

In a given sample of a gas, thermodynamic process is shown on a P-V diagram as a line or curve going from the initial state to the final state of the gas as shown in Figure 7.24. Generally, volume V of the gas is taken along x-axis and pressure P along the y-axis.

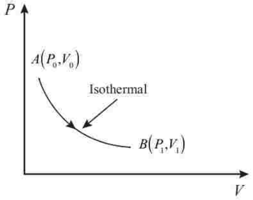


Figure 7.24 P-V diagram of an ideal gas

Point $A(P_0,V_0)$ is the initial state of the gas and point $B(P_1,V_1)$ is the final state of the gas. The curve from point A to point B represents the thermodynamic process by which the state of the gas changes. A (initial state) and B (final state) can be connected by many possible paths or processes. Each one would represent a different thermodynamic process.

7.3.2 Specific heat capacity

In thermodynamics process we are interested on how much a given amount of heat transfer change the temperature of a system. This change depends on the nature of the system. A physical quantity that describes the ability of a body to absorb heat and increase its temperature is called specific heat. Suppose a body of mass m is at temperature T and the temperature of the body changes from T to $T+\Delta T$, due to an amount of heat ΔQ absorbed by the body. The amount of heat ΔQ is given by the relation,

$$\Delta Q = mc\Delta T \qquad (7.51)$$

where c is the specific heat capacity of the material. Equation (7.51) can also be written as,

$$c = \frac{\Delta Q}{m\Delta T} \qquad (7.52)$$

Therefore, the specific heat capacity of a body is the amount of heat gained by a body of a unit mass when raised through a unit temperature difference, its unit is $\text{Jkg}^{-1}\text{K}^{-1}$.

It is found that the specific heat capacity of a substance depends on the nature of material of the substance as well as the external conditions under which heat is supplied. The two commonly used specific heats capacities are $\boldsymbol{c_p}$ and $\boldsymbol{c_v}$, the specific heat capacity at constant pressure and specific heat capacity at constant volume respectively. The c_p comes in when the substance is heated at constant pressure and c_v when the substance is heated at constant volume. It is found that c_p and c_v are quite different for gases.

The first law of thermodynamics which relates the heat supplied dQ, the change in internal energy dU and the external work done dW states that, *"In a closed system the heat supplied is equal to the change in internal energy plus the external work done"*. i.e., $dQ = dU + dW$

Let the pressure and volume be consant. The heat supplied at constant pressure is given by $mc_p dT$, the change in the internal energy is $mc_v dT$ and the external work done is PdV, it follows that;

$$mc_p dT = mc_v dT + PdV$$

From ideal gas equation,

$$PV = nRT$$

$$PdV = nRdT$$

$$PdV = \frac{m}{M}RdT$$

where, m is mass of the gas, M is its molar mass, R is universal gas constant and $\frac{R}{M} = r$ (gas constant per molar mass).

Hence,

$$mc_p dT = mc_v dT + mrdT\,; \quad c_p = c_v + r$$

where, c_p, c_v and r are measured in $\text{Jkg}^{-1}\text{K}^{-1}$.

Also, $PV = nRT$, then $P = \frac{m}{V}\frac{R}{M}T$; $P = \rho rT$. It follows that, the gas constant per molar mass can also be given by $r = \frac{P}{\rho T}$

(a) Molar specific heat capacity

The amount of heat required to raise the temperature of one mole of the material by 1°C is called molar specific heat capacity. Let n be number of moles of a substance that absorb an amount of heat Q to raise its temperature from T to $T + \Delta T$, the molar specific heat capacity C is given by;

$$C = \frac{1}{n} \times \frac{\Delta Q}{\Delta T} \qquad (7.53)$$

where $n = \frac{m}{M}$, m being the mass of the material and M its molecular weight (the number of grams in one mole).

Therefore, equation (7.53) becomes;

$$C = \frac{M}{m} \times \frac{\Delta Q}{\Delta T} \qquad (7.54)$$

(b) Specific heat capacity of gases

It is found that, the specific heat capacity of an ideal gas is independent of the nature of the gas chosen and it does not depend on its temperature. But, it depends on the physical conditions under which heat is supplied to the gas; e.g. constant volume or constant pressure.

Consider an ideal gas in a cylinder with a frictionless piston (Figure 7.25a) at an internal equilibrium temperature T, volume V and pressure P.

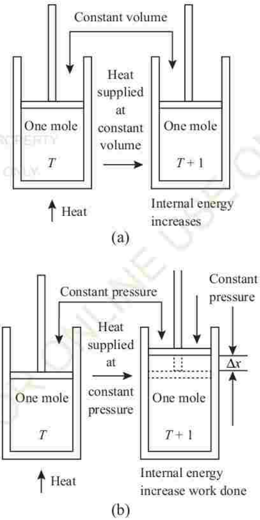


Figure 7.25 Ideal gas in a cylinder at equilibrium temperature

Let an amount of heat ΔQ be supplied to heat the gas at constant volume. This heat will increase its temperature by ΔT. This is so as heat energy ΔQ will increase the motion of molecules of the gas, thereby, increasing the internal energy U of the gas by an amount, say, ΔU, and hence its temperature, by ΔT.

Now, the kinetic energy or energy per mole of an ideal monoatomic gas is given by the equation

$$U = \frac{3}{2}RT \qquad (7.55)$$

Therefore, if the temperature increases by ΔT due to increase of internal energy ΔE, then, the internal energy at the temperature $T + \Delta T$ is

$$E + \Delta E = \frac{3}{2}R(T + \Delta T) \qquad (7.56)$$

From equations (7.55) and (7.56), the increase of energy ΔU is

$$\Delta U = \frac{3}{2}R(T + \Delta T) - \frac{3}{2}RT$$

$$\Delta U = \frac{3}{2}R\Delta T$$

This increase in the energy (ΔU) of one mole of the gas is equal to the amount of heat, ΔQ supplied to the gas at constant volume, therefore,

$$\Delta Q = \frac{3}{2}R\Delta T \text{ or } \frac{\Delta Q}{\Delta T} = \frac{3}{2}R \qquad (7.57)$$

But, as defined before, molar specific heat capacity at constant volume, C_v is,

$$C_v = \frac{1}{n} \times \frac{\Delta Q}{\Delta T}$$

for n moles of the gas, or $C_v = \frac{\Delta Q}{\Delta T}$ for 1 mole of the gas

$$C_v = \frac{3}{2}R \qquad (7.58)$$

Suppose an amount of heat ΔQ is supplied to the gas to increase its temperature from T to $T + \Delta T$ at constant pressure P, i.e. the external force F, on the piston of the cylinder containing the gas does not change during its expansion. Since the gas is at constant pressure, therefore, its volume will increase from V to $V + \Delta V$. In this process, ΔQ will be used to increase the internal energy of the gas by an amount ΔU and do some work against the atmospheric pressure P. Let the piston move through the distance Δx against the atmospheric pressure P, (Figure 7.25b),

then, $P = \frac{F}{A}$, where A is the cross-section of the piston, or $F = PA$,

Therefore, work done W, by the gas in moving the piston through Δx is, $F\Delta x = PA\Delta x$, where $A\Delta x = \Delta V$ (change in volume). This work done is equal to the extra amount of energy supplied to the gas to make the expansion possible.

Thus, the total energy ΔQ given to the gas to increase its temperature by ΔT at constant pressure is given by;

$$\Delta Q = P\Delta V + \Delta U \text{ or}$$

$$\Delta Q = R\Delta T + \Delta U \qquad (7.59)$$

(ideal gas equation, $P\Delta V = R\Delta T$)

But, ΔQ is equal to heat required to raise the temperature of one mole of the gas by ΔT at constant pressure.

Therefore, by the definition of the molar specific heat capacity at constant pressure, C_p

$$\Delta Q = \text{number of moles} \times C_p \times \text{rise in temperature}$$

or $\Delta Q = C_p \Delta T$ also,

$$\Delta U = C_v \Delta T \qquad (7.60)$$

Therefore, from equations (7.59) and (7.60), it follows that,

$$C_p \Delta T = R\Delta T + C_v \Delta T$$

$$C_p - C_v = R \qquad (7.61)$$

Equation (7.61) is called Mayer's equation. Where C_p is always greater than C_v because at constant pressure thermal energy has to be supplied not only to increase the internal energy of the gas, but also the gas does extra work against the atmospheric pressure (Figure 7.25b). The units for C_p and C_v are $\text{Jmol}^{-1}\text{K}^{-1}$. As it was shown in equation (7.58), for monoatomic gas, $C_v = \frac{3}{2}R$

Substituting this into equation (7.61), it follows that,

$$C_p = \frac{3}{2}R + R; \quad C_p = \frac{5}{2}R$$

It can easily be shown that, for a diatomic gas, $C_v = \frac{5}{2}R$ and $C_p = \frac{7}{2}R$ and for polyatomic gas $C_v = 3R$ and $C_p = 4R$. The dimensionless ratio of heat capacities is given by $\gamma = \frac{C_p}{C_v}$. Because C_p is always greater than C_v for gases, γ is always greater than unit. For monoatomic gas γ =1.67 and for diatomic gas, γ =1.4.

Example 7.17

The density of a gas is $1.775\ \text{kgm}^{-3}$ at 27 °C and $10^5\ \text{Nm}^{-2}$ pressure and its specific heat capacity at constant pressure is $0.846\ \text{kJkg}^{-1}\text{K}^{-1}$. Find the ratio of its specific heat capacity at constant pressure to that at constant volume.

The gas constant per kg of gas is given by $r = \frac{P}{\rho T}$

Since

$\rho = 1.775\ \text{kgm}^{-3}$, $T = (273+27)\text{K} = 300\text{K}$

and $P = 10^5\ \text{Nm}^{-2}$; then

$$r = \frac{10^5\ \text{Nm}^{-2}}{1.775\ \text{kgm}^{-3} \times 300\ \text{K}}$$

$$= 0.188\ \text{kJkg}^{-1}\text{K}^{-1}$$

Now $c_p - c_v = r$, $c_v = c_p - r$

$$c_v = 0.846\ \text{kJkg}^{-1}\text{K}^{-1} - 0.188\ \text{kJkg}^{-1}\text{K}^{-1}$$

$$= 0.658\ \text{kJkg}^{-1}\text{K}^{-1}$$

$$\gamma = \frac{c_p}{c_v} = \frac{0.846\ \text{kJkg}^{-1}\text{K}^{-1}}{0.658\ \text{kJkg}^{-1}\text{K}^{-1}}$$

$$\gamma = 1.29$$

Therefore, the ratio of specific heat capacity at constant pressure to that at constant volume is 1.29.

Example 7.18

What amount of heat must be supplied to 2×10^{-2} kg of nitrogen at room temperature to raise its temperature by 45°C at constant pressure? Molecular mass of nitrogen, N_2 = 28g and $R = 8.3\text{Jmol}^{-1}\text{K}^{-1}$.

Solution

Heat added, $\Delta Q = n \times R \times \Delta T$

where n is the number of moles given by;

$$n = \frac{\text{mass of nitrogen}\,(m)}{\text{molecular weght of nitrogen}\,(M)}$$

$$\Delta Q = \frac{2\times10^{-2}\ \text{kg}}{28\times10^{-3}\ \text{kg}} \times 8.3\ \text{Jmol}^{-1}\text{K}^{-1} \times 45\,°\text{C}$$

$$= 266.8\ \text{J}$$

Therefore, the amount of heat that must be supplied is 266.8 J.

Example 7.19

A typical bedroom contains about 2500 moles of air. Find the change in internal energy of the room when air is cooled from 35°C to 26°C at a constant pressure of 1atm. Treat the air as an ideal gas with $\gamma = 1.4$.

Solution

$$\gamma = \frac{C_p}{C_v},\quad \gamma = \frac{C_v + R}{C_v},\quad \gamma = 1 + \frac{R}{C_v}$$

But $C_v = \dfrac{R}{\gamma - 1}$

$$C_v = \frac{8.314\ \text{Jmol}^{-1}\text{K}^{-1}}{1.4-1} = 20.79\ \text{Jmol}^{-1}\text{K}^{-1}$$

From the relation, $dU = nc_v dT$; $\Delta U = nCv\Delta T$

$$\Delta U = 2500\ \text{mol} \times 20.79\ \text{Jmol}^{-1}\text{K}^{-1} \times (26-35)\ \text{K}$$

$$\Delta U = -4.68\times10^5\ \text{J}$$

7.3.3 Work done during thermodynamic processes

The process occurring in closed systems which do not permit the transfer of mass across their boundaries is known as non-flow process. In non-flow process, there is only work and heat transfer but there is no mass transfer into or out of the system. During the energy flow, some of the changes take place in pressure, volume, temperature, internal energy, heat, work etc.

(a) Isochoric process (constant volume)

When a gas is heated at a constant volume (i.e. fixed space), the temperature and pressure will increase (Figure 7.26). All the heating entering the system becomes internal energy. No work is done by the system, the temperature rises from T_1 to T_2 and the pressure from P_1 to P_2. Thus, $dW = 0$ and $dQ = dU = C_v(T_2 - T_1)$.

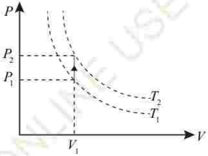


Figure 7.26 P-V diagram

(b) Isobaric process (constant pressure)

An Isobaric process is a thermodynamic process in which the pressure of an ideal gas when heated remains constant, while both its volume and temperature increase (Figure 7.27).

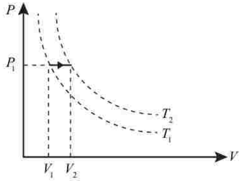


Figure 7.27 P-V diagram

The gas expands in the cylinder by heating; thus, the work is done by the gas. Also, the heat transferred changes the internal energy of the system. The relation between pressure (P), volume (V), and temperature (T) can be found from the characteristic gas equation:

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}, \text{ since } P_1 = P_2 \text{ then, } \frac{V_1}{T_1} = \frac{V_2}{T_2}$$

Work done during isobaric process

Referring to the first law of thermodynamics:

$$dQ = dU + dW$$

The work done,

$$W = P(V_2 - V_1) \quad \text{or}$$

$$W = nR(T_2 - T_1) \qquad (7.62)$$

So, equation (7.62) is the equation for work done in the isobaric process due to heat flow. Change in internal energy is $dU = nC_v dT$. The heat transfer is given by $nC_p dT = nC_v dT + PdV$

It then follows that

$$nC_p(T_2 - T_1) = nC_v(T_2 - T_1) + P(V_2 - V_1)$$

(c) Isothermal process

Isothermal process is that process in which the temperature of the working substance remains constant. In such process, heat is supplied or removed from the system at just the right rate to maintain constant temperature.

Conditions for isothermal process

(i) The gas must be held in a thin walled, highly conducting vessel, surrounded by a constant temperature bath.

(ii) The expansion or compression of the gas must take place slowly, so that the heat can pass in or out to maintain the temperature of the gas at every instant during expansion or compression.

When the temperature is constant, the pressure of a gas varies with volume and a graph which shows this variation is the isothermal curve (Figure 7.28).

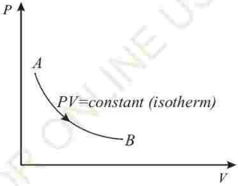


Figure 7.28 P-V curve for isothermal process

It is well known that, the pressure P and volume V of a mole of an ideal gas are related by the equation;

$$PV = RT$$

where T is the absolute temperature of the gas and R is the universal gas constant. Since in an isothermal process, temperature T is constant, then;

$$PV = \text{constant} \qquad (7.63)$$

Equation (7.63) is called the equation of the isothermal process for an ideal gas. The path of an isothermal process (called an isotherm) on the PV diagram is shown in Figure 7.28. The higher the temperature, the further the isotherm lies from the coordinate axes. When the gas expands or is compressed at constant temperature its pressure and volume change in such a way that product PV is always constant. So, if the gas expands isothermally from the initial state $A(P_1,V_1)$ to the final state $B(P_2,V_2)$, then;

$$P_1V_1 = P_2V_2 \qquad (7.64)$$

Work done during isothermal process

Assume an ideal gas undergoes isothermal expansion from state $A(P_1,V_1)$ to state $B(P_2,V_2)$ as shown in Figure 7.29.

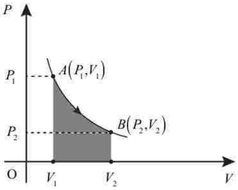


Figure 7.29 P-V diagram

The amount of work done can be determined by adding up all the small works done in small steps state A to state B.

In case of an ideal gas, an isothermal process work is done at the same rate as heat is supplied, so there is no increase of internal energy (for any ideal gas), i.e. $dU = 0$ and $dQ = dW$.

$$\int dW = \int_{V_1}^{V_2} P\,dV$$

For n moles of an ideal gas, $P = \frac{nRT}{V}$, thus,

$$W = \int_{V_1}^{V_2} \frac{nRT}{V}\,dV$$

Since the gas expands isothermally, T is constant; then,

$$W = nRT \ln\left(\frac{V_2}{V_1}\right) \qquad (7.65)$$

$$\text{or } W = 2.303\,nRT \log\left(\frac{V_2}{V_1}\right)$$

Equation (7.65) can also be expressed in terms of pressure as follows:

$$P_1V_1 = P_2V_2 \text{ and } \frac{P_1}{P_2} = \frac{V_2}{V_1}$$

$$W = nRT \ln\left(\frac{P_1}{P_2}\right) \text{ or}$$

$$W = 2.303\,nRT \log\left(\frac{P_1}{P_2}\right)$$

Example 7.20

One mole of an ideal gas which is kept at temperature of 320 K is compressed isothermally from its initial volume of 8 litres to a final volume of 4 litres. Calculate the total work done in the whole process.

Solution

Work done, $W = nRT \ln\left(\frac{V_2}{V_1}\right)$

$$W = 1 \times 8.31 \text{ Jkg}^{-1}\text{K}^{-1} \times 320 \text{ K} \times \ln\left(\frac{4\ L}{8\ L}\right)$$

$$= -1843 \text{ J}$$

Therefore, the total work done in the whole process is -1843 J.

(d) Adiabatic process

For an adiabatic expansion or compression, no heat enters or leaves the system and so $dQ = 0$. Therefore, $0 = dU + dW$ or $dW = -dU$.

All the work is done at the expense of the internal energy of the gas; the gas therefore cools. Conversely, in an adiabatic compression, the work done on the gas by an external agent increases the internal energy and the temperature of the gas rises.

Consider two isotherm in Figure 7.30 for a fixed mass of an ideal gas.

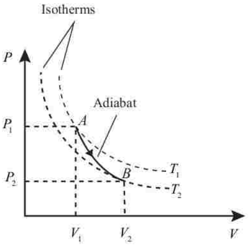


Figure 7.21 Thermal radiation

If the gas has initially a temperature T_1 and volume V_1, its state, i.e. P_1, V_1, T_1 is represented by point A on the T_1 isotherm. If it then expands adiabatically to volume V_2, so that its temperature falls to T_2, its state is now represented by point B i.e. P_2, V_2, T_2 on isotherm T_2. The curve AB relates the pressure and volume of the mass of the gas for this adiabatic change and is called an adiabat. It is steeper than the isotherm. Its equation can be shown to be;

$$PV^{\gamma} = constant$$

Where γ is the ratio of the two specific heat capacities of the gas. It works to a reversible adiabatic change for an ideal gas having a constant value of γ. Expressions for the temperature change during the reversible adiabatic process for one mole of an ideal gas can also be obtained as follows:

$$P_1V_1^{\gamma} = P_2V_2^{\gamma} \qquad (7.66)$$

also, $$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \qquad (7.67)$$

Dividing equation (7.66) by (7.67),

$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$$

$$TV^{\gamma-1} = constant$$

This gives a relation between T and V.

From equation (7.67),

$$\left(\frac{P_1V_1}{T_1}\right)^{\gamma} = \left(\frac{P_2V_2}{T_2}\right)^{\gamma} \qquad (7.68)$$

Dividing (7.68) by (7.66), it then follows that,

$$\frac{P_1^{\gamma-1}}{T_1^{\gamma}} = \frac{P_2^{\gamma-1}}{T_2^{\gamma}}$$

$$\frac{P^{\gamma-1}}{T^{\gamma}} = constant \qquad (7.69)$$

This gives a relation between P and T.

Work done during adiabatic process

Suppose there is one mole of perfect gas contained in a cylinder having insulating walls. If the gas expands adiabatically from the initial state $A(P_1,V_1)$, to the final state $B(P_2,V_2)$, the work done W by the gas during the adiabatic expansion is;

$$\int dW = \int_{V_1}^{V_2} P\,dV$$

For adiabatic processes $PV^{\gamma} = k$; where k is constant. Then

$$W = \int_{V_1}^{V_2} kV^{-\gamma}\,dV$$

$$W = \frac{1}{1-\gamma}\left(kV_2^{1-\gamma} - kV_1^{1-\gamma}\right) \qquad (7.70)$$

Since $P_1V_1^{\gamma} = P_2V_2^{\gamma} = k$, substituting k in equation (7.70),

$$W = \frac{1}{1-\gamma}\left[P_2V_2 - P_1V_1\right] \qquad (7.71)$$

If T_1 is the temperature of the initial state and T_2 is the temperature of the final state; then,

$$P_1V_1 = nRT_1 \text{ and } P_2V_2 = nRT_2 \qquad (7.72)$$

Substituting equation (7.72) to (7.71),

$$W = \frac{nR}{1-\gamma}\left(T_2 - T_1\right) \qquad (7.73)$$

Example 7.21

An ideal monatomic gas of 0.15 mole is enclosed in a cylinder at a pressure of 250 kPa and a temperature of 320 K. The gas is allowed to expand adiabatically and reversibly until its pressure is 100 kPa. Calculate the final temperature and the amount of work done by the gas. (For monoatomic gas, $\gamma = 1.67$).

Solution

From equation (7.69), $T_2 = T_1\left(\frac{P_2}{P_1}\right)^{1-\frac{1}{\gamma}}$

$$T_2 = 320\text{ K} \times \left(\frac{100\text{ kPa}}{250\text{ kPa}}\right)^{1-\frac{1}{1.67}} = 221.6\text{ K}$$

Therefore, the final temperature of the gas is 221.6 K.

work done $W = \frac{nR}{1-\gamma}\left[T_2 - T_1\right]$

$$W = \frac{0.15\text{mol} \times 8.31\text{ Jmol}^{-1}\text{K}^{-1}}{1-1.67}\left[221.6\text{ K} - 320\text{ K}\right]$$

$= 183.10$ J

Therefore, the amount of work done by the gas is 183.10 J.

(e) Applications of first law of thermodynamics

First law of thermodynamics is basically a law of energy conservation. The following are few examples of its application in real life.

Energy flow in combustion engine: When an engine burns fuel it converts the energy stored in the fuel's chemical bonds into useful mechanical work and into heat. The conservation of energy principle defined by the first law of thermodynamics states that, *"The total chemical energy stored in the fuel is converted to mechanical energy and thermal energy"*. The total mechanical energy and heat energy out (in cooling water, in oil, in exhaust, radiated to surroundings) must equal the energy available in the fuel.

Electric production system: Water energy can be harnessed by building a dam to hold back the water of a river. If you slowly release water through a small opening in the dam, you can use the driving pressure of the water to do work of turning a turbine. The work of the turbine can be used to generate electricity with the help of a generator. Some of the water energy is lost as thermal. Electricity was not created out of nothing; it is the result of transforming water energy from the river into another energy form.

Cooling systems: These systems also conserve energy. Cooling machines, such as refrigerators and air conditioners, actually use heat, simply reversing the usual process by which particles are heated. The refrigerator pulls heat (through mechanical work) from its inner compartment-the area where food and other perishables are stored and transfers it to the region outside. This is why the back of a refrigerator is warm.

Exercise 7.3

1. A cylinder contains 1 mole of oxygen at a temperature of 27 °C. The cylinder is provided with a frictionless piston which maintains a constant pressure of 1 atm on the gas. The gas is heated until its temperature rises to 127 °C.
   (a) How much work is done by the piston in the process?
   (b) What is the increase in internal energy of the gas?
   (c) How much heat was supplied to the gas?
   ($C_p = 7.03$ calmol^{-1}°C^{-1};
   $R = 1.99$ calmol^{-1}°C^{-1};
   1 cal = 4.184 J)
2. Two moles of an ideal gas are compressed in a cylinder at a constant temperature of 65.0 °C until the original pressure is tripled.
   (a) Sketch a *P*-*V* diagram for this process.
   (b) Calculate the amount of work done.
3. A cylinder contains 0.250 mol of carbon dioxide (CO_2) gas at a temperature of 17.0 °C. The cylinder is provided with a frictionless piston which maintains a constant pressure of 1.00 atm on the gas. The gas is heated until its temperature increases to 127 °C. Assume that CO_2 may be treated as an ideal gas.
   (a) Draw a *P*-*V* diagram for this process.
   (b) How much work is done by the gas in this process?
   (c) On what is this work done?
   (d) What is the change in internal energy of the gas?
   (e) How much heat was supplied to the gas?
   (f) How much work would have been done if the pressure had been 0.50 atm?
4. An ideal gas at 17 °C has a pressure of 760 mmHg and is compressed
   (a) isothermally,
   (b) adiabatically,
   until its volume is halved. Calculate in each case the final pressure and temperature of the gas ($\gamma = 1.4$).
5. A motor car tyre has a pressure of four atmospheres at a room temperature

of 27°C. If the tyre suddenly bursts, calculate the temperature of escaping air. Value of γ for air is 1.4.

6. (a) When a gas expands adiabatically, it does work on its surroundings. But if there is no heat input to the gas, where does the energy come from to do the work?

   (b) Show that for an adiabatic process in an ideal gas, the relationship between the volume (V) of the gas and temperature (T) is given by $\frac{dT}{T}+(\gamma-1)\frac{dV}{V}=0$

7. Find the minimum attainable pressure of ideal gas in the process $T=T_0+\alpha V^2$ where T_0 and α are positive constants and V is the volume of one mole of the gas.

8. The amount of heat required to raise the temperature of 3.00 mol of a polyatomic gas at constant pressure from 320 K to 370 K is 4.99 kJ. Calculate:

   (a) c_p and c_v;

   (b) The value of γ; and

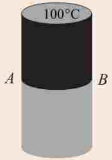
   (c) The heat required to raise the temperature of 4.00 mol from 300 K to 400 K at constant volume.

9. A gas at an initial pressure of 76 mm mercury is expanded adiabatically until its volume is doubled. Calculate the final pressure of the gas if the ratio of the principle specific heat capacities is 1.40.

Revision exercise 7

1. Explain why electrons in insulators do not contribute to its conductivity.
2. The tile floor feels colder than the wooden floor even though both floor materials are at the same temperature. Why?
3. How does cross-section area of a rod affect thermal conduction of a given material?
4. Calculate the quantity of heat conducted through 2 m^2 of brick wall of 12 cm thickness in 1 hour if the temperature on one side is 80°C, and the other side is 28°C. Thermal conductivity of brick is 0.13 $Wm^{-1}K^{-1}$.
5. A composite bar is made of a bar of copper 10 cm long, a bar of iron 8 cm long and a bar of Aluminum 12 cm long, all having the same cross-sectional area. If the extreme ends of the bars are maintained at 100°C and 10°C respectively, find the temperature at the two junctions, given that thermal conductivity of copper, iron, and aluminium is 400 $Wm^{-1}K^{-1}$, 40 $Wm^{-1}K^{-1}$ and 20 $Wm^{-1}K^{-1}$ respectively.
6. Calculate the heat flow rate through a layer of cork of 2 mm thickness and 24 cm^2 area when the temperature difference between its surfaces is 60 K. (k of cork = 0.05 $Wm^{-1}K^{-1}$)
7. Two cylinders of equal physical dimensions are placed one on top

of the other as illustrated by the following diagram:



The lower surface of the silver cylinder is kept at $0°C$ and the upper surface of the iron cylinder is kept at $100°C$. Given that, the thermal conductivity of silver is eleven times that of iron, calculate the temperature of the surface *AB*.

8. An electric heater is used in a room of total wall area of $137\ m^2$ to maintain a temperature of $20°C$ inside it when the outside temperature is $0°C$. The walls have three layers of different materials. The inner most layer is of wood of thickness $2.5\ cm$, the middle layer is of cement of thickness $1.0\ cm$ and the outermost layer is of brick of the thickness $25\ cm$. Find the power of the electric heater. Assume that there is no heat loss through the floor and ceiling. Thermal conductivity of wood, cement, and brick are $1.25\ Wm^{-1}K^{-1}$, $1.5\ Wm^{-1}K^{-1}$ and $1.0\ Wm^{-1}K^{-1}$ respectively.

9. A thin walled copper sphere of radius $5\ cm$ and mass $100\ g$ containing $100\ g$ of water is cooled to $-176°C$ by immersing it in liquid air. It is then placed inside a filling hollow sphere of expanded polythene of outer radius 10cm in a room at $20°C$. What is the value of thermal conductivity of ice if the ice just melts after 24 hours? (Specific heat capacities of ice and copper are $2.1\ kJkg^{-1}K^{-1}$ and $0.4\ kJkg^{-1}K^{-1}$ respectively and the specific latent heat of fusion of ice is $336\ kJkg^{-1}$).

10. A liquid cools from $70°C$ to $50°C$ in 4 minutes. How much time will it take to cool from $50°C$ to $40°C$? The surroundings temperature is $20°C$.

11. A patient waiting to be seen by his physician is asked to remove all his clothes in an examination room that is at $16\ °C$. Calculate the rate of heat loss by radiation from the patient, given that his skin temperature is $34\ °C$ and his surface area is $1.6\ m^2$. Assume emissivity = 0.80 and Stefan-Boltzmann constant $= 5.67\times10^{-8}\ Wm^{-2}K^{-4}$.

12. A 0.32 g of oxygen is kept in a rigid container and is heated. Find the amount of heat needed to raise the temperature from $25°C$ to $35°C$. The molar heat capacity of oxygen at constant volume is $20\ JK^{-1}mol^{-1}$.

13. A tank of volume $0.2\ m^3$ contains helium gas at a temperature of 300 K and pressure of $1.0\times10^5\ Nm^{-2}$. Find the amount of heat required to raise the temperature to 400 K. The molar heat capacity at constant volume is $3.0\ cal\ K^{-1}$. Neglect any expansion in the volume of a tank. $1\ cal = 4.184\ J$.

14. A gas has a volume of $0.02\ m^3$ at a pressure of $2\times10^5\ Pa$ and

temperature of 27 °C. It is heated at constant pressure until its volume increases to $0.03\ m^3$. If its molar heat capacity at constant volume is $0.8\ Jmol^{-1}K^{-1}$, and its molar mass is 32 g., calculate:

(a) The external work done;
(b) The new temperature of the gas; and
(c) The increase in internal energy of the gas.

15. (a) What is meant by an adiabatic change and isothermal change of a state of a gas?
(b) A gas is contained in a thin-walled metal cylinder and compressed by a piston moving with constant velocity. Explain whether the change is approximately an adiabatic or isothermal as the piston moves with a high or low velocity.

16. The resistance R_θ of a particular resistance thermometer at a Celsius temperature θ as measured by constant volume gas thermometer is given by $R_\theta = 50.00 + 0.1700\theta + 3.00 \times 10^{-4}\theta^2$. Calculate the temperature as measured on the scale of the resistance thermometer which corresponds to a temperature of 60 °C on the gas thermometer.

17. The volume V_θ of a fixed mass of mercury at temperature θ °C measured on the perfect gas scale is given by
$V_\theta = V_0(1 + 1.818 \times 10^{-4}\theta + 0.8 \times 10^{-8}\theta^2)$,
where V_0 is the volume at 0°C on the gas scale. Calculate the temperature expected on a mercury thermometer when the gas thermometer scale temperature is 40°C.

18. A copper-constantan thermocouple with its cold junction at 0°C had an e.m.f. of 4.28 mV when its other hot junction was at 100°C. The e.m.f. became 9.2 mV when the temperature of hot junction was 200°C. If the e.m.f. is related to the temperature difference θ between the hot and cold junction by the equation, $E = A\theta + B\theta^2$. Calculate:
(a) The value of A and B; and
(b) The temperature for which E may be assumed to be proportional to θ without incurring an error of more than 1%.

19. Estimate the rate at which ice melts in a wooden box 2 cm thick of inside measurements $(60 \times 60 \times 60)\ cm^3$. Assume that the outside of the box is maintained at a temperature of 27 °C and that the coefficient of thermal conductivity of wood is $0.1674\ Js^{-1}m^{-1}°C^{-1}$. Latent heat of fusion of ice is $336 \times 10^3\ Jkg^{-1}$.

20. The volume of a gas at atmospheric pressure is compressed adiabatically to half its original volume. Calculate the resulting pressure ($\gamma = 1.4$).

21. A thermos flask uses some principles learned in this chapter to maintain temperature of its content constant for at most six hours. Propose modifications to be made in the design and construction of thermos flask to extend the time it can maintain the temperature of its content constant by two hours.

Chapter Eight

Vibrations and waves

Introduction

Vibration refers to the mechanical oscillation of an object about an equilibrium point. Oscillations may be regular, such as the motion of a pendulum, or random, such as the movement of a tyre on a gravel road. Similarly, waves occur when a system is disturbed from equilibrium, and the disturbance travels, or propagates from one region of the system to another. As the wave propagates, it carries energy. Ripples on a pond, musical sounds, seismic tremours triggered by an earthquake are good examples of the wave phenomena. The concepts of vibration and wave play an important role in understanding the sound waves, light waves, microwaves, and microscopic properties of an atom, electrons, and nucleus. In this chapter, you will learn about mechanical vibrations, wave motion, sounds, electromagnetic waves, physical optics, and Doppler's effects.

8.1 Mechanical vibrations

Mechanical vibrations are oscillations in dynamic systems that repeat within a time period and can carry energy. The design of string instruments such as guitars, is based on strings vibrating at a certain frequency. Kinetic energy is converted to potential energy when the string is plucked, and potential energy is converted to kinetic energy when the string is released. An example of mechanical vibrations is the sound vibrations from human voice that transfer energy form one person's vocal cord to another person's ear drum by repeated vibrations of air molecules. In this section, you will learn the distinction between free and forced vibrations, as well as the distinction between underdamped, critically damped and overdamped vibrations. You will also derive the velocity of vibrations of mechanical vibrations.

8.1.1 Free and forced vibrations

Mechanical vibrations of a system can be considered as free or forced oscillation depending on the environment in which the system is subjected to.

(a) Free oscillations

A system is said to undergo free oscillations when the only external force acting on it is the restoring force. That is, there is no force to dissipate energy and therefore, the oscillation maintains its amplitude (Figure 8.1), thus total energy remains constant.

Therefore, in free oscillation, the system oscillates at its own natural frequency. Such case can be observed in vacuum where there is an absence of external forces except gravity. Free oscillations are sometimes referred to as un-damped oscillations. Free oscillations are ideal; in practice, the energy of a vibrating system is dissipated to the surroundings over time and the amplitude decays to zero.

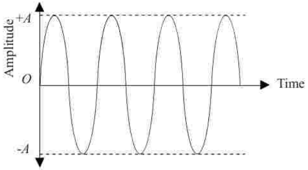


Figure 8.1 Free (un-damped) oscillations

(b) Forced oscillations

A system is said to undergo forced oscillation when it is maintained in a state of oscillation by an external periodic force of frequency other than the natural frequency of the system. Therefore, the system oscillates on a particular definite frequency and period. When the frequency of an external agent is nearly or equal to the natural frequency of the oscillating system, there is a sharp rise in the amplitude of oscillation called resonance. Resonance is useful in radio or television tuning, although it can cause annoying oscillating rattle in a car and an annoying boom or buzz of a loudspeaker.

8.1.2 Damped oscillations

The oscillation whose amplitude of vibration becomes progressively smaller is said to be damped. Damped oscillations can be underdamped, critical-damped and over-damped oscillations depending on the level of decay of its amplitude of oscillation.

