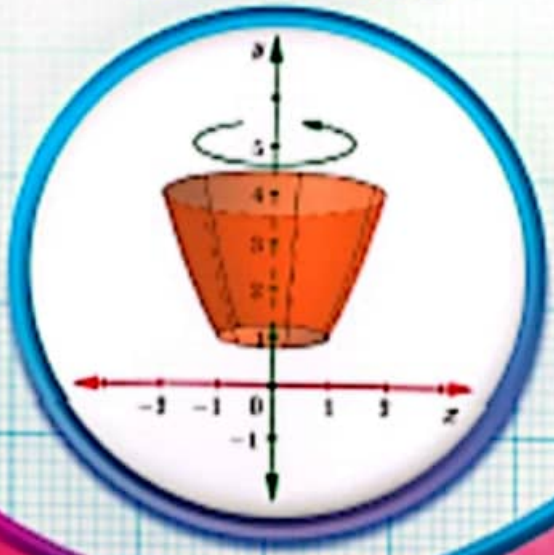
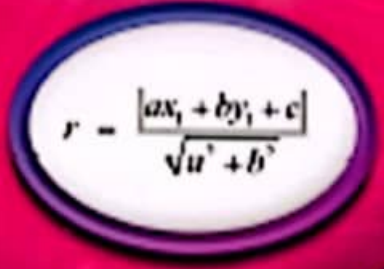
Mathematics

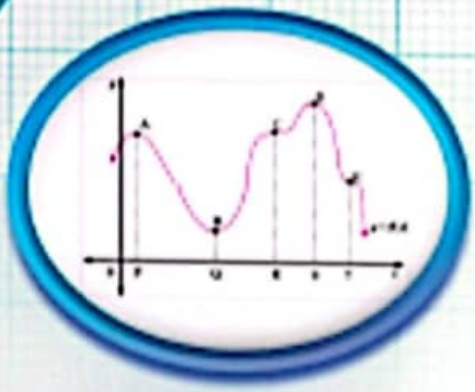
for Advanced Secondary Schools

Form Five

Student's Book







Tanzania Institute of Education

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Mathematics for Advanced Secondary Schools

Student's Book

Form Five

THE UNITED REPUBLIC OF TANZANIA
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Commissioner for Education

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Dr Aneth A. Komba
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Preface

This textbook, *Mathematics for Advanced Secondary Schools* is written specifically for Form Five Students in the United Republic of Tanzania. The book is prepaired in accordance with the 2023 Mathematics Syllabus for Advanced Secondary Education, Form V - VI issued by the Ministry of Education, Science and Technology.

The book consists of nine chapters, namely; Sets, Logic, Coordinate geometry I, Functions, Algebra, Trigonometry, Linear programming, Differentiation, and Integration. Each chapter contains illustrations, activities, and exercises. The answers for questions for proofs, show, verifications, and some illustrations are not provided. You are encouraged to do all activities and exercises together with other assignments provided. You are advised to do a project work in any theme from the covered chapters of this book. You are also required to prepare a portfolio for keeping records of different activities performed in different lessons. This will enhance your understanding and development of the intended competencies for this level.

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Chapter One

Introduction

A set is a collection of objects which have common properties. Sets are usually denoted by capital letters and their objects are listed inside the curly brackets { } separated by commas. Each object inside the set is called an element of that set. Some basic mathematical concepts, such as relations and vectors are precisely specified using theoretical concepts of sets. In this chapter, you will learn about the number of elements in a set, basic set operations, Venn diagrams, and simplification of set expressions. The competencies developed can be applied to perform various tasks such as organizing, creating, and categorizing objects.

Think

The arrangement of objects without the concept of sets.

Methods of representing sets

Sets can be represented in various ways depending on the context and the requirements of the problem. Some common methods of representing sets include descriptive or statement method, roster or listing method, and rule method or set builder notation.

Descriptive method

In this method, the elements of a set are described by words and the description is enclosed in curly brackets. For instance, if A is the set of positive even numbers less than or equal to twenty, it can be written as A = {positive even numbers less than or equal to twenty}.

Roster method

In a roster or listing method, the elements of a set are listed inside the curly brackets. The elements in a set are separated by commas. For example, if B is the set of all factors of 42, then it can be represented as B = {1, 2, 3, 6, 7, 14, 21, 42}.

Set builder notation

The set builder notation uses formulas to describe sets. The elements of the set are described by using symbols, usually letters such as x followed by a colon (:) which is read as "such that", then followed by a description of the elements. The whole description is enclosed in curly brackets. For instance, if C is a

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set of odd numbers, it can be described as $C=\{x:x=2n-1, \text{ where } n\in \mathbb{N}\}$, which is read as "C is the set of all elements x such that x is an odd positive number".

Example 1.1

Write each of the following sets in roster form.

(a) Set of integers between $-\frac{7}{2}$ and $\frac{19}{2}$.

(b) The set of all letters in the word "DISTINGUISH"

Solution

(a) Let A be a set of integers between $-\frac{7}{2}$ and $\frac{15}{2}$.

Since $-\frac{7}{2}=-3.5$ and $\frac{15}{2}=7.5$, then the integers between -3.5 and 9.5 are:

$-2, -1, 0, 1, 2, 3, 4, 5, 6$ and 7.

Therefore, in roster form

$A=\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$.

(b) Let B be a set of letters in the word "DISTINGUISH".

Note that, the repeated letters are taken only once. Thus, the word DISTINGUISH has the letters D, I, S, T, I, N, G, U, I, S, H.

Therefore,

$B=\{D, I, S, T, N, G, U, H\}$.

Example 1.2

Write the set $C=\{2, 4, 8, 16, 32, 64\}$ in set builder notation.

Solution

Clearly, $C=\{2^1, 2^2, 2^3, 2^4, 2^5, 2^6\}$

Thus, in set builder notation, the set is

$C=\{x:x=2^n, n\in \mathbb{N} \text{ and } 1\le n\le 6\}$.

Exercise 1.1

1. By using the descriptive method, describe each of the following sets:
   (a) $A=\{1, 8, 27, \supset\}$
   (b) $B=\{3, 6, 9, 12, \supset\}$
   (c) $C=\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
   (d) $D=\{\text{dog, cat, sheep, goat, cow, } \cdots\}$
2. Using the set builder notation, describe each of the following sets:
   (a) $A=\{1, 4, 9, 16, \supset\}$
   (b) $B=\{1, 8, 27, 64, \supset\}$
   (c) $C=\{-3, 3\}$
   (d) $D=\{1, 2, 3, 4, 5, \supset\}$
   (e) $E=\{2, 4, 6, 8, \supset\}$
   (f) $F=\{0, 1, \ldots\}$
   (g) $G=\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$
3. If x is an integer, use the roster method to describe each of the following sets:
   (a) $A=\{x:x^2-1=0\}$
   (b) $B=\{x:x\ge 3\}$
   (c) $C=\{x:x^2-4x+3=0\}$

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Types of sets

There are various types of sets. These include the empty or null set, equal sets, equivalent sets, subsets, finite and infinite sets, proper and improper subsets, supersets, power sets, singleton set, and universal set.

Empty or null set

An empty set is a set which does not contain any element. The empty set is denoted by $\varnothing$ and read as empty set. In roster form an empty set is denoted by $\{\ \}$.

For example, A = {three sided rectangles} is an empty set because there is no rectangle with three sides. Also, the set $B = \{x : 7 < x < 8;\ x \in \mathbb{N}\}$ is an empty set because there is no natural number between 7 and 8.

Example 1.3

Given the set $K = \{x : -10 \leq x \leq -1\}$. Find all natural numbers in K.

Solution

Let C be the set of natural number in K. Natural numbers are counting numbers 1, 2, 3, 4, ...

There are no counting numbers in the interval $-10 \leq x < -1$.

Therefore, $C = \varnothing$ or $C = \{\ \}$.

Example 1.4

Given that $P = \{1, 3, 5, 7, 9, 11\}$. Identify the set of even numbers in P.

Solution

Let E be the set of even numbers in P.

A number which is divisible by 2 without a remainder is an even number.

There are no even numbers in $P = \{1, 3, 5, 7, 9, 11\}$.

Therefore, $E = \{\ \}$ or $E = \varnothing$.

Example 1.5

If A = {Hemedi, Shedrack, Peter, Jonathan, Patrick}, find the set G which represents the names of girls in set A.

Solution

There are no girls' names in set A.

Therefore, $G = \{\ \}$ or $G = \varnothing$.

Equal sets

Two or more sets are said to be equal if they have exactly the same elements. For instance, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A = B$. The two sets are equal, meaning that the same elements in set A are found in set B. Thus, the two sets have the same number of elements. The comparison is written as $A = B$ and it is read as set A is equal to set B.

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Unequal sets are sets which have at least one non-common element. Unequal sets C and D are denoted by $C \neq D$.

For example, if C = {a, b, c, d, e, g} and D = {a, b, c, d, e, f}, then the sets C and D are not equal.

Note that, not all elements in set C are in set D and vice versa, although the two sets have the same number of elements. Therefore, $C \neq D$.

Example 1.6

Given the sets A = {Halima, Angelina, Christian, Sarapia} and B = {Halima, Angelina, Christian, Sarapia}. Are the two sets are equal?

Solution

Since each element of set A is in set B and vice-versa, then A = B.

Therefore, the two sets are equal.

Example 1.7

Given the sets C = {2, 4, 6, 8} and D = {2, 4, 6, 8, 10, 12}. Determine whether the two sets are equal.

Solution

Since set D contains the elements 10 and 12 which are not in set C, then $C \neq D$.

Therefore, the two sets are not equal.

Equivalent sets

Two sets, A and B are said to be equivalent if they have the same number of elements. The elements do not need to be the same. What matters is the sets to have one-to-one correspondence of elements. For instance, if A = {a, e, i, o, u} and B = {1, 2, 3, 4, 5}, then the number of elements in sets A and B are usually denoted by n(A) and n(B), respectively. Thus, sets A and B are equivalent because n(A) = n(B) = 5. The number of elements in a set is known as the cardinality of the set.

Set equivalence is denoted by $A \equiv B$.

Example 1.8

Given that S = {x : x is a counting number less than 7} and set G = {x : $1 \leq x \leq 6$, x is an integer}. Determine whether or not the two sets, S and G, are equivalent.

Solution

Given S = {1, 2, 3, 4, 5, 6} and G = {1, 2, 3, 4, 5, 6}.

Since n(S) = 6 and n(G) = 6, then n(S) = n(G).

Therefore, the sets S and G are equivalent.

Example 1.9

Given that D = {1, 3, 5, 7, 9, 11, 13} and G = {x : $3 \leq x \leq 9$, $x \in N$}. Are the two sets equivalent?

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Solution

Since $G = \{3, 4, 5, 6, 7, 8, 9\}$

Then, $n(D) = 7$ and $n(G) = 7$, hence $n(D) = n(G)$.

Therefore, the two sets are equivalent.

Subsets

Set A is said to be a subset of set B if all elements of A are also elements of B. The symbol for subset is $\subset$. For instance, if set F = {all English alphabets} and G = {all vowels}, then G is a subset of F. This is written as $G \subset F$.

Note that, the number of subsets in a set is given by 2^n, where n is the number of elements in the given set. The terms "proper" and "improper" subsets are used in sets to differentiate the subsets. A proper subset, denoted by the symbol $\subset$, is a subset containing some of the elements (not all elements) of the original set, whereas an improper subset, denoted by the symbol $\subseteq$, is a set containing every element in the original set. Any set has one improper subset which is the set itself. Therefore, given a set A, then $A \subseteq A$ and it is read as "set A is an improper subset of A". All other subsets of A are proper subsets. A non-empty set S has two distinct trivial subsets, namely the empty set $\varnothing$ and S itself. Other subsets of S are non-trivial subsets.

Power set

The collection of all subsets of a set is called the power set. Given set A, the power set of A is denoted by P(A), and it is read as "power set of A". Generally, if $n(A) = n$, then the number of subsets of set A is given by 2^n. Thus, $n[P(A)] = 2^n$.

Example 1.10

Given that $B = \{1, 2, 3\}$.

(a) List all the subsets of B.

(b) List all improper and proper subsets of B.

(c) Find P(B)

(d) Find n(P(B))

Solution

(a) The subsets of B are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$.

(b) Improper subset of B is $\{1, 2, 3\}$.

Proper subsets of B are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1, 3\}$, and $\{2, 3\}$.

(c) $P(B) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

(d) $n(P(B)) = 2^3 = 8$.

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Example 1.11

List all subsets of K given that $K = \{\text{Anna, Ally, Halima, John}\}$.

Solution

The subsets of K are {}, {Anna}, {Ally}, {Halima}, {John}, {Anna, Ally}, {Anna, Halima}, {Anna, John}, {Ally, Halima}, {Ally, John}, {Halima, John}, {Anna, Ally, Halima}, {Anna, Ally, John}, {Ally, Halima, John}, {Anna, Halima, John}, and {Anna, Ally, Halima, John}.

Finite and infinite sets

A finite set is a set which has countable number of elements. The elements in a finite set are finite and can be listed. An infinite set is a set with uncountable or endless number of elements. In other words, an infinite set is a set that is not finite.

For instance, if V = {a, e, i, o, u} and P = {prime numbers}, then V is a finite set because it contains countable elements while P is an infinite set because it contains uncountable elements.

Example 1.12

Identify the finite and infinite sets among the sets $A = \{2y^2 : y \in \mathbb{N}\}$, $B = \{1, 2, 3, 4, 5, 6, ...\}$, and $C = \{x : 1 \leq x \leq 20, x \text{ is an integer}\}$

Solution

The sets A and B are infinite since they have uncountable number of elements, while set C is finite because it has countable number of elements.

Singleton sets

A set containing a single element is called a singleton set. For example,

(i) $A = \{0\}$ is a singleton set

(ii) $B = \{k\}$ is a singleton set

(iii) $C = \{x : x \in \mathbb{N} \text{ and } x^3 = 27\}$ is a singleton set with a single element, 3.

(iv) $D = \{x : x \in \mathbb{Z} \text{ and } x^2 = 4\}$ is not a singleton set because $D = \{-2, 2\}$.

Universal sets

A universal set is a set containing all elements or members of all related sets without repetition. The symbol U or ξ is used to denote a universal set. However, U is the most used symbol. For instance, if A = {1, 2, a, 4, 5} and B = {1, 2, 3, a, b, c, d}, then the universal set associated with A and B is given by U = {1, 2,3,4, 5, a, b, c, d}.

Note that, $A \subset U$ and $B \subset U$.

Example 1.13

Let U = {7, 8, 9, 10, 11, 12, 13 ,14, 15, 16}. If A, B, and C are subsets of U, list all elements of the sets:

(a) $A = \{x : x \text{ is a factor of } 60\}$

(b) $B = \{x : x \text{ is an even number}\}$

(c) $C = \{x : x \text{ is an odd number}\}$

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Solution

The elements of sets A, B, and C from the universal set "U" are;

(a) $A = \{10, 12, 15\}$

(b) $B = \{8, 10, 12, 14, 16\}$

(c) $C = \{7, 9, 11, 13, 15\}$

Exercise 1.2

1. Which of the following sets are finite and which are infinite? Give reasons to support your answer.
   (a) $E = \{x : x$ is a plant on the earth's surface$\}$
   (b) $F = \{x : 10 \leq x \leq 30, x \in \mathbb{R}\}$.
   (c) $G = \{x : 10 < x < 20, x$ is not an odd number$\}$

2. If $H = \{6, 7, 8, 9, 10\}$,
   $T = \{$Tony, James, Herry, Juma$\}$, and
   $R = \{$Juma, Herry, Tony, James$\}$, which among the three sets are equivalent, equal or unequal?

3. Write True or False in each of the following:
   (a) $\{4\}$ is an element of $\{\{4\}, \{4\}\}$
   (b) $\{1\} \subseteq \{0, 1\}$
   (c) $\{1\} \subseteq \{1\}$
   (d) $\varnothing$ is an element of $\{\}$
   (e) 0 is an element of $\{\}$

4. Given the set $K = \{a, b, c, d\}$.
   (a) List all the subsets of K.
   (b) Find n(K)

5. Given the set $J = \{$dog, cat, lion, zebra$\}$. Find:
   (a) The power set, P(J)
   (b) $n[P(J)]$

6. Find the universal set associated with the sets:
   (a) V and S, if V = {all prime numbers less than 20} and $S = \{2, 3, 4, 6, 7, 8, 10, 11, 14\}$.
   (b) $A = \{x : x = 2n+1, n \in \mathbb{Z}\}$, and $B = \{x : x = 2n, n \in \mathbb{Z}\}$.

7. List the elements of each of the following sets:
   (a) $A = \{x \in \mathbb{N} : x + 7 = 4\}$
   (b) $B = \{x \in \mathbb{Z} : x + 8 = 3\}$

Basic set operations

Set operations are performed on two or more sets in order to build a relationship. There are four main set operations, namely union of sets, intersection of sets, complement of a set, and difference of sets.

Union of sets

The union of two sets A and B is the set of all elements that are either in A or in B, or in both A and B without repetition. The symbol "$\cup$" is used to denote the union of sets. In set notation, it is written as $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

For example, if $A = \{2, 4, 6, 7\}$ and $B = \{1, 2, 3, 5\}$, then

$A \cup B = \{2, 4, 6, 7\} \cup \{1, 2, 3, 5\}$

Thus, $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

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Example 1.14

Find $A \cup B$ if $A = \{a, b, c, d, e, f, g, h\}$ and $B = \{a, e, i, o, u\}$.

Solution

$A \cup B$ is the set of all elements that are either in set A or in set B or in both.
That is, $A \cup B = \{a, b, c, d, e, f, g, h\} \cup \{a, e, i, o, u\}$.

Therefore, $A \cup B = \{a, b, c, d, e, f, g, h, i, o, u\}$.

Example 1.15

Find $A \cup B$ if $A = \{3, 5, 7, 8, 10, 11, 14, 16\}$ and $B = \{1, 2, 6, 9, 12\}$.

Solution

From the given sets, $A \cup B = \{3, 5, 7, 8, 10, 11, 14, 16\} \cup \{1, 2, 6, 9, 12\}$
Therefore, $A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$.

Intersection of sets

The intersection of two sets, A and B is the set of elements that belong to both A and B. The symbol "$\cap$" is used to denote the intersection of sets. In set notation, it is written as $A \cap B = \{x : x \in A \text{ and } x \in B\}$. For instance, if $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \cap B = \{1, 3, 5, 7\}$.

If the intersection of sets is an empty set (no common elements), then the sets are said to be disjoint. Otherwise, they are joint sets.

Example 1.16

Find $A \cap B$, if $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d, e, f, g, h, i\}$.

Solution

From the given sets,

$A \cap B = \{a, e, i, o, u\} \cap \{a, b, c, d, e, f, g, h, i\}$
$= \{a, e, i\}$.

Therefore, $A \cap B = \{a, e, i\}$.

Note that, A and B have three elements in common, thus they are joint sets.

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Example 1.17

If $A=\{a, e, i, o\}$ and $B=\{b, c, f, g\}$, find $A \cap B$ and state whether or not A and B are disjoint.