(a) Under-damped oscillations

This occurs when the oscillating system overshoots (passes) the equilibrium position and oscillates with decreasing amplitude about the equilibrium position (Figure 8.2). A system is said to be under-damped if its coefficient of damping (δ) is less than 1, i.e., $\delta < 1$. A vibrating string of a guitar or oscillation of a swing are good examples of underdamping.

(b) Critically-damped oscillations

This occurs when the oscillating system is brought to equilibrium quickly without oscillating. It provides the quickest approach to zero amplitude for a damped oscillator (Figure 8.2). A system is considered critically damped when its coefficient of damping is equal to 1, i.e., $\delta = 1$. Coils of electric meters, for example, are critically-damped in such a way that, their oscillations return to equilibrium position quickly to make readings of current in a shortest possible time. Similarly, cars have dampers (shock absorber) connected parallel to or through the springs so that the suspension provides critical-damping and therefore come to rest in the shortest time possible. This provides comfortable ride, otherwise, the car would move up and down for sometime after hitting a bump on a road.

(c) Over-damped oscillations

This occurs when the oscillating system is brought to equilibrium slowly without oscillating (Figure 8.2). A system is over-damped if its coefficient of damping is greater than 1, i.e., $\delta > 1$. Swinging doors are fitted with overdamped system to control them not to overshoot the closing position and hurt someone approaching the door.

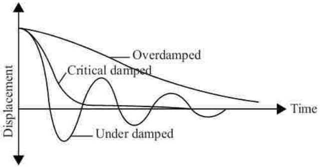


Figure 8.2 Damped oscillations

8.1.3 Velocity of vibrations of a string

A key property of a vibration is velocity. The expression for the velocity of mechanical vibration (e.g. vibrating string) can be derived using the method of dimensional analysis. Experiments show that, the physical quantities that determine the velocity of vibrations on a string are the tension T in the string and its linear mass density μ (also called mass per unit length). Therefore, the velocity v can be obtained by method of dimensions as follows:

$$v \propto T^a \mu^b$$

$$v = kT^a \mu^b \qquad (8.1)$$

(where k is a dimensionless constant)

The dimensions of v, T and μ can be written as ; LT^{-1}, MLT^{-2} and ML^{-1} respectively.

Thus, by the method of dimension analysis, equation (8.1) can be written as;

$$M^0 L^1 T^{-1} = M^{a+b} L^{a-b} T^{-2a} \qquad (8.2)$$

Solving for a and b you obtain, $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

Substituting values of a and b into equation (8.1), you obtain,

$$v = k\sqrt{\frac{T}{\mu}} \qquad (8.3)$$

The equation (8.3) shows that, the velocity of vibrations is independent of frequency of the vibrations. The velocity of the vibrations is determined by the mechanical properties of the medium. Experimentally $k = 1$, then (8.3) can be written as;

$$v = \sqrt{\frac{T}{\mu}} \qquad (8.4)$$

Therefore, the velocity of mechanical vibrations is equal to the square root of the tension per linear mass density.

8.1.4 Applications of mechanical vibrations

Mechanical vibrations have several applications in various fields of engineering including: design of machines, structures, foundations, engines, turbines, control systems and musical instruments.

In musical instruments such as piano or guitar, the forced vibrations of the struck string cause the sound board of a guitar to vibrate at the string frequency, thus increasing the volume of sound of the guitar. In addition, forced vibrations in reed instruments e.g. saxophone causes air columns to vibrate inside the instrument and amplify the sound. Electric signals force speaker cones in loudspeakers to vibrate, thus setting up air motion (longitudinal waves) which are heard as sound. Forced vibrations can have repercussion on a machine as they can cause it to vibrate in unwanted frequencies causing overheating, undue wear and misalignment. Usually, rubber mounts are used to damp the vibrations.

When the natural and forced vibrations match in frequency; resonance occurs, and the amplitude of vibration greatly increases. Resonance is used extensively in electronic circuits for tuning and phase matching. However, resonance can lead to destructions if not managed well. In engines for example, resonance is minimized using harmonic balancers, precision flywheels and cylinder firing order.

Example 8.1

A string has mass per unit length of $0.05\ \text{kgm}^{-1}$, calculate the tension in the string along which vibrations have a speed of $8\ \text{cms}^{-1}$.

Solution

$T = \mu v^2 = 0.05\ \text{kgm}^{-1} \times (0.08\ \text{ms}^{-1})^2$

Therefore,

$T = 3.2 \times 10^{-4}\ \text{kgms}^{-2}$

or $3.2 \times 10^{-4}\ \text{N}$

Exercise 8.1

1. Using examples, explain the terms: damped and forced oscillations, and resonance.
2. Describe an experiment to illustrate the behavior of a simple pendulum performing forced oscillations. Indicate the results you would expect to observe.
3. What are the factors which determine:
   (a) the period of free oscillations of a mechanical system?
   (b) the amplitude of a system performing forced oscillations?
4. The speed of a vibration is found to depend on tension T in the string, and mass per unit length μ (linear mass density). Using dimension analysis, derive the relationship between v, T and μ.
5. The still wire of $0.4\ \text{m}$ long and mass $3.0\ \text{g}$, is stretched with tension of $800\ \text{N}$. What is the velocity of vibration produced by this vibration?

8.2 Wave motion

Wave motion is a propagation of disturbances as a continous train in a regular and organized way. The surface waves on water, sound and light travel as wavelike disturbances. The simplest types of wave motion are produced by vibrations of elastic media, such as air, crystalline solids, or stretched strings. A wave transfers energy through matter or space. Mechanical waves (e.g. sound waves) require a medium for propagation, but electromagnetic waves (e.g. light) do not. Waves can take several forms, but there are two fundamental types of waves: "longitudinal" and "transverse". In a longitudinal wave, the wave propagates parallel to the direction in which the particles are disturbed. In a transverse wave, the wave propagates perpendicular to the direction in which the particles are disturbed.

Note that, a wave travelling through a medium causes disturbance (vibration) to particles of the medium. In this section, you will learn the difference between progressive and stationary waves, how to derive the expression for progressive and stationary wave motion and deduce the principle of superposition of waves.

8.2.1 Progressive and stationary waves

The motion of a wave in a medium can be bound between two fixed points or unbounded to progress around the surrounding environment. For example, when you pluck at the middle of a string fixed at both ends, the motion of the vibrations (wave) appear to be bound between the fixed ends. In contrast, when you drop a stone on a pond of water, the produced waves progress from the source outwardly toward the edge of the pond. In this case, waves can be categorized as progressive or stationary waves.

(a) Progressive waves

A wave is progressive (travelling wave) when a disturbance moves from a source to surrounding region resulting to energy transfer from one point to another, without transporting the particles of the medium in which the waves travel through. In progressive waves, one oscillating molecule transfers some of its energy to the next molecule, which then starts oscillating as well. This molecule now transfers energy to the next molecule and so on. Thus, the energy is transferred along the wave. Waves in a ripple tank, light waves and sound waves in an unbounded medium are all examples of progressive waves. Likewise, the light coming from the sun is an example of a progressive wave.

As the transverse wave propagates through the medium, the particles of the medium undergo displacements. Each particle executes the same type of vibration as the preceding one, though not at the same time. The amplitude of each particle displacement is the same, whereas the phase changes continuously. At this point, no particle is permanently at rest, but different particles attain the state of momentary rest at different instants, and the particles attain the same maximum velocity when they pass through their mean positions.

A progressive wave can propagate either in transverse or longitudinal mode. A progressive wave is transverse if the displacements of particles in a medium are perpendicular to the direction of wave propagation (Figure 8.3a). Examples of transverse progressive waves include a wave in a ripple tank and light waves. On the other hand, a progressive wave is longitudinal if the displacements of particles in a medium are along the same direction as that the wave propagates (Figure 8.1c). An example of a longitudinal progressive waves is a sound wave. The displacement of the particles in both progressive transverse and longitudinal waves can be represented in a displacement-distance graph as shown in Figure 8.3(b).

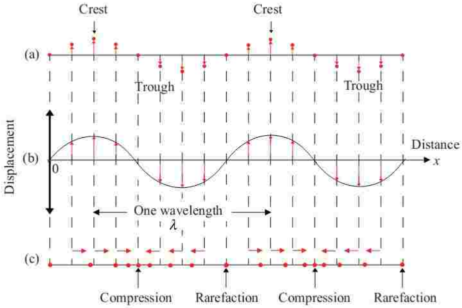


Figure 8.3 Progressive transverse and longitudinal waves

(b) Principle of superposition of waves

Consider two waves A and B travelling in a medium in opposite directions. Suppose the crest of A coincides with that of B and the trough of A with that of B (Figure 8.4a). The resultant wave C will be sum of the amplitudes of the two waves, A and B. The resultant displacement is larger than the displacement caused by individual waves A and B.

Now, suppose the crest of A coincides with the trough of B and the trough of A falls on crest of B (Figure 8.4b), then the resultant wave D is the sum of the amplitude of the two waves, A and B, and it is smaller than the displacement caused by individual waves, A and B. The observed change of displacement when two or more waves meet is based on the principle of superposition of waves which states that, *"When two or more waves pass through the same medium at the same time, the net displacement at any point is equal to the vector sum of the individual displacement at the point".*

That is;

$$y(x,t) = y_1(x,t) + y_2(x,t) + \ldots + y_n(x,t) \quad (8.5)$$

The principle depends on linearity of the wave equation. Therefore, for a medium that does not obey Hooke's law, the wave equation is not linear and this principle does not hold.

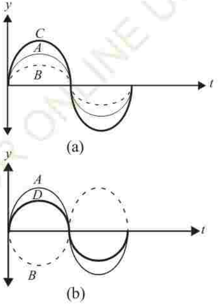


Figure 8.4 Principle of superposition of waves

An important application of the superposition principle of waves involves beats. Beats are amplitude variations in sound due to superposition of two waves of equal amplitude, but slightly different frequencies. This leads to loudness variation called beat frequency, f_b (Figure 8.5).

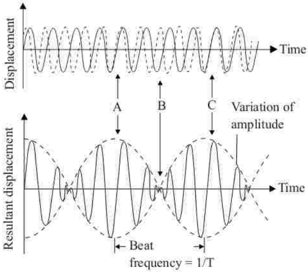


Figure 8.5 Production of beat frequency

Suppose that the two superposing waves have frequencies f_1 and f_2, such that, f_1 is slightly larger than f_2, then, the corresponding periods are T_1 and T_2 with $T_2 > T_1$. If the two waves start out in phase at time $t = 0$, then, the frequencies are in phase again when the first wave has gone through exactly one more cycle than the second wave and the observer will hear a loud sound at points *A* and *C*, whereas little sound or nothing is heard at point *B* (Figure 8.5). The loud sound at *A* and *C* happens at the value of t equals to periods of beat T_b. If n is the number of cycles of the first wave, then the respective periodic time T_b for the first and second waves is;

$$T_b = nT_1 = (n-1)T_2 \qquad (8.6)$$

Eliminating n from equation (8.6) and rearranging;

$$\frac{1}{T_b} = \frac{1}{T_1} - \frac{1}{T_2} \qquad (8.7)$$

From (8.7), the beat frequency, $f_b = \frac{1}{T_b}$, can be written as,

$$f_b = f_1 - f_2 \qquad (8.8)$$

Example 8.2

Two forks, *A* and *B*, when sounded together produce 4 beats/second. The fork *A* is in unison with 30 cm length of a sonometer wire and *B* is in unison with 25 cm length of the same wire at the same tension. Calculate the frequencies of the forks.

Solution

For a sonometer, $f \propto \frac{1}{L}$, therefore,

$$f_A = \frac{f_B L_B}{L_A} = \frac{25 \text{ cm} \times f_B}{30 \text{ cm}} = \frac{5}{6} f_B$$

From equation (8.8), beat frequency, f_B is

$$f_B = f_b + f_A = 4 \text{ beats/s} + \frac{5}{6} f_B$$

solving for f_A and f_B

Thus, $f_A = 20$ Hz and $f_B = 24$ Hz

Therefore, the frequencies of the forks *A* and *B* are 20 Hz and 24 Hz respectively.

(c) Stationary waves

When two progressive waves, with the same speed and frequency, and nearly equal amplitudes, travelling in opposite directions, are superposed on each other, they form a stationary (standing) wave. In stationary waves, energy is stored in one place that means, the waves do not transfer energy from one place to another.

If you fix one end of a long, narrow, stretched spring (slinky spring) and move the other end continuously up and down, a wave is formed. The wave is reflected at the fixed end toward the source, and the process repeats. If the frequency of shaking is increased, one or more loops of large amplitudes called stationary waves are formed. The waveform seems to stay stationary along the spring in either direction. The points along the line of propagation with amplitudes equal to zero are called nodes (N) while points with largest amplitudes are called antinodes (A) (Figure 8.6).

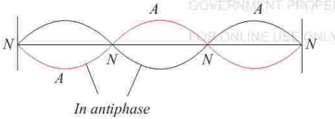


Figure 8.6 Stationary waves

In stationary waves, each particle has its own vibrational characteristic. The displacement amplitudes of the different particles are different, ranging from zero at the nodes to maximum at the antinodes. All particles in a given segment vibrate in phase but in opposite phase relative to the particles in the adjacent segment. The particles at the nodes are permanently at rest but other particles attain their position of momentary rest simultaneously.

Stationary waves can propagate either in transverse or longitudinal mode as in the case of progressive waves.

An example of transverse stationary waves is the stationary waves produced on a vibrating string of a sonometer. On the other hand, an example of a longitudinal stationary waves are the stationary waves produced in organ pipes.

8.2.2 Expression for progressive and stationary wave motion

Particles' motion under progressive and stationary waves can be considered to perform a simple harmonic motion. In this regard, their displacements and velocities can be derived based on simple harmonic motion.

(a) Displacement and velocity of progressive wave motion

Suppose a particle P is at the origin O, and oscillating with simple harmonic motion of frequency f, amplitude A and angular frequency ω. Then, the displacement y of the particle P at O with time t is given as:

$$y = A\sin\omega t \qquad (8.9)$$

where $\omega = 2\pi f$

If a wave travels from left to right, the particle P at a distance x from O will lag behind by a phase angle φ (Figure 8.7).

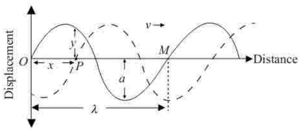


Figure 8.7 Displacement for progressive wave

Therefore, the displacement y of particle P at a distance x is given as:

$$y(x,t) = A\sin(\omega t - \varphi) \qquad (8.10)$$

A point M from O (one wavelength) has a phase difference of 2π, thus particle P from O has a phase difference of

$$\varphi = \left(\frac{x}{\lambda}\right) \times 2\pi = \frac{2\pi}{\lambda}x.$$

Therefore, equation (8.10) can be written as;

$$y(x,t) = A\sin(\omega t - kx) \qquad (8.11)$$

In equation (8.11), $k = \frac{2\pi}{\lambda}$ is known as wave number.

If the wave moves from right to left, the particle P will lead that at O by phase angle φ. The displacement y at P is given as;

$$y(x,t) = A\sin(\omega t + kx) \qquad (8.12)$$

Therefore, the displacement y of the particle moving from left to right and vice versa can be written as;

$$y(x,t) = A\sin(\omega t \pm kx) \qquad (8.13)$$

Since y represents the displacement of a particle as the wave travels, then the particle velocity v at any instant is given by;

$$v_p = \frac{dy}{dt} = A\omega\cos(\omega t \pm kx) \qquad (8.14)$$

In addition, since the displacement y remains constant as the particle moves each point, say, $P_1, P_2, \ldots,$ then the wave or phase velocity which is the rate at which the disturbance (wave) moves across the oscillator is given by;

$$v = \frac{dx}{dt} = \pm\frac{\omega}{k} \qquad (8.15)$$

Note that, equation (8.13), (8.14) and (8.15) apply for both transverse and longitudinal progressive waves.

Example 8.3

A wave travelling along a string is described by $y(x,t) = 3.3\sin(2.7t - 72.1x)$ in which y is in millimeters, x is in meters and t in seconds. What is the (i) amplitude, (ii) wavelength, (iii) period, (iv) frequency and (v) velocity of this wave?

Solution

Compare equation (8.11) with the given wave equation $y(x,t) = 3.3\sin(2.7t - 72.1x)$

(i) Amplitude $A = 3.3$ mm

(ii) Since, the wave number $k = 72.1$, then wavelength,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{72.1} = 0.087 \text{ m}$$

(iii) Since from the given equation, $2.7t = \omega t$, then

$$T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{2.7 \text{ rad s}^{-1}} = 2.33 \text{ s}$$

(iv) Frequency $f = \frac{1}{T} = \frac{1}{2.33\ \text{s}} = 0.43\ \text{s}^{-1}$

(v) Velocity of the wave $v = \frac{\omega}{k}$

Thus, $v = \frac{2.7\ \text{rad s}^{-1}}{72.1\ \text{rad m}^{-1}} = 3.74\ \text{cm s}^{-1}$

(b) Displacement and velocity of stationary wave motion

Suppose that two progressive waves of equal amplitude A and frequency f are traveling in opposite directions, the displacements y of a wave traveling to the right and left respectively are given as;

$$y_1(x,t) = A\sin(\omega t - kx) \qquad (8.16)$$

$$y_2(x,t) = A\sin(\omega t + kx) \qquad (8.17)$$

By the principle of superposition, the resultant displacement y for equations (8.16) and (8.17) is given by

$$y = y_1 + y_2$$
$$= A\sin(\omega t - kx) + A\sin(\omega t + kx) \qquad (8.18)$$

Applying the trigonometric transformation for converting the sum of two sines to a product in equation (8.18), i.e.

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, you get;

$$y(x,t) = (2A\cos kx)\sin\omega t \qquad (8.19)$$

Equation (8.19) which has function of x and t applies for both transverse and longitudinal stationary waves. The factor $\sin\omega t$ shows that the wave shape stays in the same position oscillating up and down. The factor $2A\cos kx$ shows that at each instant the shape of the string is a cosine curve. Therefore, the factor $2A\cos kx$ gives the changing amplitude of a standing wave. Equation (8.19) can be written as;

$$y(x,t) = B\sin\omega t \qquad (8.20)$$

where $B = 2A\cos kx = 2A\cos\left(\frac{2\pi x}{\lambda}\right)$ is the amplitude of oscillations of the various particles. It is found that, when $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, etc., B is maximum and equal to $2A$ which are the antinodes. In addition, when $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$, etc., B is minimum and equal to zero which are the nodes. Hence, equation (8.20) can be used to find the position of the nodes and antinodes of a stationary wave.

Example 8.4

Consider two identical plane progressive waves travelling in a string in opposite directions. If the resulting wave is given by the equation $y = 8\cos(2x)\sin(3t)$, determine the particle displacement of the two identical progressive waves.

Solution

Comparing equation (8.19) with the given equation, $A = 4$, $k = 2$ and $\omega = 3$, then, the two identical progressive waves can be written as;

$$y_1(x,t) = A\sin(\omega t - kx);\ y_1 = 4\sin(3t - 2x)$$

and

$$y_2(x,t) = A\sin(\omega t + kx);\ y_2 = 4\sin(3t + 2x)$$

Example 8.5

The wave function for a standing wave in a string is given by $y = 0.3\sin(0.25x)\cos(120\pi t)$, where x is in meters and t is in seconds.

Determine the wavelength, frequency and amplitude of the superposing waves.

Solution

Comparing equation (8.19) with the given equation:

Wavenumber $k = 0.25 = \frac{2\pi}{\lambda}$; gives, $\lambda = 25.13$ m

Angular frequency $\omega = 120\pi = 2\pi f$; gives, $f = 60$ Hz

Amplitude $2A = 0.3$; gives, $A = 0.15$ m

Therefore, the wavelength, frequency and amplitude of the superposed waves are 25.13 m, 60 Hz and 0.15 m respectively.

Stationary waves on string fixed at both ends

A stationary wave generally has no velocity, since, there is no energy transmission between nodes. But, a number of waves of different frequencies, wavelengths and velocities may superpose to form a group. Motion of such a group is called a group velocity which can be considered in stationary waves produced by a vibrating strings.

Consider a string of length L stretched between two fixed supports, then plucked at the middle point. The wave will travel in both directions and will be reflected at each end as shown in Figure 8.8. As adjacent nodes of a standing wave are $\frac{\lambda}{2}$ apart, then, wavelength of the first, second and third harmonics of the string is

$$L = \frac{\lambda_1}{2}, \frac{2}{2}\lambda_2, \frac{3\lambda_3}{2} \ldots$$

Therefore, if λ_n is the wavelength of the n^{th} harmonic, then,

$$\lambda_n = \frac{2L}{n}, \text{ where } n = 1,2,3\ldots \qquad (8.21)$$

The frequency f_n of the n^{th} harmonic is given by;

$$f_n = \frac{v}{\lambda_n} \qquad (8.22)$$

where, v is the velocity of either of the progressive waves that the wave produced.

From equation (8.21) and (8.22),

$$f_n = \frac{nv}{2L} \qquad (8.23)$$

From equation (8.23), when $n = 1$, f_1 is the frequency of the fundamental and is called the first harmonic.

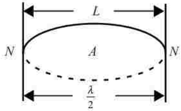
Therefore, equation (8.23) can be written as;

$$f_n = nf_1 \qquad (8.24)$$

From equation (8.24), when $n = 2$, $f_2 = 2f_1$, which is the frequency of the first overtone or second harmonic, and so on.

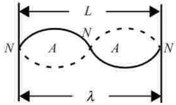
From equation (8.4) and (8.24), the frequency of the n^{th} harmonic for a vibrating string with tension T can be written as;

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad (8.25)$$



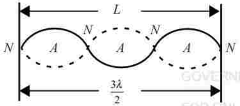
1st harmonic (fundamental)

(a)



2nd harmonic (1st overtone)

(b)



3rd harmonic (2nd overtone)

(c)

Figure 8.8 Vibration modes of a stretched string

Example 8.6

A thin wire of length 75.0 cm has a mass of 16.5 g. One end is tied to a nail and the other end is attached to a screw that can be adjusted to vary the tension in the wire. To what tension must the screw be adjusted so that a transverse wave of wavelength 3.33 cm makes 875 vibrations per second?

Solution

From equation (8.4); $v=\sqrt{\frac{T}{\mu}}$

and $v=f\lambda$

$$T=\frac{f^2\lambda^2 m}{L}$$

$$=\frac{(875\ \text{s}^{-1})^2\times(3.33\times10^{-2}\ \text{m})^2\times(16.5\times10^{-3}\ \text{kg})}{(75\times10^{-2}\ \text{m})}$$

$$T=18.7\ \text{N}$$

Therefore, the screw must be adjusted to 18.7 N in order the transverse wave of wavelength 3.33 cm making 875 vibrations per second.

Example 8.7

A piano tuner stretches a steel piano wire with a tension of 800 N. The steel wire is 0.40 m long and has a mass of 3.0 g ,

(a) What is the frequency of its fundamental mode of vibration?

(b) What is the number of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to 10207 Hz ?

Solution

(a) From equation (8.23) the fundamental mode of vibration

$$f_1=\frac{1}{2L}\sqrt{\frac{T}{\mu}}=\frac{1}{2\times0.4}\sqrt{\frac{800\ \text{N}\times0.4\ \text{m}}{3\times10^{-3}\ \text{kg}}}$$

$$=408.25\ \text{Hz}$$

Therefore, the fundamental mode of vibrations is 408.25 Hz.

(b) From equation (8.23), number of harmonics is given as;

$$n=\frac{2Lf_n}{v}=2Lf_n\sqrt{\frac{\mu}{T}}$$

$$=2\times0.4\ \text{m}\times10207\ \text{Hz}\times\sqrt{\frac{3\times10^{-3}\ \text{kg}}{0.4\ \text{m}\times800\ \text{N}}}$$

$$=25$$

Physics Form V.indd 247 23/07/2020 09:22

Therefore, there are 25 harmonics that could be heard by a person who is capable of hearing frequencies up to 10207 Hz.

Example 8.8

A string of length 2 m and mass $6.0\times10^{-4}\text{ kg}$, fixed at both ends, is under a tension of 20 N. It is plucked at a point 20 cm from one end. What would be the frequency of vibration of the string?

Solution

The plucked string will make an antinode A at the plucking point and each end of the string will be a node N. The AN (distance between A and N) is given as:

$$AN = \frac{\lambda_n}{4} = \frac{2L}{4n}, \text{ where } \lambda_n = \frac{2L}{n}$$

Thus, $n = \frac{L}{2AN} = \frac{2\text{ m}}{2\times0.2\text{ m}} = 5$

Therefore, the string will vibrate in the third harmonic.

From equation (8.25), the frequency of vibrations is

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} = \frac{5}{2\times2\text{ m}}\sqrt{\frac{20\text{ N}\times2\text{ m}}{6.0\times10^{-4}\text{ kg}}}$$

$$= 322.7\text{ Hz}$$

Example 8.9

A string under a tension of 129.6 N produces 10 beats/second when it vibrates with a tuning fork. When the tension of the string is increased to 160 N, it vibrates in unison with the same tuning fork. Calculate the frequency of the tuning fork.

Solution

From equation (8.25), the frequency of vibration can be written as $f \propto \sqrt{T}$, thus, frequency of the tuning fork f_2 relates to the frequency of vibration of the string f_1 as,

$$f_1 = \sqrt{\frac{T_1}{T_2}} \times f_2 = \sqrt{\frac{129.6\text{ N}}{160\text{ N}}} \times f_2 = 0.9f_2$$

Using equation (8.8) and a given frequency of beat $f_b = 10\text{ beat/s}$, the frequency of the tuning fork is,

$$f_2 = f_b + f_1 = 10\text{ Hz} + 0.9f_2,$$

$$f_2 = 100\text{ Hz}$$

Therefore, the frequency of vibration of the tuning fork is 100 Hz.

Exercise 8.2

1. Describe the distinguishing features between travelling (progressive) waves and stationary (standing) waves.
2. Two waves travel on the same string. Is it possible for them to have different wavelength? Explain your answer.
3. Energy can be transferred along a string by wave motion. However, in a standing wave on a string, no energy can ever be transferred past a node. Why?
4. One of the harmonic frequencies for a particular string under tension is 325 Hz. The next higher harmonic frequency is 390 Hz. What is next higher harmonic frequency after the harmonic frequency 195 Hz?

5. A nylon guitar string has a linear mass density of $7.20\ \text{gm}^{-1}$ and is under tension of $150\ \text{N}$. The fixed supports are at a distance of $90.0\ \text{cm}$ apart. The string is oscillating in third harmonic mode. Calculate:
   (a) The speed;
   (b) The wavelength; and
   (c) The frequency

   of the travelling waves whose superposition gives this standing wave.
6. What is the fastest transverse wave that can be sent along a steel wire? For safety reasons, the maximum tensile stress for steel is $7.00\times10^{8}\ \text{Nm}^{-2}$. The density of steel is $7800\ \text{kgm}^{-3}$.
7. A block of mass $5\ \text{kg}$ is hanging vertically from a free end of the rope of length $10\ \text{m}$. A transverse pulse of wavelength $0.05\ \text{m}$ is produced at the lower end of the rope. What is the wavelength of the pulse when it is $2\ \text{m}$ below the vertical support?
8. A progressive and a stationary simple harmonic wave each has the same frequency of $250\ \text{Hz}$ and the same velocity of $30\ \text{ms}^{-1}$. Calculate:
   (a) The phase difference between the two vibrating points on the progressive wave which are $10\ \text{cm}$ apart;
   (b) The equation of motion of the progressive wave if its amplitude is $0.03\ \text{m}$;
   (c) The distance between nodes in the stationary wave; and
   (d) The equation of motion of the stationary wave if its amplitude is $0.01\ \text{m}$.
9. The displacement equation of a transverse wave on a string is $y = 2.0\ \text{mm}\times\sin\left((20\ \text{m}^{-1})x-(600\ \text{s}^{-1})\right)$. The tension in the string is $15\ \text{N}$.
   (a) What is the wave speed?
   (b) Find the linear mass density of this string.
10. A string fixed at both ends is $8.40\ \text{m}$ long and has a mass of $0.120\ \text{kg}$. It is subjected to a tension of $96.0\ \text{N}$ and allowed to oscillate.
    (a) What is the speed of the waves of the string?
    (b) What is the longest possible wavelength for a standing wave?
    (c) Find the frequency of this wave.
11. A string that is stretched between two fixed supports separated by $75.0\ \text{cm}$ has resonant frequencies of $420\ \text{Hz}$ and $315\ \text{Hz}$, with no intermediate resonant frequencies.
    (a) What is the lowest resonant frequency?
    (b) What is the wave speed?
12. A piano string having a diameter of $0.90\ \text{mm}$ is replaced by another string of the same material but with diameter $0.93\ \text{mm}$. If the tension of the wire is the same as before,
    (a) what is the percentage change in the frequency of the fundamental note?
    (b) what percentage change in the tension would be necessary to restore the original frequency?

8.3 Sound waves

The most important waves in your everyday life are sound waves. The reason is that, the human ear is tremendously sensitive and can detect sound waves even of very low intensity. Sound waves are longitudinal waves in a medium and can be described in terms of pressure fluctuation. Sound waves travel through gases and liquids in the form of compression and rarefactions (Figure 8.9a). Sound can travel in solids as both longitudinal waves and transverse waves. They usually travel out in all directions from the source of sound, with an amplitude, frequency and wavelength that depends on the direction and distance from the source. The displacement and pressure on the particles in a medium is as shown in Figure 8.9 (b) and (c) respectively. At the nodes N, the particles are at the minimum displacement with maximum pressure, whereas, at the antinode A, the particles possess maximum displacement with minimum pressure. Examples of sources of sound waves include sounding brass, pipes and tuning forks. In this section, you will learn how to derive velocity of sound in materials, determine velocity of sound in air and describe the applications of mechanical vibrations and waves.

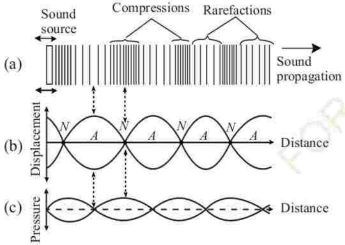


Figure 8.9 Sound wave in a material

8.3.1 Velocity of sound in materials

The speed of sound depends on both an inertia property μ of the medium (to restore kinetic energy) and an elastic property T of the medium (to store potential energy). Thus the speed of sound v can be generalized as;

$$v = \sqrt{\frac{T}{\mu}} \qquad (8.26)$$

(a) Velocity of sound in solids

When sound waves travel in a solid medium, the particles in the medium are subjected to varying stresses, with resulting strains. Thus, the speed of sound wave is governed by the Young's modulus of elasticity Y and the inertia property which depends on the density ρ "massiveness" of a bulk of a medium. Hence, equation (8.26) can be written as;

$$v = \sqrt{\frac{Y}{\rho}} \qquad (8.27)$$

Equation (8.27) can be verified by the method of dimensional analysis.

(b) Velocity of sound in liquid

The velocity of sound in liquid is also govern by equation (8.26), where elastic property is the bulk modulus K of the liquid and the inertia property is the density ρ of the liquid.

Hence, equation (8.26) for speed of sound in liquid can be written as;

$$v=\sqrt{\frac{K}{\rho}} \qquad (8.28)$$

(c) Velocity of sound in gas

The velocity of sound in a gas can be determined using equation (8.26), where the elastic property is the bulk modulus of the gas with pressure P, and the inertia property is the density ρ, of the gas. Experiments show that when a sound wave propagates through a gas, the temperature of the gas changes and therefore the propagation of a sound wave in a gas is an adiabatic process. Under the conditions in which a sound wave travels in a gas, the bulk modulus K is given by, $K=\gamma P$, where γ is the ratio of its specific heat capacities. That is;

$$\gamma=\frac{c_p}{c_v}$$

where c_p and c_v are the specific heat capacities of a gas at constant pressure and constant volume respectively.

Hence, for speed of sound in liquid shown by equation (8.26) can be written as;

$$v=\sqrt{\frac{\gamma P}{\rho}} \qquad (8.29)$$

Thus, the velocity of sound at S.T.P in a gaseous medium of density $\rho_{air}=1.29\ \text{kgm}^{-3}$, $P=760\ \text{mm Hg}$ and $\gamma=1.4$ is;

$$v=\sqrt{\frac{\gamma P}{\rho}}=\sqrt{\frac{1.4\times 0.76\ \text{m}\times 13600\ \text{kgm}^{-3}\times 9.8\ \text{ms}^{-2}}{1.29\ \text{kgm}^{-3}}}$$

$$\approx 332\ \text{ms}^{-1}$$

Therefore, the velocity of sound in a gaseous medium at S.T.P is $332\ \text{ms}^{-1}$.

Now, consider one mole of an ideal gas having volume V and pressure P at temperature T. The ideal gas equation of state, $PV=RT$, where R is a constant, if M is the molar mass of the gas, then, density $\rho=\frac{M}{V}=\frac{MP}{RT}$. From equation (8.29), the speed of sound in a gas can be written as;

$$v=\sqrt{\frac{\gamma RT}{M}} \qquad (8.30)$$

Since γ, R and M are constants for a given gas, it follows that the velocity of sound in a gas is independent of pressure, if the temperature remains constant. Then

$$v\propto\sqrt{T} \qquad (8.31)$$

Therefore, the velocity of sound in a gas is proportional to the square root of its absolute temperature.

Note that, sound travels faster in liquids than in gases and even faster in solids than in liquids because molecules are much closer in solids than in liquids and closer in liquids than in gases.

Example 8.10

The velocity of sound in a material depends on the Young's modulus Y and density ρ of the material.

(a) Use the method of dimension analysis to obtain the relationship between v and Y.

(b) Use the relation from (a) above to find the velocity of sound in

(i) a steel of density 7800 kgm^{-3} and elasticity of 2.0×10^{11} Nm^{-2}.

(ii) water of density 1000 kgm^{-3} and bulk modulus 2.04×10^{9} Nm^{-2}.

Solution

(a) Using method of dimension, $v=\sqrt{\frac{Y}{\rho}}$

(b) (i) $v=\sqrt{\frac{Y}{\rho}}=\sqrt{\frac{2\times10^{11}\ \text{Nm}^{-2}}{7800\ \text{kgm}^{-3}}}$

$=5064\ \text{ms}^{-1}$

(ii) $v=\sqrt{\frac{K}{\rho}}=\sqrt{\frac{2.04\times10^{9}\ \text{Nm}^{-2}}{1000\ \text{kgm}^{-3}}}$

$=1428.3\text{ms}^{-1}$

Therefore, the velocities of sound in a steel and water are $5064\ \text{ms}^{-1}$ and $1428.3\ \text{ms}^{-1}$ respectively.

Example 8.11

Calculate the velocity of sound in air at 100°C if the density of air at S.T. P. is $1.29\ \text{kgm}^{-3}$, the density ρ_m of mercury at 0°C is $13600\ \text{kgm}^{-3}$ and the ratio of heat capacities of air, γ is 1.41.

Solution

From equation (8.29), the velocity of sound in air v_0 at 0°C;

$$v_0=\sqrt{\frac{\gamma P}{\rho}}=\sqrt{\frac{\gamma h\rho_m g}{\rho_{air}}}$$

$$=\sqrt{\frac{1.41\times0.76\ \text{m}\times13600\ \text{kgm}^{-3}\times9.8\ \text{ms}^{-2}}{1.29\ \text{kgm}^{-3}}}$$

$=332.7\ \text{ms}^{-1}$

From equation (8.31), the velocity of sound in air at 100°C is;

$$v_{100}=\sqrt{\frac{T_{100}}{T_0}}\times v_0=\sqrt{\frac{373\ \text{K}}{273\ \text{K}}}\times332.7\ \text{ms}^{-1}$$

$=389\ \text{ms}^{-1}$

Therefore, the velocity of sound in air at 100°C is $389\ \text{ms}^{-1}$.

Example 8.12

The wavelength of the note emitted by a tuning fork with frequency of 512 Hz in air at 17 °C is 66.5 cm. If the density of air at S.T.P. is $1.293\ \text{kgm}^{-3}$, calculate the ratio of the two specific heat capacities of air. Density of mercury at S.T.P is $13600\ \text{kgm}^{-3}$.