Solution

From the given sets,

$A \cap B = \{a, e, i, o\} \cap \{b, c, f, g\}$

$= \varnothing$

Therefore, A and B are disjoint because they have no common element.

Complement of a set

The complement of a set A, denoted as A' or A^c is the set which contains all the elements of the universal set that are not in A, or it is the difference between the universal set and A. This means that, if A is a subset of a universal set, then the elements of the universal set which are not in A form the complement of A. For instance, if $U = \{a, b, c, d, ..., z\}$ and $A = \{a, b, c, d, e\}$, then $A' = \{f, g, h, ..., z\}$.

Example 1.18

Given $U=\{125, 245, 365, 475, 585\}$ and $A=\{365\}$, find A'.

Solution

Given $U = \{125, 245, 365, 475, 585\}$ and $A = \{365\}$. It follows that $A' = \{125, 245, 475, 585\}$.

Therefore, $A' = \{125, 245, 475, 585\}$.

Example 1.19

Given $U = \{a, e, i, o, u\}$, $B = \{e, i, u\}$, find B'.

Solution

Given $B = \{e, i, u\}$ and $U = \{a, e, i, o, u\}$, then $B' = \{a, o\}$.

Therefore, $B' = \{a, o\}$.

Example 1.20

Given $U = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$, $A = \{5, 7, 11, 13, 15, 17\}$, and $B = \{4, 6, 8, 10, 12, 14\}$, find:

(a) $A' \cup B'$ (b) $A \cap B'$

Solution

The compliments of sets A and B are:

$A' = \{4, 6, 8, 9, 10, 12, 14, 16, 18\}$ and $B' = \{5, 7, 9, 11, 13, 15, 16, 17, 18\}$

Therefore,

(a) $A' \cup B' = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

(b) $A \cap B' = \{5, 7, 11, 13, 15, 17\}$.

Difference of sets

The difference of two sets A and B, also known as set minus, written as $A - B$ or A/B, is the set of all elements of A that are not in B.

Consider two sets A and B, whereas $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7, 8\}$. The difference of the two sets $A - B$ is found by listing all the elements of A, and then remove all elements which are found in set

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B. The common elements of sets, A and B are 3, 4, and 5, thus $A-B=\{1, 2\}$.
The same results can be obtained by finding $A\cap B'$.
In general, $A-B=A\cap B'$ and $B-A=B\cap A'$.

Example 1.21

Given $A=\{4, 7, 8, 12, 15, 17\}$ and $B=\{4, 6, 8, 10, 12, 14\}$, find

(a) $A-B$ (b) $B-A$

Solution

(a) $A-B$ contains elements that are only in set A but not in set B, that is,

$A-B=\{7, 15, 17\}$.

(b) $B-A$ contains elements that are only in set B but not in set A, that is,

$B-A=\{6, 10, 14\}$.

Symmetric difference of sets

The symmetric difference of two sets A and B is the set $(A-B)\cup(B-A)$ and it is denoted by $A\Delta B$. It is the set of all elements which belong to either A or B but not to both. This means that, if $x\in(A\Delta B)$, then $A\Delta B=\{x: x\in A \text{ or } x\in B \text{ and } x\notin A\cap B\}$.

Example 1.22

Determine $A\Delta B$ for $A=\{1, 2, 3, 9, 12, 15\}$ and $B=\{1, 3, 9, 14, 13, 20, 12, 25\}$.

Solution

The set $A\Delta B$ is formed by the elements which belong to either set A or B but not to both sets. Thus, $A-B=\{2, 15\}$ and $B-A=\{13, 14, 20, 25\}$.

$\Rightarrow A\Delta B=(A-B)\cup(B-A)$

$=\{2, 13, 14, 15, 20, 25\}$.

Therefore, $A\Delta B=\{2, 13, 14, 15, 20, 25\}$.

Example 1.23

Determine $A\Delta B$ for A = {mango, pineaple, orange, banana, apple} and B = {orange, mango, watermelon, avocado, apple}.

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Solution

The set $A \Delta B$ is the set of all elements which belong to either set A or set B but not to both sets. That is, $A \Delta B = \{\text{pineapple, banana, watermelon, avocado}\}$.
Therefore, $A \Delta B = \{\text{pineapple, banana, watermelon, avocado}\}$.

Exercise 1.3

In questions 1 – 10, find the union and intersection of the given sets.

1. $A = \{\text{first five letters of the English alphabets}\}$, $B = \{a, b, c, d, e, f, g, h\}$
2. $A = \{\text{Even numbers}\}$, $B = \{\text{Counting numbers}\}$
3. $G = \{25, 30, 45\}$, $H = \{20, 25\}$
4. $J = \{0, \Delta, 3\}$, $K = \{\Delta\}$
5. $A = \{\text{All positive multiples of 6 less than 60}\}$,
   $B = \{\text{All positive multiples of 4 less than 48}\}$
6. $W = \{14, 16, 18, 20\}$, $Z = \{\ \}$
7. $A = \{94, 110, 120, 131, 140\}$, $B = \{94, 110, 265\}$
8. $A = \{\text{Prime factors of 72}\}$, $B = \{\text{Prime factors of 15}\}$
9. $A = \{\text{all even number less than 28}\}$, $B = \{\text{all multiples of 3 less than 27}\}$
10. $A = \{a, b, c, d\}$, $B = \{a, d, e\}$, and $C = \{\ \}$
11. Find the intersection of $M = \{a, b, c, d, e, g\}$ and $N = \{1, 2, 3, 4, 5, 6, 7\}$.
12. What is the name of sets which have no common elements?
13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $M = \{1, 2, 3, 4\}$ and $N = \{1, 3, 7, 9, 11\}$, find M' and N'.
14. If $U = \{\text{mango, orange, tomato, cabbage, pineapple, watermelon}\}$,
    $A' = \{\text{mango, watermelon, tomato}\}$, list the elements of set A.
15. If $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$, list the elements of $A \cup B$.
16. Given that $U = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $A = \{3, 4, 5, 6\}$, and $B = \{5, 6, 7, 8\}$, find each of the following:
    (a) $(A \cap B')$ (b) $B \cup A'$ (c) $B' \cap A'$ (d) $A' \cap B$
17. Given that $U = \{0, 11, 12, 13, 14, 15, 16, 17\}$, $P = \{0, 11, 13\}$, and $Q = \{15, 16, 17\}$. Write T for a true statement and F for a false statement in each of the following:

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(a) $P' = Q$ (b) $P' \cap Q' = (P \cup Q)'$ (c) $n(P \cap Q) = 1$ (d) $n(P' \cup Q') = 9$

18. If $U = \{\text{magenta, red, pupple, green, pink, blue, black, white}\}$, find A' if $A = \{\text{green, pink, white, pupple, black}\}$.

19. Given that $U = \{a, b, c, d, e, f, g, h, i, j\}$, $A = \{a, b, c, d, e, f\}$, and $B = \{a, d, e\}$, find each of the following:

(a) $A \cap B$ (c) $A' \cap B$ (e) $A' \cap B'$

(b) $A \cup B$ (d) $A \cap B'$ (f) $A' \cup B'$

20. Find the union and intersection of each of the following sets:

(a) $A = \{22, 24, 26\}$, $F = \{24, 26, 28, 30\}$

(b) $Y = \{x: 2 < x \leq 18\}$, $W = \{x: 7 \leq x < 23\}$

21. Given that $A = \{13, 14, 15\}$ and $B = \{14, 15, 16, 17, 18\}$. Determine:

(a) $A - B$ (b) $B - A$

22. If $M = \{32, 34, 36, 38, 40, 42\}$ and $K = \{33, 34, 35, 36, 37, 38, 40\}$, find:

(a) $M - K$ (b) $K - M$ (c) $(M - K) \cup (K - M)$

23. Given three sets P, Q, and R, with $P = \{8, 9, 10, 11, 12\}$, $R = \{9, 11, 14, 16, 18\}$, and Q $\{8, 9, 14, 16, 18, 20, 22\}$, find each of the following:

(a) $P - R$ (b) $R - P$ (c) $Q - P$

24. Given that $U = \{\text{all odd numbers less than } 20\}$, $A = \{7, 13, 15\}$, and $B = \{3, 5, 9\}$, find each of the following:

(a) $B - A$ (b) $(A - B)'$

25. If $X = \{2, 4, 6, 8, 10\}$, and $Y = \{15, 17, 20\}$, find $X - Y$.

26. Given $\{6, 9, 12, 15\} - \{15\} = \{6, x, y\}$, find the values of x and y.

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Representation of sets on a number line

A number line can be used to represent a set of numbers in a given interval. The number of elements within the interval can be finite or infinite depending on the nature of the set. An interval on the number line can be open, closed, half open, or half closed.

An open interval is an interval in which the end points are excluded. For example, if $a < b$, then the open interval from a to b, denoted by $a < x < b$ or (a, b) is a number line segment extending from point a to point b, excluding the end points a and b. A closed interval from a to b, denoted by $a \leq x \leq b$ or $[a, b]$ is a number line segment extending from point a to point b, including the end points a and b.

An interval can include only one end point. This kind of interval is called half-open or half-closed. A half-open interval can be left closed, and right open, denoted as $a \leq x < b$ or $[a,b)$; or it can be left open, and right closed, denoted as $a < x \leq b$ or $(a,b]$. Table 1.1 shows how sets of numbers are represented on a number line.

Table 1.1: Sets on a number line

Interval notation	Number line representation	Meaning
$\{x : a < x < b\}$ or $\{x : x \in (a, b)\}$		Left open, right open
$\{x : a \leq x \leq b\}$ or $\{x : x \in [a, b]\}$		Left closed, right closed
$\{x : a < x \leq b\}$ or $\{x : x \in (a, b]\}$		Left open, right closed
$\{x : a \leq x < b\}$ or $\{x : x \in [a, b)\}$		Left closed, right open
$\{x : x \leq a\}$ or $\{x : x \in (-\infty, a]\}$		Right closed
$\{x : -\infty < x < a\}$ or $\{x : x \in (-\infty, a)\}$		Right open
$\{x : x > a\}$ or $\{x : x \in (a, \infty)\}$		Left open
$\{x : x \geq a\}$ or $\{x : x \in [a, \infty)\}$		Left closed

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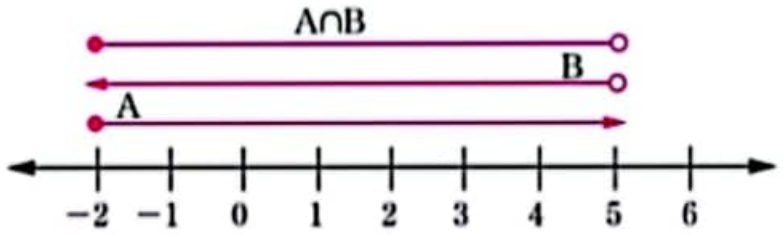
Example 1.24

Using the number line, find $A \cap B$ given that $A = \{x : x \geq -2\}$ and $B = \{x : x < 5\}$, where $x \in \mathbb{R}$.

Solution

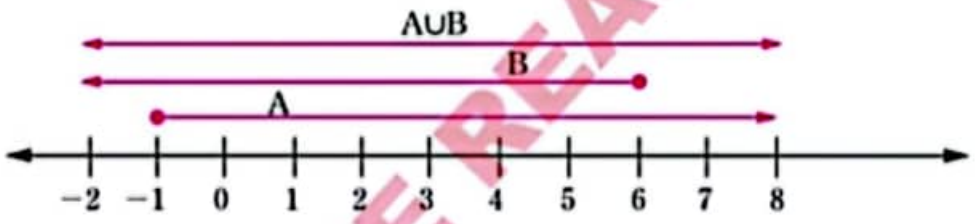


Therefore, $A \cap B = \{x : -2 \leq x < 5\}$ as represented on the number line.

Example 1.25

Use a number line to represent $A \cup B$, if $A = \{x : x \geq -1\}$ and $B = \{x : x \leq 6\}$, where $x \in \mathbb{R}$.

Solution

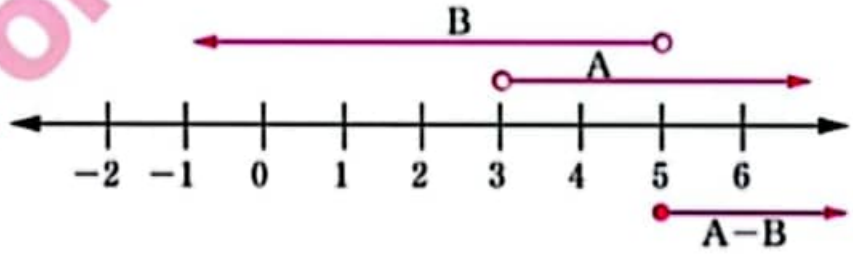


Therefore, $A \cup B = \{x : x \in \mathbb{R}\}$ as represented on the number line.

Example 1.26

Find $A - B$ by using a number line, where $A = \{x : x > 3\}$ and $B = \{x : x < 5\}$, $x \in \mathbb{R}$.

Solution



Therefore, $A - B = \{x : x \geq 5\}$ as represented on the number line.

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Exercise 1.4

1. Given $A=\{x:6<x<10\}$ and $B=\{x:x<12\}$, where $x\in\mathbb{R}$, use a number line to represent each of the following:

(a) A'	(e) $A-B$	(i) $(A-B)'$	(m) $A'-B'$
(b) B'	(f) $B-A$	(j) $(B-A)'$	(n) $(A-B)\cup A'\cap B'$
(c) $(A\cup B)'$	(g) $A-B'$	(k) $A'\cap B$	
(d) $(A\cap B)'$	(h) $(A-B')'$	(l) $A'\cap B'$	

2. If $A=\{x:8<x<20\}$ and $B=\{x:x<15\}$, where $x\in\mathbb{R}$, use a number line to represent each of the following:

(a) $A'\cap B'$	(e) $B-A$	(i) $A-B'$
(b) $(A\cup B)'$	(f) $(A-B)'$	(j) $(A'\Delta B)'$
(c) $(A\cap B)'$	(g) $(A\cup B)'\Delta(B-A)$	(k) $A'\Delta B$
(d) A'	(h) $[(A\Delta B)'\cup(A\cap B)]'$	(l) $A\Delta B$

3. If $A=\{x:8\le x\le 20\}$ and $B=\{x:x\le 18\}$, where $x\in\mathbb{R}$, use a number line to determine each of the following:

(a) $(A-B)'$ (b) $(B-A)'$ (c) $(A-B')'$

Fundamental laws of the algebra of sets

The fundamental laws of the algebra of sets are the rules that govern set operations and relationships between sets. The rules provide insight into the fundamental nature of sets. They are used in writing set expressions in the most simplified form without changing the meaning of the original expression.

Proof of the fundamental laws of the algebra of sets

The laws of the algebra of sets can be proved using basic operations. Table 1.2 shows some laws of the algebra of sets, where A, B, and C are non-empty sets.

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Table 1.2: *Laws of the algebra of sets*

S/N	Set notations	Laws
1.	$A \cup \varnothing = A$ $\quad A \cup U = U$ $A \cap U = A$ $\quad A \cap \varnothing = \varnothing$	Identity/Domination
2.	$A \cup A = A$ $A \cap A = A$	Idempotent
3.	$(A')' = A$	Double complement
4.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative
5.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative
6.	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$	De Morgan's
7.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption
8.	$A \cup A' = U$; $\varnothing' = U$ $A \cap A' = \varnothing$; $U' = \varnothing$	Complement
9.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive

Note that, two or more sets are equal if and only if they are improper subsets of each other. That is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(a) Idempotent law

Given the set A, then $A \cup A = A$ and $A \cap A = A$.

Proof

Given the set A, it follows that

$A \cap A = \{x : x \in A \text{ and } x \in A\}$. It implies $\{x : x \in A\} = A$ and

$A \cup A = \{x : x \in A \text{ or } x \in A\}$. Thus, $\{x : x \in A\} = A$.

Therefore, $A \cup A = A$ and $A \cap A = A$.

(b) Commutative law

Given two sets A and B, then $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

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Proof

To prove $A \cup B = B \cup A$, it holds prove that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$. First, prove that $A \cup B \subseteq B \cup A$. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. It implies $x \in B$ or $x \in A$ and hence $x \in B \cup A$. Thus, $A \cup B \subseteq B \cup A$. Similarly, it can be shown that $B \cup A \subseteq A \cup B$. Therefore, $A \cup B = B \cup A$. To prove $A \cap B = B \cap A$, first prove that $A \cap B \subseteq B \cap A$. Suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. It implies $x \in B$ and $x \in A$, hence $x \in B \cap A$. Thus, $A \cap B \subseteq B \cap A$. Similarly, it can be shown that $B \cap A \subseteq A \cap B$. Therefore, $A \cap B = B \cap A$.

(c) Associative law

Suppose A, B, and C are three sets, then $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof

To prove that $A \cup (B \cup C) = (A \cup B) \cup C$, first prove that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

If $x \in (A \cup B) \cup C$, then $x \in A \cup B$ or $x \in C$. It follows that ($x \in A$ or $x \in B$) or $x \in C$. Thus, $x \in A$ or $x \in B \cup C$.

Hence, $x \in A \cup (B \cup C)$.

Therefore, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, it can be proved that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$.

Activity: Identifying associative laws of sets

Individually or in a group, perform the following tasks:

1. Select three sets A, B, and C, where A and B are joint sets, as well as B and C. Also, make sure that, A and $B \cap C$ have at least one common member as well as C and $A \cap B$.
2. List the elements of $A \cap B$ and elements of $B \cap C$ in task 1.
3. List the elements of $A \cap (B \cap C)$.
4. List the elements of $(A \cap B) \cap C$.
5. Compare the lists in tasks 3 and 4.
6. What did you observe in task 5? Give reasons.

(d) Distributive law

Given three sets A, B, and C, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof

To prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, one has to prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

First prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Suppose $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Thus, $x \in A$ and ($x \in B$ or $x \in C$). It implies that $x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in [(A \cap B) \cup (A \cap C)]$.

Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

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Similarly, it can be shown that
$(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$
Therefore,
$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

(e) De Morgan's law

Given two sets A and B, then
$(A\cup B)'=A'\cap B'$ and $(A\cap B)'=A'\cup B'$.

Proof

To prove that $(A\cup B)'=A'\cap B'$, let $K=(A\cup B)'$ and $N=A'\cap B'$.

First prove that $K\subseteq N$.

Let x be an arbitrary element of K, then $x\in K\Rightarrow x\in(A\cup B)'$. This implies that, $x\notin(A\cup B)$ such that $x\notin A$ and $x\notin B$. It follows that $x\in A'$ and $x\in B'$.
$\Rightarrow x\in A'\cap B'$, hence $x\in N$.
Thus, $K\subseteq N$ ..... (i)

Next, prove that $K\subseteq N$. Let y be an arbitrary element of N, that is $y\in N\Rightarrow y\in A'\cap B'$.
This implies that, $y\in A'$ and $y\in B'$ so that $y\notin A$ and $y\notin B$. Thus, $y\notin(A\cup B)$, such that $y\in(A\cup B)$, hence $y\in K$.
Thus, $N\subseteq K$ ..... (ii)

Combination of (i) and (ii) proves that $(A\cup B)'=A'\cap B'$.

Note that, $(A\cap B)'=A'\cup B'$ can be proved in the same way.

(f) Complement law

If U is a universal set and A is a finite set, then the following hold true:
(i) $A\cup A'=U$ (iii) $\varnothing'=U$
(ii) $A\cap A'=\varnothing$ (iv) $U'=\varnothing$

Proof

(i) $A\cup A'=\{x\in U:x\in A\}\cup\{x\in U:x\notin A\}=U$
(ii) $A\cap A'=\{x\in U:x\in A\}\cap\{x\in U:x\notin A\}=\varnothing$
(iii) $\varnothing'=\{x\in U:x\notin\varnothing\}=U$
(iv) $U'=\{x\in\varnothing:x\notin U\}=\varnothing$

(g) Identity/Domination law

If U is a universal set and A is a finite set, then the following holds:
(i) $A\cup\varnothing=A$ (iii) $A\cup U=U$
(ii) $A\cap U=A$ (iv) $A\cap\varnothing=\varnothing$

Proof

(i) $A\cup\varnothing=\{x\in U:x\in A\}\cup\{x\in U:x\in\varnothing\}=A$
(ii) $A\cap U=\{x\in U:x\in A\}\cap\{x\in U:x\in U\}=A$
(iii) $A\cup U=\{x\in U:x\in A\}\cup\{x\in U:x\in U\}=U$
(iv) $A\cap\varnothing=\{x\in U:x\in A\}\cap\{x\in U:x\in\varnothing\}=\varnothing$

(h) Double complement law

If U is a universal set and A is a finite set, then $(A')'=A$.

Proof

If $x\in(A')'$, then x does not belong to A', that is $x\notin A'$. It follows that $x\in A\Rightarrow(A')'\subseteq A$.
Similarly, it can be shown that $A\subseteq(A')'$.
Therefore, $(A')'=A$.

(i) Absorption law

If A and B are two sets, then $A\cap(A\cup B)=A$ and $A\cup(A\cap B)=A$.

Proof

Given $A\cap(A\cup B)$ then $A\subseteq A$ and $A\subseteq A\cup B$.
$\Rightarrow A\cap(A\cup B)=A$

Therefore, $A\cap(A\cup B)=A$.
Also, given $A\cup(A\cap B)$, It follows that $A\subseteq A$ and $A\cap B\subseteq A$
$\Rightarrow A\cup(A\cap B)=A$

Therefore, $A\cup(A\cap B)=A$.

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Example 1.27

Use the laws of algebra of sets to simplify $(A \cup B)' \cap A$.

Solution

Given $(A \cup B)' \cap A$, then

$(A \cup B)' \cap A = (A' \cap B') \cap A$ De Morgan's law

$= A \cap (A' \cap B')$ Commutative law

$= (A \cap A') \cap B'$ Associative law

$= \varnothing \cap B'$ Complement law

$= \varnothing$ Identity law

Therefore, $(A \cup B)' \cap A = \varnothing$.

Example 1.28

By using the laws of algebra of sets, simplify $[A - (A \cap B')] \cup (A \cap B)$.

Solution

Given $[A - (A \cap B')] \cup (A \cap B)$, then

$[A - (A \cap B')] \cup (A \cap B) = [A \cap (A \cap B')'] \cup (A \cap B)$ since $A - B = A \cap B'$.

$= [A \cap (A' \cup B)] \cup (A \cap B)$ De Morgan's law

$= [(A \cap A') \cup (A \cap B)] \cup (A \cap B)$ Distributive law

$= [\varnothing \cup (A \cap B)] \cup (A \cap B)$ Complement law

$= (A \cap B) \cup (A \cap B)$ Identity law

$= A \cap B$ Idempotent law

Therefore, $[A - (A \cap B')] \cup (A \cap B) = A \cap B$.

Example 1.29

Use the laws of algebra of sets to simplify $A \cap (A \cup B)$.

Solution

Given $A \cap (A \cup B)$, then

$A \cap (A \cup B) = (A \cup \varnothing) \cap (A \cup B)$ Identity law

$= A \cup (\varnothing \cap B)$ Distributive law

$= A \cup \varnothing$ Identity law

$= A$ Identity law

Therefore, $A \cap (A \cup B) = A$.

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Exercise 1.5

1. By using the laws of the algebra of sets, prove that:
   (a) $(B \cap A) \cup (U \cap A) = A$
   (b) $(\varnothing \cup B) \cap (B \cup A) = B$

2. If A, B, and C are any three sets, show that:
   (a) $(C - A) \cup (B \cap A') = (C \cup B) - A$
   (b) $(A \cap B) \subseteq B$
   (c) $B \subseteq (A \cup B)$

3. Use the laws of the algebra of sets to simplify each of the following expressions:
   (a) $(A \cup B) \cap (A \cup C)$
   (b) $(A \cup B)' \cap (A \cap B)'$
   (c) $(\varnothing \cup A) \cap (B \cup A)$
   (d) $(A \cap B') \cap (B \cap A')$
   (e) $[A \cap (A \cap B)']'$
   (f) $(A - B) \cap B$
   (g) $(A \cup B') \cup [(A - C) \cup (B' - C)]$
   (h) $(P - M) - P$
   (i) $(A \cup B) \cap (A' \cup (A \cap B))$
   (j) $(A \cup B)' \cap (A \cup B) \cap (A \cup B)'$

4. Use the laws of the algebra of sets to prove each of the following identities.
   (a) $A - B = A - (A \cap B)$
   (b) $(A \cap C) - (B \cap C) = (A - B) \cap C$
   (c) $(A \cup B) = A \cup (B - A)$

5. Show each of the following identities by using the laws of the algebra of sets:
   (a) $A \cap B = A \cap (A' \cup B)$
   (b) $(A - B) \cap (A - C) = A - (B \cup C)$
   (c) $A = (A \cap B) \cup (A \cap B')$

Venn diagrams

Sets can be presented geometrically by drawings called Venn diagrams. Venn diagrams were introduced by the English Mathematician, John Venn. Venn diagrams consist of overlapping circles or ovals inscribed in a rectangle which represents a universal set. Each circle or oval represents a set. The points inside a circle represent elements of the set, while the points outside the boundary of the circle represent elements that are not in the set. Venn diagrams provide a quick way of showing relationships of the sets and thus, they are useful in presentations and reports.

For example, the sets, $U = \{a, b, c, d, e\}$ and $A = \{a, b, c\}$ can be presented in a Venn diagram as shown in Figure 1.1.

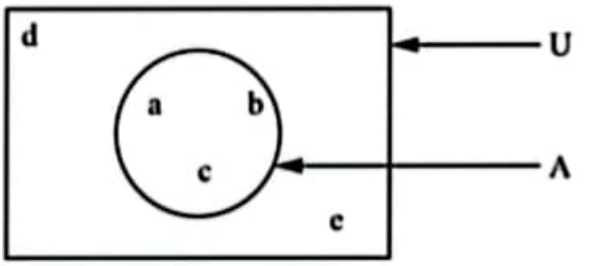

Figure 1.1: *Venn diagram*

In Figure 1.1, U is the universal set containing the elements a, b, c, d, and e while set A contains the elements a, b, and c. It is clearly seen that, A is a subset of U and A does not contain the elements d and e.

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If the sets have elements in common, then the circles overlap. Such sets are called joint sets. For example, if $A=\{a, b, c\}$ and $B=\{a, b, d\}$, then the relation betwen A and B is represented as in Figure 1.2.

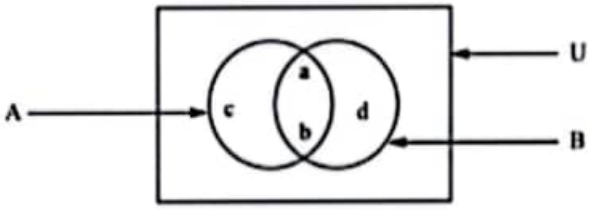

Figure 1.2: *Joint sets in a Venn diagram*

If the sets do not overlap in a Venn diagram, then the sets are disjoint. For example, if $A=\{a, b\}$ and $B=\{1, 2\}$, the relation between A and B is shown in Figure 1.3.

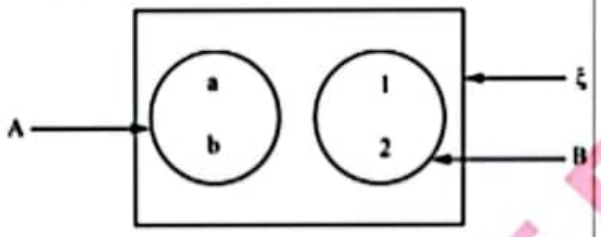

Figure 1.3: *Disjoint sets in a Venn diagram*

If all elements of set A are also elements of set B, then A is a subset of B.
For example, if $A = \{7, 8, 11\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$, then A is contained in set B or A is a proper subset of B as shown in the Venn diagram in Figure 1.4.

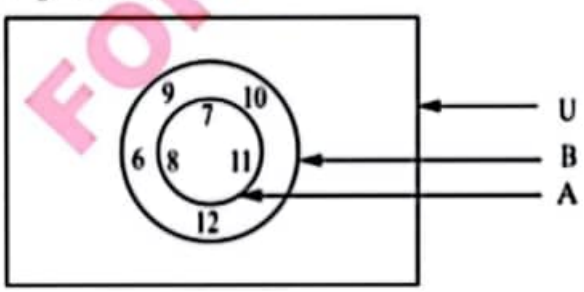

Figure 1.4: *Subsets in a Venn diagram*

Sets operations using Venn diagrams

A Venn diagram is designed in such a way that it depicts the set and how its subsets are related to each other. The number of circles and the way they intersect should depict the entire problem. The regions are then labelled according to the problem specifications. Usually, the region corresponding to a given set is shown by a shaded region.

Intersection of sets

The intersection of two or more sets is shown in a Venn diagram by shading the overlapping region of the given sets. The shaded region in Figure 1.5 shows the intersection of sets A and B.

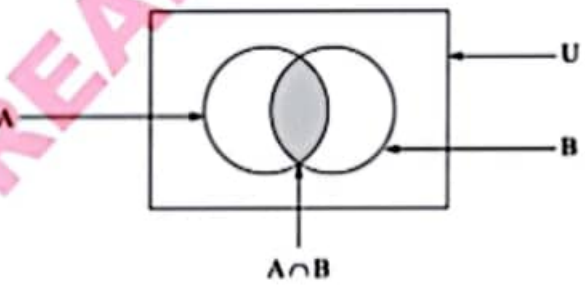

Figure 1.5: *A∩B is shaded*

Union of sets

The union of sets in a Venn diagram is represented by the combined regions of all sets under consideration. For example, in Figure 1.6, the union of sets A and B is the shaded region.

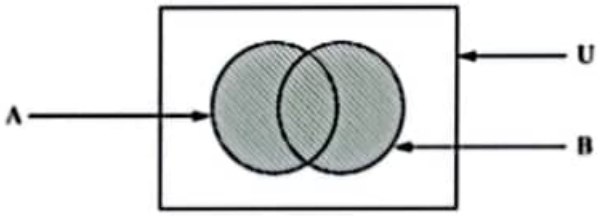

Figure 1.6: *A∪B is shaded*

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Symmetric difference of sets

The symmetric difference of two sets, A and B is also known as the disjunctive union set of elements which belong either to set A or B but not to both. In set notation, the symmetric difference of two sets, A and B is given by $A\Delta B=(A-B)\cup(B-A)$. This combination of sets is represented in a Venn diagram by the region that does not include the intersection of A and B. The shaded region in Figure 1.7 represents the symmetric difference of two sets, A and B.

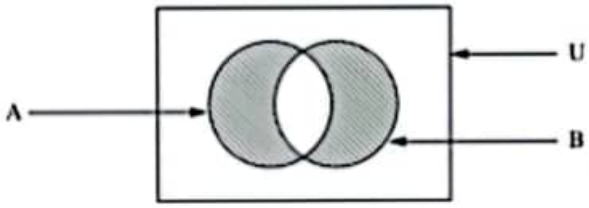


Figure 1.7: *$A\Delta B$ is shaded*

Relative difference

The relative difference of set A with respect to set B, that is $A-B$ is the region in A that does not include the elements in B. The relative difference of set A with respect to set B is the shaded region in Figure 1.8.

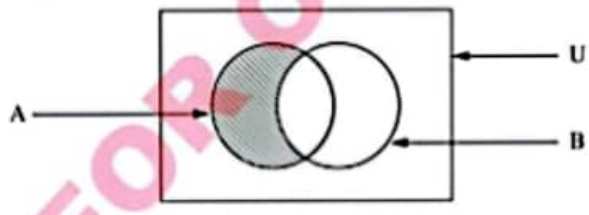


Figure 1.8: *$A-B$ is shaded*

Complement of a set

As it was described earlier, the complement of a set A, denoted by A' or A^c is the set containing the elements that are in the universal set but not in A. In the Venn diagram, A' is shown by the region outside A within the universal set as shown in Figure 1.9.

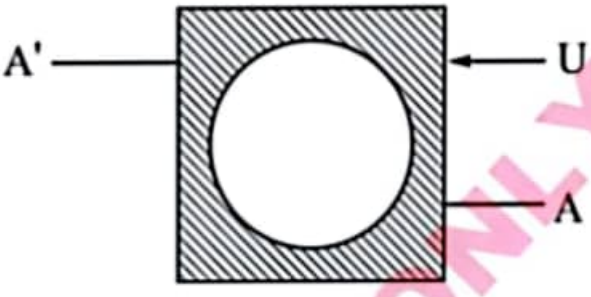


Figure 1.9: *A' is shaded*

Example 1.30

Represent each of the following set expressions in a Venn diagram:

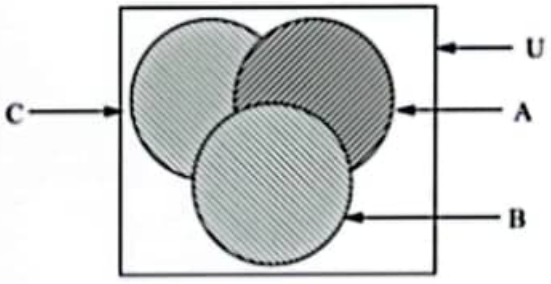
(a) $A\cap(B\cup C)$

(b) $(A\cup B)\cap(A\cup B')'$

Solution

(a) Given $A\cap(B\cup C)$.

First shade the region A and then the region $B\cup C$ as shown in the following figure.



The region representing the set $A\cap(B\cup C)$ is the intersection of the two shadings; that of A and that of $B\cup C$ as shown in the following figure.

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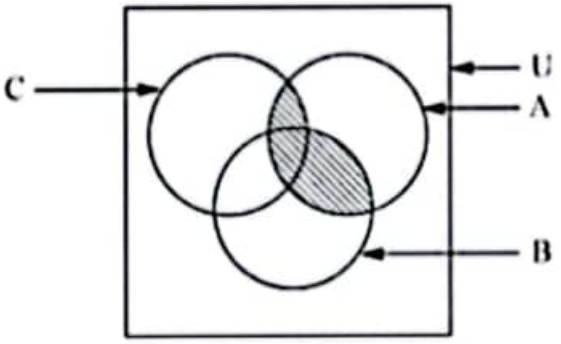
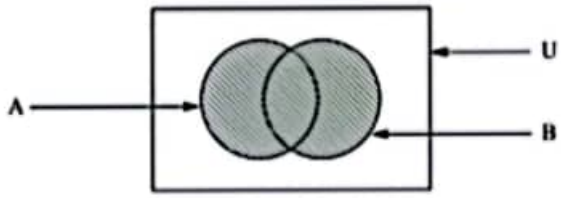


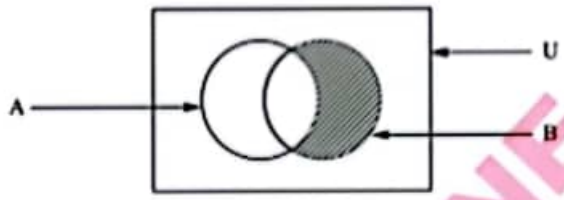
The shaded part of the Venn diagram represents $A \cap (B \cup C)$.

(b) Given $(A \cup B) \cap (A \cup B')$.

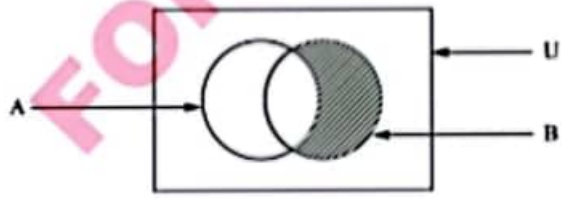
First, shade the region $A \cup B$ as shown in the following figure.



Next, shade the region $(A \cup B')'$ as shown in the following figure.



The shaded region represents $(A \cup B')'$. The region representing the set $(A \cup B) \cap (A \cup B')'$ is the intersection of the two shadings; that of $A \cup B$ and that of $(A \cup B')'$ as shown in the following figure.



The shaded region represents $(A \cup B) \cap (A \cup B')'$.

Exercise 1.6

Represent each of the following sets in Venn diagram:

1. $(A \cup B)' \cap (A \cup C)$
2. $(A \cup B)'$
3. $A' \cap B'$
4. $A \cap B \cap C$
5. $(A - B) \cup (A - C) \cup (B - C)$
6. $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
7. $(A \cup B)' \cap (A' \cap B')$
8. $(A \Delta B) \cap C$
9. $(A - B) \Delta C$
10. $A - (B \cap C)$

Number of elements/cardinality of a set

The cardinality of a set is a measure of its size, that is the number of elements in the set. Thus, if A is a finite set with n elements, then the cardinality of set A is n. Therefore, the cardinality of an empty set is zero. The cardinality of a combined set of two or more sets can be determined by using a general formula.

To derive the general formula, consider two arbitrary finite sets, A and B enclosed in a universal set as represented in Figure 1.10, where k, j, and p are the number of elements in the sets.

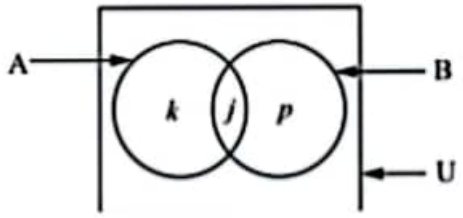


Figure 1.10: *Two sets enclosed sets in a universal set*

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From Figure 1.10, it follows that

$n(A) = k + j$

$n(B) = j + p$

$n(A \cap B) = j$

$n(A \cup B) = k + j + p$

Adding the cardinality of set A and set B gives:

$$n(A) + n(B) = (k + j) + (j + p)$$
$$= (k + j + p) + j$$
$$= n(A \cup B) + n(A \cap B)$$

Making $n(A \cup B)$ the subject gives,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Thus, if U is the universal set with subsets A and B, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This is a general formula commonly used in solving problems involving two sets.

Example 1.31

If $n(A) = 28$, $n(B) = 10$, and $n(A \cup B) = 33$, find $n(A \cap B)$.

Solution

Given $n(A) = 28$, $n(B) = 10$ and $n(A \cup B) = 33$.