Solution

From equation (8.31), (8.29) and $v=f\lambda$, the ratio of two specific heat capacities of air γ is

$$\gamma=\frac{\rho_{air}(\lambda f)^2T_0}{\rho_{Hg}hgT}$$

$$=\frac{1.293\ \text{kgm}^{-3}\times(512\ \text{Hz}\times0.665\ \text{m})^2\times273\ \text{K}}{0.76\ \text{m}\times13600\ \text{kgm}^{-3}\times290\ \text{K}\times9.8\ \text{ms}^{-2}}$$

$=1.39$

Therefore, the ratio of two specific heat capacities of air is 1.39.

8.3.2 Determination of the velocity of sound in air

When a tuning fork is struck and placed immediately over the opening of a tube of suitable length opened at one end, and whose position in the water can be raised or lowered (Figure 8.10(a)), a point is reached where the column of air in the tube vibrates with the fork and a loudest note is heard. At this point, the frequency of vibration of air column is equal to the frequency of the fork (resonance). The motion of air in the tube is a succession of plane wave pulses sent from the fork and reflected at the water surface. The condition for resonance is that the reflected wave must be out of phase with the emitted wave by π-radians. The resonance will be most powerful, if the time the pulse takes to travel to the water surface and back to the fork is exactly half the periodic time of the vibration of the fork.

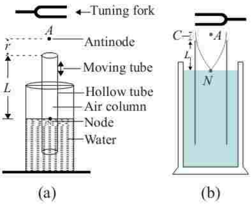


Figure 8.10 Measuring speed of sound using resonance tube

The pulse travels along the tube with constant velocity of sound in air. The condition for resonance in this case, is that, length of the air column is equal to the quarter wavelength of the sound waves (Figure 8.10b). That is,

$$L = \frac{\lambda}{4} \qquad (8.32)$$

Equation (8.32) may not be obviously accurate since the air at the open end of the pipe is free to move, causing the vibrations at the end of sounding pipe to extend a little to air outside the pipe (Figure 8.10(b)).

A correction should be done for the open end of the pipe. This correction has been calculated theoretically, and has been shown to be nearly equivalent to increasing the observed length of the resonance column by an amount equal to one half of its diameter. Introducing this correction, equation (8.32) becomes

$$L_1 + C = \frac{\lambda}{4} \qquad (8.33)$$

where C is the end correction.

Similarly, the first overtone (second harmonic) of the same tuning fork can be written as,

$$L_2 + C = \frac{3\lambda}{4} \qquad (8.34)$$

Subtracting equation (8.33) from equation (8.34) you get;

$$L_2 - L_1 = \frac{\lambda}{2} \qquad (8.35)$$

Thus, the velocity of sound in air column $(v = f\lambda)$ is written as,

$$v = 2f(L_2 - L_1) \qquad (8.36)$$

Therefore, knowing the length L_1 for fundamental and L_2 for first overtone (second harmonic) of the air columns, and the frequency of the tuning fork, the velocity of sound in air can be easily determined.

Note that, the end correction C is the length that must be added on to the length of the pipe to take account of antinodes extending beyond the open end of the pipe. Thus, for the closed pipe the length of the pipe together with its end correction is given by $L_C = L + C$.

Experiments show that $C = 0.6r$ where r is the radius of a pipe. Thus, the above equation can be written as $L_C = L + 0.6r$.

For an open pipe the length of a pipe will be $L_C =$ L+2C $= L + 1.2r$.

Activity 8.1

To determine velocity of sound in air using tuning fork of different frequencies.

Materials: Tuning forks of different known frequencies, large glass jar, glass tube, water.

Procedure

(a) Fill three quarter of the large glass jar with water

(b) Immerse the glass tube in the glass jar with water such that air column L can be created within the glass tube

(c) Strike and hold the tuning fork over the open end of the glass tube.

(d) Lower or raise the tube until a loud note is heard.

(e) Measure and record the length at which the note is loudest (Figure 8.10b).

(f) Repeat procedure (a) - (e) with the forks of different frequency to obtain six different measurements.

(g) Tabulate your results as shown in Table 8.2.

Table 8.1 Frequency of tuning fork and length

Frequency of tuning forks $f\,(Hz)$	Length of air column $L(cm)$	$\frac{1}{f}(s^{-1})$

Questions

(a) Use your results to plot a graph of the length L of the air column against $\frac{1}{f}$.

(b) Find the slope of the graph.

(c) The velocity of sound in air column for forks of different frequencies is governed by equation, $L = \frac{v}{4f} - C$.

Find the velocity of sound of air.

(d) Find the value of L-intercept and suggest its physical significance.

The resonant tube treated in the preceding discussion can be considered as a closed pipe resonator. A closed pipe or organ consists of a metal pipe closed at one end Q. When a blast of air is blown into it at the open end P (Figure 8.11a), a sound wave travels up the pipe to Q, and is reflected at this end down the pipe. So, a stationary wave is formed by superposition between the two waves.

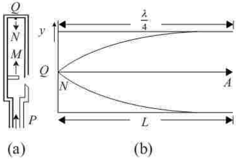


Figure 8.11 (a) Closed organ pipe, and (b) fundamental mode of closed pipe

The end Q of the closed pipe must be a node N since the layer in contact with Q must be permanently at rest. The open end P, where the air is free to vibrate, must be an antinode A (Figure 8.11b). The distance between node and antinode (Figure 8.11b) can be written as;

$$L = \frac{\lambda}{4} \quad (8.37)$$

From equation (8.37), the frequency, $f = \frac{v}{\lambda}$ of the note can be written as;

$$f = f_0 = \frac{v}{4L} \quad (8.38)$$

Equation (8.37) is the frequency of the lowest note of a closed pipe termed as fundamental frequency f_0 or first harmonic. Thus, the velocity of sound in a closed pipe for the frequency f_0 is;

$$v = 4Lf_0 \quad (8.39)$$

If a stronger blast of air is blown into a closed pipe, notes of higher frequency called overtones which are simple multiples of the fundamental frequency, f_0 can be obtained. Consider an overtone formed when air is blown into a closed pipe at high frequency. The length L can be written as;

$$L = \frac{3\lambda_1}{4} \quad (8.40)$$

From equations (8.39) and (8.40), the frequency f_1 can be written as:

$$f_1 = \frac{3v}{4L} \quad (8.41)$$

Equations (8.38) and (8.41) frequency f_1 can be written as;

$$f_1 = 3f_0 \quad (8.42)$$

Equation (8.42) is called first overtone or third harmonic of a closed pipe and is represented by Figure 8.12.

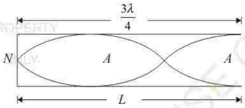


Figure 8.12 First overtone of a closed pipe

Similarly, the frequencies of the second, third, fourth (and so on) overtones of a closed pipe can be shown to be $3f_0, 5f_0$, $7f_0$ and so on, which are odd multiples of the fundamental frequency. Generally, for a closed pipe, the overtone of the higher frequencies f_n is given as:

$$f_n = (2n+1)f_0 \quad (8.43)$$

where n is the number of overtones, that is $n = 1, 2, 3...$ and $2n-1$ are the n^{th} harmonics.

Then, it follows that, the velocity v of sound of overtones of wavelength λ_n in a closed pipe in general is given as:

$$v = (2n+1) f_0 \lambda_n \qquad (8.44)$$

Example 8.13

(a) Draw a sketch diagram showing the position of nodes and antinodes for the vibration of an air column in a pipe closed at one end when giving the second overtone. Calculate the frequency of this second overtone if the effective length of the pipe is 72 cm. Use velocity of sound in air as $340\ \text{ms}^{-1}$.

(b) A small loudspeaker is placed near the open end of a pipe of length 400 mm closed at its other end. The minimum frequency at which the pipe resonate is 215 Hz.

(i) Estimate the speed of sound in the pipe.

(ii) Calculate the next highest frequency for resonance.

Solution

(a) For second overtone, $n = 2$, and substituting equation (8.38) into (8.43), the frequency f_2 is

$$f_2 = \frac{5v}{4L} = \frac{5 \times 340\ \text{ms}^{-1}}{4 \times 0.72\ \text{m}}$$
$$= 590.28\ \text{Hz}$$

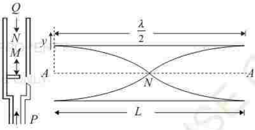
(b) (i) Using equation (8.39),

$$v = 4 \times 0.4\ \text{m} \times 215\ \text{s}^{-1}$$
$$= 344\ \text{ms}^{-1}$$

(ii) The next highest frequency occurs at the first overtone ($n = 1$). Using equation (8.41); frequency of the first overtone is,

$$f_1 = \frac{3v}{4L}$$
$$= \frac{3 \times 344\ \text{ms}^{-1}}{4 \times 400 \times 10^{-3}\ \text{m}} = 645\ \text{Hz}$$

When air is blown into an open pipe (pipe open at both ends), a wave M travels to the open end Q, the wave is reflected in the direction N on encountering the free air (Figure 8.13a). A stationary wave is therefore set up in the air within the pipe. The two ends of the pipe are both antinodes and the nodes midway of the pipe (Figure 8.13b).



(a) (b)

Figure 8.13 (a) Open organ pipe and, (b) fundamental mode of open pipe

The length L of the pipe which is the distance between consecutive antinodes (Figure 8.13b) can be written as:

$$L = \frac{\lambda}{2} \qquad (8.45)$$

Then, frequency, $f = \frac{v}{\lambda}$ of the note can be written as:

$$f = f_0 = \frac{v}{2L} \qquad (8.46)$$

Equation (8.46) gives the frequency of the lowest note of an open pipe termed as fundamental frequency f_0 or first harmonic. Thus, the velocity of sound in an open pipe for the frequency f_0 is

$$v = 2Lf_0 \qquad (8.47)$$

If a stronger blast of air is blown into an open pipe, notes of higher frequencies called overtones can be obtained which are simple multiples of the fundamental frequency f_0. Consider an overtone formed when air is blown into an open pipe at a high frequency. The length, L is given by;

$$L = \lambda_1 \qquad (8.48)$$

From equation (8.46), the frequency f_1 can be written as;

$$f_1 = \frac{v}{L} \qquad (8.49)$$

From equations (8.46) and (8.49) frequency f_1 can be written as;

$$f_1 = 2f_0 \qquad (8.50)$$

Equation (8.50) is called first overtone or second harmonic of an open pipe and is represented by Figure 8.14.

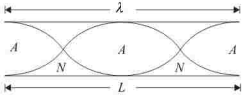


Figure 8.14 First overtone of an open pipe

Similarly, the frequencies of the first, second and third (and so on) overtones of an open pipe are; $2f_0$, $3f_0$, $4f_0$ and so on. Generally, for an open pipe, the overtone of the higher frequencies f_n is given as;

$$f_n = (n+1)f_0 \qquad (8.51)$$

where n is the number of overtones, that is $n = 1, 2, 3...$ and n are the n^{th} harmonics.

Then, it follows that, the velocity, v of sound for overtones of wavelength λ_n in an open pipe is given as;

$$v_n = (n+1)f_0\lambda_n \qquad (8.52)$$

In summary, for a given length of a pipe, the open pipe frequency is twice that of the closed pipe frequency and an open pipe gives more harmonics (odd and even) than a closed pipe (odd only).

Example 8.14

An open pipe 30 cm long and a closed pipe 23 cm long both of the same diameter, each sounding its first overtone and are in unison. What is the end correction of the pipes?

Solution

Since the two pipes are in unison, then the frequency of first overtone f_2 of an open pipe is equal to the first overtone f_3 of a closed pipe. Then from equations (8.50) and (8.42) that is $f_2 = f_3$, then,

$$\frac{v}{L_o + 2C} = \frac{3v}{4(L_c + C)}$$

Thus, the end correction C is

$$C = \frac{4L_C - 3L_O}{2}$$

$$= \frac{(4 \times 23\text{ cm}) - (3 \times 30\text{ cm})}{2} = 1\text{ cm}$$

Therefore, the end correction is 1cm, where L_c is the length of the closed pipe and L_O is the length of the open pipe.

Example 8.15

Two open ended organ pipes are sounded together in their first harmonics and are heard. If the first pipe is of length 0.80 m, what is the length of the other pipe? Ignore any end corrections and take speed of sound in air as $320\ \text{ms}^{-1}$.

Solution

From equations (8.49) and (8.50), the frequency f_s of the first pipe of length L_s is

$$f_s = \frac{nv}{2L_s} = \frac{1\times 320\ \text{ms}^{-1}}{2\times 0.8\ \text{m}} = 200\ \text{Hz}$$

and for the second pipe f_l with length L_l is

$$f_l = \frac{1}{2}\times\frac{nv}{L_l} = \frac{320}{2L_l}$$

From equation (8.8), and given beat frequency $f_b = 8\,\text{beats/s}$, $f_s - f_l = f_b$ equivalent to

$$200\ \text{s}^{-1} - \frac{320\ \text{ms}^{-1}}{2L_l} = 8;\ \ L_l = 0.83\ \text{m}$$

Then, the length of the second pipe is 0.83 m.

8.3.3 Applications of sound waves

Sound waves have wide applications in daily life. For example, forced vibrations are used to tune musical instruments such as piano, guitar, using beats between string frequencies.

Also, the architect when designing a building has to consider its acoustical demands. For example, when planning construction of a hotel or a radio station, it has to be kept in mind that the penetration of sound from one room to the other has to be negligible.

Similarly, in case of an auditorium, it has to be ensured that the sound is properly diffused and there is no echo. It is a common experience that, a sound produced in a building is reflected repeatedly by its walls and takes some time to die out. This persistence of audible sound after the source has ceased to emit it, is called the reverberation. The time taken by a sound to die out after the source has ceased to emit it is called the reverberation time. A long reverberation time can make a building sound loudy and noisy. Thus, conference halls, mosques, churches and lecture halls should have short reverberation time. The value of this time is large in an unfurnished room compared to a furnished one. The reverberation time can be reduced by applying sound absorbing surfaces on the walls and the ceiling.

In a musical concert, it can easily be experienced that a slight reverberation provides richness to music. Reverberation in a room can be controlled by having the walls covered with absorbent materials, a few open windows, a good audience and a good amount of furniture in the room.

Other important conditions for good acoustical designs of rooms is the shapes of the walls and the ceilings. Curved walls should be avoided because they cause focusing of sound, thereby, concentrating it at one point.

Exercise 8.3

1. Explain the conditions necessary for the formation of stationary waves in air.
2. Under what conditions are beats heard? Derive an expression for their frequency.
3. Calculate the frequency of the sound emitted by an open ended organ pipe 2 m long when sounding its first overtone. Speed of sound in air $v = 340\ ms^{-1}$.
4. Calculate the shortest length of a closed organ pipe which resonate with a 440 Hz tuning fork, neglecting any end correction. Speed of sound in air is $350\ ms^{-1}$.
5. An organ pipe has two successive harmonics with frequencies 1372 Hz and 1764 Hz.
   (a) Is this an open or a closed pipe? Explain.
   (b) What is the length of the pipe?
6. Two organ pipes (both ends open) give 5 beats/s at 10°C. How many beats will be heard per second at 15°C?
7. A uniform 165 N bar is supported horizontally by two identical wires, *A* and *B*. A small 185 N cube of lead is placed three quarters of the way from *A* to *B*. The wires are each 75.0 cm long and have a mass of 5.50 g. If both of them are simultaneously plucked at the center, what is the frequency of the beats that they will produce when vibrating in their fundamental?
8. (a) If the velocity of sound in air at 15°C is $342\ ms^{-1}$, calculate the velocity at 0°C and 47°C.
   (b) What is the velocity if the pressure of the air changes from 75 cm to 95 cm of mercury, the temperature remaining constant at 15°C?
9. An observer looking due north sees the flash of a gun 4 seconds before he records the arrival of the sound. If the temperature is 20 °C and the wind is blowing from east to west with a velocity of 48 km/hr, calculate the distance between the observer and the gun. Speed of sound in air at 0 °C is $330\ ms^{-1}$
10. A tuning fork vibrating with a sonometer wire 20 cm long produces 5 beats per second. Given that the beat frequency does not change if the length of the wire is changed to 21cm, calculate the frequency of the tuning fork.

8.4 Electromagnetic waves

What is light? This question has been asked by humans for centuries, but there was no answer until when electricity and magnetism were unified into electromagnetism, by Maxwell. Through the Maxwell's equations it became clear that a time-varying magnetic field acts as a source of electric field and that a time-varying electric field acts as a source of magnetic field. These fields sustain each other, forming an electromagnetic (EM) wave that propagates through space. Examples of electromagnetic waves include: visible light from different sources, waves produced by TV and radio stations, X-rays, and γ-rays.

Unlike waves on a string or sound waves in a fluid, electromagnetic waves do not require a material medium. Nonetheless, electromagnetic waves and mechanical waves have much in common and are described in much the same language. In this section, you will learn the nature of EM-waves and describe their propagation.

8.4.1 Nature of electromagnetic waves

Consider a plane EM-wave propagating in the positive x-direction in which there are no charges or conduction currents. Suppose a tube has sides parallel to the coordinate planes (Figure 8.15), then, according to Gauss's law, the total electric or magnetic flux through this tube is zero expressed as;

$$\oint \vec{E} \cdot \vec{d}_s = 0 \qquad (8.53)$$

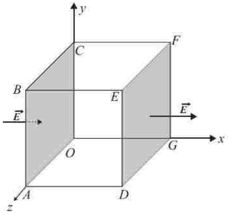


Figure 8.15 Nature of electromagnetic waves

Since it is a plane wave in x direction, the electric field $\vec{E}$ depends on x and t therefore, the net electric flux originates from this face only i.e., $OABC$ and $DEFG$. The electric flux due to field $\vec{E}_1$ is given as $\vec{E}_1 \times \text{area}\, OABC$ and flux due to $\vec{E}_2$ is $\vec{E}_2 \times \text{area}\, DEFG$.

Thus, total flux from equation (8.53) is $A \times (\vec{E}_1 - \vec{E}_2) = 0$, where A is the area of $OABC = DEFG$. Since $A \neq 0$, $\vec{E}_1 - \vec{E}_2 = 0$. There are two possible solutions here, that is $\vec{E}_1 = \vec{E}_2 \neq 0$ or $\vec{E}_1 = \vec{E}_2 = 0$. The possibility $\vec{E}_1 = \vec{E}_2$ means, the $\vec{E}$ associated with EM-wave is constant. However, constant field cannot produce wave, therefore, we ignore this possibility. The possibility, $\vec{E}_1 = \vec{E}_2 = 0$ means, there is no $\vec{E}$-field in the direction of propagation. If EM-wave propagates and there is no $\vec{E}$-field along the direction of propagation, this means, the $\vec{E}$-field propagates perpendicular to the direction of wave propagation. Similar conclusion can be drawn for magnetic $\vec{B}$ field. Since $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation, electromagnetic waves are transverse in nature.

8.4.2 Propagation of electromagnetic waves

Based on Ampere's law and Faraday's law of electromagnetic induction, a varying $\vec{E}$-field produces $\vec{B}$ field varying with time and a varying $\vec{B}$ field produces $\vec{E}$-field varying with time. This implies that, whenever there is a change in $\vec{E}$-field with time, then a $\vec{B}$ field changing with time are produced in space. Similarly, the produced $\vec{B}$ field will produce varying $\vec{E}$-field and so on. The result of alternating $\vec{E}$ and $\vec{B}$ -fields will result in an electric-magnetic disturbance that propagates through space. Using Maxwell's equations, $\vec{E}$ and $\vec{B}$ -fields in EM-waves occur in mutually perpendicular directions (Figure 8.16).

That is $\vec{E}$ and $\vec{B}$ -fields are vibrating perpendicular to each other and both perpendicular to the direction of wave propagation.

The speed of EM-waves in free space (vacuum) is given as: $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$,

where $\mu_0 = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$ and $\varepsilon_0 = 8.854 \times 10^{-12}\ \text{Fm}^{-1}$ are the respective permeability and permittivity of free space.

Thus,

$$c = \frac{1}{\sqrt{4\pi \times 10^{-7}\text{Hm}^{-1} \times 8.85 \times 10^{-12}\ \text{Fm}^{-1}}}$$

$$= 3.0 \times 10^{8}\ \text{ms}^{-1}$$

Therefore, the speed of electromagnetic wave in space is $3.0 \times 10^{8}\ \text{ms}^{-1}$. The calculated speed of EM-waves match precisely with the measured speed of light in free space (vacuum). In this case, light is an electromagnetic wave.

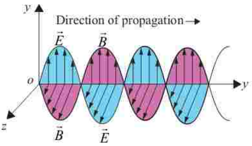


Figure 8.16 Propagation of electromagnetic waves

8.4.3 Electromagnetic spectrum

As already seen earlier, EM-waves exist in wide range of frequencies or wavelengths constituting a series known as the electromagnetic spectrum as shown in Figure 7.22. The spectrum includes gamma rays, X-rays, ultraviolet waves, visible spectrum (light), infrared, microwaves and radio waves. A spectrum of interest is the visible spectrum which consists of seven spectral lines, namely; violet, indigo, blue, green, yellow, orange and red, (Figure 8.17).

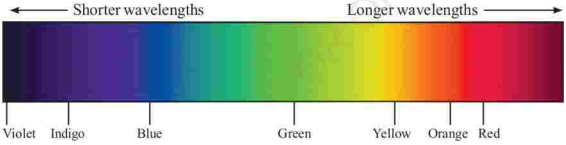


Figure 8.17 Visible light spectrum

8.4.4 Applications of electromagnetic waves

Radio waves are electromagnetic waves which vary in wavelength from a few millimeters to several kilometers. These waves are very useful in global communication. Microwaves are also used in communication for example in mobile phones, and in radar, electron spin resonance studies and in heating. Ultraviolet radiations cause fluorescence and ionization, promoting chemical reactions, affect photographic films and produce photoelectric emissions. Visible light is due to electron transition in atoms. It affects a photographic film, stimulates the retina in the eye and used in photosynthesis. Infrared radiation is due to small energy changes of an electron in an atom or molecular vibrations; it is used for heating, both in homes and in hospitals, and used in devices that emit infrared beams such as camera. X-rays are used in dentistry and medicine, for example checking damaged or fractured body parts; and gamma rays are used to destroy cancer cells.

Exercise 8.4

1. Differentiate between mechanical and electromagnetic waves.
2. Explain the nature of electromagnetic waves.
3. Describe how electromagnetic waves propagate.
4. Explain the applications of electromagnetic waves.
5. Find the speed of EM-in a medium if the relative permeability and permittivity are 1.0 and 2.25 respectively. If the speed of EM-wave in medium is given as $c = \frac{1}{\sqrt{\mu_m \varepsilon_m}}$, where c is the speed of EM waves in vacuum, and μ_m and ε_m the relative permeability and permittivity of medium respectively.

8.5 Physical optics

Optics is a branch of physics which involves the behaviour and properties of light including its interaction with matter. The branch of optics which considers light in terms of rays only is optics; the branch dealing specifically with wave behaviour is called wave optics. Physical optics deal with the physical properties and behaviour of electromagnetic waves and their interactions with matter. In physical optics or wave optics we study interference, diffraction, and polarization of light. In this section, you will learn interference, diffraction and polarization. You will learn how to treat light as waves rather than as rays leading to a satisfying description of such phenomena.

8.5.1 Interference of light

The colour seen in thin films of oil in soap bubbles and roads are due to light undergoing interference. Thus, interference refers to any situation in which two or more waves overlap in space. This section explains the necessary conditions for interference of light and the determination of the wavelength of monochromatic light. It also investigates production of interference by thin films.

Finally, it identifies the applications of interference of light.

(a) Necessary conditions for interference of light

Interference of light occurs when the waves are monochromatic (single wavelength) and the sources are coherent. This means that, they must maintain constant phase with respect to each other, therefore, they must have the same frequency of nearly or equal amplitude. The interference of light can either be constructive or destructive.

(i) Constructive interference of light

Consider two coherent sources of light, S_1 and S_2 in space and that, a point P is at a distance r_1 from S_1 and a distance r_2 from S_2 (Figure 8.18). If waves that leave S_1 and S_2 are in phase, they arrive at P in phase. When the waves reinforce each other, the amplitude of the resultant wave is the sum of the amplitudes of individual waves. This is called constructive interference. It requires that the crest of S_1 overlaps with the crest of S_2. Similarly, the trough of S_1 overlaps with the trough of S_2. For this to happen, the path difference $r_2 - r_1$ for the two sources must be an integral multiple of the wavelength λ, that is;

$$r_2 - r_1 = n\lambda \qquad (8.54)$$

where $n = 0,1,2,3,...$

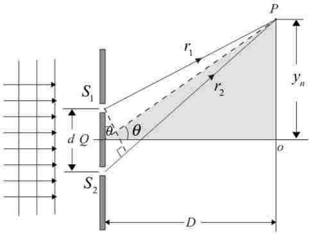


Figure 8.18 Coherent wave interfering at a point in space

(ii) Destructive interference of light

If the waves from S_1 and S_2 (Figure 8.18) arrive at point P out of phase, say, exactly a half-cycle out of phase, the crest of one wave arrives at the same time as the trough of the other. The resultant amplitude is the difference between the two individual amplitudes. If the individual amplitudes are equal, the resultant amplitude is zero. This cancellation or partial cancellation of the individual waves is called destructive interference of light. The condition for destructive interference is thus,

$$r_2 - r_1 = (n + \tfrac{1}{2})\lambda \qquad (8.55)$$

where $n = 0,1,2,3,...$

(b) Determination of wavelength of monochromatic light

The wavelength of monochromatic light can be obtained using Young's double slit or Newton's rings method. Light from a single source can be split so that parts of it emerge from two (or more) regions of space forming two (or more) coherent secondary sources.

(i) Young's double slits experiment

One of the first demonstrations of the interference of light was done by Thomas Young in his experiment called double slit experiment in 1801. Young placed a source S of a monochromatic light in front of narrow slits S_1 and S_2 close to each other. Patterns of bright and dark regions called fringes were observed on the screen (Figure 8.19). The structure of these fringes is known as double slit interference pattern.

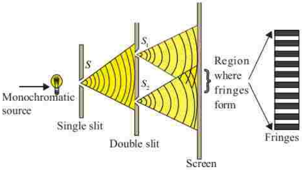


Figure 8.19 Interference by double slit

The interference pattern at any point from the double slit may be observed with a micrometer eyepiece or by placing a screen in the path of the waves. To obtain good pattern, the separation across double slits should be less than 1mm and each slit should have a width of about 0.3 mm. The distance between the double slits and the screen should lie between 50 cm and 100 cm. The source and the double slits must be parallel to produce the optimum interference pattern. The formula relating the dimensions of the apparatus and the wavelength of light may be proved as shown in Figure 8.20.

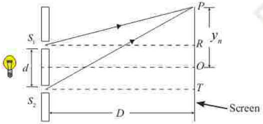


Figure 8.20 Double slit experiment

Consider the triangle S_1PR and S_2PT, using Pythagoras theorem it follows that,

$$(S_2P)^2 = D^2 + \left(y_n + \frac{d}{2}\right)^2 \quad (8.56)$$

$$(S_1P)^2 = D^2 + \left(y_n - \frac{d}{2}\right)^2 \quad (8.57)$$

Subtracting equation (8.56) from (8.57),

$$(S_2P)^2 - (S_1P)^2 = \left(y_n + \frac{d}{2}\right)^2 - \left(y_n - \frac{d}{2}\right)^2 \quad (8.58)$$

Simplifying equation (8.58) gives,

$$(S_2P + S_1P)(S_2P - S_1P) = 2dy_n \quad (8.59)$$

Note that, If the screen is at a very far distance D from the slits as compared to the slit separation d, then, the triangle S_1S_2P is very thin and therefore, $S_2P + S_1P \approx 2D$.

Thus, equation (8.59) becomes

$$S_2P - S_1P = \frac{dy_n}{D} \quad (8.60)$$

The quantity $S_2P - S_1P$ gives the path difference for the two sources S_1 and S_2.

Using equation (8.54) and (8.60), you get

$$r_2 - r_1 = S_2P - S_1P = \frac{dy_n}{D} = n\lambda; \text{ then,}$$

For bright fringes,

$$y_n = \frac{n\lambda D}{d};\ (n = 0,1,2,......) \quad (8.61)$$

Similarly, using equation (8.55) and (8.60), you get,

$$r_2 - r_1 = S_2P - S_1P = \frac{dy_n}{D} = \left(n+\frac{1}{2}\right)\lambda;$$

then for dark fringes,

$$y_n = \frac{\left(n+\frac{1}{2}\right)\lambda D}{d} \quad (8.62)$$

where $n = 0,1,2,...$ of which the first dark fringe is obtained and so on.

The fringe width (fringe spacing) which is the distance between any two consecutive bright fringes or two consecutive dark fringes can be obtained from Figure 8.18 as follows. If P is the position of the n^{th} bright fringes, then $S_2P - S_1P = n\lambda$. In practice, S_1S_2 is very small compared to PQ, it then follows that, angle PQO = angle $S_2S_1N = \theta$.

From triangle S_2S_1N; $\sin\theta = \frac{S_2N}{S_1S_2} = \frac{n\lambda}{d}$

and from triangle PQO;

$$\tan\theta = \frac{S_1S_2}{QO} = \frac{y_n}{D}.$$

Since θ is very small, $\tan\theta \approx \sin\theta \to \theta$ Therefore,

$$y_n = \frac{n\lambda D}{d} \quad (8.63)$$

If R is the neighboring $(n-1)^{th}$ bright fringe, then,

$$y_{n-1} = \frac{(n-1)D\lambda}{d} \quad (8.64)$$

The separation $y = y_n - y_{n-1}$ between successive fringes is obtained by subtracting equation (8.64) from (8.63), that is;

$$y = y_n - y_{n-1} = \frac{\lambda D}{d} \quad (8.65)$$

Therefore, the wavelength of the monochromatic waves used in the Young's experiment is;

$$\lambda = \frac{yd}{D} \quad (8.66)$$

Example 8.16

Young's experiment is performed with a light of wavelength 502 nm. Fringes are measured carefully on a screen 1.20 m away from the double slits, and the center of the 20^{th} bright fringe is found to be 10.6 mm from the centre of the central bright fringe. What is the separation of the two slits?

Solution

The separation of the two slit can be obtain from equation (8.61) as;

$$d = \frac{n\lambda D}{y_n}$$

$$= \frac{20\times502\times10^{-9}\ \text{m}\times1.20\ \text{m}}{10.6\times10^{-3}\ \text{m}}$$

$$= 1.14\times10^{-3}\ \text{m}$$

Therefore, the separation of the two slit is 1.14×10^{-3} m.

Example 8.17

Coherent light with a wavelength of 600 nm passes through two very narrow slits and an interference pattern is observed on a screen. The first order bright fringe is marked as P on the screen. What wavelength of light will the first order dark fringe be observed at point P?

Solution

Since the bright and the dark fringe is observed at the same point P, equation (8.59) is equal to equation (8.60); then,

$$\frac{n\lambda_1 D}{d} = \frac{\left(n+\frac{1}{2}\right)\lambda_2 D}{d}.$$

The wavelength λ_2 of light of the first order fringe at point P is ;

$$\lambda_2 = \frac{n\lambda_1}{n+\frac{1}{2}} = \frac{1\times 600\times 10^{-9}\text{ m}}{0+\frac{1}{2}}$$

$$= 1.2\times 10^{-6}\text{ m}$$

Where $n = 1$ for first order bright fringe and $n = 0$ for fist order dark fringes. Therefore, the wavelength of light in which the first order dark fringe observed at point P is 1.2×10^{-6} m.

Example 8.18

Coarsen light $\lambda = 51000\,\text{Å}$ in a Young's double slit experiment gives 12 fringes in 1.8 cm on a screen at 1.5 m. Find out the distance between the coherent sources.

Solution

Using equation (8.64), the distance between the coherent sources for n fringes is;

$$d = \frac{nD\lambda_n}{y}$$

$$= \frac{12\times 1.5\text{ m}\times 5100\times 10^{-10}\text{ m}}{1.8\times 10^{-2}\text{ m}}$$

$$= 5.10\times 10^{-4}\text{ m}$$

Therefore, the distance between the coherent sources for 12 fringes is 5.10×10^{-4} m.

Fringe shift

Consider a thin transparent plate of thickness t and refractive index μ being introduced in the path of one of the interfering waves (Figure 8.21).

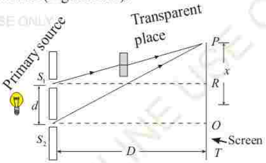


Figure 8.21 Fringe shift experiment

The effective path in air is increased by $(\mu-1)t$ due to the introduction of the plate. Therefore,

$$\text{path difference} = S_2P - \left[S_1P + (\mu-1)t\right],$$

$$S_2P - S_1P - (\mu-1)t = \frac{xd}{D} - (\mu-1)t$$

For maxima, path difference $= n\lambda$, where $n = 0,1,2,\ldots$..position of n[th] maxima is now;

$$\frac{x_n d}{D}-(\mu-1)t=n\lambda,\ x_n=\frac{D}{d}\left[n\lambda+(\mu-1)t\right]$$

Fringe width, $\beta=x_{n+1}-x_n=\frac{D\lambda}{d}$. In the absence of the plate (i.e. $t=0$), the position of the nth maxima is: $x_n=\frac{nD\lambda}{d}$.

Displacement of fringes is now given by;

$$\Delta x=\frac{D}{d}\left[n\lambda+(\mu-1)t\right]-\left[\frac{nD\lambda}{d}\right],$$

$$\Delta x=\frac{\beta}{\lambda}(\mu-1)t$$

Therefore, with an introduction of the transparent plate in the path of one of the slits, the entire fringe pattern is displaced a distance, $\frac{D}{d}(\mu-1)t$ or $\frac{\beta}{\lambda}(\mu-1)t$ towards the side on which the plate is placed providing a shift in the interference pattern given as;

$$\Delta x=\frac{D}{d}(\mu-1)t$$

(ii) Newton's rings experiment

Newton's rings are interference patterns first studied by Isaac Newton. On the basis of wave theory of light, the rings were first correctly accounted for by Thomas Young. A convex lens is placed on a plane glass plate, and a thin film of air is enclosed between the lower surface of the lens and the upper surface of the plate. The thickness of the air film is very small at the point of contact and gradually increases outwardly (Figure 8.22).

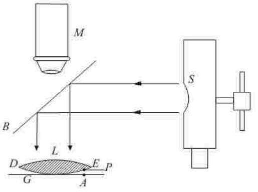


Figure 8.22 Set up for Newton's rings

A horizontal beam of light falls on glass plate B inclined at 45°. The glass plate B reflects part of the incident light towards the air film enclosed by the lens L and the plane glass plate G. The reflected beam from the air film is viewed with a microscope M. The interference between the light reflected from the lower surface of the lens DPE and the upper surface of the glass plate G results into interference pattern observed as circular fringes. Therefore, the two rays of light have net path difference of $2t$ where $t = PA$. The phase change adds an extra path of half a wavelength (that is, a crest is reflected as a trough) because there is a phase change of 180° when the wave-train is reflected at the top surface of the glass slide G. Therefore, the path difference between the two wave trains at P is $2t+\frac{\lambda}{2}$ where λ is the wavelength of the light. A bright fringe is formed at P when $2t=(2n-1)\frac{\lambda}{2}$ and the dark fringe is formed if; $2t=n\lambda$.

Therefore, the central spot (for which $t = 0$) appears dark instead of bright as the geometrical path difference is zero. The dark spot occurs because one of the interfering rays has undergone a phase shift π equivalent to $\frac{\lambda}{2}$. The theory of radius of Newton's rings can be used to determine the wavelength λ of light source if the radius of curvature a of the lower surface of the lens is known (Figure 8.23).