From

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$,

$$\Rightarrow n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$= 28 + 10 - 33$$
$$= 5$$

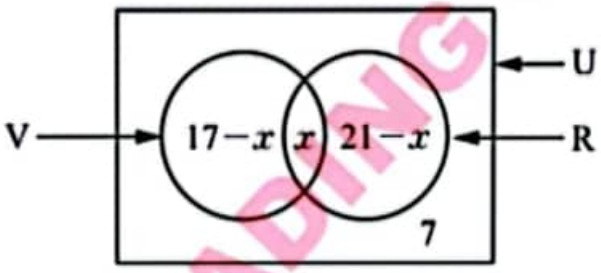
Therefore, $n(A \cap B) = 5$.

Example 1.32

There are 21 red flowers and 17 variegated flowers in a bouquet of 44 blooms. How many red and variegated flowers are there if 7 are neither red nor variegated?

Solution

Let R represents red flowers and V represent variegated flowers. The following figure shows the given information, where x is the number of elements in the intersection of the sets.



From the Venn diagram, it implies that

$n(V \cup R) + n(V \cup R)' = n(U)$.

But $n(V \cup R) = n(V) + n(R) - n(V \cap R)$.

$$\Rightarrow n(V) + n(R) - n(V \cap R) + n(V \cup R)' = n(U)$$
$$\Rightarrow (17 - x + x) + (21 - x + x) - x + 7 = 44$$
$$\Rightarrow x = 1$$

Therefore, 1 flower is both red and variegated.

Cardinality of union of three sets

The formula for the cardinality of union of two sets can be extended to suit problems involving three sets by using the laws of the algebra of sets. If A, B, and C are finite sets, then $A \cup B \cup C$ and $A \cap B \cap C$ are also finite. The cardinality of $A \cup B \cup C$ is given by;

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$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Proof

$n(A \cup B \cup C) = n[(A \cup B) \cup C]$; associative law

$= n(A \cup B) + n(C) - n[(A \cup B) \cap C]$

$= n(A) + n(B) - n(A \cap B) + n(C) - n[(A \cap C) \cup (B \cap C)]$

$= n(A) + n(B) - n(A \cap B) + n(C) - [n(A \cap C) + n(B \cap C)] + n[(A \cap C) \cap (B \cap C)]$

$= n(A) + n(B) - n(A \cap B) + n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

Therefore,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

Example 1.33

Given that $n(A) = 10$, $n(B) = 7$, $n(C) = 9$, $n(A \cap B) = 4$, $n(B \cap C) = 3$, $n(A \cap C) = 3$, and $n(A \cap B \cap C) = 1$, find $n(A \cup B \cup C)$.

Solution

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Substituting the values in the formula gives;

$$\begin{aligned} n(A \cup B \cup C) &= 10 + 7 + 9 - 4 - 3 - 3 + 1 \\ &= 27 - 10 \\ &= 17 \end{aligned}$$

Therefore, $n(A \cup B \cup C) = 17$.

Example 1.34

A factory has 45 men and 45 women workers. A survey showed that 60 workers in the factory were machine operators, 42 workers were cleaners, 30 of the men were operators, 30 of the men were cleaners, 24 of the men were both operators and cleaners, and 12 of the women were neither operators nor cleaners. How many workers in the factory were both operators and cleaners?

Solution

Let M and C represent the sets of men and women who are machine operators and cleaners, respectively, and x represents the number of women who are both machine operators and cleaners, then the information is summarized in the following figure:

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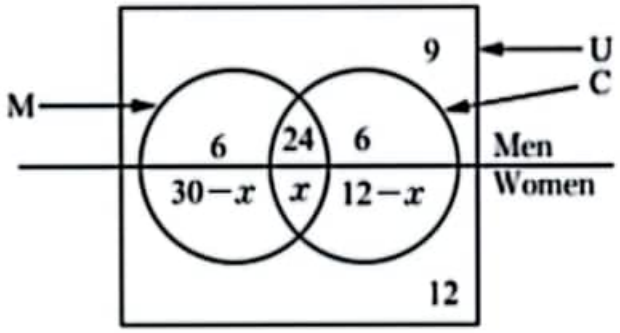


It is clear that,

$(30 - x) + x + (12 - x) + 12 = 45$

$30 + 12 + 12 - x = 45$

$54 - x = 45$

$x = 54 - 45$

Thus, $x = 9$.

Hence, $n(M \cap C) = 24 + 9 = 33$.

Therefore, the number of workers who were both cleaners and operators is 33.

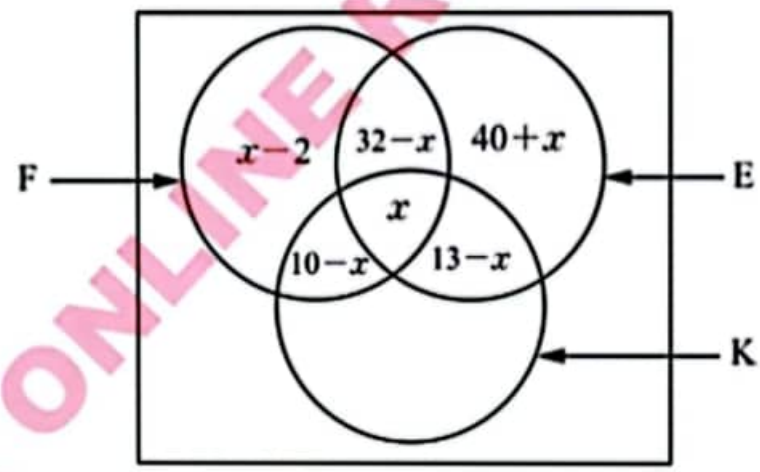
Example 1.35

In a certain city, 85% of the people speak English, 40% speak French, and 20% speak Kiswahili. Also, 32% speak English and French, 13% speak English and Kiswahili, and 10% speak French and Kiswahili. If every person speaks at least one language, find the percentage of people who speak all the three languages.

Solution

Let E, F, and K represent the sets of people who speak English, French, and Kiswahili, respectively.

Let x be the percentage of people who speak all three languages. The Venn diagram which summarizes the given information is shown as follows:



The total percentage is 100, that is $n(E \cup F \cup K) = 100\%$.

From the Venn diagram, it follows that

$100 = x - 2 + x - 3 + 40 + x + 10 - x + 13 - x + 32 - x + x$

$\Rightarrow 100 = 90 + x$

$\Rightarrow x = 10$

Therefore, the percentage of people who speak all the three languages is 10.

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Example 1.36

A survey of 300 summer TV shows customers found that, most customers watched at least one of the three types of TV shows, namely comedy, story, and action.

Let C represents the comedy TV shows customers, S represents the story TV shows customers, and A represents the action TV shows customers.

If $n(C) = 156$, $n(S) = 106$,

$n(A) = 133$, $n(C \cap S \cap A) = 8$,

$n(C \cap S) = 53$, $n(S \cap A) = 41$,

$n(C \cap A) = 87$:

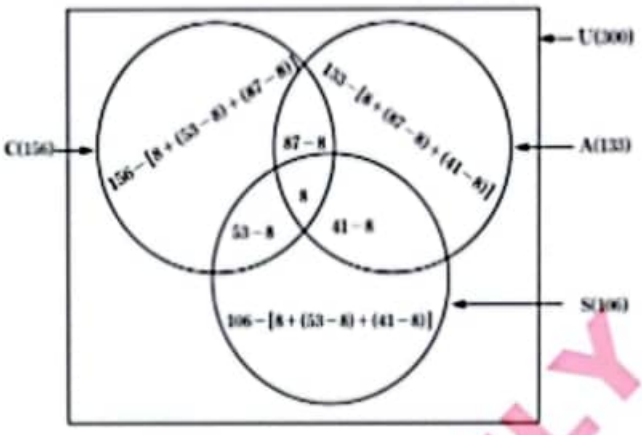
(a) How many customers did not watch comedy or stories, or action TV shows?

(b) How many customers watched comedy or action TV shows?

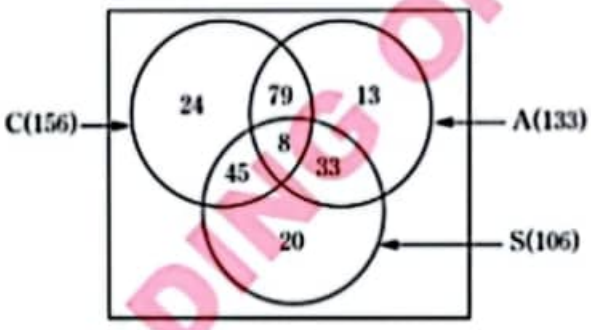
(c) How many customers watched story TV shows only?

Solution

The given information can be represented in the Venn diagram as follows:



Simplification yields



(a) Let U be the universal set, then the number of customers who did not watch comedy, stories, or action TV shows is given by;

$$n(A \cup S \cup C)' = n(U) - n(A \cup S \cup C)$$

$$= 300 - (20 + 45 + 8 + 33 + 13 + 79 + 24)$$

$$= 300 - 222$$

$$= 78$$

Therefore, 78 customers did not watch comedy, stories, or action TV shows.

(b) The number of customers who watched comedy or action TV shows is given by;

$$n(A \cup C) = 45 + 33 + 8 + 13 + 79 + 24$$

$$= 202$$

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Therefore, 202 customers watched comedy or action TV shows.

(c) Number of customers who watched story TV shows only is,

$n(S) - n(S \cap A) - n(S \cap C) - n(S \cap A \cap C)$

$= 106 - 33 - 45 - 8 = 20$

Therefore, 20 customers watched story TV shows only.

Example 1.37

In a certain class, there are 15 girls who like chocolate, 5 girls who are white, and 6 girls who are intelligent. Every white girl likes chocolate, 3 intelligent girls do not like chocolate. If 2 girls are both white and intelligent,

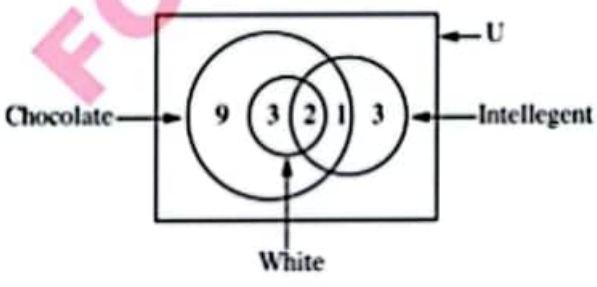
(a) present the above information in a Venn diagram,

(b) find the number of girls in the class,

(c) find the number of girls who are white but not intelligent.

Solution

(a) The given information can be represented in a Venn diagram as follows:



(b) Number of girls in the class

$= 9 + 3 + 2 + 1 + 3$

$= 18$

Therefore, the number of girls in the class is 18.

(c) 3 girls are white but not inteligent.

Example 1.38

In a certain school, there are 20 students who are girls and 20 who are boys. There are 15 students who study Mathematics, 16 students who study Chemistry. 10 girls study Mathematics and 11 boys study Chemistry. If 8 girls and 7 boys study neither Mathematics nor chemistry, find the number of students who study:

(a) Both subjects,

(b) Mathematics but not Chemistry,

(c) Only one subject,

(d) At least one subject.

Solution

Let: C represent the students who study Chemistry

M represent the students who study Mathematics

Given that,

Number of girls = 20

Number of boys = 20

$n(M) = 15$

$n(C) = 16$

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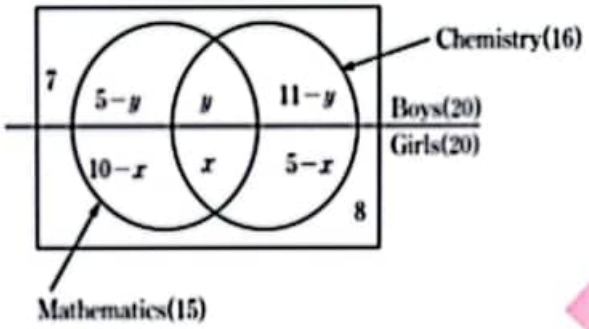
Number of girls who study Mathematics =10
Number of boys who study Chemistry = 11
8 girls study neither subject
7 boys study neither subject.

Let y be the number of boys who study both Mathematics and Chemistry and be the number of girls who study both Mathematics and Chemistry.

The given information can be represented in a Venn diagram as follows:



Using the Venn diagram, the value of y is obtained by equating the sum of all members to the total number of boys; that is,

$5-y+y+11-y+7=20$

$\Rightarrow 23-y=20$

$\Rightarrow y=23-20$

Thus, $y=3$.

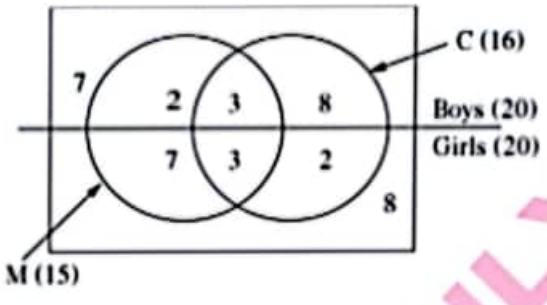
From the Venn diagram, the value of x is obtained by equating the sum of all members to the total number of girls; that is,

$10-x+x+5-x+8=20$

$\Rightarrow 23-x=20$

$\Rightarrow x=23-20$

$\Rightarrow x=3$



Using the Venn diagram, it follows that:

(a) $n(M\cap C)=3+3=6$ students
Therefore, 6 students study both subjects.

(b) $n(M\cap C')=2+7=9$ students
Therefore, 9 students study Mathematics but not Chemistry.

(c) Number of students who study only one subject
$=n(M\cap C')+n(M'\cap C)+n(M\cap C)$
$=2+7+8+2=19$ students
Therefore, 19 students study only one subject.

(d) Number of students who study at least one subject
$=n(M\cap C')+n(M'\cap C)+n(M\cap C)$
$=2+7+3+3+8+2$
$=25$ students
Therefore, 25 students study at least one subject.

Exercise 1.7

In questions 1 to 5, find:

(a) $n(A\cap B)$ (c) $n(A\cup B)$
(b) $n(A\Delta B)$ (d) $n(A-B)$

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1. $A = \{6, 7, 4, 3\}$ and $B = \{3, 7, 4, 10\}$.
2. $A = \{x : 3 \leq x \leq 10\}$ and $B = \{x : x \leq 6\}$; where $x \in \mathbb{N}$.
3. $A = \{0\}$ and $B = \{5, 10\}$.
4. $A = \{6\}$ and $B = \{4, 14, 6\}$.
5. $B = \{6\}$ and $A = \{14, 6\}$.
6. Given that $n(A \cup B)' = 63$, $n(A') = 30$, and $n(B') = 40$, find $n(A \cap B)'$.
7. Given that $n(A \cap B) = 50$, $n(A \cap B') = 100$, and $n(A' \cap B) = 50$. If A and B are the only subsets of the universal set, find:
   (a) n(A) (b) n(B)
8. In a certain hospital, 150 patients were diagnosed to be suffering from at least one disease as follows: 93 had Covid-19, 90 had Malaria, 64 had Tuberculosis, 32 had Tuberculosis and Covid-19, 60 had Covid-19 and Malaria, 25 had Malaria and Tuberculosis. Find the number of patients who had all the three diseases.
9. A survey on the type of crops grown in a certain village revealed that, out of 210 families, 140 grow rice, 75 grow maize, 48 grow rice and maize only, 22 grow millet and maize, 14 grow rice and millet only. The number of families who grow rice only is twice the number of those who grow millet only and 7 families grow none of the crops. Determine the number of families growing;
   (a) all three crops
   (c) at most one crop
   (b) exactly one crop
   (d) rice or maize but not millet.
10. Given that $A = \{x \in \mathbb{R} : x \leq -1\}$, $B = \{x \in \mathbb{R} : -2 \leq x \leq 3\}$, and $C = \{x \in \mathbb{R} : x \geq -2\}$. Represent on a number line the solution which defines each of the following sets:
    (a) $A \cap B$
    (b) $A \cap B \cap C$
    (c) $(A \cap B \cap C)'$
    (d) $(A \cap B') \cup C'$

Chapter summary

1. A set is a collection of objects which have common properties.
2. Sets can be presented in various forms. The common forms are descriptive or statement, roster or listing, and set builder form.
3. There are different types of sets, namely; the empty set or null set, equal sets, equivalent sets, finite and infinite sets, singleton sets, power sets, proper and improper subsets, and universal sets.

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4. The basic operations of sets include union of sets, intersection of sets, complement of sets, and difference of sets.

5. The common laws of the algebra of sets are; Identity law, idempotent law, double complement law, commutative law, associative law, De Morgan's law, absorption law, distributive law, and complement law.

6. If A and B are non empty sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

7. If A, B, and C are non empty sets, then
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

8. The symmetric difference of two sets, A and B is defined by,
$A \Delta B = (A - B) \cup (B - A)$.

Revision exercise 1

1. Let $A = \{0, 4, 8, 12, 16\}$, $B = \{0, 2, 4, 6, 8\}$, and $C = \{0, 6, 12, 18\}$. Find each of the following:

(a) $A \cup B \cup C$ (c) (i) $n(A \cup B \cup C)$ (ii) $n(A \cap B \cap C)$

(b) $A \cap B \cap C$

2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 9, 10\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, verify that:

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) $n(A \cap B)' = n(A \cap B') + n(B \cap A') - n(A' \cup B')' + n(A')$.

3. How many subsets does each of the following sets have?

(a) $\{1, 2, 3\}$ (b) $\{6, 7, 8, 9\}$ (c) $\{10, 12, 14, 16, 18, 20\}$

4. (a) Represent each of the following sets in a Venn diagram:

(i) $B - A'$ (iv) $A \cap B \cap C'$ (vii) $A' \cap (B \cup C)$

(ii) $(A \cap B) \cup (B' \cap C)$ (v) $(A \cup B) \cap C'$ (viii) $A' \cap B' \cap C$

(iii) $A \cap B' \cap C$ (vi) $A' \cap B' \cap C'$ (ix) $A \cap (B' \cup C)$

(b) Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by using Venn diagrams.

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5. If $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{d, e, f, g\}$, find each of the following sets:

(a) $B - C$ (b) $A - C$ (c) $A - B$ (d) $C - B$ (e) $B - A$ (f) $C - A$

6. Given that $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 2, 4, 7, 6\}$, and $C = \{2, 5, 6\}$, verify that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

7. Determine whether each of the following is true or false:

(a) $A \Delta B = B \Delta C$

(b) $(A \Delta B) \Delta B = A$

(c) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(d) $A \Delta C = (A \cup C) - (A \cap C)$

(e) $A \Delta C = (A - C) \cup (B - A)$

(f) $(A - B) - C = A - (B - C)$

(g) $A - C = C - B$

(h) $(A - C) \Delta (B - C) = B$

8. (a) Let E be the set of all positive integers less than 50, and A and B be subsets of E such that:

$A = \{$Elements of E which are multiples of 5$\}$.

$B = \{$Elements of E wich are multiples of 4$\}$.

(i) List all the elements of A, B, $A \cap B$, and $A \cup B$.

(ii) Describe in words the members of $A \cap B$.

(iii) Find n(A), n(B), and $n(A \cap B)$.