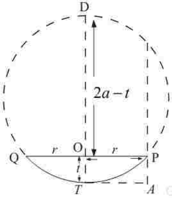


Figure 8.23 Radius of Newton's rings

From the theorem of intersecting chords $(\text{TO}) \times (\text{OD}) = (\text{QO}) \times (\text{OP})$ which gives,

$$2at - t^2 = r^2 \qquad (8.67)$$

But t^2 is very small compared to $2at$ as a is large, then (8.67) becomes $2at = r^2$ and

$$2t = \frac{r^2}{a} \qquad (8.68)$$

$$\text{or } 2t = \frac{D^2}{4a} \qquad (8.69)$$

where r is the radius of the ring and D is its corresponding diameter. Combining equation $2t = n\lambda$ and $2t = \frac{r^2}{a}$, the dark fringes can be obtained when,

$$n\lambda = \frac{r_n^{\,2}}{a} \qquad (8.70)$$

where $n = 1, 2, 3\ldots$.

Combining equation $2t = (2n-1)\frac{\lambda}{2}$ and $2t = \frac{r^2}{a}$, bright fringes can be obtained when,

$$(2n-1)\frac{\lambda}{2} = \frac{r_n^{\,2}}{a} \qquad (8.71)$$

where $n = 1, 2, 3\ldots$

In this measurement, the diameters of the rings are used rather than their radii because it is difficult to locate the exact center of the central spot. In addition, on counting the order of the dark rings, the central ring (spot) is not counted. Using equation (8.68), (8.69) and (8.70),

$$D_n^{\,2} = 4a\lambda n \qquad (8.72)$$

If $D_n^{\,2}$ for several rings are plotted against n, then the wavelength λ of light can be determined from the slope of the plot (Figure 8.24). The value of a can be measured with a spherometer and this value is used with that of slope, to calculate λ.

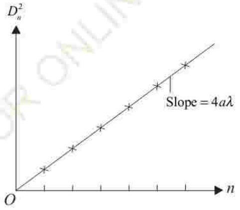


Figure 8.24 A plot to determine λ from measurements on Newton's rings

(c) Production of interference by thin films

Easily observed interference effect is produced by reflections from thin transparent film like soap bubbles and oil films. The observed bright colours are due to interference. The interference of reflected light from the front and back surfaces of soap films in a loop is shown in Figure 8.25.

Figure 8.25 Thin film of a soap bubble

The films are very thin at the top where it appears dark and increases in thickness towards the bottom where the interference fringes are obtained. The loop is kept vertical so that, the film is slightly wedge-shaped due to its own weight. A source of monochromatic light is placed behind the camera during taking the photograph in Figure 8.25. The incident light is placed nearly at right angle to the film surface.

The light from the soap film arrives at the camera after reflection from the film's front and back surface. Two features may be observed: one, there is no reflected light from the top area of the film which appears black, secondly, horizontal interference fringes occur below the dark region.

Consider reflections which occur at the two surfaces of the film. Figure 8.26(a) indicates a light beam at nearly normal incident on a transparent film with air in either side. The incident beam (1) is split into two beams at the film's front surface, a reflected beam (2) and transmitted beam (3). The transmitted beam (3) split into two beams at the back of the film; reflected beam (4) and transmitted beam (5). The reflected beam (4) is further split to obtain transmitted beam (6) and reflected beam not indicated in the diagram. Beam (2) and beam (6) are coherent, since they originate from one beam, and they have the same wavelength.

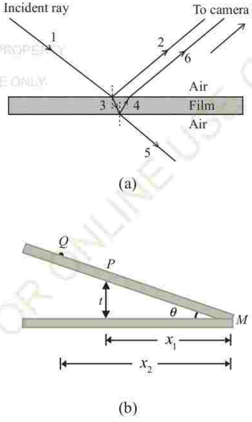


Figure 8.26 Reflections from the front and back surfaces of a film

From Figure 8.26 (b), t is the thickness of the air wedge at P. Since the incidence ray at P is nearly normal, the path difference between rays at P is $2t$. At M where the thickness is zero dark fringes followed by alternate bright and dark fringes are obtained.

For dark fringes $2t = n\lambda$, where $n = 0, 1, 2, 3...$, it follows that when $n = 0$ first dark fringe, $n = 1$ second dark fringe, $n = 2$ third dark fringe and so on.

For bright fringes $2t = (n - \frac{1}{2})\lambda$, where $n = 1, 2, 3...$, it follows that for $n = 1$ first bright fringe, $n = 2$ second bright fringe and so on.

The angle θ of the wedge can be obtained by taking the tangent of the angle at the point where the fringe is formed. For the dark fringe at P, $\tan\theta = \frac{t}{x_1}$. When the angle θ approaches zero, $\tan\theta \approx \theta$ hence $\theta = \frac{t}{x_1}$.

Consider the n^{th} and $(n+k)^{th}$ dark fringes at P and Q respectively.

$2\theta x_1 = n\lambda$ and $2\theta x_2 = (n+k)\lambda$.

Subtracting the two equations results to $2\theta(x_2 - x_1) = k\lambda$ if $k = 1$, $x_2 - x_1 = \omega$ where ω is the width of fringe, then $2\theta\omega = \lambda$.

Therefore, the wedge angle, $\theta = \frac{\lambda}{2\omega}$.

Example 8.19

A piece of wire of diameter 0.050 mm and two thin glass strips are available to produce the air wedge. If a total of 200 fringes are produced, what is the wavelength of the light used?

Solution

The wavelength of the light is given by;

$$\lambda = \frac{2t}{n} = \frac{2 \times 0.05 \times 10^{-3} \text{ m}}{200}$$

$$= 5.00 \times 10^{-7} \text{ m}$$

Therefore, light of wavelength 5.00×10^{-7} m was used.

Example 8.20

An air wedge is made by separating two plane sheets of glass by a fine wire at one end. When the wedge is illuminated normally by a light of wavelength 590 nm, a fringe pattern is observed in reflected light. The distance measured between the centre of the first bright fringe and the centre of the eleventh bright fringe is 8.1 mm. Calculate the angle of the air wedge.

Solution

Using vector resolution, $\tan\theta = \frac{h}{x}$, where $h = \frac{n\lambda}{2}$.

Therefore,

$$\theta = \tan^{-1}\left(\frac{n\lambda}{2x}\right)$$

$$= \tan^{-1}\left(\frac{10 \times 590 \times 10^{-9} \text{ m}}{2 \times 8.1 \times 10^{-3} \text{ m})}\right) = 0.021°$$

The angle of the air wedge is 0.021°.

Example 8.21

In a Newton's rings experiment, the diameter of the 15th dark ring was found to be 0.59 cm and that of the 5th ring was 0.336 cm. If the radius of curvature of the Plano-convex lens is 100 cm, calculate the wavelength of the light used.

Solution

Using equation (8.72) for a given n and m fringes;

$$D_{n+m}^2 = 4a\lambda(n+m)$$

Then,

$$\lambda = \frac{D_{n+m}^2 - D_n^2}{4ma}$$

$$\lambda = \frac{(0.59\times10^{-2}\text{ m})^2 - (0.336\times10^{-2}\text{ m})^2}{4\times10\times100\times10^{-2}\text{ m}}$$

$$= 5.88\times10^{-7}\text{ m}$$

Therefore, the wavelength of light used is 5.88×10^{-7} m.

Example 8.22

In a Newton's rings experiment, the diameter of the 12^{th} dark ring changes from 1.50 cm to 1.35 cm when a liquid is introduced between the lens and the plate. Calculate the refractive index of the liquid.

Solution

The refractive index μ of liquid can be obtained by taking the ratio of the square of the change of diameter of curvature of the ring D'_n before introducing the liquid to that of the liquid D_n after introducing.

Thus, $\mu = \left(\frac{D'_n}{D_n}\right)^2 = \left(\frac{1.50\text{ m}}{1.35\text{ m}}\right)^2 = 1.23$

Therefore, the refractive index of the liquid is 1.23.

(d) Applications of interference of light

One of the most spectacular applications of interference is the hologram. Light from a laser, which is completely coherent, falls on an object and is reflected in all directions. Some of the reflected light lands on a photographic plate, where it interferes with light coming directly from the laser. This interference produces a complex set of fringes of maxima and minima, recorded on the photographic plate. To see the hologram, light of the same wavelength is allowed to fall on the developed photographic plate. This produces further interference, allowing you to see a three-dimensional image of the original object. Holograms are used extensively in scientific measurement and data recording, but their striking three-dimensional images have made them important in art and design: they are on most credit cards.

The phenomenon of light waves interference at oily or filmy surfaces has applications in areas relating to optics: sunglasses, lenses for binoculars or cameras and even visors for astronauts. In each case unfiltered light could be harmful or at least inconvenient for the user, thus, the destructive interference eliminates certain colours and unwanted reflections.

Exercise 8.5.1

1. Explain the effects on the fringe spacing (width) when the following happens in Young's double slit experiment:
   (a) The Young's apparatus is immersed in water;
   (b) The distance between the slits is reduced;
   (c) The source of light is moved closer to slits;
   (d) The screen is brought closer to the slits; and
   (e) A thin transparent plate is placed in front of one of the slits.
2. A Young's double slit experiment is carried out with the light of wavelength 5000 Å. If the distance between the slits is 0.2 mm and the screen is at 200 cm from the slits. Calculate the distance of the third bright fringe and third dark fringe from the central bright fringe.
3. The distance between the two consecutive dark fringes is 8.0 mm when light of wavelength 630 mm is incident on a pair of slits. A second monochromatic light is used and the dark fringes are 7.0 mm apart. What is the wavelength of second light?
4. In Young's double slit experiment, a total of 23 bright fringes occupying a distance of 3.9 mm were visible in a travelling microscope. The microscope was focused on a plane which was 3 cm from the double slit and the wavelength of the light being used was 5.5×10^{-7} m. What was the separation of the double slit?
5. Coherent light that contains two wavelengths, 660 nm and 470 nm, passes through two narrow slits 0.300 m apart. The interference pattern is observed on a screen 5.0 m from the slits. What is the distance on the screen between the first order bright fringes for the two wavelengths?
6. In an experiment using Young's slits the distance between the center of the interference pattern and the length of the bright fringe on either side is 3.44 cm and the distance between the slits and the screen is 2.00 m. If the wavelength of the light used is 5.89×10^{-7} m, determine the slit separation.
7. When monochromatic light is reflected from two flat glass plates, with a wedge-shaped air film of small angle between them, a pattern of bright and dark lines can be seen. Explain in detail:
   (a) Why this pattern is produced;
   (b) How the separation of the lines depends upon the angle of the wedge; and
   (c) The effect of filling the space between the plates with a transparent liquid.
8. An air wedge is formed using two optically plane glass plates of length 15 cm each by placing a human hair of diameter 0.006 cm at one of their ends. When the air wedge

is illuminated by a monochromatic light of $\lambda = 5890$ Å, fringes are formed. Calculate the fringe width.

9. In Newton's rings experiment, rings are formed by reflected light of wavelength 5895 Å with a liquid between the plane and the curved surface. If the diameter of the 5th bright ring is 3 mm and the radius of curvature of the curved surface is 100 cm, calculate the refractive index of the liquid.
10. A set of Newton's rings was produced between one surface of a biconvex lens and a glass plate using green light of wavelength 5.46×10^{-5} cm. The diameters of particular bright rings of orders of interference m and $m+10$, were found to be 5.72 mm and 8.10 mm respectively. When the space between the lens surface and the plate was filled with liquid, the corresponding values were 4.95 mm and 7.02 mm respectively. Determine the radius of curvature of the lens surface and the refractive index of the liquid.

8.5.2 Diffraction of light

Diffraction is the ability of a wave to spread out in wave-fronts as it passes through a small aperture or around a sharp edge (Figure 8.27). The light that comes out from a narrow torch head shines a very wide area, and sound that comes out of a very thin whistle is heard over a very wide region due to diffraction phenomenon. Diffraction can be described as the bending of light when it strikes a barrier that has an aperture or an edge. The bending of light becomes more pronounced when the size of the obstacle's edge or aperture is comparable to the wavelength of light. The amount of diffraction (the sharpness of the bending) increases with increasing wavelength and decreases with decreasing wavelength. In fact, when the wavelength of the waves is smaller than the obstacle, no noticeable diffraction occurs. This section explains the necessary conditions for diffraction of light to occur and the principle of diffraction grating. It also describes how to determine the wavelength of monochromatic light by the diffraction method. Finally, it describes the applications of diffraction of light.

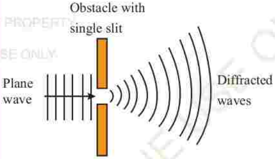


Figure 8.27 Diffraction of waves

(a) Necessary conditions for diffraction of light

Diffraction of light occurs if the size of obstacle is comparable to the wavelength of light (4×10^{-7}m to 7×10^{-7}m). If the size of opening or obstacle is close to this limit, diffraction of light can be observed. If the source and obstacle are kept far apart from each other, the incident wave-fronts on the diffracting obstacle are plane.

A plane wave can also be produced by using a converging lens . This diffraction is called Fraunhofer diffraction. On the other hand, if the source and the screen are close to each other, the wave-fronts are spherical and the wave-front leaving the obstacle are also spherical. This diffraction is called Fresnel diffraction. However, in this section, you will deal only with Fraunhofer diffraction.

Fraunhofer diffraction occurs when the light source and the screen are effectively at infinite distances from the aperture or obstacle causing the diffraction such that all rays are considered parallel (Figure 8.28(a)). A distant source can be used to provide the incident wave fronts, but it is often more convenient to use a source which is placed at the focal point of a converging lens, (convex) to obtain the parallel rays, (Figure 8.28(b)). Likewise, a converging lens is used to converge parallel rays onto a nearer screen.

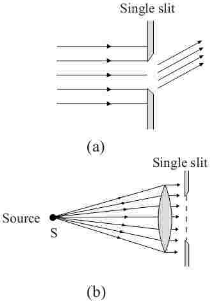


Figure 8.28 Fraunhofer diffraction

(b) Determination of wavelength using Fraunhofer diffraction

Consider a slit of width, a, splits into two $2n$ (where $n = 1,2,3...$) equal parts of width $\frac{a}{2n}$ (Figure 8.29).

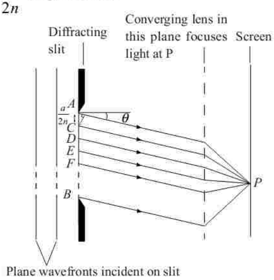


Figure 8.29 Fraunhofer diffraction by single slit

If λ is the wavelength of the light used and θ is the direction to the normal such that, $AN = \frac{\lambda}{2}$, the wave from A will be completely out of phase with that from C. Thus, waves from A and C will interfere destructively at point P on the screen. Similarly, light from each point between A and C, and that from corresponding point between C and D will interfere destructively. This happens for every pair of sections such as DE and EF. Hence, the light will not diffract in those directions θ, which are such that, $AN = \frac{\lambda}{2}$ that is,

$AC\sin\theta = \frac{\lambda}{2}$, then

$\frac{a}{2n}\sin\theta = \frac{\lambda}{2}$, therefore, a condition for dark fringe is;

$$a\sin\theta = \pm n\lambda \qquad (8.73)$$

where $n = 1, 2, 3...$ and the $\pm$ sign shows that, there are symmetric dark fringes above and below point P.

Note that, equation (8.73) may be used for the other segment of the slits. The $\sin\theta = 0$ corresponds to a bright band, where light from the entire slit arrives at P is in phase, giving the central maximum. The positions of other maxima are placed approximately mid-way between the minima, and they are less intense than the central maximum.

From Figure 8.18, $\tan\theta = \frac{y_n}{D}$ and for small angle θ, $\tan\theta \approx \sin\theta \to \theta$; then,

$$y_n = \frac{n\lambda D}{d} \qquad (8.74)$$

Equation (8.74) is valid when $y_n <<< D$.

Example 8.23

A laser light of wavelength 633 nm is passed through a narrow slit and the diffraction pattern on a screen 6.0 m away, shows that distance between centre of the first minima on either side of the central bright fringe is 32 mm. Calculate the width of the slit.

Solution

Using equation (8.74), the slit width is;

$$d = \frac{n\lambda D}{y_n} = \frac{1\times 633\times10^{-9}\ \text{m}\times 6\ \text{m}}{\frac{32}{2}\times10^{-3}\ \text{m}}$$

$$= 2.4\times10^{-4}\ \text{m}$$

Therefore, the slit width is 2.4×10^{-4} m.

(c) The principle of diffraction grating

Diffraction grating is an arrangement of a large number of closely spaced parallel slits, all with the same width and equally spaced between their centre, ruled on glass or polished metal. The lines scatter the incident light and are mostly opaque whereas, the space between them transmit light and acts as a slit. For a grating of N slits, where each slit is narrow compared to the wavelength λ of the incoming plane wave, its diffraction pattern spreads out nearly uniformly. If the slits are equally spaced and the wave from one slit is in phase with that of adjacent slit, then, the wave is also in phase with those from other slits.

Suppose, a plane wave of monochromatic light of wavelength λ falls on a transmission grating in which the grating spacing (slit separation) is d (Figure 8.30).

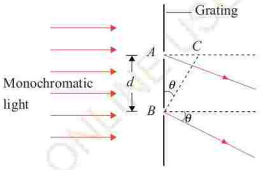


Figure 8.30 Diffraction grating

Consider wavelets coming from points A and B at an angle θ. The path difference AC between the wavelets is $d\sin\theta$.

Therefore, for constructive interference,

$$d\sin\theta = n\lambda \qquad (8.75)$$

where n is an integer giving the order of the spectrum. For $n=0$, $\theta=0$, then, the central maximum is observed (zero order image). The first order, second order and so on, are given by $n=1,2,3...$, but these are less bright than for $n=0$ (Figure 8.31) the number of lines per metre of the gratings is given by; $N=\dfrac{1}{d}$.

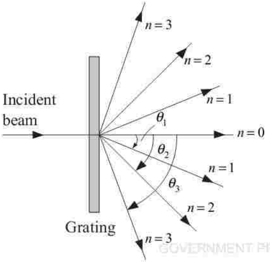


Figure 8.31 Orders of images produced by diffraction grating

Example 8.24

Light of wavelength $5890\,\text{Å}$ is incident normally on a grating with a spacing of 2.00×10^{-6} m What is the angle to the normal of;

(a) first order principal maximum,

(b) a second order principal maximum?

Solution

Using equation (8.75),

(a) The first order principle maximum, $n=1$ is;

$$\theta=\sin^{-1}\left(\frac{n\lambda}{d}\right)=\sin^{-1}\left(\frac{1\times5890\times10^{-10}\text{ m}}{2\times10^{-6}\text{ m}}\right)=17.1^\circ$$

(b) The second order principle maximum, $n=2$

$$\theta=\sin^{-1}\left(\frac{n\lambda}{d}\right)=\sin^{-1}\left(\frac{2\times5890\times10^{-10}\text{ m}}{2\times10^{-6}\text{ m}}\right)=36.1^\circ$$

Therefore, the angles to the normal of first order and second order principle maximum are 17.1° and 36.1° respectively.

Example 8.25

A special grating has 1000 lines per cm. Determine the maximum number of orders one can get when light of $\lambda=6328\text{ Å}$ falls on the grating.

Solution

Using (8.75), maximum value of n is obtained for $\sin\theta=1$, therefore,

$$n=\frac{1}{N\lambda}$$

$$=\frac{1}{1000\times100\text{ m}^{-1}\times6328\times10^{-10}\text{ m}}$$

$$=15.8$$

Therefore, since n must be an integer, then the maximum number of order is 15.

Example 8.26

A diffraction grating produces a second order maximum at 50.6° to the normal when being illuminated normally with light of wavelength 644 nm. Calculate the number of lines per millimetre of the grating.

Solution

The number of lines per mm of the grating, $N=\dfrac{1}{d}$ is

$$N = \frac{1}{d} = \frac{\sin\theta}{n\lambda}$$

$$= \frac{\sin(50.6°)}{2 \times 644 \times 10^{-9}\ \text{m} \times 1000}$$

$$= 600 \text{ lines per mm}$$

The number of lines are 600 lines per mm.

(d) Wavelength of monochromatic light by diffraction method

The wavelength of a monochromatic light can be measured by a combination of diffraction grating and spectrometer. Once the angular position, θ, of one of the principal maxima (produced by grating) is measured and the grating spacing d is known; then the wavelength of monochromatic wave can be determined.

Activity 8.2

Measuring wavelength by using diffraction grating

Materials

Source of monochromatic light (e.g. Sodium lamp), spectrometer and diffraction grating.

Procedure

(a) Adjust the eyepiece of the telescope so that the cross-wires are sharply focused.

(b) Focus the telescope for parallel light using a distant object. (There should be no parallax between the image seen in the telescope and the cross-wires seen through the eyepiece).

(c) Place the sodium lamp in front of the collimator.

(d) Level the turntable of the spectrometer if necessary.

(e) Looking through the telescope, focus the collimator lens and adjust the width of the slit until a clear narrow image is seen.

(f) Place the diffraction grating on the turntable at right angles to the beam.

(g) Move the telescope to the right until the cross wires are centered on the first bright image. Take the reading θ_1 from the scale on the turntable. (Use magnifying lens and lamp to see the scale more easily).

(h) Move the telescope back through the centre and then to the first bright image on the left. Take the reading θ_2 from the scale. The setup of the experiment is shown in Figure 8.32.

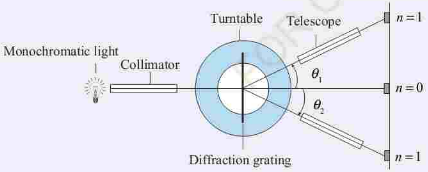


Figure 8.32 Measurement of wavelength using diffraction grating

Questions

From the obtained readings,

1. Calculate;
   (a) θ using, $\theta = \dfrac{\theta_2 - \theta_1}{2}$
   (b) the distance d between the slits using $d = \dfrac{1}{N}$, where N is the number of lines per metre on the grating
   (c) the wavelength of the monochromatic light using equation (8.75).
2. Repeat this for different orders n and get an average value for the wavelength.

Applications of diffraction

The knowledge that light undergoes diffraction has found a number of uses in science and technology. For example, the knowledge of diffraction is used in production of three dimensional holograms (3D). discovery of X-rays diffraction provided the means for studying the atomic structure of crystals and polymers. In 1952 scientists used X-rays diffraction to photograph DNA.

Exercise 8.5.2

1. Diffraction effects for sound waves and water waves are readily observed but not for light. Explain.
2. Why is a diffraction grating better than a two slit set up for measuring wave lengths of light?
3. A laser light of $\lambda = 6328$ Å illuminates a 0.4 mm wide slit. Find the width of the central maxima on a screen kept at a distance of 16 m.
4. Parallel rays of light with wavelength 620 nm pass through a slit covering a lens with a focal length of 40.0 cm. The diffraction pattern is observed in the focal plane of the lens, and the distance from the centre of the central maximum to the first minimum is 36.5 cm. Calculate the width of the slit. Do not use small angle approximation.
5. Parallel beams of two wavelengths 5890 Å and 896 Å of sodium vapour lamp fall on a diffraction grating having 6000 lines/cm. Estimate the dispersion produced by the grating on the two wavelengths, in the first order spectra.
6. Light consisting of two wavelengths which differ by 160 nm passes through a diffraction grating with 2.5×10^5 lines per metre. In the diffracted light, the third order of one wavelength coincides with the fourth order of the other. What are the two wavelengths and at what angle of diffraction does this coincidence occur?
7. Light of wavelength 600 nm is incident normally on diffraction grating of width 20.0 mm on which 10.0×10^3 lines have been ruled. Calculate the angular positions of various orders.
8. A rectangular piece of glass $2\,\text{cm} \times 3\,\text{cm}$ has 18000 evenly spaced lines ruled across its whole

surface, parallel to the shorter side, to form a diffraction grating. Parallel rays of light of wavelength 5×10^{-5} cm fall normally on the grating. What is the highest order of spectrum in the transmitted light?

9. Light of wavelength 535 nm falls normally on a diffraction grating. Find its grating spacing if the diffraction angle 35° corresponds to one of the principal maxima and the highest order of spectrum is equal to 5.

10. In a certain experiment using normal incidence, the readings for the angle of diffraction in the second order spectrum for the two sodium D lines were, $D_1 = 42°$, $D_2 = 42°$; If the wavelength of D_1 line is 5896 Å, find the number of lines per centimeter of the grating D_1 line and the wavelength of the D_2 line.

8.5.3 Polarization of light

Polarization is a characteristic of all transverse waves. If all the vibrations of a transverse wave are in a single plane which contains the direction of propagation of the wave, such a wave is said to be plane-polarized (or linearly polarized). Therefore, polarization is the process of making waves vibrate in only one plane. Observations and experiments show that light is a transverse electromagnetic wave and therefore it can exhibit polarization.

As discussed previously (section 8.4.2), an electromagnetic wave is due to fluctuating electric $\vec{E}$ and magnetic $\vec{B}$ fields that are perpendicular to each other and to the direction of propagation. The vibrations of $\vec{E}$ and $\vec{B}$ fields of light can be restricted in particular direction in a plane perpendicular to the propagation of light. This process is called polarization of light.

(a) Methods of producing plane polarized light

There are different ways of producing polarization. These include use of polaroid, reflection, double refraction and scattering.

(i) Polarization by polaroid

The most common method of polarization is the use of polaroid filters (Figure 8.33). Polaroid filters are made of materials which are capable of blocking one of the two planes of vibration of an electromagnetic wave. When un-polarized light strikes the filter, the parallel vibrations are allowed to pass through the polaroid while the perpendicular vibrations are absorbed. For this case, polaroid used to polarize light is called a polarizer. When un-polarized light with intensity I_0 is incident normally on a polarizer, the intensity of the emerging beam from the polaroid sheet is $\frac{I_0}{2}$. This is because the vibrations of electric vector E which are parallel to the polaroid transmission axis are allowed to pass while the perpendicular ones are absorbed.

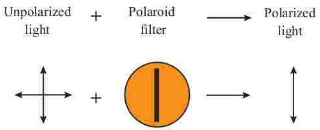


Figure 8.33 Polarized light by polaroid

(ii) Polarization by reflection

When unpolarized natural light is incident on a glass surface at an angle, each of the vibrations of the incident light can be resolved into a component parallel to the glass surface and a component perpendicular to the surface. The light due to the components parallel to the glass is reflected, but the remainder of the light is refracted into the glass. If metallic surfaces are used, the light reflected vibrates in various planes and hence, the light will continue in its un-polarized state. On the other hand, non-metallic planes (glass) will reflect most of the vibrations at a single plane parallel to the plane of incidence. When light hits the material and crosses the interface, the atoms absorb the light temporarily and the electron starts vibrating in the direction of the electric field of the refracted ray. This gets re-emitted with an electric field vector which is perpendicular to the direction of propagation of the wave. Thus, light reflected by the glass is plane polarized (Figure 8.34).

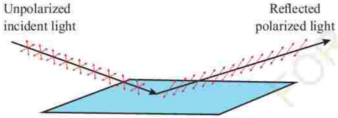


Figure 8.34 Plane polarized wave by reflection

Polarization of light by reflection is done by reflecting off the unpolarized light in a nonmetallic reflecting surface. The amount of polarization will depend on the angle of incidence of the light and the composition of the material used for the reflecting surface. The reflection coefficient of light will go to zero between the angles of 0° to 90°, since the electric field goes parallel to the plane of incidence. At this angle, the reflected light will become linearly polarized.

Unpolarized light can also undergo polarization by reflection off nonmetallic surfaces. The extent to which polarization occurs is dependent upon the angle at which the light approaches the surface and upon the material that the surface is made of. Metallic surfaces reflect light with a variety of vibrational directions; such reflected light is un-polarized. However, nonmetallic surfaces such as asphalt roadways, snowfields and water reflect light such that there is a large concentration of vibrations in a plane parallel to the reflecting surface. A person viewing objects by means of light reflected off nonmetallic surfaces will often perceive a glare if the extent of polarization is large. Fishermen are familiar with this glare since it prevents them from seeing fish that lie below the water. Light reflected off a lake is partially polarized in a direction parallel to the water's surface. Fishermen know that the use of glare-reducing sunglasses with the proper polarization axis allows for the blocking of this partially polarized light. By blocking the plane-polarized light, the glare is reduced and the fisherman can easily see fish located under the water.

(iii) Polarization by double refraction

Polarization can also occur by the refraction of light. Refraction occurs when a beam of light passes from one material into another. At the interface of the two materials, the path of the beam changes its direction. The refracted beam acquires some degree of polarization. Most often, the polarization occurs in a plane perpendicular to the surface. The polarization of refracted light is often demonstrated by using a unique crystal that serves as a double-refracting crystal. The light is split into two beams when entering the crystal (Figure 8.35).

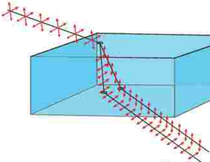


Figure 8.35 Polarization by double refraction

If an object is viewed by looking through the crystal, two images will be seen. The two images are the result of the double refraction of light. Both refracted light beams are polarized-one in a direction parallel to the surface and the other in a direction perpendicular to the surface.

Since these two refracted rays are polarized with a perpendicular orientation, a polarizing filter can be used to completely block one of the images. If the polarization axis of the filter is aligned perpendicular to the plane of polarized light, the light is completely blocked by the filter; meanwhile, the second bright image appears. If the filter is then turned $90°$ in either direction, the second image re-appears and the first image disappears.

(iv) Polarization by scattering

The scattering of light off air molecules produces linearly polarized light in the plane which is normal to the incident light. When light strikes the atoms of a medium, it will often set the electrons of those atoms into vibrations. The vibrating electrons then produce their own electromagnetic wave that is radiated outward in all directions (Figure 8.36).

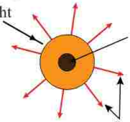


Figure 8.36 Polarization by scattering

The generated wave strikes neighbouring atoms, forcing their electrons into vibrations at the same original frequency. The vibrating electrons in turn produce another electromagnetic wave that is once more radiated outward in all directions. The absorption and reemission of light waves cause the light to be scattered about the medium. Polarization by scattering is observed as light passes through the atmosphere. The light that is partially polarized by scattering, contributes to the blueness of the skies.

(b) Brewster's Law

Polarization can be achieved by allowing the light ray to fall on a surface of a transparent medium in such a way that, the reflected ray makes an angle of 90° with the refracted ray. This is the Brewster law named after a Scottish physicist, Sir David Brewster, who first proposed it in the year 1811.

When a light ray is incident on a surface at an angle i_p, (called Brewster angle), part of the ray is fully polarized by reflection at an angle i_p and another is partially polarized by refraction to an angle $r = 90° - i_p$ (Figure 8.37).

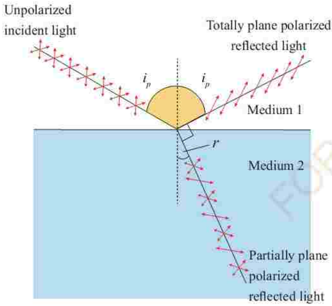


Figure 8.37 Polarization by reflection

Applying Snell's law;

$$\eta_1 \sin i_p = \eta_2 \sin r$$

where η_1 and η_2 are the refractive indices of media 1 and 2, i_p and r are the polarizing and refracted angles respectively.

$$i_p + 90° + r = 180°,$$

therefore, $r = 90° - i_p$.

So,

$$\eta_1 \sin i_p = \eta_2 \sin(90° - i_p)$$

$$= \eta_2 \cos i_p$$

$$\frac{\eta_2}{\eta_1} = \frac{\sin i_p}{\cos i_p} = \tan i_p \quad (8.76)$$

From equation (8.76), if medium 1 is air or vacuum $\eta_1 = 1$, then,

$$\eta_2 = \tan i_p \quad (8.77)$$

Equation (8.77) is Brewster's law. The law states that, *"Maximum polarization of reflected beam occurs for an angle of incidence i_p given by $\tan i_p = \eta$, refractive index of glass"*. For a glass of refractive index $\eta_g = \eta_2 = 1.5$, $i_p = 57°$.

Therefore, at the incident angle of 57°, the light which is reflected from the surfaces of glass is plane polarized and at angles of incidence other than 57°, the reflected light is partially plane polarized.

(c) Optical activity of solution

The concentration of solutions (e.g. sugar) can be measured using a saccharimeter. This is done by measuring the refractive index or the angle of polarization of optically active solution. The saccharimeter consists of a polarimeter. A polarimeter is an instrument used to measure an optical activity. The optical activity depends on various factors including the concentration of the solution, temperature, length of the tube containing the solution and wavelength of the light passing through the solution (Figure 8.38).

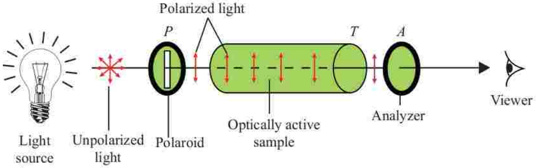


Figure 8.38 Saccharimeter

Saccharimeter is used in food processing, brewing and alcohol industries. The sample of the solution is placed in the tube *T*, and un-polarized light emerging from the light source falls on the polarizer *P*. The polarizer *P* produce polarized light that get into the optically active solution in tube *T*. The viewer can observe the polarized light by the solution through the analyzer *A*. ***Note that,*** the optical activity measurement is carried out in a dark room.

(d) Applications of polarization

Polarization has wide applications in various fields including glare-reducing sunglasses, transparent plastics and entertainment industry. When light is reflected from a flat surface (e.g. water), it tends to become horizontally polarized. Watching at such polarized (glare) light become annoying and reduce visibility. Polaroid sunglasses provide a superior glare protection. The sunglasses contain a special filter that blocks such reflected polarized light, hence reducing the glare and improving visibility. However, polaroid sunglasses may not be required when skiing down hills as they may block light reflected from ice patches. In addition, sunglasses may reduce visibility of images displaced on liquid crystal display (LCD) or light-emitting diode display (LED) found on a dashboard of some cars or teller machines.

In industry, polaroid filters are used to perform stress analysis tests on transparent plastics. As light passes through a plastic, each colour of visible light is polarized with its own orientation. If such a plastic is placed between two polarizing plates, a colourful pattern is revealed. As the top plate is turned, the colour pattern changes as new colours become blocked and the formerly blocked colours are transmitted. In addition, three Dimensional (3D) movies are produced and shown by means of polarization. Movies are (3D), actually two movies being shown at the same time through two projectors. The two movies are filmed from two slightly different camera locations. Each individual movie is then projected from different sides of the audience onto a metal screen. The movies are projected through a polarizing filter. The polarizing filter used for the projector on the left may have its polarization axis aligned horizontally while the polarizing filter used for the projector on the right would have its polarization axis aligned vertically. Consequently, there are two slightly different movies being projected onto a screen. Each movie is cast by light that is polarized with an orientation perpendicular to other movie. The audience then wears glasses that have two polaroid filters. Each filter has a different polarization axis; one is horizontal and the other is vertical. The result of this arrangement of projectors and filters is that, the left eye sees the movie that is projected from the right projector while the right eye sees the movie that is projected from the left projector. This gives the viewer a perception of depth (3D perception).

Exercise 8.5.3

1. Calculate the polarizing angle of glass, water and diamond with refractive indices; 1.53 , 1.33 and 2.42 respectively.
2. (a) What is meant by polarization and angle of polarization?

   (b) Calculate, the angle of polarization for water of refractive index 1.33.
3. Explain what is meant by double refraction. Describe how you could demonstrate experimentally that two refracted beams produced from a single beam by a piece of calcite are plain polarized at right angles to each other.
4. Explain why polaroid sunglasses are effective in reducing glare.
5. (a) Use a sketch diagram to show how you would demonstrate that a beam of light is completely polarized.

   (b) A parallel beam of un-polarized light is incident at an angle of $58°$ on a plane glass surface and the reflected beam is completely polarized. What is the refractive index of the glass, and the angle of refraction of the transmitted beam?

8.6 The Doppler Effect

Perhaps you have noticed how the sound of a vehicle's horn changes as the vehicle moves past you. The frequency of the sound you hear as the vehicle approaches you is higher than the frequency you hear as it moves away from you. This apparent change in pitch, due to the relative motion between the source and the observer, was first explained by an Austrian scientist, Christian Doppler and is known as Doppler effect.