(b) Find the elements of sets A, B, and C given that

$A \cup B = \{p, q, r, s\}$, $A \cup C = \{q, r, s, t\}$, $A \cap B = \{q, r\}$, and $A \cap C = \{q, s\}$.

9. Given $A = \{x : x \in \mathbb{R}, x \geq 2\}$ and $B = \{x : x \in \mathbb{R}, -2 < x \leq 5\}$, represent the solution of each of the following sets on a number line.

(a) $A' \cup B$ (b) $A' \cap B$ (c) $(A \cap B)'$ (d) $A - B$

10. By using the laws of the algebra of sets, simplify each of the following expressions:

(a) $A \cup (A \cap B')'$

(b) $(A - B) - (A \cap B')$

(c) $A \cap (A' \cap B)'$

(d) $B \cap (B \cup A)$

(e) $(A \cup B) \Delta B'$

(f) $A \cap (A' \cup B')$

(g) $\left(((A \cup B) \cap C)' \cup B'\right)' \cup C$

(h) $(X \cap Y') \cup (X \cup Y')$

(i) $[A - (B - A)] - [A \cup (B \cap A)]$

(j) $(A' \cap B) \Delta A$

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11. (a) Use appropriate laws of the algebra of sets to simplify $A \cap (A \cup B)'$.

(b) If A, B, and C are equivalent sets of which A and C are disjoint sets, prove that $n(A \cup B \cup C) = 3n(A) - n(A \cap B) - n(B \cap C)$.

12. If $U = \{1, 2, 3, ..., 10\}$, and $B = \{2, 4, 6\}$, show that:

(a) $B \Delta B = \varnothing$ (b) $B \Delta U = B'$ (c) $B \Delta \varnothing = B$ (d) $B \Delta B' = U$

13. If A, B, and C are any three sets, prove that:

(a) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

(b) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

(c) $U \cap A \subseteq (A \cup B \cup C)$

(d) $[(A \cap B') - C] \subseteq A - C$

(e) $(A \cap C') \cap (C \cap B') = \varnothing$

(f) $(B \cap A') \cup (C - A) = (B \cup C) \cap A'$

(g) $A \cap B' \subseteq A$

(h) $(B \cap C') \cup (B' \cup C)' = B \cap C'$

(i) $(A \cap B \cap C)' = A' \cup B' \cup C'$

14. Out of 880 boys in a school, 224 play cricket, 240 play hockey, and 336 play basketball. Of the total, 64 play basketball and hockey, 80 play cricket and basketball, and 40 play cricket and hockey, 24 boys play all the three games. Determine the number of boys who;

(a) do not play any game,
(b) play only one game,
(c) play cricket or hockey but not basketball,
(d) play basketball and hockey but not cricket.

15. A class contains 15 boys and 15 girls. There are 20 students who take Science and 14 students take Mathematics. Of the boys, 10 take Science and 10 take Mathematics, 8 of the boys take both Science and Mathematics, and 4 of the girls take neither Science nor Mathematics. Find the number of students in the class who take;

(a) both Science and Mathematics,
(b) Mathematics but not Science,
(c) exactly one subject,
(d) neither of the subjects.

16. In a certain high school, 100 students were asked to mention the foreign languages they learn. Their responses showed that, 45 students learn Spanish, 28 learn French, 22 learn Chinese, 12 students learn Spanish and French, 8 learn Spanish

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and Chinese, 10 learn French and Chinese, and 30 students do not learn any language. Find the number of students who learn;
(a) all the three languages,
(b) exactly one of the three languages,
(c) at most two of the three languages.

17. In an examination, each student passed at least one of the three subjects. It is given that, 40% of the students passed in Mathematics, 45% in Chemistry, and 55% in Physics. If 10% passed in Mathematics and Chemistry, 20% in Chemistry and Physics, and 15% in Physics and Mathematics, determine the pass percentages in;
(a) all the three subjects
(b) at least two subjects,
(c) Chemistry or Physics but not Mathematics.

18. The spare parts manufactured by a certain factory were subjected to three types of defects A, B, and C. A sample of 4,000 items were inspected and it was found that 6.2% had defect A, 7.4% had defect B, 8.2% had defect C, 2.2% had defects A and B, 2.6% had defects B and C, 2.0% had defects A and C, 1.2% had all the three defects. Find the percentage of items which had,
(a) none of the defects,
(b) at least one of the defects,
(c) not more than one defect.

19. Out of 35 students in a certain school, 2 students study Physics, Chemistry and Mathematics. It is given that 6 of them study only Physics and Chemistry, 5 students study only Physics and Mathematics, and 4 students study only Chemistry and Mathematics. The number of students who study only one subject is the same for all the three subjects. If every student studies at least one of the three subjects, find the number of students who study:
(a) Mathematics only.
(b) Physics or Chemistry but not Mathematics.
(c) At most two subjects.

20. Given two sets, A and B such that $n(A) = 10$ and $n(B) = 15$, find:
(a) The maximum and minimum values of $n(A \cup B)$,
(b) The maximum and minimum values of $n(A \cap B)$.

21. In a group of 72 students who were surveyed, it was found that each student liked at least one fruit as follows: 29 students liked mangoes, 41 liked bananas, 48 liked oranges, 22 liked oranges and bananas, 16 liked bananas

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and mangoes, and 20 liked oranges and mangoes. Find the number of students who liked,

(a) mangoes but not bananas or oranges,

(b) all the three fruits,

(c) oranges or bananas but not mangoes,

(d) exactly two fruits.

22. Simplify each of the following expressions by using the laws of the algebra of sets:

(a) $(A \cup B) - (A - B)$

(b) $(A' \cap B') \cap (A \cap B)'$

(c) $A \cap (A \cup B) - (A - B)$

(d) $(A \cap (A' \cup B)) \cup (B \cap (A' \cup B'))$

23. (a) If A is a subset of B and B is a subset of C, prove that A is a subset of C.

(b) Given that

$A = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$,

$B = \{x \in \mathbb{R} : x > 3\}$, and

$C = \{x \in \mathbb{R} : x > 3\}$, find the expression for each of the following sets:

(i) $(A' \cup B') \cap C$

(ii) $A' \cap B \cap C$

24. (a) Use a Venn diagram to express each of the following sets:

(i) $(B - A)'$

(ii) $(A \cap B) \cup (B' \cap C)$

(iii) $(A \cap B' \cap C) \cup (A' \cap B \cap C')$

(b) By using the laws of the algebra of sets, simplify $(A - B)' \cap (A \cup B)$.

25. In a certain teachers' college, there are 110 trainees taking Economics (E), History (H), and Geography (G), where $n(E \cup H) = 70$, $n(H \cup G) = 80$, $n(G \cup E) = 85$, $n(H) = 40$, $n(G) = 55$, $(H \cap E \cap G) = 5$, and $n(E) = 50$.

Find the number of trainees who take;

(a) History and Geography

(b) History and Geography but not Economics

(c) History or Geography but not Economics

(d) exactly two subjects

26. In a certain village, there are 21 boys who are football fans, 7 boys like music, and 9 boys are cricket fans. All boys who like music are also football fans and 4 boys who are cricket fans are not football fans. If 3 boys like music and are cricket fans,

(a) present this information in a Venn diagram,

(b) determine the number of boys who are football or cricket fans.

(c) find the number of boys who like music but are not cricket fans.

(d) find the number of boys who are both football and cricket fans.

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Chapter Two

Introduction

The term "logic" originates from the Greek word "Logos" meaning idea, word, thought, argument, account, reason, or principle. Logic can be defined as a science or an art which deals with the study of truth, principles of correct reasoning, and making good decisions. In this chapter, you will learn about statements, logical connectives, laws of the algebra of propositions, validity of arguments, and electrical networks. The competencies developed have a number of applications in real life situations such as to distinguish between valid and invalid arguments, correct reasoning and making proper decisions in daily life activities, constructions of circuit diagrams in the field of electronics, and making judgements in the field of law, among many other applications.

Think

Solving problems without logical reasoning.

Concept of logic

Logic in real life refers to correct reasoning. In mathematics, it is the study of truth and how the truth value of a statement or proposition can be obtained from mathematical deduction. In logic, the interest is not on the statements themselves, but how true and false statements are related to each other. In most cases, propositions are represented by letters like p, q, r, s, and so on. Propositions have to be either true (T) or false (F) but not both.

Statements

In mathematics, a statement or proposition can be defined as a declarative sentence which is either "true" or "false" but not both. For example, "The sum of interior angles of a rectangle is 180° ". The truth value of this statement is false.

The following sentences are statements:

(i) $6+2=9$

(ii) $3+2=5$

(iii) Tanga is the capital of Tanzania.

(iv) Lusaka is the largest city in Tanzania.

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The truth value of the statement in (ii) is true, whereas the truth value of statements in (i), (iii), and (iv) is false.

Note that, not all sentences are statements. For instance;

(i) What is your name?
(ii) Good night daughter.
(iii) $x-7=12$
(iv) $2x+y=5z$

The sentences in (i) and (ii) are not statements because they are not declarative sentences, and the sentences in (iii) and (iv) are not statements because they are neither true nor false. Activity 2.1 strengthens the knowledge on mathematical statements.

Activity 2.1: Identifying a statement

Individually or in a group, perform the following tasks:

1. Construct any six sentences of your choice.
2. From the sentences constructed in task 1, identify mathematical and non-mathematical statements.
3. Share the results you obtained in task 2 with other students.
4. Discuss with your fellow students how to differentiate between a mathematical and a non-mathematical statements.
5. Discuss the challenges you faced in task 2.

Sentences which involve exclamative words, questioning sentences, and instructional sentences are not mathematical sentences. If a sentence mentions a particular place or a particular person it becomes a statement. For instance, "Is Mount Kilimanjaro in Tanzania?" is not a statement, but "Mount Kilimanjaro is in Tanzania" is a statement. In writing mathematical statements, the letters $p, q, r, s, \ldots$ are usually used to denote propositions in order to reduce the requirement of writing long sentences. The truth value of a true proposition is denoted by T and the truth value of a false proposition is denoted by F.

Types of mathematical statements

There are two types of mathematical statements, namely; simple mathematical statements and compound mathematical statements. A simple mathematical statement is formed by one declarative sentence that is either true or false or is a statement which cannot be brocken into two or more statements. For instance, the statement "Asha is swimming" is a simple statement, while a compound statement is formed by more than one declarative sentence. The statement "If Elia studies hard, then he will pass the test" is a compound statement, since it is formed by two simple statements which are "Elia studies hard" and "he will pass the test".

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Truth table

A truth table is a table that is used to show the validity of a compound statement depending on the truth values of its simple statements. The number of cases that describe the given compound statement depends on the number n of propositions. For instance, if the compound statement has 1 proposition $(n = 1)$, then the number of cases is 2. If it has 2 propositions $(n = 2)$, then the number of cases is $2^2 = 4$, and if it has 3 propositions $(n = 3)$, then the number of cases is $2^3 = 8$.

Generally, if there are n propositions, then the number of cases is 2^n.

Table 2.1 shows the truth tables of the statement formed by 1, 2, and 3 propositions.

Table 2.1(a): *Number of cases for one proposition p*

	p
Case 1	T
Case 2	F

Table 2.1(b): *Number of cases for two propositions, p and q*

	p	q
Case 1	T	T
Case 2	T	F
Case 3	F	T
Case 4	F	F

Table 2.1(c): *Number of cases for three propositions p, q, and r*

	p	q	r
Case 1	T	T	T
Case 2	T	T	F
Case 3	T	F	T
Case 4	T	F	F
Case 5	F	T	T
Case 6	F	T	F
Case 7	F	F	T
Case 8	F	F	F

Negation of a statement

Negation of a statement is a mathematical statement that has the opposite truth value from the original statement. For instance, the statement "It is not p" is called the negation of p, mathematically it is written as "$\sim p$" and it is read as "negation of p". One way of writing a negation statement is to put the word "not" with the verb. Another way is to precede it with the phrase "it is not true that" or "it is false that", "it is not the case that". For instance, the negation of the proposition "Today is Friday" is "Today is not Friday" or "It is not true that Today is Friday" or "It is false that Today is Friday". Similarly, the negation of the proposition "Issa does not love Rita" is "Issa loves Rita".

Example 2.1

Draw a truth table of a mathematical statement $\sim p$.

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Solution

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

Exercise 2.1

1. Which of the following sentences are statements?
   (a) $26-5=21$.
   (b) $7(8\sqrt{2})+8=36$.
   (c) A square has all its sides equal.
   (d) Will you come tommorow?
   (e) Given that PQR is a right-angled triangle with a right angle at vertex Q and if M is the midpoint of the hypotenuse, then the line segment connecting vertex Q to M is half the length of the hypotenuse.
2. State which of the following sentences is a simple or compound statement:
   (a) Five is less than eight.
   (b) John eats rice or meat.
   (c) Dodoma is the capital of Tanzania.
   (d) Juma is singing and Thomas is reading a novel.
   (e) Ester will pass the examination if and only if she studies hard.
3. Draw the truth table whose number of propositions is:
   (a) $n=3$
   (b) $n=4$
4. Write the negation of each of the following propositions.
   (a) Pili is not a woman.
   (b) It is raining now.
   (c) Tanzanians maintain their good culture.
   (d) Summer comes after spring.
   (e) Industries are friendly to the environment.
   (f) Tomorrow is Saturday.

Logical connectives

There are many ways of combining simple statements to form compound statement. Logical connectives are symbols used to connect two or more propositions to form a compound statement. Logical connectives include conjunction, disjunction, conditional (implication), and biconditional (double implications).

Conjunction

In mathematics, a conjunction is a statement formed by connecting two logical statements p and q with a connector "and", symbolically denoted by a wedge symbol " $\wedge$ " and it reads as "and". When the values of two propositions p and q are combined together by the connector, it is expressed symbolically as $p \wedge q$ and it is read "p and

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q". A conjunction has truth value true if and only if both of its statements are true, otherwise it is false. In other words, a conjunction of two statements is true only when each sub-statement is true. The truth table for $p \wedge q$ is shown in Table 2.2.

Table 2.2: *Truth table for* $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.2

Write the conjunction of the propositions p and q, where p and are the propositions "Halima likes cold coffee" and "Alex likes milkshake", respectively. Hence, comment on its truth value.

Solution

Given the following propositions:

p: Halima likes cold coffee.

q: Alex likes milkshake.

The conjunction is written as $p \wedge q$. The proposition $p \wedge q$ is a compound statement "Halima likes cold coffee and Alex likes milkshake". The formed compound statement $p \wedge q$ is true when p and q are both true, otherwise it is false.

Example 2.3

Write the conjunction of the propositions p and q, where p is the proposition "Mwanaidi likes singing" and q is the propositon "Tumaini likes swimming". Hence, comment on its truth value.

Solution

Given the following propositions:

p: Mwanaidi likes singing.

q: Tumaini likes swimming.

The conjunction is given as $p \wedge q$. The conjunction $p \wedge q$ is a compound statement, "Mwanaidi likes singing and Tumaini likes swimming". The compound statement $p \wedge q$ is true if and only if p and q are both true, otherwise it is false.

Example 2.4

Determine the truth value in each of the following statements:

(a) $8+25=43$ and $72\sqrt{8}\leq 9$.

(b) 2 is a prime number but 4 is not a prime number.

(c) Dodoma is the capital of Tanzania and it is false that Mwanza is a city in Tanzania.

Solution

(a) Let p: $8+25=43$.

q: $72\sqrt{8}\leq 9$.

The truth value of $8+25=43$ is false(F) and the truth value of

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$72\sqrt{8} \leq 9$ is true (T), then the truth value of $p \wedge q$ is $F \wedge T = F$.

Therefore, the truth value of $p \wedge q$ is false.

(b) Let p: 2 is a prime number.
q: 4 is not a prime number.

The truth value of "2 is a prime number" is true (T) and the truth value of "4 is not a prime number" is true (T), then the truth value of $p \wedge q$ is $T \wedge T = T$.

Therefore, the truth value of $p \wedge q$ is true.

(c) Let p: Dodoma is the capital of Tanzania.
q: Mwanza is a city in Tanzania.

Thus, $\sim q$ is the statement "It is false that Mwanza is a city in Tanzania". The the truth value of p is true (T) and the truth value of $\sim q$ is false (F), then the truth value of $p \wedge \sim q = T \wedge F = F$.

Therefore, the truth value of $p \wedge \sim q$ is false.

Example 2.5

Write the conjunction of the propositions p and q, when p is the proposition "Subira likes novels" and q is a proposition "Jack likes movies". Hence, comment on its truth value.

Solution

Given the following:

p : Subira likes novels.
q : Jack likes movies.

The conjunction is written as $p \wedge q$.

The proposition $p \wedge q$ is the compound statement "Subira likes novels but Jack likes movies". This compound statement is true when p and q are true, otherwise it is false.

Disjunction

Disjunction is a compound statement which is formulated by using a connector "or" and it is denoted by inverted wedge symbol "$\vee$". When two statements p or q are disjoined together, the combination is symbolically expressed as "$p \vee q$". The truth value of a disjunction is false if and only if both statements are false, otherwise it is true. The truth table for $p \vee q$ is as shown in Table 2.3.

Table 2.3: *Truth table for* $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2.6

If p is the proposition "John eats beans" and q is the proposition "John

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eats meat", write the statement in each of the following:

(a) $p \vee q$ (b) $\sim(p \vee q)$

Solution

(a) "John eats beans or meat".

(b) "It is false that John eats beans or meat".

Example 2.7

If p is the proposition "A giraffe eats meat" and q is the proposition "A lion eats meat", write the truth value in each of the following statements:

(a) $p \vee q$ (c) $\sim(p \wedge q)$

(b) $p \wedge q$ (d) $\sim(p \vee q)$

Solution

Given p is the proposition "A giraffe eats meat", its truth value is false (F), and q is the proposition "A lion eats meat" and its truth value is true (T), then

(a) The truth value of $p \vee q$ is

$F \vee T = T$.

Therefore, the truth value of $p \vee q$ is true.

(b) The truth value of $p \wedge q$ is

$F \wedge T = F$.

Therefore, the truth value of $p \wedge q$ is false.

(c) The truth value of $\sim(p \wedge q)$ is

$\sim(F) = T$.

Therefore, the truth value of $\sim(p \wedge q)$ is true.

(d) The truth value of $\sim(p \vee q)$ is

$\sim(T) = F$.

Therefore, the truth value is false.

Conditional statements (implications)

A relationship between two statements in which the second statement is a logical consequence of the first statement is referred to as a conditional or implication statement. If p and q are propositions, then the implication is written as "$p \rightarrow q$", and its truth value is false when p is true and q is false, otherwise it is true. The proposition "$p \rightarrow q$" is read as p implies q, if p then q, or sometimes, it is read as q if p. In this case, p is called the hypothesis (or antecedent) and q is called the conclusion (or consequent). The truth table of $p \rightarrow q$ is as shown in Table 2.4.

Table 2.4: *Truth table for* $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The following phrases are used to represent the condition statement, "$p \rightarrow q$".

(i) "If p then q"

(ii) "p is sufficient for q"

(iii) "If p, q"

(iv) "a necessary condition for p is q"

(v) "q if p"

(vi) "q when p"

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(vii) "p implies q"

(viii) "p only if q"

(ix) "q is necessary for p"

(x) "q whenever p"

(xi) "q follows from p"

(xii) q provided that p

(xiii) q unless $\sim p$.