Doppler effect in sound is different from that in light. In the case of light, the Doppler effect is symmetric i.e. the apparent frequency is the same for the two cases, either when a source moves towards a stationary observer or when an observer moves towards a stationary source. In this section, you will compare the Doppler effects in sound and light and discuss its applications in real life.

8.6.1 Doppler Effect for Sound

When either sound source or observer moves, or both of them move, the observer will notice a change of pitch of sound. From this effect the observer can determine whether the source of sound is approaching or receding. Then, the speed of either the source or observer can be estimated. Doppler effect has many important applications as it depends on things moving. It can generally be used to determine the apparent frequency of an object approaching to or receding from an observer.

(a) Source moving towards a stationary observer

When the source of sound moves, the waves in front of it are compressed while those behind it are stretched. Therefore, a moving source affects wavelength of the wave. Consider a source S moving towards the stationary observer O, with velocity u_s, the frequency of the sound f is compressed in smaller distance $(v-u_s)$ because S moves a distance u_s towards O per second (Figure 8.39).

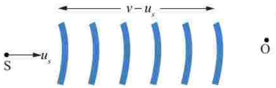


Figure 8.39 Source moving towards stationary observer

Then, it follows that, the apparent wavelength, $\lambda' = \frac{v-u_s}{f}$. It follows that, if v is velocity of sound in air, the apparent frequency f' is given as,

$$f' = \frac{v}{\lambda'} = \left(\frac{v}{v-u_s}\right)f \qquad (8.78)$$

Since $(v-u_s)$ is less than v, then the apparent frequency f' is greater than the frequency of the sound f, thus, there is an apparent increase in frequency when a source is moving towards an observer.

(b) A source moving away from a stationary observer

In this case, the sound wave (f wave) is moving away from O per second occupy a distance $(v+u_s)$ as shown in Figure 8.40.

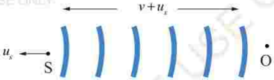


Figure 8.40 Source moving away from a stationary observer

The wavelength λ' of the waves at O is therefore, $\lambda' = \frac{v+u_s}{f}$, hence the apparent frequency, is

$$f' = \frac{v}{\lambda'} = \left(\frac{v}{v+u_s}\right)f \qquad (8.79)$$

Since $v+u_s$ is greater than v, then the apparent frequency f' is less than the frequency of the sound f; thus, there is an apparent decrease in frequency when a source is moving away an observer.

Example 8.27

A train standing at the outer signal of a railway station blows a whistle of frequency 400 Hz in still air.

(a) What is the frequency of the whistle for a platform observer when the train,

(i) approaches the platform with a speed of $10\ \text{ms}^{-1}$?

(ii) recedes from the platform with a speed of $10\ \text{ms}^{-1}$?

(b) What is the speed of sound in each case? Speed of sound in still air is $340\ \text{ms}^{-1}$.

Solution

(a) Source is moving

(i) Since the train (source) is approaching the observer, therefore, the source moves in the direction of the waves,

$$f' = \left(\frac{v}{v-u_s}\right)f,$$

$$f' = \frac{340\ \text{ms}^{-1}}{(340-10)\ \text{ms}^{-1}} \times 400\ \text{Hz}$$

$$= 412\ \text{Hz}$$

(ii) Since the train recedes away from the observer, therefore, the source and the wave move in opposite directions.

$$f' = \left(\frac{v}{v+u_s}\right)f,$$

$$f' = \frac{340\ \text{ms}^{-1}}{(340+10)\ \text{ms}^{-1}} \times 400\ \text{Hz}$$

$$= 388.6\ \text{Hz}$$

(b) The speed of sound in either case will be $340\ \text{ms}^{-1}$ as it does not depend on either the motion of the source or the observer or both.

(c) An observer moving towards a stationary source

In this case, the velocity of the sound wave relative to O is given by, $v + u_o$ (Figure 8.41). Hence the apparent frequency f' is given by $f' = \frac{v+u_o}{\lambda}$, where $\lambda = \frac{v}{f}$,

thus,

$$f' = \left(\frac{v+u_o}{v}\right)f \qquad (8.80)$$

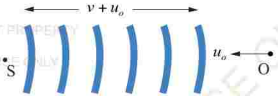


Figure 8.41 Observer moving toward stationary source

Since $(v+u_o)$ is greater than v, then, the frequency f' heard by the observer is higher than f. This implies that the listener moving towards the stationary source hears a frequency that is higher than the source frequency.

(d) An observer moving away from stationary source

In this case, the wavelength of the waves reaching O is unchanged and is given by $\frac{v}{f}$. The velocity of the sound waves relative to O is now $v - u_o$ (Figure 8.42).

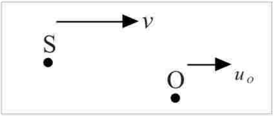


Figure 8.42 Observer moving away from stationary source

Hence the apparent frequency f' is given by $f' = \frac{v - u_o}{\lambda}$, where $\lambda = \frac{v}{f}$, then

$$f' = \left(\frac{v - u_o}{v}\right) f \qquad (8.81)$$

Since $(v - u_o)$ is less than v, then the apparent frequency f' is less than the frequency of the sound f, thus, there is an apparent decrease in frequency when a source is moving away from an observer.

Example 8.28

Two stationary sources A and B each emit notes of frequency 392 Hz. A listener is between the two sources and is moving towards B with a speed of $15\ \text{ms}^{-1}$. What is the beat frequency detected by the listener? Velocity of sound in still air is $340\ \text{ms}^{-1}$.

Solution

The frequencies detected by a listener away from A and towards B are

$$f_A = \left(\frac{v - u_o}{v}\right) f \text{ and } f_B = \left(\frac{v + u_o}{v}\right) f$$

respectively.

The beat frequency f' is given as;

$$f' = f_B - f_A,\ f' = \left(\frac{2u_o}{v}\right) f,$$

$$= \frac{2 \times 15\ \text{ms}^{-1}}{340\ \text{ms}^{-1}} \times 392\ \text{Hz} = 34.6\ \text{Hz}$$

Therefore the beat frequency detected by a listener is 34.6 Hz.

(e) Both source and observer moving

In this case, both wavelength and frequency will be affected as both source and observer are moving. When source and observer are moving towards each other, the apparent wavelength is $\lambda' = \frac{v - u_s}{f}$ and change in velocity,

$v' = v + u_o$; then,

$$f' = \frac{v'}{\lambda'} = \left(\frac{v + u_o}{v - u_s}\right) f \qquad (8.82)$$

Since $(v + u_o)$ is greater than $(v - u_s)$, then, the frequency f' heard by the observer is higher than the frequency f of the source. This implies that the listener hears a frequency that is higher than the source frequency.

When source and observer are moving away from one another, the apparent wavelength is $\lambda' = \frac{v + u_s}{f}$ and $v' = v - u_o$;

therefore,

$$f' = \frac{v'}{\lambda'} = \left(\frac{v - u_o}{v + u_s}\right) f \qquad (8.83)$$

Since $v - u_o$ is less than $v + u_s$, then the apparent frequency f' is less than the frequency of the sound f, thus, there is an apparent decrease in frequency when a source and observer are moving away from each other.

When both observer and source are moving in the same direction with source behind the observer, the apparent wavelength $\lambda' = \frac{v - u_s}{f}$, then the change in velocity $v' = v - u_o$.

Therefore,

$$f' = \frac{v'}{\lambda'} = \left(\frac{v - u_o}{v - u_s}\right) f \qquad (8.84)$$

The apparent frequency f' heard by the observer will depends on the value of $\frac{v - u_o}{v - u_s}$. If u_o is less than u_s, then, apparent frequency f' will be greater than the frequency f of the source. On the other hand, if u_o is greater than u_s, then, apparent frequency f' will be less than the frequency f of the source. When both observer and the source are moving in the same direction with source in front of the observer, the apparent wavelength $\lambda' = \frac{v + u_s}{f}$, then apparent frequency, $f' = \frac{v + u_o}{\lambda}$.

Therefore,

$$f' = \frac{v'}{\lambda'} = \left(\frac{v + u_o}{v + u_s}\right) f \qquad (8.85)$$

The apparent frequency f' heard by the observer will depends on the value of $\frac{v + u_o}{v + u_s}$. If u_o is greater than u_s, then, apparent frequency f' will be greater than the frequency f of the source. On the other hand, if u_o is less than u_s, then, apparent frequency f' will be less than the frequency f of the source.

(f) Source moving at an angle to the line joining the source and the observer

If the source moves at an angle to the observer, the apparent frequency changes continuously. The frequency heard by the observer depends on the situation of source at the time of emission and not at the time of hearing.

Consider a source moving along a line AB while the observer O is stationary (Figure 8.43).

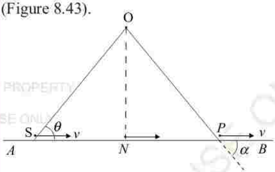


Figure 8.43 Oblique Doppler effect

When the source is moving at an angle θ to the line joining the source and the observer, the apparent wavelength $\lambda' = \frac{v - u_s \cos\theta}{f}$, then the apparent frequency,

$$f' = \left(\frac{v}{v - u_s \cos\theta}\right) f \qquad (8.86)$$

Since v is greater than $v - u_s \cos\theta$, then, the frequency f' heard by the observer is higher than the frequency f of the source. This implies that the listener hears a frequency that is higher than the source frequency.

When the source reaches P, the component of u_s is $u_s\cos\theta$ away from stationary observer. Thus, when the source is moving at an angle θ away from the stationary observer, the apparent wavelength $\lambda' = \frac{v + u_s\cos\theta}{f}$, then the apparent frequency,

$$f' = \left(\frac{v}{v + u_s\cos\theta}\right)f \qquad (8.87)$$

Since $v + u_s\cos\theta$ is greater than v, then, the frequency f' heard by the observer is less than the frequency f of the source. This implies that the listener hears a frequency that is lower than the source frequency.

Note that, the motion of the observer affects only velocity v of the waves arriving at the observer, while the motion of the source affects only the wavelength of the waves reaching the observer.

Example 8.29

A police car's siren emits a sinusoidal wave with frequency $300\ \text{Hz}$. The speed of sound is $340\ \text{ms}^{-1}$ and the air is still. Find the wavelength of the waves in front of and behind the siren if the car is moving at $30\ \text{ms}^{-1}$.

Solution

The source in front of the observer provides the value of wavelength,

$$\lambda' = \frac{v - u_s}{f},$$

$$\lambda' = \frac{(340 - 30)\ \text{ms}^{-1}}{300\ \text{s}^{-1}} = 1.03\ \text{m}$$

The wavelength of the observer behind the car is given as:

$$\lambda' = \frac{v + u_s}{f},$$

$$\lambda' = \frac{(340 + 30)\ \text{ms}^{-1}}{300\ \text{s}^{-1}} = 1.23\ \text{m}$$

Therefore, the wavelength of the waves in front of and behind the siren are $1.03\ \text{m}$ and $1.23\ \text{m}$ respectively.

Example 8.30

A horn of frequency $900\ \text{Hz}$ is sounded by a car travelling towards a cliff and normal to the cliff with a velocity of $20\ \text{ms}^{-1}$. Calculate the beat frequency of the horn sound as heard by the car driver. Velocity of sound in air is $320\ \text{ms}^{-1}$.

Solution

The car driver will hear beats with frequency, f_b due to the actual frequency of the horn, f, and the frequency, f_2, of the reflected sound from the cliff. Since the apparent wavelength is $\lambda' = \frac{v - u_s}{f}$, then, the apparent frequency is:

$$f' = \frac{v'}{\lambda'} = \left(\frac{v}{v - u_s}\right)f,$$

$$f' = \frac{320\ \text{ms}^{-1}}{(320 - 20)\ \text{ms}^{-1}} \times 900\ \text{Hz}$$

$$= 960\ \text{Hz}$$

The frequency f_b is the one that reaches the cliff. This "frequency" will be reflected back to the car driver.

Therefore, the cliff acts as a source of sound and the car driver is the listener. The apparent frequency f_2 heard by the driver from the cliff is:

$$f_2 = \left(\frac{v+u_o}{v}\right) f',$$

$$f_2 = \frac{(320+20)\ \text{ms}^{-1}}{320\ \text{ms}^{-1}} \times 960\ \text{Hz}$$

$$= 1020\ \text{Hz}$$

$$f_b = f_2 - f,$$

$$f_b = 1020\ \text{Hz} - 900\ \text{Hz} = 120\ \text{Hz}$$

Therefore, the beat frequency of the horn sound heard by the car driver is $120\ \text{Hz}$.

8.6.2 Doppler Effect for Light

The theory of the Doppler's effect in light is different from that in sound. In the case of light, the Doppler effect is symmetric i.e. the apparent frequency is the same for the two cases, either when a source moves towards a stationary observer or when an observer moves towards a stationary source. Doppler effect for light waves depends on only one velocity, the relative velocity v between source and observer measured from the reference frame of either.

Suppose a star is moving with a velocity v away from the earth and emits light of wavelength λ. If the frequency of the vibrations is f cycles per second, then f waves are emitted in one second, where $c = f\lambda$ and c is the velocity of light in vacuum. Owing to the velocity v, the f waves occupy a distance $(c+v)$. Thus, the apparent wavelength λ' to an observer on the earth in line with the star's motion is,

$$\lambda' = \frac{c+v}{f} = \left(\frac{c+v}{c}\right)\lambda,\ \lambda' - \lambda = \frac{v}{c}\lambda$$

$$\lambda' - \lambda = \frac{v}{c}\lambda \qquad (8.89)$$

Since $\frac{c+v}{c} > 1$, it follows that, λ' is greater than λ when the star is moving away from the earth, i.e. there is a shift or displacement towards the red end of the spectrum.

If the star is moving towards the earth with a velocity v, the apparent wavelength λ is given by

$$\lambda' = \frac{c-v}{f} = \left(\frac{c-v}{c}\right)\lambda,\quad \lambda' = \left(1 - \frac{v}{c}\right)\lambda$$

$$\therefore \lambda - \lambda' = \frac{v}{c}\lambda$$

Since $\frac{c-v}{c} < 1$, it follows that, λ' is less than λ when the star or planet is moving towards the earth, i.e. there is a shift or displacement towards the blue end of the spectrum.

8.6.3 Applications of Doppler Effect

The Doppler effect has a number of applications with regard to the sensing of movement. For instance, Meteorologists use Doppler radar to track movement of storm systems. By detecting the direction and velocity of raindrops or hail, for instance, Doppler radar can be used to determine the motion of winds and, thus predicts weather patterns that will follow in the next minutes or hours. Moreover, Doppler radar can do more than simply detecting a storm in progress: Doppler technology also aids meteorologists by interpreting wind direction, as an indicator of coming storms. The radar also uses radiowaves to determine the location and velocity of the distant moving objects, such as aeroplanes, jets, etc. for navigation purposes. Police officers use Doppler effect to calculate the speed of moving cars by measuring the shift in frequency of microwaves reflected by it. Consider a car (observer) moving with the speed v towards a stationary police (source). The frequency, f' received by the car is given by,

$$f' = (\frac{c+v}{c}) \times f \qquad (8.90)$$

This frequency is reflected by the car as the moving source of velocity v towards the stationary police (observer). The frequency noted by police is given by,

$$f'' = (\frac{c}{c-v}) \times f' \qquad (8.91)$$

Substituting equation (8.90) into (8.91) gives,

$$f'' = (\frac{c+v}{c-v}) \times f$$

The change in frequency Δf observed by the police is,

$$\Delta f = f'' - f \qquad (8.92)$$

which results to

$$\frac{\Delta f}{f} = \frac{2v}{c-v}$$

Since v is very small compared to c, then $(c-v) \approx c$

$$\frac{\Delta f}{f} = \frac{2v}{c} \rightarrow v = \frac{c\Delta f}{2f} \qquad (8.93)$$

Similarly, physicians and medical technicians apply it to measure the rate and direction of blood flow in a patient's body, along with ultra-sound. A beam of ultrasound is pointed towards an artery, and the reflected waves exhibit a shift in frequency, because the blood cells act as moving sources of sound waves.

Consider a ultrasonic transducer emitting waves of frequency, f and velocity, v incident on the blood vessel of cross-sectional area, A at an angle θ with the vessel in which blood cells flow with speed, u as shown in Figure 8.44.

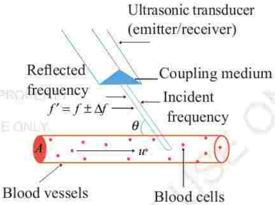


Figure 8.44 Blood flow measurement

The speed, u of blood cells is given by; $u = \frac{v\Delta f}{2f}$ where Δf is the shift in frequency.

If the speed, u is resolved along the direction of the ultrasonic transducer, the equation becomes,

$$u\cos\theta = \frac{v\Delta f}{2f}; \text{ so that, } u = \frac{v\Delta f}{2f\cos\theta}$$

The blood flow rate, Q is such that $Q = Au$

Example 8.31

In the measurement of blood flow in a patient, ultrasound of frequency of 10.0 MHz is incident at an angle 30° to the blood vessel and a Doppler shift in frequency of 8.8 kHz is observed. If the velocity of ultrasound can be taken as 2.2 kms^{-1} and the diameter of blood vessel is 0.8 cm. Calculate:

(a) Blood flow velocity; and
(b) Volume flow rate of blood.

Solution

(a) From, $u = \dfrac{v\Delta f}{2f\cos\theta}$

$$= \frac{2.2\times10^3\,\text{ms}^{-1}\times8.8\times10^3\,\text{Hz}}{2\times10\times10^6\cos30°} = 1.12\ \text{ms}^{-1}$$

Therefore, the blood flow velocity is 1.12 ms^{-1}

(b) Volume flow rate,

$$Q = Au;\ Q = \frac{\pi d^2}{4}\times u$$

$$Q = \frac{3.14\times(8\times10^{-3})^2\text{m}^2\times1.12\ \text{ms}^{-1}}{4}$$

$$= 5.63\times10^{-5}\text{m}^3\text{s}^{-1}$$

Therefore, the volume flow rate is $5.63\times10^{-5}\text{m}^3\text{s}^{-1}$

Bats use the Doppler effect phenomena to hunt for prey. Note that as a bat flies, it navigates by emitting whistles and listening for the echoes. When it is chasing down food, its brain detects a change in pitch between the emitted whistle, and the echo it receives. This tells the bat the speed of its prey, and the bat adjusts its own speed accordingly.

The Doppler effect in light can also be used to determine the speed of distant stars and extra solar planets by using the measurement of the wavelength of the spectral lines they emit. If the shift is towards the red, the star is receding from the earth; if it is towards the blue, the star is approaching the earth.

The Doppler effect has also been used to measure the speed of rotation of the sun. Photographs are taken of the east and west edges of the sun; each contains absorption lines due to elements such as iron vaporized in the sun and also some absorption lines due to oxygen in the earth's atmosphere. When the photographs are put together so that the oxygen lines coincide, the iron lines are displaced relative to each other. This shows that in one case the edge of the sun approaches the earth and in the other, the opposite edge recedes from the earth.

The Doppler effect in light can also be used to measure plasma temperature. At plasma temperatures, molecules of the glowing gas move away and towards the observer with very high speeds and owing to Doppler effect, the wavelength λ of a particular spectral line is apparently changed.

One edge of the line corresponds to an apparently increased wavelength λ_1 and the other edge to an apparent decreased wavelength λ_2. The line is thus observed to be broadened. If v is the velocity of the molecules, then,

$$\lambda_1 = \left(\frac{c+v}{c}\right)\lambda \text{ and } \lambda_2 = \left(\frac{c-v}{c}\right)\lambda$$

Thus, the width of line is given as;

$$\lambda_1 - \lambda_2 = \frac{2v\lambda}{c}$$

The width of the line can be measured by a diffraction grating, and as λ and c are known, the velocity v can be calculated. From kinetic theory of gases, the velocity v of the molecules is roughly the root mean square velocity, $v \approx v_{rms} = \sqrt{\frac{3RT}{M}}$, where T is the absolute temperature, R is the molar gas constant and M is the molar mass of one molecule.

Therefore, the absolute temperature of the plasma can be found.

Example 8.32

Two stars of equal mass move in a circular orbit of radius r about their common centre of mass. Observations in the plane of the orbit show that the wavelength of a spectral line from one of the stars varies between 599.9 nm and 600.1 nm in the course of one revolution.

(a) Calculate the speed v of the star in its orbit.

(b) If the orbital period T of the stars is 3.5×10^6 s, calculate the orbital radius r.

Solution

(a) Using the relation $\lambda_1 - \lambda_2 = \frac{2v\lambda}{c}$,

$$v = \left(\frac{\lambda_1 - \lambda_2}{2\lambda}\right)c,$$

$$v = \frac{(600.1 - 599.9)\ \text{nm}}{2\times600\ \text{nm}} \times 3\times10^8\ \text{ms}^{-1}$$

$$= 5\times10^4\ \text{ms}^{-1}$$

(b) $$r = \frac{vT}{2\pi} = \frac{5\times10^4\ \text{ms}^{-1}\times3.5\times10^6\ \text{s}}{2\pi}$$

$$= 2.78\times10^{10}\text{m}$$

Example 8.33

The wavelength of the yellow sodium line 5896 Å emitted by a star is red-shifted to 6010 Å. What is the component of the star's recessional velocity along the line of sight? For small recessional speeds, you may use a formula for Doppler effect analogous to that of sound (speed of light is $3\times10^8\ \text{ms}^{-1}$).

Solution

In this case,

$$\lambda' = \left(\frac{c+v}{c}\right)\lambda,\quad v = \left(\frac{\lambda' - \lambda}{\lambda}\right)\times c$$

$$v = \frac{(6010 - 5896)\times10^{-10}\ \text{m}}{5896\times10^{-10}\ \text{m}}\times3\times10^8\ \text{ms}^{-1}$$

$$= 5.8\times10^6\ \text{ms}^{-1}$$

Therefore, the star's recessional velocity v is $5.8\times10^6\ \text{ms}^{-1}$.

Exercise 8.6

1. Two sources of sound are emitting waves of wavelengths 5 m and 5.5m. If the velocity of sound is $340\ \text{ms}^{-1}$, what is the number of beats that will be produced?
2. Suppose that a source at rest is emitting sound having frequency of 800 Hz. Calculate the frequency observed when a listener moving with a velocity of $25\ \text{ms}^{-1}$ is
   (a) approaching, and
   (b) receding from the source.

3. A source is emitting a sound with a frequency of 400 Hz. A listener hears a sound with a frequency of 380 Hz. If the speed of sound $= 342\ \text{ms}^{-1}$, what is the speed and direction if
   (a) the source is in motion, and
   (b) a listener is in motion?
4. A sound source and a listener are both at rest on the earth, but a strong wind is blowing from the source towards the listener. Is there a Doppler effect? Why or why not?
5. A red shift is observed in the light received on earth from some galaxies.
   (a) Explain what a red shift is and how it occurs.
   (b) What useful information can be obtained from the red shift of a particular galaxy?
6. Many bats use the Doppler effect for detecting obstacles and prey. State what could be deduced about the obstacle if a bat detected a reflected wave of frequency less than that emitted.
7. A light of wavelength 6560 Å comes from a hydrogen atom in a distant star. Find the speed and direction of the star if there is to be an increase of 10% in its observed wavelength.
8. Show that when a source emitting sound waves of frequency f moves towards a stationary observer with velocity u, the observer hears a note of frequency $\frac{fv}{(v-u)}$, where v is the velocity of sound.

Revision exercise 8

1. Compute the velocity of waves on a string under a tension of 36 N and having a linear density of $6.25 \times 10^{-4}\ \text{kgm}^{-1}$.
2. A stationary observer is standing at a distance l from a straight railway track and a train passes with uniform velocity v sounding a whistle with frequency f_0. Taking the velocity of sound as u, derive a formula giving the observed frequency f as a function of time. At which position of the train will $f = f_0$? Give a physical interpretation of the result.
3. Determine which gaseous source would have less Doppler broadening, a mercury lamps at 200 °C or a Krypton lamps at 0 °C. Relative atomic mass of mercury is 200, relative atomic mass of Krypton is 84.
4. A police car chases a speeder along a straight road towards a cliff. Both vehicles move at 160 km/h. The siren on the police car produces sound at a frequency of 100 Hz. Calculate the Doppler shift in the frequency heard by the driver in a car behind the police car, moving at 120 km/h towards the cliff. (velocity of sound in air is 330 m/s)
5. A train approaches a stationary observer alongside the railway line while blowing a whistle of frequency 1000 Hz. After passing alongside an observer, the apparent frequency changes in the ratio 14:15. Estimate the speed of the train given that the speed of sound in air is 340 m/s.

6. It has been found experimentally that, the frequency of a fundamental note produced by a resonant tube, is affected by the end correction r and the air temperature θ. Show that the frequency is related to r and θ as;

$$f = \frac{v_0}{4(L+r)}\sqrt{1+\frac{\theta}{273}}$$, where v_0 is the velocity of sound at STP and L is the length of the resonant tube.

7. Interference can occur in thin films. Why is the line between "thin" and "thick" important with regards to the film? Explain your reasoning.

8. Newton's rings are formed with light of wavelength 5.89×10^{-5} cm between the curved surface of a plane convex lens and a flat glass plate in perfect contact. Find the radius of the 20^{th} dark ring from the centre if the radius of curvature of the lens surface is 100 cm. How will this ring move and what will its radius become if the lens and the plate are slowly separated to a distance of 5.00×10^{-4} cm apart?

9. A two slits Young's experiment is done with a monochromatic light of wavelength 6000 Å. The slits are 2 mm apart and the fringes are observed on a screen placed 10 cm away from the slits and it is found that, the interference pattern shifts by 5 mm when a transparent plate of thickness 0.5 mm is introduced in the path of one of the slits. What is the refractive index of the transparent plate?

10. Explain why
    (a) light can be polarized but sound cannot.
    (b) it is necessary to use satellite for long distance TV transmission.

11. Describe an experimental arrangement to observe interference of light. How would you use this experiment to determine a value for the wavelength of the light used?

12. An air wedge is formed between two glass plates which are in contact at one end and separated by a piece of thin metal foil at the other end. Calculate the thickness of the foil if 30 dark fringes are observed between the ends when light of wavelength 6×10^{-7} m is incident normally on the wedge.

13. With the aid of diagrams, differentiate between Fresnel and Fraunhofer diffraction.

14. Calculate the bulk modulus of air from the fact that, the speed of sound in air is 331.5 ms^{-1}. The density of air is 1.3 kgm^{-3}.

15. The wavelength of mercury green light is 5461 Å in vacuum. If this light meets a glass surface and is partly reflected and partly transmitted, what is the frequency and wavelength of reflected and transmitted light? The velocity of light in vacuum is 2.998×10^{8} ms^{-1} and that in glass is 1.898×10^{8} ms^{-1}.

16. The transverse displacement of a string (clamped at both ends) is given by: $y(x,t) = 4\cos\left(\frac{\pi x}{3}\right)\sin(40\pi t)$, where x is in cm and t in s.
    (a) Does the function represent a travelling wave or a stationary wave?
    (b) Interpret the wave as a superposition of two waves travelling in opposite directions.
    (c) What are the wavelength, frequency and speed of propagation of each wave?
17. Explain why the note emitted by a stretched string can easily be distinguished from that of a tuning fork with which it is in unison.
18. Explain the colour of a thin film in a white light and show that films which appear bright in reflected light, appear dark in transmitted light.
19. Determine how fast you would have to go through a red light to have it appear green. Take 620 nm as wavelength of red light and 450 nm as the wavelength of green light.
20. A whistle emitting a sound of frequency 440 Hz is tied to a string of 1.5 m length and rotated with an angular velocity of 20 rad s^{-1} in the horizontal plane. Calculate the range of frequencies heard by an observer stationed at a large distance from the whistle. Take velocity of sound to be 330 ms^{-1}.

Chapter Nine

Electrostatics

Introduction

Have you ever thought what causes charge to build up in a thundercloud? Why is it that when a plastic ruler is rubbed with wool it attracts paper scraps? These questions and other experiments help to explain why when some kinds of materials are rubbed against each other produce electric charges. For example, when glass and silk are rubbed against each other, they repel each other. However, when a glass rubbed with silk is brought close to ebonite rubbed with fur, they attract each other. The electrification of materials is due to the transfer of electrons from one material to another. In the process of electrification, charges are conserved and quantized. In this chapter, you will learn about the electric fields, electric potential, and capacitance.

9.1 Electric field

To visualize how a charge, or collection of charges, influences the region around it, the concept of an electric field is used. The electric field E is analogous to g, the acceleration due to gravity which in reality is gravitational field. Hence electric field is a region around a charged particle or object within which a force would be exerted on other charged particles or objects. A charge Q sets up an electric field in the space around it. If another charge q is brought near Q; then, electric field of Q exerts a force on q. Therefore, the electric field due to a point charge Q is defined as *"the space around the charge in which any other charge experiences an electrostatic force"*. The concept of electric field is described by a quantity called electric field intensity. The electric field intensity at a point is the force experienced by a unit test charge placed at that point. In this section, you will deal with different aspects of electric fields. These include Coulomb's law, electric field intensity of a point charge and electric field intensity of simple symmetrical charge distribution.

9.1.1 Coulomb's Law

Experiments show that charges interact by exerting forces on each other. A general rule called fundamental law of charges states that, *"Like charges repel and unlike charges attract"*. Charles de Coulomb investigated the interaction forces of charged particles and found out that, "*The electrostatic force between two point charges is directly proportional* *to the product of their magnitudes and inversely proportional to the square of the distance between their centres".*

Consider two point charges Q_1 and Q_2 held at a distance r apart in a medium (Figure 9.1).

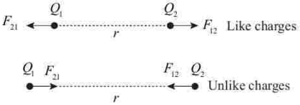


Figure 9.1 Interactions between point charges

From Coulomb's findings,

$$F \propto \frac{Q_1Q_2}{r^2}$$

$$F = k\frac{Q_1Q_2}{r^2} \qquad (9.1)$$

where k is a constant of proportionality called electrostatic force constant whose numerical value depends on the nature of medium where the charges are placed. Its value and SI unit is approximately to $9.0\times10^9\ \text{Nm}^2\text{C}^{-2}$ in a free space or a vacuum, and the constant k is given by

$$k = \frac{1}{4\pi\varepsilon_o}$$

where ε_0 is another constant, called absolute permittivity of free space with a value of $8.854\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}$. Therefore, Coulomb's Law can now be written as,

$$F = \frac{Q_1Q_2}{4\pi\varepsilon_o r^2} \qquad (9.2)$$

For a non-free space material, we have relative permittivity given by;

$$\text{Relative permittivity}(\varepsilon_r) = \frac{\text{permittivity of material}(\varepsilon_m)}{\text{absolute permittivity of air}(\varepsilon_0)}$$

Vector form of Coulomb's Law

Consider Figure 9.1. Let $\vec{F}_{21}$ be force on Q_2 due to Q_1, $\vec{F}_{12}$ be force on Q_1 due to Q_2 and $\hat{r}$ be unit vector pointing from Q_1 to Q_2. According to Coulomb's law,

$$\vec{F} = k\frac{Q_1Q_2}{r^2}\hat{r} \qquad (9.3)$$

where $\hat{r} = \frac{\vec{r}}{r}$. Therefore, equation (9.3) becomes $\vec{F} = k\frac{Q_1Q_2}{r^3}\vec{r}$, i.e., force $\vec{F}_{12} = -\vec{F}_{21} = \vec{F}$.

Experiments show that when two (or more) charges exert forces simultaneously on another point charge, the total force acting on that charge is the vector sum of the forces that the two (or more) charges would exert individually.

This property is called the superposition principle.The net force acting on one point charge due to a number of interacting charges is given as, $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n$.

Generally, the magnitude of net force acting on a point charge is given by,

$$F = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}$$

Example 9.1

How many electrons must be removed from a piece of metal to give it a positive charge of 1×10^{-7} C?

Solution

Using the relation that, $Q = ne$, then

$$n = \frac{Q}{e} = \frac{1\times10^{-7}\ \text{C}}{1.6\times10^{-19}\ \text{C}}$$

$$= 6.25\times10^{11}\ \text{electrons}$$

Example 9.2

The distance between the electron and the proton in a hydrogen atom is 5.3×10^{-11} m. Calculate the electrostatic force of attraction between them.

Solution

From Coulomb's law,

$$F = k\frac{Q_1Q_2}{r^2}$$

$$= \frac{9.0\times10^{9}\ \text{Nm}^2\text{C}^{-2}\times\left(1.6\times10^{-19}\ \text{C}\right)^2}{\left(5.3\times10^{-11}\ \text{m}\right)^2}$$

$$= 8.2\times10^{-8}\ \text{N}$$

Example 9.3

Two point charges are located on the x–axis of a coordinated system: $Q_1 = +1.0$ nC is at $x = +2.0$ cm, and $Q_2 = +3.0$ nC at $x = +4.0$ cm. What is the total force exerted by Q_1 and Q_2 on a charge $Q_3 = +5.0$ nC at $x = 0$ cm?

Solution

Figure 9.2 (a) shows the condition for the problem,

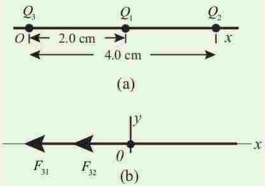


Figure 9.2 Point charges

Let, F_{31} be electrostatic force on Q_3 due to Q_1 and F_{32} be electrostatic force on Q_3 due to Q_2.

From Coulomb's law, $F_{31} = k\frac{Q_1Q_3}{(r_{13})^2}$; thus,

$$F_{31} = \frac{9\times10^{9}\ \text{Nm}^2\text{C}^{-2}\times1\times10^{-9}\ \text{C}\times5\times10^{-9}\ \text{C}}{\left(2.0\times10^{-2}\ \text{m}\right)^2}$$

$$= 1.13\times10^{-4}\ \text{N}$$

and $F_{32} = k\frac{Q_2Q_3}{(r_{23})^2}$,

$$= 9\times10^{9}\ \text{Nm}^2\text{C}^{-2}\times\frac{3.0\times10^{-9}\ \text{C}\times5\times10^{-9}\ \text{C}\times5\times10^{-9}\ \text{C}}{\left(4.0\times10^{-2}\ \text{m}\right)^2}$$

$$= 8.44\times10^{-5}\ \text{N}$$

From free body diagram for Q_3 in figure 9.2(b);

$$\sum F = F_{32} + F_{31},$$

$$\sum F = 8.44\times10^{-5}\ \text{N} + 1.13\times10^{-4}\ \text{N}$$

$$= 1.97\times10^{-4}\ \text{N}$$

Therefore, the total force exerted by Q_1 and Q_2 on a charge Q_3 is 1.97×10^{-4} N to the left of Q_3.

Example 9.4

Three point charges of $2\,\mu C$, $3\,\mu C$, and $4\,\mu C$ are placed on the vertices of an equilateral triangle of side 0.2 m. Calculate the magnitude of force on the $4\,\mu C$ charge due to other charges.

Solution

Consider Figures 9.3 (a) and (b),

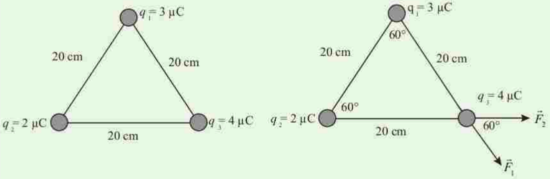


Figure 9.3 Point charges at vertices of an equilateral triangle

The free body diagram for the $4\,\mu C$ charge is shown in Figure 9.3 (b), $\vec{F_1}$ is the electrostatic force on $4\,\mu C$ due to $3\,\mu C$, $\vec{F_2}$ is the electrostatic force on $4\,\mu C$ due to $2\,\mu C$.

Considering the magnitudes:
Net force vertically;

$$\sum F_y = -F_1 \sin 60° + F_2 \sin 0° = -F_1 \sin 60°$$

$$= -9\times10^9\ \text{Nm}^2\text{C}^{-2} \times \frac{4\times10^{-6}\ \text{C}\times 2\times10^{-6}\ \text{C}}{(0.2\ \text{m})^2}\sin 60°$$

$$\therefore \sum F_y = -1.56\ \text{N}$$

Net force horizontally,

$$\sum F_x = F_1 \cos 60° + F_2$$

$$\sum F_x = 9\times10^9\ \text{Nm}^2\text{C}^{-2} \times \frac{4\times10^{-6}\ \text{C}\times 2\times10^{-6}\ \text{C}\times\cos 60° + 3\times10^{-6}\ \text{C}\times 4\times10^{-6}\ \text{C}}{(0.2\ \text{m})^2}$$

$$\therefore \sum F_x = 3.6\ \text{N}$$

$$F = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2},\ F = \sqrt{\left((3.6\ \text{N})^2 + (-1.56\ \text{N})\right)^2} = 3.92\ \text{N}$$

Therefore, magnitude of the force is 3.92 N.

9.1.2 The electric field of a point charge

The strength of an electric field at a point in space is determined by placing a small charged body (test charge) at the point. If the charge experiences a force, then there is an electric field at that point and its strength is given by

$$\vec{E} = \frac{\vec{F}}{q} \qquad (9.4)$$

where $\vec{F}$ is the electrostatic force experienced by a test charge, q is the magnitude of the test charge and $\vec{E}$ is the strength of the electric field at a point where q is placed. This field is produced by charge other than q. Thus, the electric field strength at a point is defined by equation (9.4) as the electric force per unit test charge experienced by a charge at that point.

For example, a source charge q_0 is at O in space, a test charge q is at point P at distance r from 0 (Figure 9.4).

Figure 9.4 Relationship between electric field and electrostatic force

The magnitude F of the force is given by Coulomb's law;

$$F = \frac{|q_o q|}{4\pi\varepsilon_o r^2}$$

From equation (9.4), the magnitude E of the electric field at P is,

$$E = \frac{|F|}{q} = \frac{|q_o q|}{4\pi\varepsilon_o r^2 q} = \frac{q_0}{4\pi\varepsilon_0 r^2} \qquad (9.5)$$

The electric field vector ($\vec{E}$) is in the direction of the force on a unit positive test charge.

Electric field lines

An electric field line is a path along which a positive test charge would move if it is free to do so. Electric field lines always originate from a positive charge and terminate at a negative charge. The tangent to the line at a point gives the direction of the electric field intensity at that point. Therefore, electric field lines are sometimes called electric lines of force see (Figure 9.5). The electric lines of force never cross each other, leave and enter the surface of a conductor at right angle.

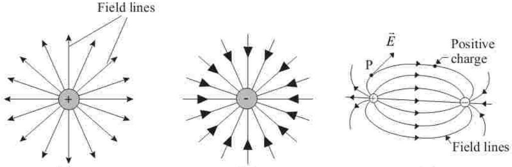


Figure 9.5 Electric field lines

Example 9.5

Two point charges $2\ \mu C$ each are placed $20\ cm$ apart. What is the electric field at the midpoint on the line connecting them?

Solution

Consider Figure 9.6.

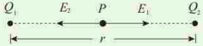


Figure 9.6 Two point charges

The net field E, is given as .

$$\sum E = E_1 + (-E_2) = E_1 - E_2.$$

$$\sum E = k\frac{Q_1}{r_1^2} - k\frac{Q_2}{r_1^2}, \text{ but } Q_1 = Q_2 \text{ and } r_1 = r_2$$

hence $Q_1 - Q_2 = 0$

The net electric field at point P is

$$\sum E = 0.$$

Example 9.6

A point charge of 3.3nC is placed in a medium of relative permittivity of 5. Calculate the electric field intensity at a point 10cm from the charge.

Solution

Using the relation that:

$$\varepsilon_r = \frac{\varepsilon_m}{\varepsilon_o} \text{ then, } \varepsilon_o \varepsilon_r = \varepsilon_m$$

$$E = \frac{Q}{4\pi\varepsilon_m r^2} = \frac{Q}{4\pi\varepsilon_0 \varepsilon_r r^2} = \frac{kQ}{\varepsilon_r r^2}$$

$$E = 9\times10^9\ \text{Nm}^2\text{C}^{-2} \times \frac{3.3\times10^{-9}\ \text{C}}{5\times\left(10.0\times10^{-2}\ \text{m}\right)^2}$$

$$= 594\ \text{NC}^{-1}$$

The electric field intensity is

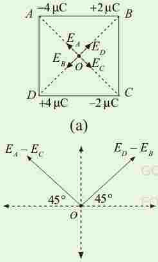
$E = 594\ \text{NC}^{-1}$.

Example 9.7

Four point charges of –4 μC, +2 μC, –2 μC and +4 μC are placed on corners *ABCD* of a square respectively. Determine the strength of the electric field at the centre of a square of side 2 m.

Solution

Figure 9.7 (a) shows four point charges at corners of the square.



(a)

(b)

Figure 9.7 Four charges placed at the corners of a square

From the free body diagram for point *O* in Figure 9.7(b),

Vertically,

$$\sum E_y = (E_A - E_C)\sin\theta + (E_D - E_B)\sin\theta$$

$$\sum E_y = (E_A - E_C + E_D - E_B)\sin\theta$$

$$\sum E_y = \frac{k\sin\theta}{r^2}\left((Q_A + Q_D) - (Q_C + Q_B)\right)$$

where $r = \sqrt{2}$ m, $\theta = 45°$ and

$$k = 9\times10^9\ \text{Nm}^2\text{C}^{-2}$$

Substituting variables by the numerical values, you get

$$\sum E_y = 12728\ \text{NC}^{-1}$$

Horizontally,

$$\sum E_x = (E_D - E_B)\cos\theta - (E_A - E_C)\cos\theta$$

$$\sum E_x = (E_D + E_C - E_A - E_B)\cos\theta$$

$$\sum E_x = \frac{k\cos\theta}{r^2}(Q_D + Q_C - Q_A - Q_B)$$

Substituting variables by the numerical values, you get

$$\sum E_x = 0\ \text{NC}^{-1}$$

$$\sum E = \sqrt{\left(\sum F_x\right)^2 + \left(\sum F_y\right)^2}$$

$$E_0 = \sqrt{\left(0\ \text{NC}^{-1}\right)^2 + \left(12728\ \text{NC}^{-1}\right)^2}$$

$$= 12728\ \text{NC}^{-1}$$

The net electric field at O will be 12728 NC^{-1} upwards.

9.1.3 Electric field due to continuous charge distribution

In practice we deal with charge distribution on bodies, for example along a line, over surface or volume. We speak of linear charge density *λ*, surface charge density *σ*, and volume charge density *ρ*, given by:

$$\lambda = \frac{\text{charge}}{\text{length}} = \frac{dQ}{dl},\ \sigma = \frac{\text{charge}}{\text{surface area}} = \frac{dQ}{dA}$$

and $\rho = \dfrac{\text{charge}}{\text{volume}} = \dfrac{dQ}{dV}$

These quantities describe the amount of charge per unit length, per unit area and per unit volume respectively. In calculating electric field caused by continuous charge distribution, you consider the distribution to consist of an infinitesimal charge elements dQ.

(a) Electric field due to a line of charge

The electric field of a line of charge is found by superposing the point charge fields of infinitesimal charge elements dQ. Consider the diagram in Figure 9.8.

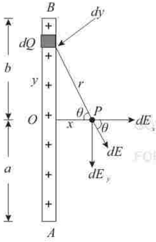


Figure 9.8 Electric field due to a line of charge

The electric field at point P, due to the charge element dQ is given as,

$$dE = k\frac{dQ}{r^2}$$

The net horizontal field at P is given by $dE_x = dE\cos\theta$. Thus,

$$dE_x = k\frac{dQ}{r^2}\left(\frac{x}{r}\right)$$

where $dQ = \lambda dy, \lambda =$ linear charge density. The net vertical field component at P is zero (0), due to field cancellation effect when elements of charge are considered from both sides of the line, about O.

The total field at P is given by,

$$E_p = \int_{-a}^{b} dE_x = k\lambda\int_{-a}^{b}\frac{xdy}{(x^2+y^2)^{\frac{3}{2}}}$$

Using calculus techniques to integrate the equation you obtain

$$E_p = k\frac{\lambda}{x}\left(\frac{b}{(x^2+b^2)^{\frac{1}{2}}}+\frac{a}{(x^2+a^2)^{\frac{1}{2}}}\right) \quad (9.6)$$

As the limits a and b approach infinity, equation (9.6) approaches the infinite line of charge expression given as

$$E_p = \frac{2k\lambda}{x} \quad or \quad E_P = \frac{\lambda}{2\pi\varepsilon_0 x}$$

Electric flux and Gauss's Law

From the discussion on electric field lines, it was shown that electric fields can be described by the lines of force. Since the density of lines increases near the charge where the electric field strength ($\vec{E}$) is high, then $\vec{E}$ at a point can be given as the "number of lines per unit area" through a surface perpendicular to the lines of force at that point. Consider Figure 9.9.

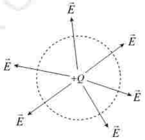


Figure 9.9 Electric flux through an area A due to a positive point charge

The number of lines of force crossing normally a given surface give the values to a quantity called electric flux, denoted as ϕ. Therefore, flux is given as,

$$\phi = \vec{E} \cdot \vec{A} \qquad (9.7)$$

$$\phi = E\cos\theta \times A \qquad (9.8)$$

where θ is measured from a normal to surface.

Therefore maximum flux is obtained when the angle between $\vec{E}$ and $\vec{A}$ is zero, that is when $\vec{E}$ and $\vec{A}$ are parallel. If the electric field $\vec{E}$ is not uniform or if A is part of curved surface, then the surface is divided into small area element dA and the equation (9.8) is integrated to obtain the total flux as:

$$\phi = \int_s E\cos\theta \, dA \qquad (9.9)$$

The integral in equation (9.9) is called surface integral of the component E over the area, or surface integral of $\vec{E} \cdot d\vec{A}$.

Example 9.8

A disc of radius 0.10 m is oriented with its area vector at 30° to a uniform electric field E of magnitude 2.0×10^{-3} NC^{-1}. Calculate the electric flux through the disc.

Solution

Figure 9.10 shows condition for the problem.

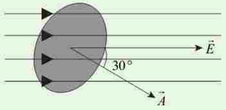


Figure 9.10 A disc in a uniform electric field

Using equation (9.8),

$$\phi = \left(2.0 \times 10^{3}\ \text{NC}^{-1}\right) \times \pi \times \left(0.1\ \text{m}\right)^2 \times \cos 30°$$

$$= 54\ \text{N m}^2\text{C}^{-1}$$

Gauss's Law

Gauss's law states that, "*The total electric flux through any closed surface is proportional to the net electric charge inside that surface*". i.e. The electric flux through any closed surface is equal to the total charge inside divided by ε_0.

Suppose the surface encloses several charges $Q_1, Q_2, ..., Q_n$. Let Q_T be the total charge enclosed by the surface; $Q_T = Q_1 + Q_2 + ... + Q_n$. Also let $\vec{E}$ be the total electric field at the position of the surface element dA.

Then Gauss's law can be written in mathematical form as,

$$\phi = \frac{Q_T}{\varepsilon_o} \qquad (9.10)$$

Combining equations (9.9) and (9.10) gives the general form of Gauss's law:

$$\oint E\cos\theta dA = \frac{Q_T}{\varepsilon_o} \text{ or } \oint \vec{E} \cdot d\vec{A} = \frac{Q_T}{\varepsilon_o}$$

When using Gauss's law, there are some important steps to follow:

(i) Draw an imaginary surface called Gaussian surface to enclose the charge. This charge may be a single point charge, a collection of point charges or a given charge distribution.

(ii) Divide the Gaussian surface into smaller area element dA such that $d\vec{A}$ and $\vec{E}$ at that particular position are parallel.

(iii) Carry an integration over the entire surface to get the total electric flux.

Gauss's law is useful in determining electric fields when the charge distribution is characterized by a high degree of symmetry. The following examples show ways of choosing the Gaussian surface over which the surface integral is applied to determine the electric field. In choosing the surface, always take advantage of the symmetry of the charge distribution so that you can remove E from the integral and solve for it.

(b) Electric field due to a sphere of charge

When charge is distributed over a sphere of radius R the electric field due to that charge will have spherical symmetry. However the flux passing through any closed surface of any shape is always equal to $\frac{Q}{\varepsilon_0}$. That is, the field will have the same value at equal distance r from the centre of the sphere. For a solid insulating sphere, the electric field inside and outside are different. Consider a uniformly charged non-conducting (insulating) solid sphere with centre O, radius R, volume charge density ρ and total charge Q.

(i) Electric field outside the insulating sphere ($r > R$)

Because the charge distribution is spherically symmetric, select a spherical Gaussian surface of radius r, concentric with the sphere. Then find the electric field intensity at a point outside the solid sphere, distance r from the sphere surface (Figure 9.11).

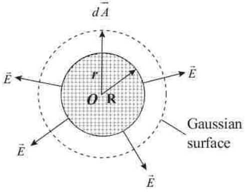


Figure 9.11 Electric field outside the insulating sphere

Since it is an insulating sphere, charge will reside entirely in the volume. The net charge $Q = \frac{4}{3}\pi R^3 \rho$. Since E is constant and normal to the spherical Gaussian surface of radius r, the surface integral equals, $E \times 4\pi r^2$. Therefore, $E \times 4\pi r^2 = \frac{Q}{\varepsilon_0}$. At a distance $r > R$, the electric field is identical to that of a point charge, Q at the centre of the sphere. Hence,

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad (9.11)$$

(ii) Electric field inside the insulating sphere ($r < R$)

Since the sphere is not conducting material, a charge Q' will reside inside the sphere as well. For a radius $r < R$, Gaussian surface will enclose less than the total charge and electric field will be less (Figure 9.12).

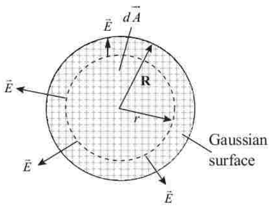


Figure 9.12 Electric field inside an insulating solid sphere

Therefore, the field E at a distance r from the centre of the sphere is still spherically symmetric and is given by

$$E = \frac{Q'}{4\pi\varepsilon_o r^2}$$

Note that the charge Q' is obtained from the ratio of volumes as follows:

$$\frac{Q'}{V'} = \frac{Q}{V};\ Q' = \frac{V'Q}{V} \text{ then, } Q' = \frac{Qr^3}{R^3}$$

$$E = \frac{Qr}{4\pi\varepsilon_o R^3} \qquad (9.12)$$

(c) Solid conducting sphere

Consider a thin spherical conducting (metal) shell of radius R charged uniformly with charge Q. Electric field E can be obtained by considering outside and inside the sphere as follows:

(i) Electric field outside the sphere
$(r > R)$

Since charges reside entirely on the surface of a solid conductor, the electric field will have spherical symmetry and will be given by equation (9.11) same as for point charge. This implies that, E outside a uniformly charged conducting sphere is the same as if charge were concentrated as a point charge at the centre of the sphere.

(ii) Electric field inside the sphere
($r < R$)

Since there is no charge within a concentric spherical Gaussian surface of a conducting charged shell, then the net flux equals zero from Gauss's law. From symmetry therefore; the electric field is zero inside the spherical charged shell. Hence, the electric field due to uniformly charged sphere is zero at all points inside the sphere.

(d) Hollow conducting sphere

Consider a hollow sphere of inner radius R_1 and outer radius R_2 charged uniformly with total charge Q (Figure 9.13). What will be the electric field at any point, a distance r from the centre.

(i) Electric field outside the sphere
$(r > R_2)$.
From Gaussian,

$$E \times 4\pi r^2 = \frac{Q}{\varepsilon_o};\quad E = \frac{kQ}{r^2}$$

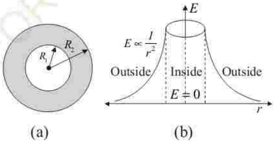


Figure 9.13 Electric field variation inside and outside a hollow sphere

Therefore, the electric field will resemble that of a point change for same reason that the charge resides entirely on the surface of a sphere and the field will have spherical symmetry.

(ii) Electric field inside the sphere ($r < R_1$). The field will be zero. No net charge can reside inside a conductor.

(e) Hollow insulating sphere

Consider a hollow insulating sphere with uniform charge density ρ. Its inner and outer radii are R_1 and R_2 respectively. The expressions for the magnitude of the electric field in the regions are:

(i) Electric field outside the sphere ($r > R_2$). The field will be given by equation (9.11) for similar reasoning.

(ii) Electric field inside the sphere ($r < R_1$). Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero, application of Gauss's law shows that $E = 0$ in the region r.

(iii) Electric field within the sphere ($R_1 < r < R_2$). From Gauss's law,

$$E \times 4\pi r^2 = \frac{4}{3\varepsilon_0}\pi \times \left(r^3 - R_1^3\right)\rho,$$

but $$\rho = \frac{Q}{\frac{4\pi}{3}\left(R_2^3 - R_1^3\right)}$$

Therefore,

$$E = \frac{1}{4\pi\varepsilon_0 r^2}\frac{\left(r^3 - R_1^3\right)}{\left(R_2^3 - R_1^3\right)}Q$$

(f) Electric field due to plane charge distribution

Consider an infinity plane sheet with charge Q distributed uniformly. The electric field intensity at a distance d from the sheet of charge can be obtained by choosing Gaussian surface, say cylindrical surface with radius r (Figure 9.14a).

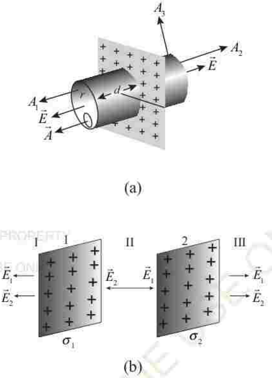


Figure 9.14 (a) Electric field due to infinity plane sheet of charge and (b) Parallel sheet of charges

There are three surfaces, two end surfaces (A_1 & A_2) and one side surface (A_3). Therefore the total flux is given by;

$$\phi = \oint_{A_1} EdA_1\cos\theta_1 + \oint_{A_2} EdA_2\cos\theta_2 + \oint_{A_3} EdA_3\cos\theta_3$$

where $\theta_1 = \theta_2 = 0°$, $\theta_3 = 90°$, also, $A_1 = A_2 = A$, which is a circular area.

Then from Gauss's law,

$$\frac{Q}{\varepsilon_o} = 2E\oint_A dA$$

hence,

$$\frac{Q}{\varepsilon_o} = 2 \times EA, \quad E = \frac{Q}{2A\varepsilon_o} = \frac{\sigma}{2\varepsilon_o} \qquad (9.13)$$

Equation (9.13) shows that; electric field intensity due to infinity sheet of charge is independent of distance from the sheet. But it depends on the density of charge distribution only. Consider the two parallel sheets with charge densities σ_1 and σ_2 (Figure 9.14b).

Electric field intensity due to two parallel infinity sheet of charge at three regions (I, II, III) is as follows;

The electric fields due to two sheets is the vector sum of individual sheets:

Region I: Two vectors acting in negative direction

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$E = \frac{-\sigma_1}{2\varepsilon_o} + \frac{-\sigma_2}{2\varepsilon_o} = \frac{-1}{2\varepsilon_o}(\sigma_1 + \sigma_2)$$

Region II: Two vectors acting in opposite directions

$$\vec{E} = \vec{E}_1 + (-\vec{E}_2)$$

$$E = \frac{-\sigma_1}{2\varepsilon_o} + \frac{-\sigma_2}{2\varepsilon_o} = -\frac{1}{2\varepsilon_o}(\sigma_1 + \sigma_2)$$

Region III: Two vectors acting in the positive direction

$$\vec{E} = \vec{E}_1 + \vec{E}_2,$$

$$E = \frac{\sigma_1}{2\varepsilon_o} + \frac{\sigma_2}{2\varepsilon_o} = \frac{1}{2\varepsilon_o}(\sigma_1 + \sigma_2)$$

Special case: When the charge distribution is the same, i.e. $\sigma_1 = \sigma_2 = \sigma$:

Region I: Two vectors acting in the negative direction

$$E = \frac{-\sigma}{2\varepsilon_o} + \frac{-\sigma}{2\varepsilon_o} = \frac{-\sigma}{\varepsilon_o}$$

Region II: Two vectors acting in opposite directions

$$E = \frac{\sigma}{2\varepsilon_o} + \frac{-\sigma}{2\varepsilon_o} = 0$$

Region III: Two vectors acting in the positive direction

$$E = \frac{\sigma}{2\varepsilon_o} + \frac{\sigma}{2\varepsilon_o} = \frac{\sigma}{\varepsilon_o}$$

Note that, A parallel plate capacitor which is an arrangement of metal plates connected in parallel and separated from each other by some distance, behaves like a plane with charges distributed uniformly and its electric field can be calculated using the concepts applied in the plane charge distribution.

Exercise 9.1

1. Explain the following observation in relation to electrostatics:
   (a) The free electrons in a metal are gravitationally attracted towards the earth. Why do they all not settle to the bottom of the conductor like sediment settling at the bottom of a river?
   (b) Bits of paper are attracted to an electrified comb or rod even though they have no net charge. How is this possible?

2. Two small plastic spheres are given positive electrical charges. When they are 15.0 cm apart, the repulsive force between them has a magnitude of 0.220 N . Find the charge on each sphere,
   (a) if the two charges are equal.
   (b) if one sphere has four times the charge of the other.
3. A negative charge of −0.55 μC exerts an upward force of 0.20 N on an unknown charge which is 0.30 m directly below it.
   (a) What is the unknown charge (magnitude and sign)?
   (b) What is the magnitude and direction of the force that the unknown charge exerts on the −0.55 μC charge?
4. A small insulating sphere is given a charge of +15 μC and a second sphere of equal size is given a charge of −10 μC. The two spheres are allowed to touch each other and then separated 20 cm apart. Assuming air as the medium, what force exists between them?
5. Two identical small spheres with mass m are hang from silk threads of length L . Each sphere is given a charge Q . Show that when the two charged spheres are at equilibrium, the distance d between their centers is given as,

$$d=\left(\frac{Q^2L}{2\pi\varepsilon_o mg}\right)^{\frac{1}{3}}$$

   State the assumption used to arrive at your answer.
6. Two charges, one of 2.5 μC is placed at the origin and the other of −3.50 μC is placed 0.60 m on the x-axis. Find the position on the x-axis where the net force on a small charge $+q$ would be zero.
7. A point charge Q of +8 nC is placed at the origin of the $x-y$ coordinate system. Determine the electric field strength at a point $P(x, y)=(1.2\text{m}, -1.6\text{m})$.
8. Two tiny spheres, of mass 6.80 mg each carry charges of equal magnitude, 72.0 nC but opposite sign. They are tied to the same ceiling hook by light strings of length 0.53 m . When a horizontal uniform electric field E directed to the left is turned on, the spheres hang at rest with the angle θ between the strings equal to 50°.
   (a) Which sphere has a positive charge?
   (b) What is the magnitude of the electric field?
9. A particle of mass 1kg and charge 0.01 C is placed on an inclined plane making an angle 30° with the horizontal. The incline and the charged particle are placed in a uniform horizontal electric field of 100 N C^{-1}. What should be the coefficient of static friction for the particle not to slide down the incline?
10. ABC is an equilateral triangle whose side is 1 m . Two point charges of 2 μC and −2 μC are placed at corner A and B respectively. Determine the magnitude and direction of the electric field E at corner C .

11. Two horizontal parallel plates 10 mm apart have a potential difference (*p.d.*) of 1000 volts between them, the upper plate being at positive potential. If a negatively charged oil drop of mass 4.8×10^{-15} kg is held stationary between the plates, find the number of electrons on the drop (Use $e=1.6\times10^{-19}$ C).
12. Show that the magnitude of the electric field at any point perpendicular to the plane of an infinite plane sheet of charge is given as $E=\dfrac{\sigma}{2\varepsilon_o}$, where σ is the surface charge density.
13. A small charge q is placed at a midpoint on the line connecting two other charges of equal magnitude and sign. The charge q is displaced along the line and released, prove that it performs simple harmonic motion.

9.2 Electric potential

Just like a mass has potential energy in the gravitational field, electric charge has electrostatic potential energy in the electrostatic field. There is electric potential energy associated with interacting charges.

The electric potential is useful, since it provides an alternative to the electric field in electrostatic problems. This section discusses concept of electric potential, potential due to charge distribution and motion of a charged particle in a uniform electric field.

9.2.1 The concept of electric potential

When a positive test charge Q is moved in an electric field against such field, work has to be done to overcome the electrostatic repulsion. The work done to move a unit positive test charge is called the electric potential. Therefore, an electric potential is a property of an electric field at a given point defined as being numerically equal to the work done in bringing a unit positive test charge from infinity to that point against the electrostatic field.

Electric potential due to a point charge

Consider a point charge $+Q$ such as proton placed at point O as shown in Figure 9.15. This charge sets up an electrostatic field which extends up to infinity. Suppose a unit positive test charge q is at a point P a distance r from O. If q is moved from point B to point A, work has to be done.

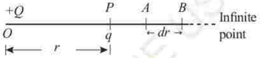


Figure 9.15 Electric potential due to a point charge

The work done by an external agent on moving charge q a small distance dr from B towards A is equal to the work done by electrostatic force F on moving a charge from A to B if charge Q does not accelerate.

$W_{BA}=-\int_B^A F\,dr$, where $F=\dfrac{kQq}{r^2}$. Thus,

$$W_{BA}=-kQq\int_B^A r^{-2}\,dr$$

The negative sign is important because it implies that, the test particle loses potential energy when moving to $r = 0$.

From the definition of electric potential,

$$V_{BA} = \frac{W_{BA}}{q} = -kQ\int_B^A r^{-2} dr$$

Therefore,

$$V_{BA} = kQ\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \quad (9.14)$$

Equation (9.14) gives the electrostatic potential difference between points A and B. It is the work an external agent has to do to carry a unit positive charge from B to A.

When calculating electrostatic potential, a reference point is always chosen at infinity where the electric potential is zero. Therefore, if point B is at infinity, equation (9.14) changes to:

$$V_A = \frac{kQ}{r_A} \quad (9.15)$$

The unit of potential and potential difference is volt (V) or JC^{-1}. It should also be noted that electrostatic potential is a scalar quantity and therefore the potential of a point due to a group of point charges is the algebraic sum of the (separate) potentials due to each charge. Remember that, potential due to a positive charge is positive and that due to a negative charge is negative.

Electrostatic potential energy *(U)* of a point in the electrostatic field is numerically equal to the work done in bringing a positive charge from infinity to that point. That is,

$$U_{A\infty} = W_{A\infty} \quad (9.16)$$

However, electric potential is potential energy per unit charge. Thus,

$$U_{A\infty} = W_{A\infty} = qV_{A\infty} \rightarrow U_A = qV_A$$

$$\therefore U = q\Delta V \quad (9.17)$$

Example 9.9

Two point charges $+20\ \mu C$ and $-20\ \mu C$ are placed 20 cm apart. Calculate the electrostatic potential at a point midway on the line connecting the two charges.

Solution

Consider Figure 9.16.

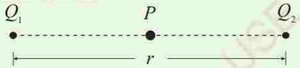


Figure 9.16 Point charges

Using the principle of superposition of electric potentials,

$$V_p = V_{Q_1} + V_{Q_2},\ V_p = \frac{kQ_1}{\frac{r}{2}} + \frac{kQ_2}{\frac{r}{2}},$$

hence $V_p = \frac{2k}{r}(Q_1 + Q_2)$.

$$= \frac{2 \times 9 \times 10^9\ \text{Nm}^2\text{C}^{-2}}{20 \times 10^{-2}\ \text{m}} \times \left(20 \times 10^{-6}\ \text{C} + (-20 \times 10^{-6}\ \text{C})\right)$$

$$= 0\ \text{V}$$

Electrostatic potential at a point midway is 0 V.

Example 9.10

Three charges of $2\ \mu C, -3\ \mu C$ and $4\ \mu C$ are placed on the three corners A, B, and C of an equilateral triangle of sides $2\ m$ respectively. Determine the electric potential at a point half way between AB.

Solution

Figure 9.17 shows the conditions of the problem.

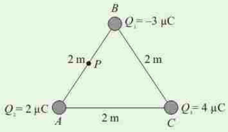


Figure 9.17 Point charges placed at corners of a triangle

Using the principle of superposition of electric potentials.

$$V_p = V_A + V_B + V_C$$

$$V_p = \frac{kQ_A}{r_A} + \frac{kQ_B}{r_B} + \frac{kQ_C}{r_C}; \text{ where}$$

$$r_A = \overline{AP}, \quad r_B = \overline{PB} \quad and \quad r_C = \overline{CP}$$

$$Q_A = Q_2, \quad Q_B = Q_1, \text{ and } Q_C = Q_3$$

$$V_p = k\left(\frac{Q_A}{r_A} + \frac{Q_B}{r_B} + \frac{Q_C}{r_C}\right)$$

$$V_p = 9\times10^9\ Nm^2C^{-2}\left(\frac{2\times10^{-6}\ C}{1\ m} + \frac{-3\times10^{-6}\ C}{1\ m} + \frac{4\times10^{-6}\ C}{\sqrt{3}\ m}\right)$$

$$= 11784.6\ V$$

Therefore, the electric potential at the midway of AB is 11784.6 V.

Example 9.11

Two positive point charges of $10\ \mu C$ and $8\ \mu C$ respectively are $10\ cm$ apart. Find the work done in bringing them to a separation of $6\ cm$.

Solution

Suppose the $8\ \mu C$ charge is fixed in position. Then the potential difference between $6\ cm$ mark and $10\ cm$ mark is, $\Delta V = V_A - V_B$, then,

$$\Delta V = kQ\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

Therefore the work done in moving charge $q = 10\ \mu C$ up to a distance $6\ cm$ is,

$$\Delta W = q\Delta V = kQq\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$\Delta W = 9\times10^9\ Nm^2C^{-2} \times 8\times10^{-6}\ C \times 10\times10^{-6}\ C \times \left(\frac{1}{6\times10^{-2}\ m} - \frac{1}{10\times10^{-2}\ m}\right) = 4.8\ J$$

Hence, the work done in bringing the charge a distance 6 cm apart is 4.8 J.

Example 9.12

Two charges of 3 nC and −3 nC are fixed in positions 3 cm apart. A dust particle with a mass of 5 μg and a charge of 2 μC starts from rest and moves in a straight line from a point 1.0 cm from the 3 nC charge to a point 1.0 cm from the −3 nC. What is the speed of the dust particle at the second location?

Solution

Figure 9.18 shows the conditions for the problem.

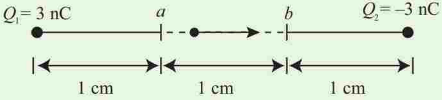


Figure 9.18 Two charges at fixed positions

From conservation of mechanical energy, $K_a + U_a = K_b + U_b$, where K_a and U_b are kinetic and potential energies respectively. Since the particle start from rest $K_a = 0$.

$$K_b = U_a - U_b,\ K_b = \frac{1}{2}mv_b^2 = U_a - U_b \text{ where } U_a - U_b = q(V_a - V_b)$$

$$v_b = \sqrt{\frac{2q(V_a - V_b)}{m}} \qquad (i)$$

Solving for V_a and V_b;

$$V_a = 9.0\times10^9 \text{ Nm}^2\text{C}^{-2} \times 3\times10^{-9}\text{ C}\times\left(\frac{1}{1.0\times10^{-2}\text{ m}} + \frac{-1}{2.0\times10^{-2}\text{ m}}\right) = 1350 \text{ V}$$

$$V_b = 9.0\times10^9 \text{ Nm}^2\text{C}^{-2} \times 3\times10^{-9}\text{ C}\times\left(\frac{1}{2.0\times10^{-2}\text{ m}} + \frac{-1}{1.0\times10^{-2}\text{ m}}\right) = -1350 \text{ V}$$

$$V_a - V_b = 1350 \text{ V} - (-1350 \text{ V}) = 2700 \text{ V} \qquad (ii)$$

From equation (i), $V_b = \sqrt{\frac{2\times2\times10^{-6}\text{ C}\times2700\text{ V}}{5\times10^{-6}\text{ kg}}} = 1469.69\text{ ms}^{-1}$.

Therefore, velocity of a dust particle at b is about 1469.69ms^{-1}.

Relationship between electric field and electric potential difference

When electric field at a point is produced by a charge distribution, the electric potential at the point in that field can easily be calculated using the expression of E at that point. Suppose a test charge q_0 in a uniform electric field E is moving from point a to b (with constant speed) distance dr apart. The work done on q_0 along displacement ab by electric force is given by

$$W_{ab} = \int_a^b \vec{F}\cdot d\vec{r} = \int_a^b q_0\vec{E}\cdot d\vec{r}.$$

Thus, the potential difference $\frac{W_{ab}}{q_0} = \int_a^b \vec{E}\cdot d\vec{r}$.

The work done per unit charge is given as,

$$\frac{W_{ab}}{q_0} = V_a - V_b$$

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{r} \qquad (9.18)$$

Therefore, the general relationship between electric potential and electric field intensity is

$$V_a - V_b = -\int E\,dr \text{ or } dV = -Edr$$

so that

$$E = -\frac{dV}{dr}.$$

The displacement of charge will be in the direction opposite to the direction of electric field. Consequently, the angle between force on charge and the displacement of charge is 180°. The angle between the two vectors is 180°, thus $\vec{E}\cdot d\vec{r} = Edr\cos180° = -Edr$. Setting the potential at b to be zero (b at infinity) the potential at a point is given by $V_a = \int_a^\infty \vec{E}\cdot d\vec{r}$.

9.2.2 Electric potential due to charge distribution

Electric potential at a point due to continuous charge distribution depends on the geometry of distribution. The electric potential due to uniform distributed charge can be obtained by using the relation $V_a - V_b = -\int E\,dr$. In practice there exists a variety of distribution geometries, for example along a line and over a surface or volume.

(a) Electric potential due to infinite line of charge

The electrostatic potential due to an infinite line of charge is calculated using the relation $V_a - V_b = -\int E\,dr$. But the electric field due to a line charge distribution is given by, $E = \frac{\lambda}{2\pi\varepsilon_o r}$.

The electric potential is then,

$$V_a - V_b = \int_{r_a}^{r_b} \frac{\lambda}{2\pi\varepsilon_o r} dr$$

$$V_a - V_b = \frac{\lambda}{2\pi\varepsilon_o} \ln\left(\frac{r_b}{r_a}\right) \qquad (9.19)$$

The expression of equation (9.19) can be used for a conducting cylinder of charge. Suppose R is the radius of such a cylinder. Then, the potential at a point A, distance r from the axis of a cylinder for which $r > R$ is

$$V = V(r) - V(R) \text{ but, } V(R) = 0$$

$$V = -\int_R^r E\,dr - \int_R^r \frac{\lambda}{2\pi r\varepsilon_0} dr$$

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln(r)\right]_R^r = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{R}{r}\right) \qquad (9.20)$$

Therefore,inside the conducting cylinder, $E = 0$, and $V = 0$ as on the cylinder's surface.

(b) Electric potential due to finite line of charge

Consider electrostatic potential due to a finite line of charge. Suppose a positive electric charge Q is distributed uniformly along a line of length $2a$ as shown in Figure 9.19.

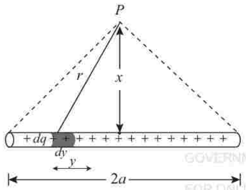


Figure 9.19 Electric potential due to finite line of charge

The electric potential at point P is,

$$V_p = \int_{-a}^{a} k\frac{dQ}{r}, \text{ where } r = \left(y^2 + x^2\right)^{\frac{1}{2}}.$$

Then,

$$V_p = \int_{-a}^{a} k\frac{\lambda dy}{\left(y^2 + x^2\right)^{\frac{1}{2}}}$$

Using integration techniques, you get

$$V_p = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{\sqrt{a^2 + x^2} + a}{\sqrt{a^2 + x^2} - a}\right) \qquad (9.21)$$

(c) Electric potential due to charged conducting sphere

Suppose a total charge Q is placed on the solid sphere of radius R (Figure 9.20a). The electrostatic potential at a point distance r, such that $r > R$ is given by;

$$V = \frac{kQ}{r} \qquad (9.22)$$

This is because for a sphere of charge, an electric field at a point outside such a sphere resembles that of a point charge.