Example 2.8

Write each of the following statements in symbolic form:

(a) If the car is gone, then Leah has left.

(b) If you get a degree, then you can get a job.

Solution

(a) Let p : The car is gone.

q : Leah has left.

Therefore, $p \rightarrow q$.

(b) Let p : You get a degree.

q : You can get a job.

Therefore, $p \rightarrow q$.

Example 2.9

Let p: Angel goes abroad.

q: Angel has a passport.

Write down a verbal sentence for each of the following:

(a) $p \rightarrow \sim q$

(b) $\sim p \rightarrow \sim q$

(c) $\sim p \rightarrow q$

Solution

(a) If Angel goes abroad, then she does not have a passport.

(b) If Angel does not go abroad, then she does not have a passport.

(c) If Angel does not go abroad, then she has a passport.

Biconditional statements

A biconditional statement is a statement which is written in the form "p if and only if q". It is denoted by a double-headed arrow "$\leftrightarrow$". If p and q are propositions, then the biconditional proposition is written as "$p \leftrightarrow q$" which is read as "p if and only if q". A biconditional statement is said to be true if and only if both parts have the same truth value, otherwise it is false. The truth table of $p \leftrightarrow q$ is as shown in Table 2.5.

Table 2.5: *Truth table for* $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Also, the truth value of $p \leftrightarrow q$ is true if the implication $(p \rightarrow q)$ is true and $(q \rightarrow p)$ is also true. The truth value of the statement $(p \rightarrow q) \wedge (q \rightarrow p)$ in Table 2.6 is the same as that of $p \leftrightarrow q$ shown in Table 2.5.

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Table 2.6: *Truth table for* $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that, if p and q have the same truth value, then $p \leftrightarrow q$ is true, if p and q have opposite truth values, then $p \leftrightarrow q$ is false. The following phrases are used to represent the biconditional statement, "$p \leftrightarrow q$":

(i) "q if and only if p"
(ii) "p if and only if q"
(iii) "p is a necessary and sufficient condition for q"
(iv) "If p then q and conversely"
(v) "p double implication q"
(vi) "p implies q and q implies p"

Example 2.10

Write each of the following statements in symbolic form:

(a) You will pass the examination if and only if you work hard.
(b) I will give you my car if and only if you know how to drive.

Solution

(a) Let p : You will pass the examination.
q : You will work hard.
Therefore, $p \leftrightarrow q$.

(b) Let p : I will give you my car.
q : You know how to drive.

Therefore, $p \leftrightarrow q$.

Example 2.11

Determine the truth value in each of the following statements:

(a) $2+9=12$ if and only if 2 is a factor of 12.
(b) It is not true that $6+14=20$ if and only if $18 \times 5 = 90$.

Solution

(a) Let p : $2+9=12$.
q : 2 is a factor of 12.
The truth value of p is false (F) and the truth value of q is true (T).
Thus, the truth value of $p \leftrightarrow q$ is $F \leftrightarrow T = F$.
Therefore, the truth value of $p \leftrightarrow q$ is false.

(b) Let p : $6 + 14 = 20$.
q : $18 \times 5 = 90$.
The truth value of p is true (T) and the truth value of q is true (T).
The truth value of $p \leftrightarrow q$ is $T \leftrightarrow T = T$.
Thus, the truth value of $\sim(p \leftrightarrow q)$ is $\sim(T) = F$.
Therefore, the truth value of $\sim(p \leftrightarrow q)$ is false.

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Exercise 2.2

1. Determine the truth value in each of the following statements:
   (a) Paris is in France and Mombasa is in Uganda.
   (b) $6 < 9$ if and only if $-6 \div 2 > -2$.
   (c) If $2+4=6$ and $6-5<1$, then $\sqrt{71} < 8.5$.
2. Let p: She is tall and q: She is beautiful. Write each of the following statements in symbolic form:
   (a) She is tall and beautiful.
   (b) She is tall but not beautiful.
   (c) It is false that she is short or beautiful.
   (d) She is neither tall nor beautiful.
   (e) She is tall or she is short and beautiful.
3. Construct a truth table for the proposition
   $(\sim p \leftrightarrow \sim q) \leftrightarrow \sim (p \rightarrow q)$.
4. Let p: It is cold and q: It rains. Write each of the following statements in symbolic form:
   (a) A necessary condition for it to be cold is that it rains.
   (b) A sufficient condition for it to be cold is that it rains.
   (c) It is not true that it is not cold if and only if it does not rain.
5. Given that p: You like Physics, q: You like Chemistry, and r: You like Biology. Give a verbal sentence which describes each of the following:
   (a) $(p \vee q) \wedge \sim r$
   (b) $(p \wedge q) \vee (\sim p \wedge \sim r)$
   (c) $\sim (p \wedge \sim r)$
6. Construct a truth table for the proposition $(p \wedge q) \rightarrow (p \vee q)$.
7. Express each of the following statements in symbolic form:
   (a) It is false that "Grace speaks French but not German".
   (b) Grace does not speak French or she does not speak German.
8. Construct a truth table for the proposition $\sim p \leftrightarrow (q \rightarrow r)$.
9. Determine the truth value in each of the following statements:
   (a) Either $69 \times 12 = 13 \times 36$ or $12 - 18 \neq -6$.
   (b) If 3 is an odd number, then neither 3 is a prime number nor an even number.
   (c) $\sqrt[3]{64} = \sqrt[4]{256}$ if and only if $2+4=6$ and $16-10=56-50$.
10. Express each of the following compound statements in symbolic notation using the letters P, Q, and R .

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(a) Either the manufactured drug is not faulty and accepted by the Tanzania Food and Drug Authority (TFDA) or the drug is faulty and it is not accepted by TFDA.

(b) If Jonathan is a member of a social committee, then the committee is strong. The committee is strong if and only if Jonathan's argument is accepted by other members. Therefore, Jonathan's argument is not accepted and the committee is not strong.

11. If p stands for "Halima is poor" and q stands for "Halima works hard", write statements to represent each of the following propositions:
(a) $q \vee p$ (e) $\sim p \rightarrow \sim q$
(b) $\sim(p \wedge q)$ (f) $\sim p \vee \sim q$
(c) $p \rightarrow q$ (g) $q \leftrightarrow p$
(d) $p \wedge q$ (h) $\sim q \vee (p \wedge q)$

12. If p stands for "Swimming in the pool is not dangerous" and q stands for "Few people have drowned in the swimming pool". Write a sentence in each of the following logical expressions:
(a) $\sim p$
(b) $\sim q \leftrightarrow p$
(a) $\sim p \rightarrow q$

13. Construct a truth table for each of the following compound statements:
(a) $(p \wedge q) \wedge \sim (p \wedge q)$
(b) $(q \rightarrow p) \leftrightarrow (p \rightarrow q)$
(c) $(p \vee q) \wedge r$
(d) $(p \vee q) \rightarrow \sim r$
(e) $[(\sim p \leftrightarrow q) \wedge r] \rightarrow t$

Converse, inverse, and contrapositive

A conditional statement can be written in its converse, inverse, or contrapositive.

Converse of a conditional statement

The converse of a conditional statement "$p \rightarrow q$" is written by interchanging the roles of the statements. Therefore, the converse of $p \rightarrow q$ is the statement "$q \rightarrow p$" as shown in Table 2.7.

Table 2.7: *Truth table for* $q \rightarrow p$

p	q	$q \rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Inverse of a conditional statement

The inverse of the conditional statement "$p \rightarrow q$" is written by negating the original conditional statement. Thus, the inverse of $p \rightarrow q$ is the implication "$\sim p \rightarrow \sim q$" as shown in Table 2.8.

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Table 2.8: *Truth table for* $\sim p \rightarrow \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Contrapositive of a conditional statement

The contrapositive of the conditional statement "$p \rightarrow q$" can be written by exchanging the roles of the inverse or by negating the converse of the given conditional statement. Hence, the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$, and it has the same truth table as $p \rightarrow q$, as shown in Table 2.9.

Table 2.9: *Truth table for* $\sim q \rightarrow \sim p$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example 2.12

Write the converse, contrapositive, and inverse of the statement "A child cries whenever it is raining".

Solution

Let p : A child cries.

q : It is raining.

Symbolically the given statement is $q \rightarrow p$ and its converse is $p \rightarrow q$, that is "If a child cries, then it is raining".

Its contrapositive is $\sim p \rightarrow \sim q$, that is "If a child does not cry, then it is not raining".

Its inverse is $\sim q \rightarrow \sim p$, that is "If it is not raining, then a child does not cry".

Example 2.13

Write the converse, contrapositive, and inverse of the statement "If Flora is happy, then she eats food and drinks milk".

Solution

Let p : Flora is happy.

q : Flora eats food and drinks milk.

Its converse is $q \rightarrow p$. That is, "If Flora eats food and drinks milk, then she is happy".

Its contrapositive is $\sim q \rightarrow \sim p$. That is, "If Flora does not eat food and drink milk, then she is not happy ".

Its inverse is $\sim p \rightarrow \sim q$. That is, "If Flora is not happy, then she does not eat food and drink milk".

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Example 2.14

Given the proposition
$[(p \to q) \wedge (q \to r)] \to (p \to r)$,
write down the;
(a) converse,
(b) inverse,
(c) contrapositive.

Solution

(a) Its converse is
$(p \to r) \to [(p \to q) \wedge (q \to r)]$

(b) Its inverse is
$\sim[(p \to q) \wedge (q \to r)] \to \sim(p \to r)$

(c) Its contrapositive is
$\sim(p \to r) \to \sim[(p \to q) \wedge (q \to r)]$

Exercise 2.3

1. Write the inverse, converse, and contrapositive in each of the following propositions:
   (a) If tomorrow is Saturday, then Paul will go to the beach.
   (b) If it is raining, then the shop is closed.
   (c) Industries are environmentally friendly if they are in harmony with the surroundings.
   (d) If ABC is an equilateral triangle, then it is a right-angled triangle.
   (e) If $f(x)$ is a rational function, then it has asymptotes.
2. Given the statement $p \to (q \to \sim p)$. Construct the truth table for;
   (a) Its converse.
   (b) Its contrapositive.
3. Given the statement "If two vectors are orthogonal, then their dot product is zero". Write the verbal sentence for its;
   (a) inverse,
   (b) converse,
   (c) contrapositive.
4. In each of the following statements, write down the corresponding contrapositive statement:
   (a) If Halima has courage, then she will win.
   (b) It is necessary to be strong in order to be a sailor.
   (c) To be a square is a sufficient condition for a geometrical figure to be a rectangle.
5. Let p : Kiswahili is interesting,
   q : Physics has applications, and
   r : Physics depends on Mathematics.

   Write in symbolic form the converse, inverse, and contrapositive of the statement "If Kiswahili is interesting, then Physics has applications and depends on Mathematics".

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6. In each of the following write:
   (a) The contrapositive of the inverse of $p \rightarrow q$.
   (b) The inverse of the converse of $\sim p \rightarrow (q \rightarrow \sim r)$.
   (c) The converse of the contrapositive of $(p \leftrightarrow q) \rightarrow (\sim p \leftrightarrow \sim r)$.

Logic symbols dominance

In algebra, whenever arithmetic operations $+, -, \times$, and $\div$ are used in an expression, then the expression is evaluated by applying the BODMAS rule. That is, Bracket of, Division, Multiplication, Addition, and Subtraction. In logical expressions, brackets are also used. However, if brackets in logical expressions are many, then the order shown in Table 2.10 should be followed. For instance, when evaluating $p \wedge q \vee r$, work out "$\wedge$" first, that is $(p \wedge q) \vee r$, and not $p \wedge (q \vee r)$.

Table 2.10: *Order of dominance in logic expression*

Connective	$\sim$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
Priority	1st	2nd	3rd	4th	5th

Activity 2.2: Recognising the order of dominance in logic expressions

Individually or in a group, perform the following tasks:

1. Using the logical connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, construct five propositions of your choice.
2. From the constructed propositions in task 1, suggest the order of evaluation using the order of dominance shown in Table 2.10.
3. Share your results with other students for more inputs.
4. With your fellow students discuss the challenges you have faced in task 2, if any.

Logical equivalences

Two or more logical expressions are logically equivalent if and only if they have the same truth table. This means, if p and q are propositions that are equivalent, then $p \leftrightarrow q$ will have the truth value T in all cases. A logical expression or statement having only truth values true (T) in the last column of its truth table is called a tautology. The notation "$p \equiv q$" denotes that p and q are logically equivalent.

Example 2.15

Show that $p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.

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Solution

The expressions $p \to q$ and $\sim p \vee q$ are logically equivalent if $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology as shown in the following table.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the truth values in the column of $(p \to q) \leftrightarrow (\sim p \vee q)$ in the truth table contain only the truth values T, then it is a tautology.
Therefore, $p \to q$ and $\sim p \vee q$ are logically equivalent.

Note that, if a compound proposition or a logical expression has truth value "F" for all combinations of the truth values of the proposition variables it is called a contradiction. In other words, a contradiction is the negation of a tautology. For instance, in Example 2.15, it was shown that $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology. Thus, its negation is a contradiction. A contradiction statement is also known as a fallacy.

Therefore, $\sim[(p \to q) \leftrightarrow (\sim p \vee q)]$ is a contradiction.

Example 2.16

Use a truth table to verify that the statement $(\sim p \to q) \wedge \sim(p \vee q)$ is a contradiction.

Solution

Given the statement $(\sim p \to q) \wedge \sim(p \vee q)$, then its truth table is shown as follows.

p	q	$\sim p$	$\sim q$	$\sim p \to q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p \to q) \wedge \sim(p \vee q)$
T	T	F	F	T	T	F	F
T	F	F	T	T	T	F	F
F	T	T	F	T	T	F	F
F	F	T	T	F	F	T	F

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From the table, all truth values in the last column are false.
Therefore, the statement is a contradiction.

Exercise 2.4

Determine whether or not the statements in each of the following pairs are logically equivalent:

1. $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$
2. $(\sim p \vee \sim q)$ and $p \vee q$
3. $(p \wedge q)$ and $\sim(p \rightarrow \sim q)$
4. $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$
5. $(q \rightarrow r) \vee (p \rightarrow r)$ and $(p \wedge q) \rightarrow r$
6. $(p \leftrightarrow q)$ and $(p \rightarrow q) \wedge (q \rightarrow p)$
7. $p \leftrightarrow q$ and $(\sim p \leftrightarrow \sim q)$
8. $p \rightarrow r$ and $(p \wedge r) \vee (\sim p \wedge \sim r)$
9. $\sim(p \leftrightarrow r)$ and $p \leftrightarrow \sim r$
10. $\sim p \vee (\sim p \wedge q)$ and $\sim p \wedge \sim q$

Laws of the algebra of propositions

The laws of the algebra of propositions are a set of fundamental principles that govern the manipulations and evaluation of logical expressions. Table 2.11 presents the laws of the algebra of propositions with their names.

Table 2.11: *Laws of algebra of propositions*

Equivalent statements	Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent
$p \wedge T \equiv p$ $p \vee F \equiv p$ $p \vee T \equiv T$ $p \wedge F \equiv F$	Identity
$\sim(\sim p) \equiv p$	Double negation
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative
$\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive
$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$	Complement (Negation)
$p \rightarrow q \equiv \sim p \vee q$ $p \leftrightarrow p \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $\equiv (\sim p \vee q) \wedge (\sim q \vee p)$	Condition and Bi-conditional

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Example 2.17

Use the laws of the algebra of propositions to simplify $\sim(p \vee q) \vee (\sim p \wedge q)$.

Solution

$\sim(p \vee q) \vee (\sim p \wedge q)$	$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$	De Morgan's law
	$\equiv \sim p \wedge (\sim q \vee q)$	Distributive law
	$\equiv \sim p \wedge \mathrm{T}$	Complement law
	$\equiv \sim p$	Identity law

Therefore, $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$.

Example 2.18

Use the laws of the algebra of propositions to show that $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

Solution

The two expressions, $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent if and only if $\sim(p \vee (\sim p \wedge q))$ can be simplified to $\sim p \wedge \sim q$.

$\sim(p \vee (\sim p \wedge q))$	$\equiv \sim p \wedge \sim(\sim p \wedge q)$	De Morgan's law
	$\equiv \sim p \wedge [\sim(\sim p) \vee \sim q]$	De Morgan's law
	$\equiv \sim p \wedge (p \vee \sim q)$	Double negation law
	$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$	Distributive law
	$\equiv \mathrm{F} \vee (\sim p \wedge \sim q)$	Complement law
	$\equiv \sim p \wedge \sim q$	Identity law

Since $\sim(p \vee (\sim p \wedge q))$ has been simplified to $\sim p \wedge \sim q$, then $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

Example 2.19

Using the laws of the algebra of proposition, show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

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Solution

If the statement is a tautology by using the laws of the algebra of propositions, it can be simplified to an equivalent statement "T". Otherwise, it is not a tautology.

$(p \wedge q) \rightarrow (p \vee q) \equiv \sim (p \wedge q) \vee (p \vee q)$	Conditional law
$\equiv (\sim p \vee \sim q) \vee (p \vee q)$	De Morgan's law
$\equiv \sim p \vee p \vee \sim q \vee q$	Commutative law
$\equiv (\sim p \vee p) \vee (\sim q \vee q)$	Associative law
$\equiv T \vee T$	Complement law
$\equiv T$	Idempotent law

Since, the last result is T, then $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Exercise 2.5

1. Use the laws of the algebra of propositions to establish whether or not each of the following propositions is a tautology:
   (a) $(p \wedge \sim q) \rightarrow \sim q$
   (b) $p \vee [(p \wedge q) \leftrightarrow p]$
   (c) $[(p \rightarrow \sim q) \wedge (\sim p \wedge q)] \rightarrow q$
   (d) $[(p \rightarrow \sim q) \wedge (r \rightarrow q) \wedge q] \rightarrow r$
   (e) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
2. Using the laws of the algebra of propositions, simplify the proposition $\sim [(p \vee q) \vee (\sim p \wedge q)]$.
3. Use the laws of the algebra of propositions to show that $(P \wedge Q) \vee [\sim R \wedge (Q \wedge P)] \equiv P \wedge Q$.
4. Simplify each of the following by using the laws of the algebra of propositions:
   (a) $q \rightarrow (\sim p \rightarrow \sim q)$
   (b) $\sim (p \vee q) \vee (\sim p \wedge q)$
   (c) $p \vee (p \wedge q)$
   (d) $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$
5. Use the laws of the algebra of propositions to simplify the proposition $[(p \rightarrow (q \vee \sim r)) \wedge (q \rightarrow (p \wedge r))] \rightarrow [((q \vee \sim r) \wedge (p \wedge r)) \rightarrow r]$
6. Using the laws of the algebra of propositions, simplify each of the following propositions:
   (a) $(\sim p \rightarrow q) \wedge \sim (p \vee q)$
   (b) $(p \leftrightarrow q) \wedge (\sim p \leftrightarrow q)$
   (c) $[(p \vee q) \wedge (p \rightarrow r) \wedge (\sim p \rightarrow r)] \rightarrow q$
7. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.