Inside a conducting sphere, $V_a = V_b$

That is the electric potential inside, is the same everywhere and is equal to its value on the surface which is, $V = \frac{kQ}{R}$.

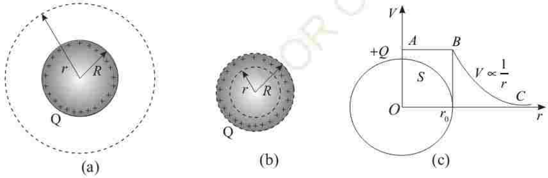


Figure 9.20 Variation of electric potential with distance inside and outside a charged sphere

Equipotential

An equipotential is a three dimensional region in space where every point in it is at the same electric potential. In practice there are equipotential surfaces and equipotential volumes. A line that traces all points on an equipotential is called an equipotential line.

(i) It is important to note that equipotential lines are always perpendicular to electric field lines. No work is required to move a charge along an equipotential, since $\Delta V = 0$.

(ii) Equipotential lines never intersect or touch each other.

That means that at a particular region the electric potential (V) is constant which creates an equipotential surface.

The quantity $\frac{dV}{dr}$ shows how the potential changes with distance, and is called the potential gradient. The diagram in Figure 9.21 shows equipotential and electric lines of force for point charges (dashed lines are equipotential lines while solid lines are electric field lines).

(a) Constant electric field (b) Point charge (c) Electric dipole

Figure 9.21 Equipotential and electric field lines of capacitor, point charge, and dipole

9.2.3 Motion of a charged particle in a uniform electric field

When a particle of mass m and charge Q is placed in a uniform electric field of strength E, the field will exert force QE on the charge. If it is the only force on the particle, the particle will accelerate uniformly. ***Note that,*** if the particle is negatively charged, it will accelerate in the direction opposite to that of the electric field. If the particle has a positive charge, its acceleration is in the direction of the electric field (Figure 9.22).

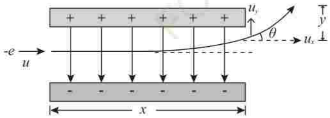


Figure 9.22 An electron projected horizontally in a uniform electric field

Suppose that an electron of charge $-e$ is projected horizontally into a uniform electric field (Figure 9.22) with an initial velocity u. Then, using kinematics equation in two dimensions:

Case1: Vertical motion

The electron experiences electrostatic force in the vertical direction

$$F_y = eE = ma_y$$

$$a_y = \frac{eE}{m} \qquad (9.23)$$

From first equation of linear motion

$$v_y = u_y + a_y t$$

Since the electron is horizontally projected; $u_y = 0$. Therefore,

$$v_y = a_y t$$

Using equation (9.23), you obtain

$$v_y = \left(\frac{eE}{m}\right)t \qquad (9.24)$$

From second equation of motion

$$y = \frac{1}{2}\frac{eE}{m}t^2 \qquad (9.25)$$

Case 2: Horizontal motion

From first equation of motion

$v_x = u_x + a_x t$, where $u_x = u$, $a_x = 0$

Hence $v_x = u$ implying constant velocity

From second equation of motion,

$s_x = x = u_x t + \frac{1}{2}a_x t^2$ but $u_x = u$, $a_x = 0$,

then

$$x = ut \qquad (9.26)$$

Combining equation (9.25) and equation (9.26) gives

$$y = \frac{1}{2}\frac{eE}{mu^2}x^2$$

Since e, E, m and u are constants, then,

$$y = kx^2$$

Hence, the trajectory of the charged particle in the electric field is a parabola.

Note that, for a charged body to move undeflected in electric field $mg = EQ$.

Example 9.13

An electron enters horizontally in the region of a uniform vertically downward electric field with a velocity of $3.0\times10^6\ \text{ms}^{-1}$. The electric field strength E is $200\ \text{NC}^{-1}$ and the horizontal width of the field is $0.1\ \text{m}$.

(a) Find the acceleration of the electron while it is in the field.

(b) Find the time it takes the electron to travel through the field.

Solution

(a) Using equation (9.23)

$$a_y = \frac{1.6\times10^{-19}\ \text{C}\times200\ \text{NC}^{-1}}{9.11\times10^{-31}\ \text{kg}}$$

$$= 3.51\times10^{13}\ \text{ms}^{-2}$$

Therefore, acceleration of the electron while in the field is $3.51\times10^{13}\ \text{ms}^{-2}$.

(b) Using equation (9.26)

$$t = \frac{x}{u} = \frac{0.1\ \text{m}}{3.0\times10^6\ \text{ms}^{-1}} = 3.33\times10^{-8}\ \text{s}$$

Hence, the time taken by the electron through the field is $3.33\times10^{-8}\ \text{s}$.

Exercise 9.2

1. Expalin why there is a need of studying electric potential.
2. Briefly explain why electric-field lines must be perpendicular to equipotential surfaces.
3. Which way do electric-field lines point, from high to low potential or from low to high? Explain your reasoning.
4. A conducting sphere is to be charged by bringing in positive charge a little at a time until the total charge is Q. The work required for this process is assumed to be proportional to Q^2. Is this correct? Explain.
5. A high voltage d.c power line falls on a car. So the entire metal body of the car is at a potential of 10.0 V with respect to the ground. Explain what happens to the passengers:
   (a) When they are sitting in the car; and
   (b) When they step out of the car.
6. If the electric field is zero throughout a certain region of space, is the potential necessarily zero in that region? If not, what can be said about the potential?
7. Two concentric spheres of radii R and r had similar charges with equal surface charge densities. Determine the electric potential at their common centre.
8. An electron is accelerated through a potential difference of 100 V. Calculate its speed.
9. An infinite plane sheet of charge density $10^{-8}\,\text{Cm}^{-2}$ is held in air. In this situation, how far apart are two equipotential surface whose potential difference is 5 V?
10. Three charges $Q, +q$ and $+q$ are placed to the vertices of an equilateral triangle of length L. If the actual electrostatic energy of the system is zero, find the value of Q.
11. Electric field strength at a point due to a point charge is $70\,\text{NC}^{-1}$ and electric potential at that point is $10\,\text{JC}^{-1}$. Calculate the magnitude of the point charge.
12. A point charge $Q = 240\,\mu\text{C}$ is held stationary at the origin. If a second point charge $Q_2 = -4.34\,\text{C}$ moves from the point $(x, y) = (0.15\,\text{m}, 0\,\text{m})$ to the point $(x, y) = (0.25\,\text{m}, 0.25\,\text{m})$, how much work is done by the electric force on Q_2?
13. An electric dipole of charges $+2.0\,\text{nC}$ and $+2.0\,\text{nC}$ separated by a distance of 0.1 mm is placed in vacuum. Calculate the electric field strength and electric potential at a point P on the perpendicular bisector of the dipole such that P is 10 cm from the centre of the dipole. Calculate the work done on placing a charge of $+2$ nC at point P.
14. Two point charges $Q_1 = 2.4\,\mu\text{C}$ and $Q_2 = -6.5\,\mu\text{C}$ are $+0.10$ m apart. Point A is midway between them; point B is 0.08 m from Q_1 and 0.02 m from Q_2. Calculate the work done by the electric field on charge $Q_1 = 2.4\,\mu\text{C}$ that travels from point B to point A.
15. A small sphere with a mass of 1.50 g hangs by a thread between two

parallel vertical plates 5.00 cm apart. The plates are insulated and have uniform surface charge densities $+\sigma$ and $-\sigma$. The charge on the sphere is $Q = 8.9\ \mu C$. Calculate the potential difference between the plates that will cause the thread to assume an angle of $30°$ with the vertical.

16. A point charge Q is located at the centre of a thin ring of radius R with uniformly distributed charge $-Q$. Find the magnitude of the electric field strength E at a point lying on the axis of the ring at a distance x from its centre, if $x >> R$.

9.3 Capacitance

Capacitors are circuit components used for storing charges. To store energy in a capacitor, electrons (charges) are transferred from one plate to the other so that one plate has a net negative charge and the other has an equal amount of positive charge. This process is called charging of a capacitor.

It requires energy to do the work of moving these charges through the resulting potential difference. The work done is stored as electric potential energy in the capacitor. ***Note that,*** at each instant of time, the net charge of a capacitor is ideally zero. This section describes types of capacitor, factors affecting capacitance of a capacitor, effective capacitance for series and parallel arrangement, charging and discharging of a capacitor and energy stored in a capacitor.

9.3.1 Types of capacitor

A capacitor is a device consisting of two or more parallel conductive (metal) plates not connected or touching each other, but are electrically separated either by air or by some form of a good insulating material such as waxed paper, mica, ceramic, plastic or some form of a liquid gel. The insulating layer between the plates is commonly named the dielectric. The types of capacitors range from those with very small size and storage capacity used in oscillators or radio circuits, up to large capacity metal-can type, used in high voltage power correction and smoothing circuits. The comparisons between different types of capacitors are generally made with regards to the dielectric material used between the plates.

Types of capacitors include air capacitor, paper capacitor, electrolyte capacitor and mica capacitor (Figure 9.23). For example, a paper capacitor uses paper dielectric while an air capacitor uses air as dielectric material. Capacitors can have fixed or variable capacity value (Figure 9.24). Variable capacitors are used in tuned circuits such as in AM radios. A basic fixed value type of capacitor consists of two plates made from metallic foil, that are separated by different insulating materials, having good dielectric properties.

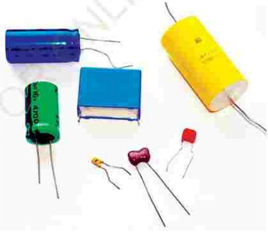


Figure 9.23 Types of capacitor

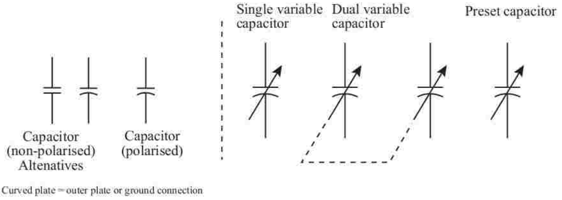


Figure 9.24 Symbols of fixed and variable capacitors

Capacitance of capacitors

The amount of charge a capacitor can store depends on the value of capacitance of the capacitor. Capacitance (C) is measured in the basic unit of the Farad (F). How do we charge capacitors?

Suppose a parallel plate capacitor with plates M and N is connected to a battery of e.m.f. V. When the switch S is open as in Figure 9.25, the capacitor plates are neutral. When the switch is closed, the electrons from plate M will be driven by the battery and start accumulating on plate N. This movement of electrons creates potential difference between plates M and N. The motion will stop only when the voltage across capacitor C (V_c) becomes equal to V, less the potential difference across R.

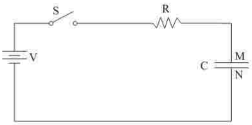


Figure 9.25 Capacitor charging circuit

It should be noted that the energy required to transfer electrons from one plate to the other is provided by the battery. It is this energy which the capacitor stores between its plates. The electrons do not cross the gap between plates due to presence of dielectric material between the plates.

Experiments show that the quantity of charge Q on a capacitor C in Figure 9.25 is linearly proportional to the potential difference between the conductors; that is $Q \propto V$. Therefore, $Q = kV$ where k is a constant of proportionality called capacitance; denoted by C,

$$C = \frac{Q}{V} \qquad (9.27)$$

The capacitance C of a capacitor is defined as the ratio of the magnitude of the charge on either plate to the magnitude of the potential difference between the plates. It is the amount of charge required to raise a unit potential difference between its plates. Capacitance is measured in farad (F).

1 Farad = 1 coulomb/volt.

Capacitance of a parallel plate capacitor

Consider a capacitor consisting of two parallel conducting plates, each of area A separated by a distance d that is small compared to the dimensions of plates. When the plates are charged, the electric field between the plates is uniform (Figure 9.26). Each plate is connected to one terminal of a battery, which acts as a source of potential difference. Let us assume the plate acquire charge Q.

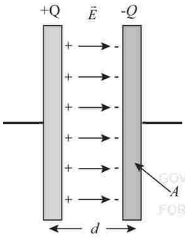


Figure 9.26 Charged plates of a parallel plate capacitor

The magnitude of the uniform electric field due to charge Q on the plate with surface area A is given by; $E=\frac{\sigma}{\varepsilon_o}=\frac{Q}{A\varepsilon_o}$.

Since the field is uniform, the potential difference between the plates is $V = Ed$.

It then follows that; $V=\frac{Qd}{A\varepsilon_o}$.

Then,

$$\frac{Q}{V}=\frac{A\varepsilon_o}{d} \qquad (9.28)$$

Comparing equation (9.27) and (9.28) you will get,

$$C=\frac{A\varepsilon_o}{d} \qquad (9.29)$$

If the capacitor is made of any N parallel plates, then,

$$C=\frac{(N-1)A\varepsilon_o}{d}$$

Thus, the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation. Generally, the capacitance of a capacitor depends on the geometry of the plates (their size, shape, and relative position) and the medium (such as air, paper, or plastic) between them.

9.3.2 Factors affecting capacitance of a capacitor

From equation (9.29), the capacitance of the capacitor depends on surface area of the plate, distance between the plates and dielectric material used.

(a) Surface area

The capacitance of a capacitor increases as surface area increases. That is capacitance is proportional to the surface area of the plate. The larger the surface area the larger the charge accumulated. Therefore, the construction of capacitors is such that, the separation between parallel plates is small but area is large enough.

(b) Distance between the plates

Capacitance of a capacitor is inversely proportional to the distance between the plates. That is the larger the distance the smaller the capacitance of the capacitor and vice versa.

(c) Dielectric material

Capacitance of a capacitor is direct proportional to the dielectric constant of the material used between the plates. Dielectric constant of the material is the ratio of capacitance of a capacitor with dielectric material to capacitance without dielectric material (with air or vacuum).

Without dielectric material $C_0 = \frac{A\varepsilon_o}{d}$ and with dielectric material $C = \frac{A\varepsilon}{d}$. Therefore, dielectric constant $\varepsilon_r = \frac{C}{C_o}$, $\varepsilon_r = \frac{\varepsilon}{\varepsilon_o}$.

Also, ε_r is called the relative permittivity of the dielectric material. Some materials offer less opposition to field flux for a given amount of electric field force. Materials with a greater permittivity allow for more field flux (offer less opposition), and thus a greater collected charge, for any given amount of field force (applied voltage).

Action of dielectric material

If a dielectric material contains polar molecules, they will generally be in random orientation when no electric field is applied. Application of electric field polarizes the material by orienting the dipole moments of the polar molecule. The dipoles are urged in the direction of the field, and the electrons in the opposite direction (Figure 9.27).

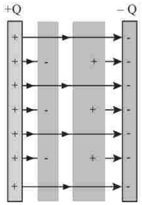


Figure 9.27 Effect of a dielectric on capacitance of a capacitor

Thus, each polarized molecule has an excess of positive charge on one end, and an excess of negative charge on the other end. These charges are of opposite signs to the charges on the plates, and so reduce the potential difference between the plates (compare Figures 9.27 and 9.28).

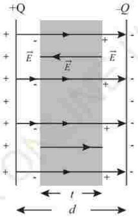


Figure 9.28 Capacitor with partially filled dielectric

The induced electric field E_i reduces the electric field between the plates and therefore reduces the potential difference eventually the capacitance increases.

Consider the dielectric material of permittivity ε partially filling between the parallel plate capacitor as shown in Figure 9.28. The voltage across the plates is $V = Ed$. Hence, $V = V_{air} + V_{dielectric}$, $V = E(d-t) + E_i t$. Therefore,

$$V = E(d-t) + \frac{E}{\varepsilon_r}t \qquad (9.30)$$

where $E_i = \frac{E}{\varepsilon_r}$ and t is the thickness of the dielectric material.

Using the relationship of E and Q and simplifying the equation (9.30);

$$\frac{Q}{V} = \frac{A\varepsilon_o}{(d-t) + \frac{t}{\varepsilon}}$$

Since $\frac{Q}{V}$ is the capacitance of a capacitor, then

$$C = \frac{A\varepsilon_o}{(d-t) + \frac{t}{\varepsilon_r}}$$

If there is no dielectric material between the plates (plates in vacuum) $t = 0$, then $C_o = \frac{A\varepsilon_o}{d}$.

If the dielectric slab fills completely the space between the plates that is, $d = t$, then, $C = \frac{A\varepsilon_o\varepsilon_r}{d}$ hence, $C = C_o\varepsilon_r$.

Example 9.14

A parallel plate capacitor having plate area $100\ \text{cm}^2$ and separation $1.0\ \text{mm}$ holds a charge of $0.12\ \mu\text{C}$ when connected to a 120 V battery. Find the dielectric constant of the material filling the gap.

Solution

From equation (9.29),

$C_o = \frac{A\varepsilon_o}{d}$, then capacitance with no dielectric material is,

$$C_o = \frac{8.854\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2} \times 100\times10^{-4}\ \text{m}^2}{1.0\times10^{-3}\ \text{m}}$$

$$= 8.854\times10^{-11}\ \text{F}$$

The capacitance with dielectric material,

$$C = \frac{Q}{V} = \frac{0.12\times10^{-6}\ \text{C}}{120\ \text{V}} = 1\times10^{-9}\ \text{F}$$

The dielectric constant is;

$$\varepsilon_r = \frac{C}{C_o} = \frac{1\times10^{-9}\ \text{F}}{8.854\times10^{-11}\ \text{F}} = 11.3$$

Therefore, the dielectric constant of the material is 11.3.

9.3.3 Combinations of capacitors

A capacitor with certain capacitance may be required in practice although they may not be commercially available. The required specification can be obtained by combining capacitors; many combinations are possible but the simplest combinations are a series connection or a parallel connection. In these combinations two or more capacitors are often involved.

Capacitors in series

A schematic diagram of a series connection is shown in Figure 9.29. Three uncharged capacitors of capacitances C_1, C_2 and C_3 are connected across a battery of constant potential V. Let the charge Q be placed on the plates of each capacitor.

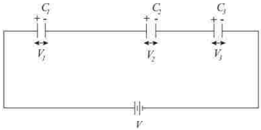


Figure 9.29 Capacitors in series

It follows that, $V = V_1 + V_2 + V_3$. From equation (9.27), it can be shown that;

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3},\ V = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

$$\frac{V}{Q} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

The quantity, $\frac{Q}{V}$ is the equivalent capacitance C_{eq} of a single capacitor of which charge Q and the potential difference V is the same as for the combination. Therefore, the combination of capacitors can be replaced by an equivalent capacitor of capacitance C_{eq} obtained by;

$$\frac{1}{C_{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

Generally, for n number of capacitors connected in series its effective (equivalent) capacitance is given by;

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n} \text{ or } \sum_{i=1}^{n} \frac{1}{C_i}$$

For two capacitors in series, the effective capacitance is,

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

Capacitors in parallel

Capacitors are said to be connected in parallel if the potential difference for all individual capacitors is the same.

Figure 9.30 shows the capacitances C_1, C_2 and C_3 connected in parallel across a battery of constant potential V.

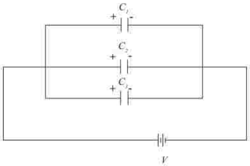


Figure 9.30 Capacitors in parallel

Let charge on capacitors C_1, C_2 and C_3 be Q_1, Q_2 and Q_3 respectively. The total charge Q of the combination, and thus the total charge on the equivalent capacitor is

$$Q = Q_1 + Q_2 + Q_3$$

$$Q = C_1 V + C_2 V + C_3 V$$

$$\frac{Q}{V} = C_1 + C_2 + C_3$$

The quantity $\frac{Q}{V}$ is the equivalent capacitance C_{eq} of the capacitors. Thus,

$$C_{eq} = C_1 + C_2 + C_3$$

Generally, for n capacitors connected in parallel, its effective (equivalent) capacitance is given by,

$$C_{eq} = C_1 + C_2 + C_3 + \ldots + C_n \text{ or}$$

$$C_{eq} = \sum_{i=1}^{n} C_i$$

Example 9.15

Two capacitors of capacitances $C_1 = 6.0\ \mu\text{F}$ and $C_2 = 3.0\ \mu\text{F}$ are connected in series across a battery of $18\ \text{V}$. Find

(a) the equivalent capacitance.
(b) the charge for each capacitor.
(c) the potential difference for each capacitor.

Solution

(a) From equation, $C_{eq} = \dfrac{C_1C_2}{C_1 + C_2}$;

$$C_{eq} = \frac{6\times10^{-6}\ \text{F}\times3\times10^{-6}\ \text{F}}{(6+3)\times10^{-6}\ \text{F}}$$

$$= 2\times10^{-6}\ \text{F}$$

The equivalent capacitance of the two capacitors is $2\times10^{-6}\ \text{F}$.

(b) Using $Q = CV$,

$Q = C_{eq}V$

$Q = 2\times10^{-6}\ \text{F}\times18\ \text{V} = 3.6\times10^{-5}\ \text{C}$

Therefore, the charge of each capacitor is $2\times10^{-6}\ \text{F}$

(c) Using $V = \dfrac{Q}{C}$,

$$V_1 = \frac{Q}{C_1} = \frac{3.6\times10^{-5}\,C}{6\times10^{-6}\,F} = 6\ \text{V}$$

$$V_2 = \frac{Q}{C_2} = \frac{3.6\times10^{-5}\ \text{C}}{3\times10^{-6}\ \text{F}} = 12\ \text{V}$$

Therefore, the potential difference across capacitor C_1 and C_2 are $6\ \text{V}$ and $12\ \text{V}$ respectively.

Example 9.16

In Figure 9.31 determine (a) the equivalent capacitance, (b) the total charge on each capacitor.

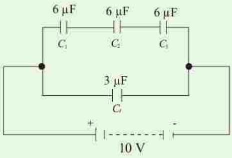


Figure 9.31 Capacitors in series and parallel

Solution

(a) From Figure 9.31 capacitors C_1, C_2 and C_3 are in series. Thus their equivalent capacitance is $\dfrac{1}{C'} = \dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3}$. It then follows that,

$$\frac{1}{C'} = \frac{1}{6\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}} = \frac{1}{2\ \mu\text{F}},$$

hence, $C' = 2\ \mu\text{F}$

Now, capacitors C_1, C_2, C_3 are replaced by Q'. This will make C' and C_4 to be connected in parallel. Then,

$C_{eq} = C' + C_4 = (2+3)\ \mu\text{F} = 5\ \mu\text{F}$

Therefore, total capacitance is $5\ \mu\text{F}$.

(b) Since capacitors C_1, C_2, and C_3 are in series connection; then, they will have the same charge: Q'. Therefore, the charge Q' on each of capacitors C_1, C_2, and C_3 is $Q' = C'V$.

$Q' = 2\times10^{-6}\ \text{F}\times10\ \text{V} = 20°\text{C}$

The charge on capacitor C_4 is

$Q_4 = C_4V$

$Q_4 = 3\times10^{-6}\,\text{F}\times10\,\text{V} = 30\,\mu\text{C}$

9.3.4 Energy stored in a capacitor

Once the charging of a capacitor has begun, the addition of electrons to the negative plate involves doing work against the repulsive forces of electrons which are already there. Equally the removal of electrons from the positive plate requires work to be done against the attractive forces on the positive charges on that plate. The work which is done is stored in the form of electrical potential energy.

To find the energy stored, suppose that a capacitor of capacitance C is already charged to a potential difference V and that the charge on its plate is Q. If a small charge dQ is to be increased on the capacitor plates, the work dW is needed to do the transfer. Hence,

$$dW = VdQ$$

The total work W needed to increase the capacitor charge dQ from zero to Q is such that;

$$W = \int_0^W dW = \int_0^Q V\,dQ$$

$$W = \int_0^Q \frac{Q}{C}dQ = \frac{Q^2}{2C} \qquad (9.31)$$

The other expressions for energy stored in the capacitor are

$$W = \frac{CV^2}{2} \text{ or } W = \frac{QV}{2}$$

This work done is stored as electric potential energy between the plates of the capacitor.

Therefore, equations for energy stored in the capacitor is $\frac{CV^2}{2} = \frac{1}{2}QV = \frac{Q^2}{2C}$

Energy density of electric field

The energy density denoted by η is defined as the electric potential energy stored per unit volume of the electric field (volume between the plates).

$$\text{Energy density}, \eta = \frac{\text{total energy stored}}{\text{volume of electric field}}$$

If the plate area is A and the separation is d, the energy density of a parallel plate capacitor is

$$\eta = \frac{\frac{1}{2}CV^2}{Ad} = \frac{1}{2}\frac{\left(\frac{\varepsilon_o A}{d}\right)(Ed)^2}{Ad}$$

$$\therefore \eta = \frac{1}{2}\varepsilon_o E^2 \qquad (9.32)$$

Example 9.17

Calculate the magnitude of the electric field required to store 1.00 J of electric potential energy in a volume of 1.0 m^3 in vacuum.

Solution

Using equation (9.32),

$$E = \sqrt{\frac{2\eta}{\varepsilon_o}} = \sqrt{\frac{2\times1.0\ \text{Jm}^{-3}}{8.85\times10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}}}$$

$$= 4.75\times10^5\ \text{Vm}^{-1}$$

The magnitude of electric field is $4.75\times10^5\ \text{Vm}^{-1}$.

Example 9.18

An uncharged capacitor of capacitance $C_1 = 8\ \mu F$ is connected to a power supply and charged to a potential difference $V = 120\ V$. The power supply is disconnected. If another uncharged capacitor of capacitance $C_2 = 4\ \mu F$ is connected to the capacitor C_1, determine

(a) the final potential difference across each capacitor;
(b) the final charge on each capacitor;
(c) the initial and the final energy of the system; and
(d) account for the energy difference.

Solution

(a) From conservation of charges,

$C_1V_1 + C_2V_2 = (C_1 + C_2)V_f$ where V_f is the final common potential difference.

$V_f = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$ since $V_2 = 0$; then

$$V_f = \frac{C_1V_1}{C_1 + C_2} = \frac{8\ \mu F \times 120\ V}{(8+4)\ \mu F} = 80\ V$$

Therefore, the final potential voltage is $80\ V$.

(b) The final charges on each capacitor,

$$Q_1 = C_1V_f = 8\times10^{-6}\ F \times 80\ V = 640\ \mu C$$

$$Q_2 = C_2V_f = 4\times10^{-6}\ F \times 80\ V = 320\ \mu C$$

The final charges in capacitor C_1 and C_2 are $640\ \mu C$ and $320\ \mu C$ respectively.

(c) The initial energy is the one stored in capacitor C_1,

$$U_i = \frac{1}{2}C_1V_o^2 = 0.5\times8\times10^{-6}\ F\times(120\ V)^2$$
$$= 0.058\ J$$

Therefore, the initial energy is $0.058\ J$.

The final energy is the one stored in capacitors C_1 and C_2. That is,

$$U_f = \frac{1}{2}(C_1 + C_2)V_f^2$$
$$= \frac{1}{2}(8+4)\times10^{-6}\ F\times(80\ V)^2$$
$$= 0.0384\ J$$

Therefore, the final energy is $0.0384\ J$.

(d) The final energy U_f is less than the initial energy U_i; this is because the difference in energy was converted to other forms of energy like thermal energy on the connecting wires and plates of the capacitors.

9.3.5 Charging and discharging of a capacitor

When a voltage is applied across the terminals of a capacitor the potential cannot rise to its final value instantaneously. As the charge builds up it tends to repel addition of further charge. The rate at which a capacitor can be charged or discharged depends on its capacitance and the resistance of the circuit through which it is being charged or discharged. This fact makes a capacitor to be a very useful component in timing circuits needed in variety of circuits ranging from clocks to computers.

Charging of a capacitor

Suppose the capacitor in Figure 9.32 was initially uncharged; then potential difference V_c across it is zero at $t = 0$.

The switch S is closed and the capacitor starts charging and the voltage V_c increases and the potential difference across the resistor R V_R decreases.

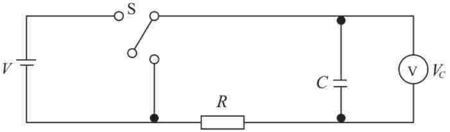


Figure 9.32 Charging of a capacitor

At an instant of time t, $V = V_R + V_c$, then

$$V = IR + \frac{Q}{C} \text{ and } I = \frac{V}{R} - \frac{Q}{RC} = \frac{VC - Q}{RC}$$

Note that, current I (dQ/dt) is the rate at which positive charge arrives at the top (positive) plate of the capacitor.

$$I = \frac{dQ}{dt} = \frac{VC - Q}{RC}, \quad \frac{dQ}{VC - Q} = \frac{dt}{RC}$$

Integrating both sides;

$$\int_0^Q \frac{dQ}{Q - VC} = -\frac{1}{RC}\int_0^t dt,$$

$$\ln\left(\frac{Q - VC}{-CV}\right) = -\frac{t}{RC}, \text{ which gives}$$

$$\frac{VC - Q}{CV} = e^{-\frac{t}{RC}}. \text{ Therefore,}$$

$$Q = VC\left(1 - e^{-\frac{t}{RC}}\right) \qquad (9.33)$$

Initially, at $t = 0$, $Q = VC\left(1 - e^0\right) = 0$

At an infinity time of charging $Q = VC$ which is the final maximum charge on plates of the capacitor when fully charged. The quantity RC in equation (9.33) is called the time constant, denoted by τ. It is a measure of how quickly the capacitor charges. When τ is small, the capacitor charges quickly, when τ is large, the charging takes more time.

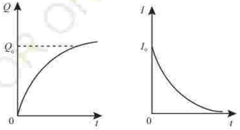
An alternative to equation (9.33) is that of instantaneous current I instead of charge Q. Taking time derivative of Q in equation (9.33),

$$I = \frac{dQ}{dt} = \frac{V}{R}e^{-\frac{t}{RC}} \text{ where } \frac{V}{R} = I_0 \text{ (this is}$$

maximum possible current)

$$I = I_o e^{-\frac{t}{RC}} \qquad (9.34)$$

Equations (9.33) and (9.34) can be represented graphically as shown in (Figure 9.33).



(a) (b)

Figure 9.33 Variation of (a) charge versus time and (b) current versus of charging capacitor

Discharging a capacitor

Suppose a fully charged capacitor is connected to a resistor with an open switch (Figure 9.34). The switch is closed at $t = 0$, at this time the charge on the plates of a capacitor is Q_o. The capacitor then discharges through the resistor, and eventually decreases to zero.

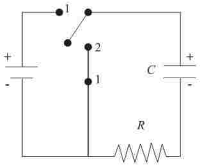


Figure 9.34 Discharging a capacitor

At an instant of time t; $V_R + V_c = 0$, $V_R = -V_c$ hence, $IR = -\frac{Q}{C}$. But current is the rate of change of flowing charge, hence, $\frac{dQ}{dt} = -\frac{Q}{RC}$. It follows that, $\int_{Q_o}^{Q} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC}$; integrating and rearranging you get;

$\ln\left(\frac{Q}{Q_o}\right) = -\frac{t}{RC}$. Applying the exponent to both sides;

$$Q = Q_o e^{-\frac{t}{RC}} \qquad (9.35)$$

The instantaneous currents I, is the time derivative of Q in equation (9.35),

$$I = \frac{dQ}{dt} = -\frac{Q_o}{RC} e^{-\frac{t}{RC}},$$

$$I = -I_o e^{-\frac{t}{RC}} \qquad (9.36)$$

Graphically, equations (9.35) and (9.36) are as shown in Figure 9.35.

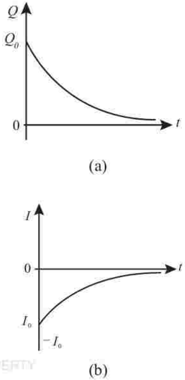


Figure 9.35 Variation of (a) charge versus time and (b) current versus time of a discharging capacitor

Example 9.19

A 10 MΩ resistor is connected in series with an uncharged 1.0 µF capacitor and battery with e.m.f. 12.0 V.

(a) What is the time constant?

(b) What fraction of the final charge is on the capacitor at $t = 46$ s?

(c) What fraction of the initial current is still flowing at $t = 46$ s?

Solution

(a) Time constant:

From $\tau = RC$,

$$\tau = 10 \times 10^6\ \Omega \times 1.0 \times 10^{-6}\ \text{F} = 10\ \text{s}$$

Therefore, time constant is 10 seconds.

(b) Using equation (9.33),

$$\frac{Q}{CV}=\left(1-e^{-\frac{46}{10}}\right)=0.99$$

The fraction of the final charge on the capacitor is 0.99.

(c) Fraction of the initial current, can be obtained using the relation that

$$I=I_o e^{-\frac{t}{RC}}$$

Thus,

$$\frac{I}{I_o}=e^{-\frac{46}{10}}=0.01$$

Therefore, the fraction of the initial current still flowing is 0.01.

Example 9.20

Uncharged capacitor is connected in series with a resistor and a source with e.m.f. of 110 V . Just after the circuit is completed, the current through the resistor is 6.5×10^{-5} A. The time constant for the circuit is 5.2 s.

(a) What is resistance of the resistor?

(b) Find capacitance of the capacitor.

Solution

(a) The resistance of the resistor

Initially (at $t=0$), the current through the resistor is

$$I_o=\frac{V}{R},\ R=\frac{V}{I_o}=\frac{110\ \text{V}}{6.5\times10^{-5}\ \text{A}}$$

$$=1.7\times10^{6}\ \Omega$$

Resistance of the resistor is $1.7\times10^{6}\ \Omega$

(b) The capacitance of a capacitor.

Since $\tau=RC$; then

$$C=\frac{\tau}{R}=\frac{5.2\ \text{s}}{1.7\times10^{6}\ \Omega}=3.1\times10^{-6}\ \text{F}$$

The capacitance of the capacitor is $3.1\ \mu$F.

Example 9.21

A $5.0\ \mu$F capacitor is charged by a 12 V supply and is then discharged through a $2.0\ \text{M}\Omega$ resistor.

(a) What is the charge on the capacitor at the start of the discharge?

(b) Find the charge and p.d across the capacitor after 5 seconds from when the discharge started.

(c) What is the current in the circuit after the 5 seconds?

Solution

(a) The charge on the capacitor at the start of the discharge.

$$Q_o=VC,$$

$$Q_o=12\ \text{V}\times5\times10^{-6}\ \text{F}=60\ \mu\text{C}$$

Therefore, the initial charge is $60\ \mu$C.

(b) After 5 seconds

From $Q=Q_o e^{-\frac{t}{RC}}$

$$Q=60\times10^{-6}\ \text{C}\times e^{-\left(\frac{5}{2\times10^{6}\times5\times10^{-6}}\right)}=36.4\ \mu\text{C}$$

Therefore, the charge on the capacitor after 5 seconds is $36.4\ \mu$C.