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Arguments

An argument is a series of connected propositions that form a definite statement. Arguments are normally made up of two parts, the initial statements, called "premises", followed by a last statement called "conclusion". For instance, "If I go to the cinema, I will not do my homework. I did my homework. Therefore, I did not go to the cinema".

This is an example of an argument. The first two statements *"If I go to the cinema, I will not do my homework"* and *"I did my homework"* are premises, and the last statement *"Therefore, I did not go to the cinema"* is the conclusion. Remember, an argument is valid if and only if whenever all the premises are true, the conclusion is also true.

Activity 2.3: Identifying premises and conclusion of an argument

Individually or in a group, perform the following tasks:

1. Construct at least four mathematical sentences of your choice.
2. Using the mathematical sentences constructed in task 1, identify statements that form some arguments, hence state the premises and conclusion in each of the argument constructed.
3. Formulate the mathematical argument, state the premises and conclusion drawn from the argument.
4. Share your results with other students for more inputs.

Testing the validity of an argument

If P_1, P_2, and P_3 are premises and q is the conclusion of an argument, then the full argument is written as $P_1 \wedge P_2 \wedge P_3 \therefore q$

In testing the validity of the argument, connect the premises P_1, P_2, and P_3 with a conjunction $\wedge$, then connect the premises with the conclusion by an implication sign $(\rightarrow)$. If the compound statement formed is a tautology, then the argument is valid, otherwise it is not valid, that is $[P_1 \wedge P_2 \wedge P_3] \rightarrow q$ must be a tautology for the argument in question to be valid. The argument can be valid for false premises and/or conclusion.

Example 2.20

Use the laws of algebra of proposition to verify the validity of the following argument:

If I study, then I will not fail the examination. If I do not play football, then I will study. But I failed the examination. Therefore, I played football.

Solution

Let p: I study.

q: I will fail the examination.

r: I play football.

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The premises and conclusion of the argument are written as:
$p \to \sim q, \sim r \to p, q \therefore r$

Connecting the premises and conclusion gives,

$[(p \to \sim q) \wedge (\sim r \to p) \wedge q] \to r$

$\equiv [(\sim p \vee \sim q) \wedge (r \vee p) \wedge q] \to r$	Implication law
$\equiv [q \wedge (\sim p \vee \sim q) \wedge (r \vee p)] \to r$	Commutative law
$\equiv [((q \wedge \sim p) \vee (q \wedge \sim q)) \wedge (r \vee p)] \to r$	Distributive law
$\equiv [((q \wedge \sim p) \vee F) \wedge (r \vee p)] \to r$	Complement law
$\equiv [(q \wedge \sim p) \wedge (r \vee p)] \to r$	Identity law
$\equiv [(q \wedge (\sim p \wedge (r \vee p))] \to r$	Associative law
$\equiv [q \wedge ((\sim p \wedge r) \vee (\sim p \wedge p))] \to r$	Distributive law
$\equiv (q \wedge ((\sim p \wedge r) \vee F)) \to r$	Complement law
$\equiv [q \wedge (\sim p \wedge r)] \to r$	Identity law
$\equiv \sim (q \wedge \sim p \wedge r) \vee r$	Implication law
$\equiv (\sim q \vee p \vee \sim r) \vee r$	De Morgans law
$\equiv (\sim q \vee p) \vee (\sim r \vee r)$	Associative law
$\equiv (\sim q \vee p) \vee T$	Complement law
$\equiv T$	Identity law

Since the truth value is a tautology, then the argument is valid.

Example 2.21

Test the validity of the following argument:
If I read my textbook, I will understand how to do my homework. I did not understand how to do my homework. Therefore, I did not read my textbook.

Solution

Let p : I read my textbook.
q : I will understand how to do my homework.

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The premises and conclusion of the argument are written as: $p \rightarrow q;\ \sim q \therefore \sim p$.
Joining the premises and conclusion gives $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$. Its truth table is shown as follows.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(\ddot{u} \rightarrow\ ) \wedge \sim$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the truth values in the last column are all true, then it is a tautology.
Therefore, the argument is valid.

Example 2.22

Test whether or not the following argument is valid.
Either Halima reaches home early or the traffic jam is not there. The traffic jam is not there. Therefore, Halima does not reach home early.

Solution

Let p: Halima reaches home early.

q : The traffic jam is there.

The premises and conclusion of the argument can be written as $p \vee \sim q;\ \sim q \therefore \sim p$.
Connecting the premises and conclusion gives $[(p \vee \sim q) \wedge \sim q] \rightarrow \sim p$. Its truth table is shown as follows.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge \sim q$	$[(p \vee \sim q) \wedge \sim q] \rightarrow \sim p$
T	T	F	F	T	F	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

The truth values in the last column are not all true.

Therefore, the argument is not valid.

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Exercise 2.6

Test whether or not each of the following arguments is valid:

1. If a girl is cute, then she is not humble. The girl is not humble. Therefore, she is cute.
2. If it rains or snows, then my rooftop leaks. My rooftop is leaking. Therefore, it is raining and snowing.
3. I wash the glasses or I do not drink juice. I drink juice. Therefore, I wash the glasses.
4. My car tyres will get a puncture if I do not change the tyres regularly. My car tyres got a puncture. Therefore, I did not change my car tyres regularly.
5. You will not be treated with respect if you are not humble. You are not treated with respect. Therefore, you are not humble.
6. If you are kind to a cat, then it will be your friend. You were not kind to a cat. Hence, it is not your friend.
7. If Nairobi is the capital of Tanzania, then it is in Tanzania. Nairobi is not in Tanzania. Therefore, Nairobi is not the capital of Tanzania.
8. If I am illiterate, then I can't read and write. I cannot read but I can write. Thus, I am not illiterate.
9. My finger nails will become dirty if I plant flowers. My finger nails are not dirty. Therefore, I did not plant flowers.
10. People will not fail to understand me if I speak to them. They did not understand me. Therefore, I did not speak to them.

Electrical networks

An electrical network is an arrangement of a battery, lamp, and switches in series or in parallel connection using wires. Electrical switches are used in electrical networks to allow or stop the flow of electric current.

Electrical switches and statements

When a switch is "ON", the electric current can flow between terminals and the lamp will be switched "ON". Otherwise, the switch is "OFF" and current cannot flow. The terms "CLOSED" and "OPEN" are also used to mean "ON" and "OFF", respectively.

Series and parallel switches

Connections of electrical switches can be done in series or in parallel.

Series switches

If p and q are two switches connected in series, then the logical expression to represent the switches is the conjunction "$p \wedge q$". This means that both p and q must be closed for electricity to flow in the electrical network. If both switches are open or one is open while the other

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switch closed, electric current will not flow. Figures 2.1(a) and 2.1 (b) show the series connection of the open and closed switches, respectively. In this case T_1 and T_2 are terminals bounding the switches.

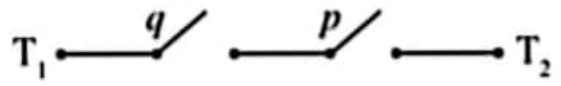

Figure 2.1(a): *Open switches in series connection*

Figure 2.1(b): *Closed switches in series connection*

Parallel switches

If two switches, p and q are connected in parallel, the logical expression that represents the two switches is the disjunction, that is "$p \vee q$". This means that either p or q or both p and q must be closed for electric current to flow in the electrical network, otherwise the electric current will not flow. Figures 2.2 (a) and 2.2 (b) show the parallel connection of open switches and closed switches, respectively, where T_1 and T_2 are terminals bounding the switches.

Figure 2.2(a): *Open switches in parallel connection*

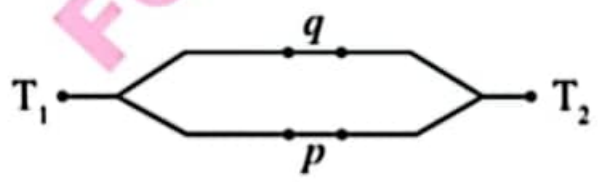

Figure 2.2(b): *Closed switches in parallel connection*

In some applications, the switches can be connected in opposite positions. This means that, if one switch is open, the other must be closed, and vice versa. Two switches that have opposite positions are called complementary switches. If these switches are connected in series, electric current will not flow between two terminals. If they are in parallel, electric current will flow between the two terminals. Figures 2.3(a) and 2.3(b) show the two cases.

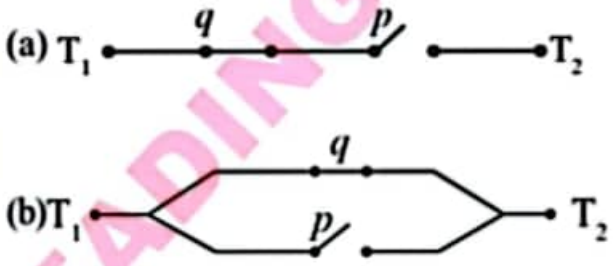

Figure 2.3: *Switches connected in different states*

Logic can be used in constructing and simplifying networks. It can also be used in reducing the switches in the circuit, and hence perform the same function as a complex circuit. This may be done using the idea of equivalence of propositions. In simplifying a complex electrical network, two steps are important to follow:

1. Construct a compound proposition of the network.
2. Simplify the compound proposition using the laws of the algebra of propositions.

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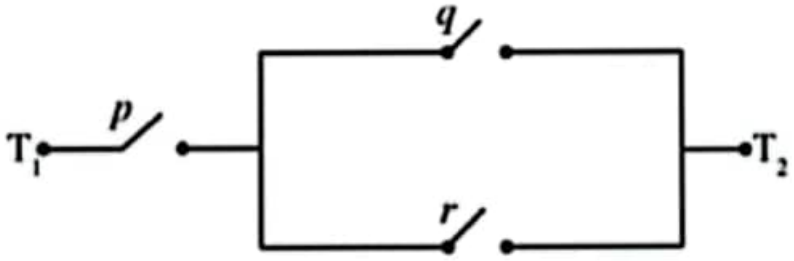
Example 2.23

Draw an electrical network for $p \wedge (q \vee r)$.

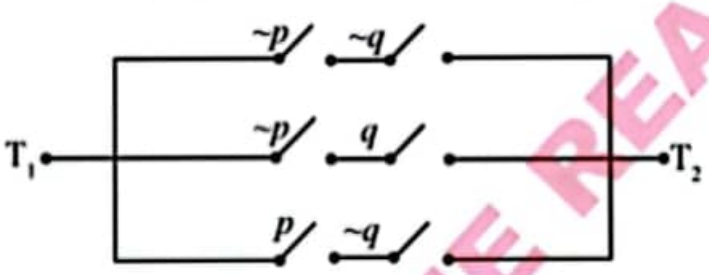
Solution

The following is the electric network for $p \wedge (q \vee r)$.



Example 2.24

Draw a simplified electric network using the following circuit.



Solution

The compound statement for the electrical network is

$(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$.

Thus, $(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \equiv \sim p \wedge (\sim q \vee q) \vee (p \wedge \sim q)$ Distributive law

$\equiv (\sim p \wedge T) \vee (p \wedge \sim q)$ Complement law

$\equiv \sim p \vee (p \wedge \sim q)$ Identity law

$\equiv (\sim p \vee p) \wedge (\sim p \vee \sim q)$ Distributive law

$\equiv T \wedge (\sim p \vee \sim q)$ Complement law

$\equiv \sim p \vee \sim q$ Identity law

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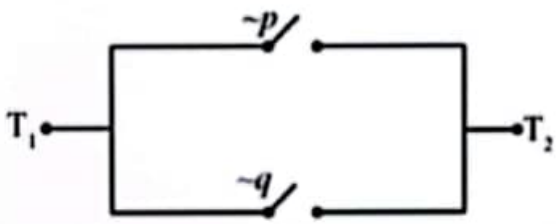
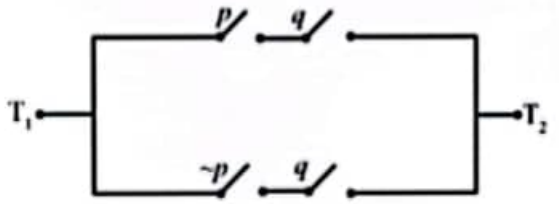
Therefore, the following is a simplified electric network.



Example 2.25

Draw an electrical network which is simpler than the following circuit.



Solution

The compound statement of the electrical network is $(p \wedge q) \vee (\sim p \wedge q)$.

Simplify the statement as follows.

$(p \wedge q) \vee (\sim p \wedge q)$

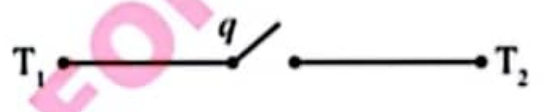
$\equiv (q \wedge p) \vee (q \wedge \quad p)$ Commutative law

$\equiv \ddot{u} \wedge (\ \vee \sim\ )$ Distributive law

$\equiv q \wedge T$ Negation law

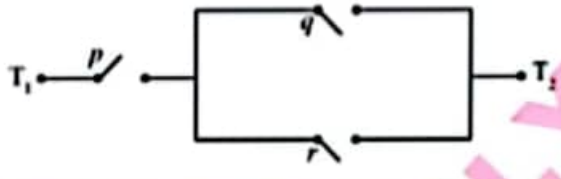
$\equiv q$ Identity law

Therefore, the following is a simplified electric network.



The electrical network indicates that the electric current can flow between T_1 and T_2 when q is closed regardless of whether p is closed or not.

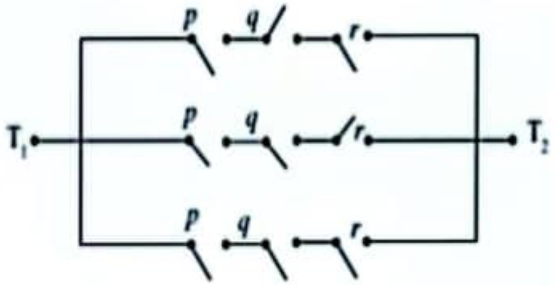
Note that, sometimes the switch is drawn opening downward to indicate a negation. Negation is not placed in such drawings as shown in the following electric network for $p \wedge (\sim q \vee \sim r)$.



Exercise 2.7

1. Draw an electrical network in each of the following compound statements:
   (a) $p \wedge q$
   (b) $p \wedge \sim q$
   (c) $p \wedge (q \vee r) \vee s$
   (d) $(p \wedge q) \vee (p \wedge r)$
   (e) $(p \vee q) \wedge (r \vee s) \vee t$
   (f) $(\sim p \vee r) \wedge (\sim p \wedge \sim r)$
   (g) $p \vee q \wedge \sim p$
   (h) $\sim (p \vee q) \vee (q \wedge p)$
   (i) $[\sim (p \vee q) \vee (q \vee p)] \wedge r$

2. Draw a simplified diagram for the following electrical network.



3. Write down the compound statement in each of the following electrical network:

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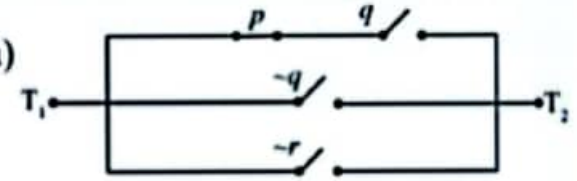
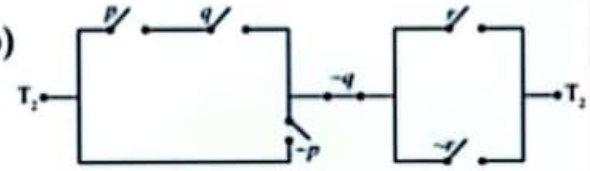
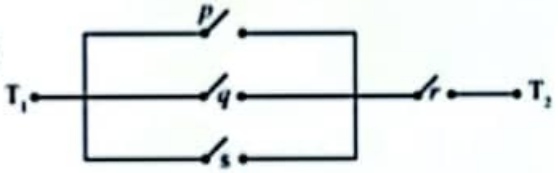
(a)



(b)

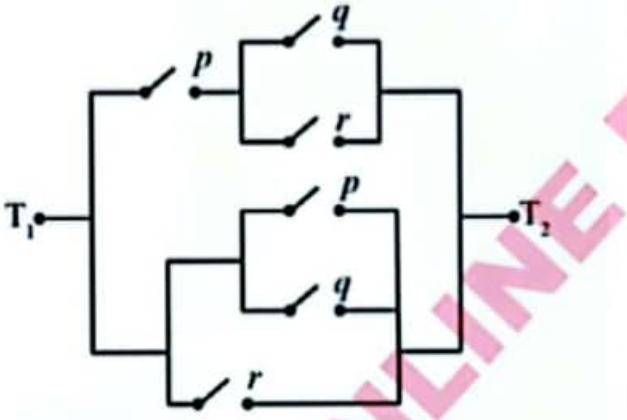


(c)



4. Draw simplified electrical networks for the electrical networks in question 3.

5. Draw a simplified electrical network for the following.



Construction of a compound proposition from a truth table

It is possible to determine an equivalent compound proposition corresponding to a given truth table. The steps are as follows: First, identify the true values in the last column of the truth table. Next, construct basic conjunctions for all true values in the last column. Finally, connect the conjunctions by disjunction(s).

Example 2.26

Construct a compound proposition equivalent to the following truth table.

p	q	r	B
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

Solution

Locate the true values in the last column (column B). The basic conjunctions are written for each true value T as follows:

p	q	r	B	Basic conjunction
T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$p \wedge \sim q \wedge \sim r$
F	T	T	F	
F	T	F	T	$\sim p \wedge q \wedge \sim r$
F	F	T	T	$\sim p \wedge \sim q \wedge r$
F	F	F	F	

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The required equivalent proposition is obtained by joining the three "basic conjunctions by the symbol "$\vee$", that is $(p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$

Exercise 2.8

From questions 1 to 5 write down the compound proposition corresponding to its last column in the following tables:

1.

p	q	S
T	T	F
T	F	T
F	T	F
F	F	F

2.

p	q	G
T	T	T
T	F	F
F	T	F
F	F	T

3.

p	q	L
T	T	F
T	F	F
F	T	F
F	F	T

4.

p	q	M
T	T	T
T	F	T
F	T	F
F	F	T

5.

p	q	D
T	T	T
T	F	F
F	T	F
F	F	F

6. Construct the compound sentence for s_1 and s_2 having the following truth table, hence use the laws of algebra of propositions to simplify the resulting statements s_1 and s_2.

p	q	r	s_1	s_2
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	T	F
F	F	F	F	F

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Chapter summary

1. Logic is a science or an art which deals with the study of truth, principles of correct reasoning, and making good decisions.
2. A mathematical statement is a declarative sentence which can be either "true" or "false", but not both.
3. The truth value of a true proposition is denoted by the letter "T" and the truth value of a false proposition is denoted by the letter "F".
4. There are two types of statements, namely; simple statements and compound statements.
5. The number of cases that describe a given compound statement depends on the number of propositions contained in the compound statement.
6. Negation of a statement is written by putting the word "not" with the verb, or to begin the sentence with the phrase "It is not true that" or "It is false that".
7. A conjunction is a compound statement which is comprised of two or more propositions joined by the "AND" operator.
8. A disjunction is a compound statement which comprises of two simple statements formed by joining the statements with the "OR" operator.
9. An implication is a type of compound statement which is formed by joining two simple statements with the logical implication connective.
10. A double implication, denoted by a double-headed arrow "$\leftrightarrow$" is a compound statement formed by the combination of a conditional statement with its converse. Such a statement is also known as a biconditional statement. The truth value of the biconditional statement $p \leftrightarrow q$ is true when the simple statements p and q are both true or both false. Otherwise, it is false.