Then the p.d. is,

$$V_c=\frac{Q}{C}=\frac{36.4\ \mu\text{C}}{5\ \mu\text{F}}=7.3\ \text{V}$$

(c) The magnitude of the current in the circuit is given by, $I = I_o e^{-\frac{t}{RC}}$

$$I = 6\ \mu A \times e^{-\frac{5}{10}} = 3.64\ \mu A$$

Exercise 9.3

1. A 500 μF capacitor with a charge of 300 μC is discharged through a resistor.
   (a) What is the initial discharge current?
   (b) What is the current after 20 s ?
2. Describe the charging and discharging of a capacitor through a resistor.
3. (a) (i) State Coulomb's law of electrostatics
   (ii) Calculate the force between two charged particles each carrying a charge of 1.0 C when they are placed in air 1m apart. Use the result to define the SI unit of charge.
   (b) Two identical copper spheres A and B are situated at a distance of 6 cm apart and each carries a charge of +6 μC . A third identical copper sphere C is first touched with A and then with B . Thereafter, sphere C is placed 2 cm from sphere B. Find the resultant force on sphere C.
   (c) (i) Define electric field intensity and state two possible SI Units.
   (ii) An electric dipole consists of 10 μC and −10 μC charges separated by 2 cm . Find the electric field intensity at a distance of 8 cm from the −10 μC charge on the coaxial axis on the side of the 10 μC .
   (iii) Two point charges A and B are situated 8 cm apart. Point A has a charge of $-3q$ and B has a charge of $+2q$. Where should a particle C having a charge of $-q$ be placed so that it does not experience a resultant electric force?
4. Derive an expression for the energy stored in a capacitor of capacitance C having a charge Q on its plates and hence or otherwise deduce the energy stored by a parallel plate capacitor per unit volume in terms of the electric field intensity E and the permittivity of free space ε_o.
5. A charged capacitor of capacitance 4 F is connected in series with a resistance R , micro ammeter and key. What do you think will happen when the key is closed? If the time taken for the charge to reduce to half its maximum value is found to be 1.3×10^{-3} seconds, determine
   (a) the circuit time constant.
   (b) the time taken for the charge on the capacitor to reduce to 1.25×10^{-3} C from 7.5×10^{-3} C.
6. Two identical charged spheres of charge 93 μC are suspended by light inelastic strings of equal length of 29 cm . The strings make an angle of 40° with each other. When suspended in a liquid of density

800 kgm^{-3}, the angle remains the same. If the density of the spheres is 2600 kgm^{-3}, determine the relative permittivity of the liquid.

7. Two capacitors of capacitances 2 μF and 3 μF are charged to p.d. of 100 V and 250 V respectively. Find:
   (a) The energy stored in each capacitor.
   (b) The loss in energy if the capacitors are connected together by wires with plates of similar charges joined. Account for the loss of energy.

8. (a) (i) Explain what is meant by dielectric constant?
   (ii) A sheet of paper 40 mm wide and 0.015 mm thick between metal foil of the same width is used to make a 2.0 μF capacitor. If the dielectric constant of the paper is 2.5, what length of paper is required?
   (b) Two capacitors of capacitances 3.0 μF and 5.0 μF are connected to form a potential divider with a 5.0 μF capacitor across a 6.0 V battery. If the input voltage supply is 12.0 V what are the two possible amounts of energy stored in a 3.0 μF capacitor?

9. (a) The plates of a parallel plate air capacitor consisting of circular plates each of radius 10 cm, placed 2 mm apart, are connected to the terminals of an electrostatic voltmeter. The system is charged to give a reading of 100 V on the voltmeter scale. The space between the plates is then filled with oil of dielectric constant 4.9 and the voltmeter reading falls to 24 V. Calculate the capacitance of the voltmeter. You may assume that the voltage recorded by the voltmeter is proportional to the scale reading.
   (b) Two capacitors C_1 and C_2 are connected in series and then charged with a battery. The battery is disconnected and C_1 and C_2 still in series are discharged through an 80 MΩ resistor. The time constant for the discharge is found to be 4.8 seconds. Calculate:
   (i) The capacitance of C_1 and C_2 in series; and
   (ii) The capacitance of C_1 if C_2 has a capacitance of 10 μF.

Revision exercise 9

1. Two charges $q_1 = 2\times10^{-5}$ C and $q_2 = 4\times10^{-5}$ C are held at a distance $d = 1$ m apart. Calculate the force exerted by these two charges on a charge $Q = 10^{-5}$ C if it is placed half way between them, and locate the point between the two charges where the net force vanishes.

2. Charges q_1, q_2, q_3 and q_4 are placed at the corners of a square of side $a = 2$ m. If $q_1 = q_2 = q_3 = Q = 1$C and $q_4 = -Q$. Find the electric field at the center of the square.

3. The electric field just above the earth's surface is known to be $E_e = 130\ \text{NC}^{-1}$. Assuming that this field results from a spherically symmetrical charge distribution over the earth, find the total charge Q_e on the earth. (the earth's radius, $R_e = 6400$ km).

4. Two masses m with equal charges Q are suspended by light strings of length l from a fixed point. If the strings hang at θ to the vertical, show that $Q^2 = (4\pi\varepsilon_o)4l^2 mg\sin^2\theta\tan\theta$

5. A cylinder of radius R has uniform charge density $\rho\,\text{cm}^{-3}$.

   (a) Show that the magnitude of the electric field E directed anywhere is $E(r) = \dfrac{R^2\rho}{2r\varepsilon_o}$.

   (b) Plot E as a function of r the distance from the axis of the cylinder.

6. A charge of 5.0 μC is placed at 0 cm mark of a meter stick and a charge of −4.0 μC is placed at the 50 cm mark.

   (a) What is the electric field at the 30 cm mark?

   (b) At what point along a line connecting the two charges is the electric field zero?

7. Two large horizontal parallel metal plates are 2.0 cm apart in vacuum and the upper is maintained at positive potential relative to the lower so that the field strength between them is $2.5\times10^5\ \text{Vm}^{-1}$.

   (a) What is the p.d between the plates?

   (b) If an electron of charge 1.6×10^{-19} C and mass 9.1×10^{-31} kg is liberated from rest at the lower plate, what is the speed on reaching the upper plate?

8. The force of 3.2×10^{-2} N is required to move a charge of 42 μC in an electric field between two points 25 cm apart. What potential difference exists between the two points?

9. A charge of 5.0 nC is at (0,0) m and a second charge of −2 nC is at (3,0) m. If the potential is taken to be zero at infinity:

   (a) What is the electric potential at point $P(0,4)$ m ?

   (b) What is the potential energy of a 1.0 nC charge at point P ?

(c) What is the work required to bring a charge of $1.0\ \text{nC}$ from infinity to point P? and

(d) What is the total potential of the three charge system?

10. Suppose the two plates of a capacitor have different areas. When the capacitor is charged by connecting it to a battery, do the charges on the two plates have equal magnitude, or may they be different? Explain your reasons.

11. A parallel-plate capacitor is charged by being connected to a battery and is kept connected to the battery. The separation between the plates is then doubled.

(a) How does the electric field change?

(b) How does the charge on the plates change?

(c) How does the total energy change?

12. (a) Explain the differences between dielectric strength and dielectric constant.

(b) The dielectric constant of water is approximately 81, larger than most insulators. Explain briefly why water is not commonly used as a dielectric in capacitors.

(c) Liquid dielectrics that have polar molecules (example water) always have dielectric constants that decrease with increasing temperature. Why?

13. A parallel plate capacitor has plates with the area of $0.2\ \text{m}^2$ and separation of $0.01\ \text{m}$. The capacitor is charged to a potential difference of $200\ \text{V}$ and the power supply is disconnected. A dielectric slab ($\varepsilon_r = 4$) of thickness $5\ \text{mm}$ is inserted between the plates. Calculate:

(a) The final charge on each plate;

(b) The final potential difference between the plates; and

(c) The final energy in the capacitor.

14. The area of each plate of a parallel plate capacitor is $0.6\ \text{m}^2$, and the distance between the two plates is $2\ \text{mm}$.

(a) What is its capacitance?

(b) What will be its new capacitance if half the space between the plates is filled with mica ($\varepsilon_r = 8$).

15. A parallel-plate capacitor is located horizontally so that one of its plates is submerged into a liquid while the other is above the liquid surface. The permittivity of the liquid is equal to ε_m; its density is equal to δ. To what height will the level of the liquid in the capacitor rise after its plates get a charge of surface density 5?

16. A parallel-plate capacitor has space between its plates filled with two slabs of thickness $\frac{d}{2}$ and dielectric constants k_1 and k_2, d is the plate separation of the capacitor.

Show that the capacitance of the capacitor is given by:

$$C = \frac{2\varepsilon_o A}{d}\left(\frac{k_1 k_2}{k_1 + k_2}\right)$$

17. By considering energy of an isolated, charged parallel plate capacitor, obtain an expression for the force between its plates.
18. When a capacitor, battery, and a resistor are connected in series does the resistor affect the maximum charge stored on the capacitor? Why?
19. Verify that the time constant RC has units of time.
20. The plates of a capacitor of capacitance 2.0 μF carry opposite charge of 10 mC. The plates are connected across a 5.0 MΩ resistor.
    (a) Find the charge flowing through the resistor during the time interval of 2.0 s.
    (b) Find the amount of heat generated in the resistor during the same interval.
21. (a) What is an equipotential surface?
    (b) Sketch the form of the equipotential surface and the electric lines of force for:
    (i) A point charge;
    (ii) A charged conducting sphere; and
    (iii) A pair of parallel conducting plates when one plate has a negative charge and the other has an equal positive charge.
22. In towns and cities around the country, dust is one of the biggest challenges in offices and residential houses especially in areas where buildings and people are crowded. Design a system that is safe and can reduce dust particles in rooms (Hint: Consider electrostatic precipitators).
23. Soil moisture content is an important aspect in agriculture. Measurement of soil moisture content can be done using capacitors as soil moisture sensor. Design a soil moisture sensor and suggest how to calibrate it. (Hint: Soil moisture can be considered as dielectric material).

Table of some physical constants

Name	Symbol	Constant
Acceleration due to gravity	g	$9.8\ ms^{-2}$
Avogadro's number	N_A	$6.023\times10^{23}\ mol^{-1}$
Boltzmann's constant	k_B	$1.38\times10^{-23}\ JK^{-1}$
Density of air	ρ_a	$1.29\ kg/m^3$
Density of fresh water	ρ_w	$1000\ kg/m^3$
Electron rest mass	m_e	$9.1\times10^{-31}\ kg$
Electronic charge	e	$1.602\times10^{-19}\ C$
Gravitational constant	G	$6.673\times10^{-11}\ Nm^2kg^{-2}$
Mass of the earth	M_E	$5.98\times10^{24}\ kg$
Mass of the moon	M_m	$7.35\times10^{22}\ kg$
Mass of the Sun	M_s	$1.99\times10^{30}\ kg$
Mean density of earth	ρ_E	$5.522\times10^{3}\ kgm^{-3}$
Molar heat capacity of air at constant pressure	C_p	$29.1\ JK^{-1}mol^{-1}$
Molar heat capacity of air at constant volume	C_v	$20.8\ JK^{-1}mol^{-1}$
Permeability of free space	μ_0	$4\pi\times10^{-7}\ Hm^{-1}$
Permittivity of free space	ε_o	$8.854\times10^{-12}\ Fm^{-1}$
Radius of the earth	r_E	$6.4\times10^{6}\ m$
Radius of the moon	r_M	$1.74\times10^{6}\ m$
Radius of the sun	r_S	$6.96\times10^{8}\ m$
Refractive index of glass	η	1.5
Specific heat capacity, water	C_w	$4.2\ kJkg^{-1}K^{-1}$
Specific latent heat of fusion, *ice*	L_f	$336\ kJkg^{-1}$
Specific latent heat of vaporization, steam	L_v	$2268\ kJkg^{-1}$
Speed of light in vacuum	c	$2.998\times10^{8}\ ms^{-1}$
Speed of sound in air (at 0°C)	v	$3.32\times10^{2}\ ms^{-1}$
Standard atmospheric pressure	p_A	7.6×10^{2} mmHg or 1.013×10^{5} Pa
Stefan-Boltzmann constant	σ	$5.671\times10^{-8}\ Wm^{-2}K^{-4}$
Triple point of water	T_r	$0.01°C = 273.16\ K$
Universal gas constant	R	$8.31\ Jmol^{-1}K^{-1}$

Glossary

Absolute zero temperature
It is zero point ($T = 0$ K or -273.15 °C) temperature at which system of molecules (such as a quantity of a gas, a liquid, or a solid) has its minimum possible total energy (kinetic plus potential).

Adhesion force
The force of attraction between the molecules of the different substances

Adiabatic process
Thermodynamic process in which there is no heat exchange between the system and surrounding

Amplitude
The maximum displacement from the equilibrium position during a cycle of periodic motion; also, the height of a wave

Angle of contact The angle made between the contact surface and tangential line on liquid meniscus

Angle of projection
An angle made between the horizontal direction and the initial velocity of a projectile

Angular momentum
The product between radius and linear momentum of a rotating rigid body

Beat The oscillation of wave amplitude that results from the superposition of two sound waves with almost identical frequencies

Blackbody radiation
Radiation emitted by a blackbody at a given temperature

Capacitance
An ability of a capacitor to store charge or the ratio of an object's stored charge to its electric potential difference

Capacitor
An electric device used to store charge that is made up of two conductors separated by an insulator

Centre of mass
A point at which all the mass of the body assumed to be concentrated

Centrifugal force
The apparent force (made-up force) that seems to pull on a moving object, but does not exert a physical outward push on it, and is observed only in rotating frames of reference

Centripetal acceleration
The centre-seeking acceleration of an object moving in a circle at a constant speed

Centripetal force
The net force exerted toward the centre of the circle that keep an object in uniform circular motion and causes it to have a centripetal acceleration

Coefficient of friction
The ratio of the friction force to the normal force

Coefficient of restitution
The ratio between relative velocity of separation to relative velocity of approach

Cohesion
The force of attraction between the molecules of the same substance

Composite conductor
A conductor made by joining two or more conductors

Crest High point of a wave

Critical angle
A certain angle of incidence in which the refracted light ray lies along the boundary between two media

Critical velocity
The maximum velocity of streamline flow which when exceeds, the flow become turbulence

Damped oscillation
An oscillation whose amplitude decreases over time due to dissipation of energy

Dielectric constant
The ratio between capacitance with dielectric material to that without dielectric material

Diffraction grating
A device for producing spectra by diffraction and for measurement of wavelength

Diffraction
A spread out of waves after passing in an aperture or sharp edge

Dimension formula
An expression or formula which shows how the fundamental quantities are represented in relation with other physical quanties

Doppler Effect
The change in frequency due to relative motion between a source of wave and an observer

Elastic collision
Type of collision in which its total kinetic energy is conserved

Elasticity
Ability of a material to retain its original shape and size after a removal of deforming force

Electric field
The region around a point charge in which a brought test charge can experience electrostatic force

Electric flux
The amount of electric field that penetrates a certain area

Electric potential
Work done in moving a unit test charge from infinity to a point in the electric field

Emissivity
The ratio of the rate of radiation from a particular surface to the rate of radiation from an equal area of an ideal radiating surface at the same temperature

Equilibrium
A state at which the net force and net torque on an object equal zero

Equipotential lines
Lines that illustrate every point at which a charged particle would experience a given potential

Equipotential surface
The surface with the same electric potential

Error A deviation from exact or true value

Escape velocity
The minimum velocity a body may be projected so that it escapes from the earth's gravitational force influence completely

Excess pressure
A difference between the inside and outside pressure of a bubble

Extension
An increase in length produced by a deforming force

Field A property of a region of space that can affect objects found in that particular region

Fluid A substance that can flow such as liquid and gas

Forced convection
The convection in which the transfer of heat energy from hot body is facilitated by an external agent such as fanning

Free fall
The motion of a body when air resistance is negligible and the motion can be considered due to the force of gravity alone

Free-body diagram
A physical model (a picture) that represents the forces acting on a system

Friction
A force acting parallel to two surfaces in contact; if an object moves, the friction force always acts opposite the direction of motion

Fundamental interval
The distance between the lower and upper fixed points

Gravitation
The force of attraction between two bodies that tend to pull them towards each other

Gravitational field The field that surrounds any objects with mass; equals the universal gravitational constant, times the mass of the object, divided by the square of the distance from the object's centre

Ice point
The equilibrium temperature of ice and water at standard pressure

Impulse
The product of force and time or simply is the change in linear momentum

Inclined plane
A plane oriented at any angle with the horizontal

Inelastic collision
A collision in which kinetic energy is not conserved, as opposed to an elastic collision, in which the total kinetic energy of all objects is the same before and after the collision

Inertia
Ability of a body to resist change of state of linear motion

Interference
A combination of wave fronts to form secondary wave front

Isobaric process
A thermodynamic process which occur at constant pressure

Isochoric process
Thermodynamic process which occur at constant volume

Isolated system
A system for which there are no external interactions. For such a system there is no transfer of momentum into or out of the system.

Isothermal process
A thermodynamic process which occur at constant temperature

Lagged conductor
An insulated thermal conductor

Lamina A rectangular sheet

Laminar flow
A steady flow attained if each particle of the fluid follows a smooth path and fairly slowly in straight lines with constant speed

Least count
The smallest value which can be measured accurately by an instrument

Liquid-in-glass thermometer
The thermometer which uses liquid as thermometric substance for example; mercury and alcohol

Measurement
A comparison between an unknown quantity and a standard

Moment of inertia
Ability of a rigid body to resist change of state of rotational motion

Momentum
The quantity of motion that an object has, equal to an object's mass multiplied by that object's velocity

Natural (free) convection
The convection in which the transfer of heat energy from a hot body occurs naturally without an external agent

Oblique collision
The collision which occurs at an angle

Orbit A circular path described by an object around another

Oscillation
To and fro motion about a fixed point

Physical quantity
A property of a material that can be quantified by measurement

Pitch The highness or lowness of a sound wave, which depends on the frequency of vibration

Point charge
A charge which is considered to be a source of electric field

Polarization
A process of restricting transverse waves to vibrate in one plane

Precision
A characteristic of a measured value describing the degree of exactness of a measurement

Progressive waves
Travelling waves that transfer energy from one point to another

Projectile
A body moving in air or space under the influence of gravitational force

Pyrometer
An instrument for measuring high temperatures using thermal radiations emitted by a hot source

Radius of gyration
The distance between the axis of rotation and the point where the mass of a body considered to be concentrated so that its moment of inertia about that point remain the same

Random error
An error which has an equal chance of being positive or negative about the mean value

Restoring force
A force that restores an oscillating object to its equilibrium position

Rigid body
A body that retains its shape and size when subjected to external force

Rotational motion
Type of motion for which the particles in an object follow different circular paths centred on a straight line called the axis of rotation

Static friction
A resistive force to a body just before it starts moving

Stationary waves
Waves which propagate without transferring energy

Strain An extension produced per unit length

Streamline flow
A flow of fluid at constant velocity or speed

Stress
A deforming force per unit cross section area

Surface energy
The work done by surface tension in changing a unit surface area of a liquid

Surface tension
An elastic tendency of a fluid surface which makes it acquire the possible minimum surface area

Systematic error
An error which is constant in one direction

Tension
A force acting along stretched material.

Terminal velocity
The maximum constant velocity attained by object moving through a fluid

Test charge
A charge which experience electrostatic force when placed in electric fields

Thermal conduction
Transfer of heat in solids

Thermal conductivity
A measure of ability of material to conduct heat through it

Thermal convection
The transfer of heat in a fluid by actual movement of molecules

Thermal resistance
Ability of material to resist conduction of heat through it

Thermocouple
An electromotive thermometer used to measure temperature

Thermodynamic scale
Scientific standard scale adopted for measuring temperature

Thermodynamics
A branch of physics deals with interaction between heat and other forms of energy

Thermometer
A device used to measure temperature

Thermometric
A physical property of an object that changes in a measurable way as temperature changes

Thermometry
A branch of science which deals with the measurement of temperature

Torque The moment of force which produce turning effect

Trajectory
A path described by a projectile

Triple Point of water
The temperature at which pure ice, water and water vapour coexist in equilibrium

Turbulent flow
The movement of fluid with fluctuating velocity or speed and direction

Ultrasound
High-frequency sound waves (above the range of human hearing) used to probe the interior of the body, much as X rays do

Uniform circular motion
The movement of an object or particle trajectory at a constant speed around a circle with a fixed radius

Un-lagged material
A non-insulated thermal conductor

Viscosity
Ability of a fluid to resist relative motion of its layers and a motion of an object that flows through it

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Answers

Chapter One

Exercise 1.1

2. $M^{-1}T^{2}$, L^{2}
4. $L^{-\frac{1}{2}}T^{2}$

Exercise 1.2

2. (a) 0.042 (b) 1.50 (c) 0.028 (d) 2.8 %
3. 7%
4. (9.1 ± 0.9)°C
5. $(9.74 \pm 0.26)\,ms^{-2}$

Revision exercise 1

5. (3.1 ± 0.2) cm
6. 8.4%
7. 1.49%

Chapter Two

Exercise 2.1

4. (a) 4.85 m (b) 4.54 seconds
5. (b) 30 N
6. (a) $P \geq 1.08M$ (b) $P \geq 2.16M$
7. (b) $2\,ms^{-2}$
8. (a) 3.92 N (b) 4.62 N (c) 6.93 N
9. (a) 8400 N (b) $112500\,Nm^{-2}$
   (c) $6.25\,ms^{-1}$, 6328.125 J
10. (c) $2.004\,ms^{-1}$

Exercise 2.2

3. (b) 60°
5. (a) 20.4°, 69.6°
   (b) $27.1\,ms^{-1}$, 3.91 s
6. (b) $17.29\,ms^{-1}$
9. (a) 57.3 m

Revision exercise 2

7. (a) 25.46 N, 42.4 N
10. $9.77\,ms^{-2}$
11. 7.87 N, 20.4 cm
12. 0.4 N
13. 2.86 seconds, 14.29 m
14. No, 18.56 ms^{-1} North
15. $40.3\,ms^{-1}$, 30° to the horizontal
16. 59842 kg
17. (a) $8000\,kgs^{-1}$ (b) $6907.8\,ms^{-1}$
18. (b) 3414.21 N
    (c) (i) $1.4\,ms^{-2}$
    (c) (ii) $7\,ms^{-2}$, $4.2\,ms^{-2}$, $1.4\,ms^{-2}$
    (iii) 33.6 N, 16.8 N
21. (b) 323.5 m
22. 18.2 seconds
23. 330.24 m, $68.8\,ms^{-1}$
24. (b) 63.4°

Chapter Three

Exercise 3.1

3. (a) $3.13\,ms^{-1}$ (b) $1.28\,ms^{-1}$
4. (b) 18.69 m
5. (a) $3.33\,rad\,s^{-1}$ (b) 266.67 N
6. 8.1°

7. (a) 16.04 N (b) 2.72 s
8. $88\,ms^{-1}$, $9.28\,ms^{-1}$
10. 45.6°
11. $2.8\,ms^{-1}$, 17.6 N

Exercise 3.2

6. (a) 2 Hz (b) $3.16\,ms^{-2}$ (c) $0.25\,ms^{-1}$
7. (a) $19.6\,kgs^{-2}$ or $19.6\,Nm^{-1}$
   (b) 0.63 s, 1.58 Hz (c) $4.9\,ms^{-2}$
   (d) $0.49\,ms^{-1}$
8. (a) $1.05\times10^{2}\,Nm^{-1}$ (b) $M = 36\,kg$
9. (a) 0.13 m (b) 0.159 Hz (c) 6.28 s
10. (b) 1.1 s

Exercise 3.3

7. 0.5%
9. $2.22\times10^{-6}\,Nkg^{-1}$
10. 3.13×10^{8} J
11. $1.69\,Nkg^{-1}$
14. $2.39\,kms^{-1}$
15. (a) 380.03 N
16. 6.02×10^{24} kg
17. 250 N
18. 1.16×10^{11} J
19. $2.66\times10^{4}\,ms^{-1}$

Revision exercise 3

6. $2.1\,ms^{-1}$
8. 0.31
9. 22.5°
10. $4.34\,rads^{-1}$
11. (a) 2 J (b) $2.83\,ms^{-1}$ (c) 0.5 J, 1.5 J
    (d) $40\,ms^{-2}$
12. $9.96\,ms^{-2}$, 4.45 m
13. (b) 32 s, 7×10^{-2} m, $0.28\,ms^{-2}$,
14. 0.31 N, 3.9 N
15. $1.62\,Nkg^{-1}$, $-2.8\times10^{6}\,Jkg^{-1}$
16. 3.4×10^{10} J
18. 3.67×10^{5} m

Chapter Four

Exercise 4.2

5. (a) $2\times10^{-4}\,kgm^{2}$ (b) $8\times10^{-4}\,kgm^{2}$
6. (a) $2.6\,kgm^{2}$ (b) $20.7\,kgm^{2}$
   (c) $1.6\times kgm^{2}$
7. (a) $7\,kgm^{2}$ (b) $8\,kgm^{2}$ (c) $3.5\,kgm^{2}$
8. $1.58\times10^{7}\,gcm^{2}$
9. $9.8\times10^{44}\,gcm^{2}$

Exercise 4.3

1. (a) $8\,kgm^{2}$ (b) $24\,kgm^{2}$
2. (a) $4\times10^{-3}\,kgm^{2}$ (b) $8\times10^{-3}\,kgm^{2}$
   (c) $2\times10^{-3}\,kgm^{2}$
3. $4.1\,kgm^{2}$
4. (a) $2.083\times10^{-5}\,kgm^{2}$
   (b) $1.302\times10^{-4}\,kgm^{2}$
   (c) $1.5\times10^{-1}\,kgm^{2}$
5. $1.25\times10^{-1}\,kgm^{2}$

Exercise 4.4

2. 0.65 m
3. 18 cm
5. 52.9 cm

Exercise 4.5

6. (a) 1.03 s (b) 21 revolutions
7. (a) 24 Nm (b) $0.036 rads^{-2}$
   (c) $1.07 ms^{-2}$
8. $2.36 ms^{-1}$
9. (a) 11.36 N, $7.57 ms^{-2}$, $9.53 ms^{-1}$
   (b) $9.53 ms^{-1}$

Exercise 4.6

2. (a) $5 kgm^2s^{-1}$ (b) 0.4 Nm
3. $17.5\ kgm^2s^{-1}$
4. $4.1 rads^{-1}$
5. $5.4\times10^6\ kgm^2s^{-1}$
6. (a) $2.67\ ms^{-1}$ (b) $0.67\ ms^{-1}$
   (c) $-1.001\ rads^{-1}$

Revision exercise 4

7. $2.29 kgm^2$
8. 4.04%
13. (a) $5 rads^{-2}$ (b) 1.5Nm (c) 115
14. (a) 1.5 Nm (b) 42.39 J
    (c) $20.59 rads^{-1}$
16. 15.78 s
18. $0.16 kgm^2$
20. 52.9cm
21. (a) $0.013 rads^{-2}$ (b) $3.14 rads^{-1}$
    (c) 235.5 s (d) 118
22. $4.8 ms^{-1}$
23. 0.3m
24. $226.08 rads^{-2}$
25. 4.85 s, 800 revolutions
26. (a) $6\ rads^{-2}$, $3 rads^{-1}$
    (b) $63 rads^{-1}$

Chapter Five

Exercise 5.1

5. 11.1
6. $3.19 ms^{-1}$
7. $64 ms^{-1}$

Exercise 5.2

3. 165 mm
4. $10 ms^{-1}$
5. $\frac{l}{\sqrt{2\pi}}$

Exercise 5.3

3. $1.98 ms^{-1}$
4. $31.2 ms^{-1}$

Revision exercise 5

2. (a) $1.98 ms^{-1}$ (b) 2.8 m
4. $15.8 ms^{-1}$
6. (c) 0.941 m of water
7. $318.5 ms^{-1}$
9. 15 N
17. $3.77\times10^{-5} m^3s^{-1}$

Chapter Six

Exercise 6.1

2. 10 cm
6. 2.62 cm

7. (a) 1.88×10^{-5} J (b) 10 mm
8. 9.4×10^{-3} Nm^{-1}
9. 1.01 mm

Exercise 6.2

3. (b). 64 N
4. (b). (i) 5×10^{-4}
   (ii) 1×10^{11} Nm^{-2} (iii) 0.025 J
5. (c) 6.4×10^{6} Nm^{-2}, 0.06, 1.1×10^{11} Nm^{-2}
6. 0.04 J, 0.08 J
7. $1 m^3$, $4.5 kgm^{-3}$
8. 6.0×10^{9} Nm^{-2}
9. 2.9×10^{-8} m
11. 47.3 g

Exercise 6.3

2. $2.64 ms^{-1}$
3. (a) 6.02×10^{-3} moles
   (b) 3.63×10^{21} molecules
   (c) $482.7 ms^{-1}$
5. $707.1 ms^{-1}$
6. $1843.6 ms^{-1}$, $460.9 ms^{-1}$

Revision exercise 6

5. 2.57 cm
6. 9 mm
7. 7.2
8. 5 cm
12. $1.0 \times 10^{5} Nm^{-2}$
13. $1.45 \times 10^{2} Nm^{-2}$
16. 0.5 mm, 0.25 mm
17. 11.28 gcm^{-3}
18. (a) 1.5 (b) 6.0×10^{-3} m, 4.0×10^{-3} m
    (c) 780 N
20. (a) 1.1×10^{-3} m (b) 1.5×10^{-2} J
21. (a) 50 N
    (b) 1.8×10^{-3} m, 4.4×10^{-2} J
    (c) 0.85×10^{-3} m
    (d) 0.084 m from B
25. (a) 4.62 moles (b) $240.3 ms^{-1}$
26. (a) 6.20×10^{-21} J
    (b) 2.33×10^{5} m^2s^{-2}
    (c) $483.2 ms^{-1}$

Chapter Seven

Exercise 7.1

1. 60 °C
3. 384.8 °C
4. 57.8 °C
5. 87.08 °C
6. 369.2 K
7. (a) 369.4 K (b) 369.4 K
   (c) 368.9 K
8. 50 °C
9. 17.3 °C

Exercise 7.2.1

1. (b) $5000 Js^{-1}$ or 5000 W
2. $41.12 Js^{-1}$, 37.13 °C
3. 16 cm
4. 38.25 m

5. (a) $650\,Wm^{-2}$
(b) $2.0\times10^{-6}\,ms^{-1}$
7. 51°C

Exercise 7.2.2

2. 10 minutes
3. (a) 24°C (b) 42.7°C
4. (a) 47°C (b) 15.4 minutes
5. 7.5°C/minutes

Exercise 7.2.3

3. (a) 0.71 (b) 2.42
4. 1933 K
5. $98.09\,Js^{-1}$
6. 5749 K

Exercise 7.3

1. (a) 832.6 J (b) 2.1×10^{3} J
(c) 2.9×10^{3} J
2. (b) −6171.5 J
3. (b) 228.25 J, (d) 684.75 J
(e) 913 J (f) 113.99 J
4. (a) 1520 mmHg, 17°C
(b) 2006 mmHg, 110°C
5. −71.2°C
8. (a) $33.2\,Jmol^{-1}K^{-1}$, $24.9\,Jmol^{-1}K^{-1}$
(b) 1.3 (c) 9960 J
9. 28.8 mmHg

Revision exercise 7

4. 405.6 kJ
5. 97.3°C, 75.5°C
6. $3.6\,Js^{-1}$
7. 8.3°C
8. 9903 W
9. $0.036\,Wm^{-1}K^{-1}$
10. 3 minutes
11. 138.4 W
12. 2 J
13. 10^{4} J
14. (a) 2 kJ (b) 450 K (c) 192.8 J
16. 56.4°C
17. 39.89°C
18. (a). $3.96\times10^{-5}\,V\,°C^{-1}$,
$3.2\times10^{-8}\,V\,°C^{-2}$
(b) $12.5°C^{-1}$
19. $1.45\times10^{-3}\,kgs^{-1}$
20. $2.673\times10^{5}\,Nm^{-2}$

Chapter Eight

Exercise 8.1

5. $326.6\,ms^{-1}$

Exercise 8.2

4. 260 Hz
5. (a) $144.34\,ms^{-1}$ (b) 0.6 m
(c) 240.5 Hz
6. $299.57\,ms^{-1}$
7. 0.01 m

8. (a) $\frac{5\pi}{3}$ rad (c) 6 cm
(d) $0.01\sin 500\pi t$
9. (a) $30\,ms^{-1}$ (b) $0.017\,kgm^{-1}$
10. (a) $82\,ms^{-1}$ (b) 16.8 m
(c) 4.88 Hz
11. (a) 105 Hz (b) $157.5\,ms^{-1}$
12. (a) 3.2% (b) 6.78%

Exercise 8.3

3. 170 Hz
4. 0.2 m
5. (b) 11cm
6. 5.044 beats/second
7. 27.5 beats/second
8. (a) $333\,ms^{-1}$, $360.5\,ms^{-1}$
(b) $384.9\,ms^{-1}$
9. 1367.6 m
10. 100 Hz

Exercise 8.4

5. $2.0\times10^{8}\,ms^{-1}$

Exercise 8.5.1

2. 10 mm ,12.5 mm
3. 551.25 nm
4. 0.09 mm
5. 3.2 μm
6. 0.034 mm
8. 0.74 mm
9. 1.174
10. 1.51m , 1.33

Exercise 8.5.2

3. 5.06 mm
4. 6.79×10^{-7} m or 679 nm
5. 20.7°, 3.1°
6. 640 nm, 480 nm, 28.7°
7. 17.5°, 36.9° and 64.2°
8. 2
9. 374 lines per mm
10. 569×10^{3} lines per metre, 5898Å

Exercise 8.5.3

1. 56.8°, 53.1°, 67.5°
2. (b) 53.1°
5. (b) 1.6, 32°

Exercise 8.6

1. 6.18 beats/second
2. (a). 859 Hz (b) 741 Hz
3. (a) $18\,ms^{-1}$ (b) $17.1\,ms^{-1}$
7. $3\times10^{7}\,ms^{-1}$

Revision exercise 8

1. $240\,ms^{-1}$
4. 30.16 Hz
5. $11.7\,ms^{-1}$
8. 3.43 mm, 3.16 mm
9. 1.2
12. 8.7×10^{-6} m
14. $1.43\times10^{5}\,Nm^{-2}$
15. 5.5×10^{14} Hz, 3457 Å

16. 6cm, 20Hz, $120\,cms^{-1}$
19. $8.2\times10^{7}ms^{-1}$
20. 403.3 Hz to 484 Hz

Chapter Nine

Exercise 9.1

2. (a) $7.42\times10^{-7}C$
   (b) $3.7\times10^{-7}C$, $1.48\times10^{-6}C$
3. (a) $+3.6\times10^{-6}C$
   (b) 0.2N, downward
4. 1.4 N
6. 0.25 m from 2.5°C
7. $50\,NC^{-1}$, $28.1\,NC^{-1}$
8. (b) $4.32\times10^{2}NC^{-1}$
9. 0.55
10. $1.8\times10^{4}NC^{-1}$,
    60° below x-direction
11. 3

Exercise 9.2

8. $5.9\times10^{6}ms^{-1}$
9. 4.43 mm
11. $1.59\times10^{-10}C$
12. $3.6\times10^{7}J$
13. $1.29\,NC^{-1}$, 0 V, 0 J
14. 4.6 J
15. 47.68 V

Exercise 9.3

1. (a) $3\mu A$ (b) $2.5\mu A$
3. (a) (ii) 9×10^{9} N (b) 379.7 N
   (c) (ii) $1.09\times10^{7}NC^{-1}$
   (c) (iii) 35.6 cm on the right of +2q
5. (a) $1.88\times10^{-3}s$
   (b) $3.36\times10^{-3}s$
6. 1.44
7. (a) $10^{-2}J$, $9.4\times10^{-2}J$
   (b) $9.4\times10^{-2}J$
8. (a) (ii) 33.90 mm
   (b) $8.44\times10^{-5}J$, $1.28\times10^{-4}J$
9. (a) $3.22\times10^{-11}F$
   (b) (i) $6\times10^{-8}F$
   (ii) $6.04\times10^{-8}F$

Revision exercise 9

1. −7.2 N, 0.41 m
2. $9\times10^{9}NC^{-1}$ in y direction
3. $5.92\times10^{5}C$
6. (a) $1.4\times10^{6}NC^{-1}$ to the right
   (b) 4.24 m from the 0 cm mark
7. (a) 5.0 kV (b) $4.2\times10^{7}ms^{-1}$
8. 190.4 V
9. (a) 7.65 V (b) $7.65\times10^{-9}J$
   (c) $7.65\times10^{-9}J$
   (d) $-2.2\times10^{-8}J$
13. (a) $3.54\times10^{-8}C$ (b) 125 V
    (c) $2.2\times10^{-6}J$
14. (a) $2.66\times10^{-9}F$ (b) $4.72\times10^{-9}F$
20. (a) 8.19 mC (b) 1.64 J

Index

P

Q

R

S

T

U

V

W

Y

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