Revision exercise 2

1. "Is it true that science subjects are poorly performed by most students?" Is this a mathematical statement or not? Give a reason.
2. Write the converse, contrapositive, and inverse of each of the following statements:
   (a) If a person is 20 years old, then he is an adult.
   (b) If today is Friday, then I have a test.
   (c) If you buy our clothes, then you are attractive.
   (d) If today is Saturday, then it is a holiday.

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(e) If Nuru is intelligent, then she will pass the examination.

3. Construct a truth table for each of the following statements:

(a) $(p \vee q) \rightarrow q$

(b) $q \rightarrow (q \vee r)$

(c) $[\sim (q \vee r)] \leftrightarrow [\sim q \wedge r]$

(d) $[(p \rightarrow q) \wedge (q \vee r) \wedge p] \rightarrow r$

(e) $[p \rightarrow [(q \vee \sim r)]] \rightarrow (p \wedge q)$

(f) $p \wedge (q \vee r)$

(g) $[\sim (p \wedge q) \vee (p \vee q)]$

(h) $p \vee \sim q \rightarrow q$

(i) $\sim r \rightarrow (p \rightarrow q)$

4. Without using a truth table, determine whether or not each of the following propositions is a tautology:

(a) $\sim (r \rightarrow t) \rightarrow \sim r$

(b) $p \rightarrow (p \vee q)$

(c) $(r \wedge t) \rightarrow (r \rightarrow t)$

(d) $\sim (r \rightarrow t) \rightarrow r$

5. Suppose that $S_1 \leftrightarrow S_2$ is a tautology. What can you say about the sentences S_1 and S_2?

6. Use truth tables to show whether or not each of the following statements is a tautology:

(a) $\sim (\sim p \vee p) \rightarrow \sim (\sim q \vee q)$

(b) $((p \rightarrow q) \vee (p \rightarrow r)) \leftrightarrow (p \rightarrow (q \vee r))$

(c) $(p \rightarrow q) \rightarrow \sim (p \vee \sim q)$

(d) $\sim (\sim p \vee \sim q) \rightarrow \sim (p \rightarrow \sim q)$

(e) $(\sim p \wedge (p \rightarrow q)) \leftrightarrow \sim q$

(f) $((p \leftrightarrow q) \wedge (p \rightarrow r)) \leftrightarrow ((p \rightarrow (q \wedge r))$

(g) $(p \vee (q \wedge q)) \leftrightarrow p$

(h) $(p \wedge q) \rightarrow ((\sim r \wedge p) \vee r)$

(i) $((p \wedge q) \rightarrow (q \wedge r)) \rightarrow (r \wedge s)$

(j) $(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$

7. If p and q represent any two sentences, and s represents a tautology, which of the following pairs of sentences are equivalent to each other?

(a) $p \wedge s$; p

(b) $\sim s \vee p$; p

(c) $s \rightarrow p$; p

(d) $p \vee s$; p

(e) $p \vee (q \vee \sim s)$; $q \vee p$

8. If H_1 and H_2 are equivalent sentences, determine whether or not the sentences, $H_1 \rightarrow H_2$ and $H_2 \rightarrow H_1$ are also equivalent to each other. Explain your answer.

9. Using truth tables, determine whether or not the following pairs of statements are logically equivalent:

(a) $p \wedge (q \vee r)$ and $[p \rightarrow (q \vee \sim r)] \rightarrow (p \wedge q)$

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(b) $\sim(p\wedge\sim q)\wedge(\sim q)$ and $\sim(p\vee q)$ (d) $(p\wedge q)\vee(\sim p\wedge\sim q)$ and $p\leftrightarrow q$

(c) $(p\rightarrow q)\vee(p\rightarrow r)$ and $p\rightarrow(q\vee r)$

10. Which of the following pairs of sentences are equivalent?

(a) $\sim p;\ p$ (d) $p\rightarrow q;\ \sim p\vee q$ (e) $p\rightarrow(q\rightarrow r);\ p$

(b) $p\wedge q;\ \sim(p\vee\sim q)$ (c) $(p\wedge q)\wedge r;\ r$ (f) $\sim(p\rightarrow q);\ p\wedge\sim q$

11. Using the laws of the algebra of propositions, simplify each of the following expressions:

(a) $[(p\rightarrow\sim q)\wedge q]\rightarrow\sim p$ (c) $[(p\rightarrow\sim q)\wedge(r\rightarrow q)\wedge q]\rightarrow r$

(b) $(p\wedge q)\rightarrow(p\vee q)$ (d) $[(p\rightarrow q)\wedge(\sim q)]\rightarrow\sim p$

12. State whether or not each of the following compound statements is a tautology:

(a) $\big((p\rightarrow q)\wedge(q\rightarrow r)\big)\leftrightarrow(p\rightarrow r)$ (b) $\sim[(p\wedge\sim r\wedge\sim q)]\wedge(\sim p\vee q\vee\sim r)$

13. Let p be "Lightness is clever", q be "Lightness is polite", and r be "Lightness is humble":

(a) Write the verbal sentence representation of each of the following:

(i) $\sim(p\rightarrow\sim q)\vee r$ (ii) $\sim p\vee(p\wedge q)$

(b) Write each of the following compound statements in symbolic form:

(i) Lightness is either clever or polite, but she is not polite if she is clever.

(ii) Lightness is either polite or humble, but she is not clever.

(iii) Lightness is neither polite nor humble as long as she is not clever.

14. Test the validity of each of the following arguments:

(a) If I am clever, then I know how to reason. I do not know how to reason. Therefore, I am not clever.

(b) Maize grows well whenever there is rain. If the maize grows well, there is no famine. But there is famine. Therefore, there is no rain.

(c) If Amos goes to school, then he must have a good life. But he will not have a good life if he fails examinations. Therefore, Amos must go to school.

(d) The game will be off if it rains or if there are no players. The game was on. Therefore, it did not rain.

(e) If my smart phone crashes, I will loose all my contacts. I have not lost all my contacts. Therefore, my smart phone has not crashed.

(f) $q;\ r;\ \sim q\rightarrow p\ \therefore\ \sim r$

(g) $p\rightarrow\sim q,\sim r\vee q,r,\mapsto\sim p$

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(h) $p \to q, q \vee r \therefore \sim r \to \sim p$

(i) $r \to q; r \to p, q \therefore \sim p$

(j) $p \to q, q \vee r \therefore r$

15. Construct an electrical network corresponding to each of the following propositions:

(a) $[(p \wedge q \wedge r) \vee s] \wedge t$

(b) $[p \vee (p \wedge r) \vee s] \wedge t$

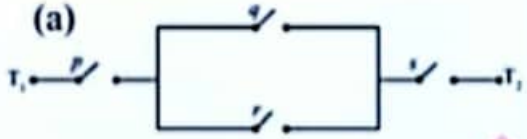
(c) $p \vee (\sim q \wedge r) \vee p$

(d) $(p \vee (q \wedge r)) \vee s$

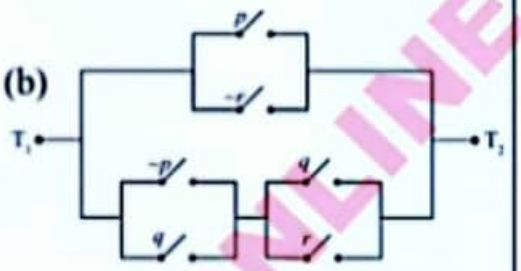
(e) $p \vee (q \wedge (r \wedge s))$

16. Write down the compound statement corresponding to each of the following electrical networks:

(a)



(b)



(c)

17. Find the compound statement having the truth values shown in columns "M and S" in the following truth tables:

(a)

p	q	M
T	T	F
T	F	T
F	T	T
F	F	F

(b)

p	q	r	S
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

18. Use the laws of the algebra of propositions to show whether or not each of the following statements is a tautology:

(a) $[(p \to q) \wedge \sim q] \to \sim p$

(b) $[p \wedge (\sim p \to q)] \vee r$

19. If the end column of the truth table of the statement $s \equiv (p \vee q) \vee [\sim r * (p \to q)]$ has all true values true except the last but one, what is the logical symbol of *?

20. In each of the following statements, draw a simple corresponding electrical networks.

(a) $\sim q \wedge [r \vee (p \wedge q) \vee (\sim p \wedge r)]$

(b) $(p \to q) \wedge (p \vee q)$

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Chapter Three

Coordinate geometry I

Introduction

Coordinate geometry is a branch of geometry in which the positions of the points of geometric figures are defined with the help of ordered pairs of numbers called coordinates. In this chapter, you will learn about the angle between two lines, perpendicular distance of a point from a line, locus of a moving point, ratio theorem, and circles. The competencies developed will help you to solve problems in the fields of trigonometry, calculus, and dimensional geometry. Also, in real life, coordinate geometry is used in space activities such as air transport, map projection, describing the position of objects, and in many other applications.

Think

The World without a coordinate system.

The angle between two lines

Given two lines, L_1 and L_2 whose slopes are m_1 and m_2, respectively. The angle of intersection between the two lines can be obtained in terms of their slopes. Consider two intersecting lines, L_1 and L_2 as shown in Figure 3.1.

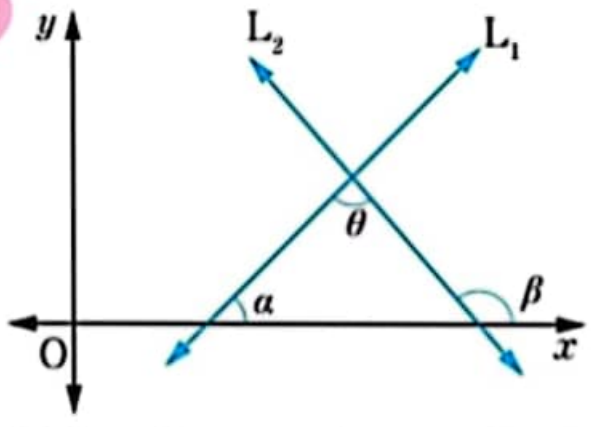

Figure 3.1: *Angle of intersection between two lines,* L_1 *and* L_2

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Let α be the angle between L_1 and the x-axis, β be the angle between L_2 and the x-axis, and θ be the acute angle formed between the two lines, L_1 and L_2.

In Figure 3.1, since
$\alpha + \theta + 180° - \beta = 180°$, then $\theta = \beta - \alpha$.

Introduce tangent on both sides of the equation $\theta = \beta - \alpha$ to obtain,
$\tan\theta = \tan(\beta - \alpha)$.

But. $\tan\theta = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$.

Also, $\tan\alpha$ = slope of L_1 and
$\tan\beta$ = slope of L_2.

Thus, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

Therefore, $\theta = \tan^{-1}\left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Example 3.1

Find the acute angle between the lines $2x + 4y = 15$ and $3y - 2x + 6 = 0$.

Solution

Let L_1 be the line $2x + 4y = 15$ and L_2 be the line $3y - 2x + 6 = 0$, then $m_1 = -\dfrac{1}{2}$ and $m_2 = \dfrac{2}{3}$, respectively.

The acute angle between the lines L_1 and L_2 is obtained from the formula;

$$\theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left|\frac{\frac{2}{3} - \left(-\frac{1}{2}\right)}{1 + \frac{2}{3}\left(-\frac{1}{2}\right)}\right|$$

$$\Rightarrow \theta = \tan^{-1}(1.75) = 60.26°.$$

Therefore, the angle between the lines L_1 and L_2 is 60.26°.

Example 3.2

Two straight lines are represented by the equations $3x - 2y = 5$ and $4x + 5y = 1$, respectively. Find the tangent of the angle between the lines.

Solution

Let L_1 be the line $3x - 2y = 5$ and L_2 be the line $4x + 5y = 1$, then $m_1 = \dfrac{3}{2}$ and $m_2 = -\dfrac{4}{5}$, respectively.

The tangent of the angle between the two lines is given by;

$\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$. Thus,

$$\tan\theta = \left(\frac{-\frac{4}{5} - \left(\frac{3}{2}\right)}{1 + \left(\frac{3}{2} \times -\frac{4}{5}\right)}\right)$$

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$$= \left(\frac{-\frac{23}{10}}{-\frac{1}{5}} \right) = \frac{23}{2}$$

Thus, $\tan\theta = 11.5$.

Therefore, the tangent of the angle between the two lines is 11.5.

Example 3.3

Find the equation of the line passing through the point (4, 6) and inclined with the line $2x - y = 2$ at 45°.

Solution

Suppose L_2 represents the line $2x - y = 2$. Thus, $m_2 = 2$.

From $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$,

where $\theta = 45$Y, it follows that

$$\tan 45\text{Y} = \frac{2 - m_1}{1 + 2m_1}$$

But $\tan 45\text{Y} = 1$.

Thus, $1 = \frac{2 - m_1}{1 + 2m_1}$.

$\Rightarrow 1 + 2m_1 = 2 - m_1$

$\Rightarrow m_1 = \frac{1}{3}$

Hence, the slope of the line is $\frac{1}{3}$.

The equation of the line is given by;

$$\frac{1}{3} = \frac{y-6}{x-4}$$

$\Rightarrow x - 4 = 3y - 18$

$\Rightarrow y = \frac{1}{3}x + \frac{14}{3} \Rightarrow x - 3y = -14$

Therefore, the equation of the line is $x - 3y = -14$.

Exercise 3.1

1. Find the obtuse angle between the lines L_1: $8x + 6y - 13 = 0$ and L_2: $-6x + 4y + 1 = 0$.
2. Find the acute angle between the line L_1 which passes through the points A(2, 2) and B(8, 6), and the line L_2 which passes through the points C(0, 3) and D(6, – 3).
3. Find the interior angles in each of the triangles having the following vertices:
   (a) A(– 4, –5), B(–1, 1), C(0, –3)
   (b) P(– 2, – 1), Q(4, – 3), R(1, 2)
   (c) L(– 4, 4), M(– 3, 1), N(6, 2).
4. Find the equation of the line through the point (2, 3) which makes an angle 45Y with the line $2x - y + 4 = 0$.
5. Let $nx + my = 8$ and $3nx - 2my = 6$ be equations of two lines which intersect at the point (3, 1). Find the values of m and n, hence determine the angle between the lines at the point of intersection.
6. Determine the angles formed between the lines with equation $2x^2 + 5xy - 12y^2 = 0$.

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7. Two lines passing through the point (4, 6) intersect each other at an angle of 30°. If the slope of one of the lines is 4, find the equation of the other line.

8. Show that the tangent of the angle between the lines $qx + py = pq$ and $qx - py = pq$ is $\frac{2pq}{p^2 - q^2}$.

9. Find the equations of two lines passing through the point $(3, -2)$ and are inclined at an angle of $\frac{\pi}{4}$ with the line $6x - 4y + 3 = 0$.

10. Show that the equations of the straight lines passing through the point $(-2, 3)$ which make an acute angle of 45° with the line $3x + 5y + 8 = 0$ are $y + x - 1 = 0$ and $y - x - 1 = 0$.

Perpendicular distance of a point from a line

The shortest distance of a point $P(x_1, y_1)$ from a given line is the perpendicular distance. In a practical situation, it is common to have a line and a point from the line.

Consider a point $P(x_1, y_1)$ and the line $ax + by + c = 0$. The formula for finding the perpendicular distance r of the point $P(x_1, y_1)$ from the line can be derived as follows.

Suppose $Q(x, y)$ is any point on the line such that $\overline{PQ}$ is perpendicular to the line $ax + by + c = 0$ as shown in Figure 3.2.

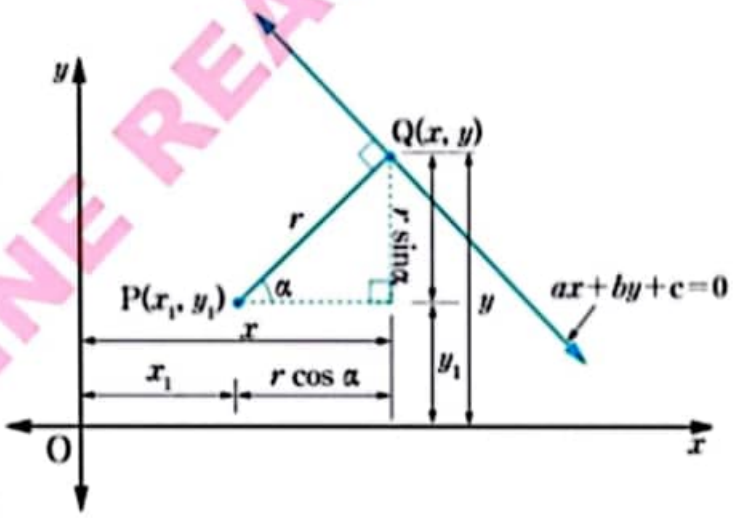

Figure 3.2: *Perpendicular distance of a point from a line*

The slope of the line represented by $ax + by + c = 0$ is $-\frac{a}{b}$. For two perpendicular lines, the slope of one of the lines is the negative of the reciprocal of the slope of the other line, or the product of the slopes of two perpendicular lines equals -1. Therefore, the slope of the line $\overline{PQ}$ is $\frac{b}{a}$. Let α be the angle between the line $\overline{PQ}$ and the horizontal axis. The tangent of the angle α equals the slope of the line $\overline{PQ}$, that is, $\tan\alpha = \frac{b}{a}$. Consider the right-angled triangle in Figure 3.3.

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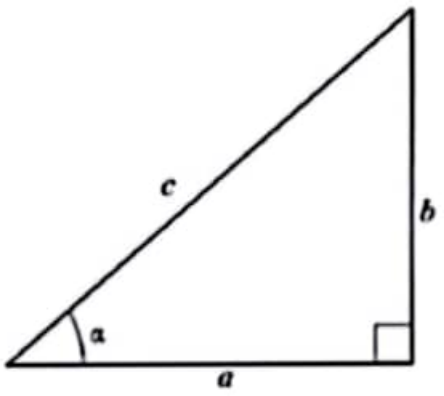


Figure 3.3: *Right-angled triangle*

From Figure 3.3, using trigonometric ratios,

$\cos\alpha = \frac{a}{c}$ and $\sin\alpha = \frac{b}{c}$.

Using the Pythagoras' theorem,

$c^2 = a^2 + b^2$

$\Rightarrow c = \sqrt{a^2 + b^2}$

$\Rightarrow \cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$.

But x and y are the coordinates of Q. Thus,

$x = x_1 + r\cos\alpha$ and $y = y_1 + r\sin\alpha$.

Substituting the values of $\cos\alpha$ and $\sin\alpha$ in x and y, it follows that $x = x_1 + \frac{ar}{\sqrt{a^2 + b^2}}$ and $y = y_1 + \frac{br}{\sqrt{a^2 + b^2}}$.

The point $Q(x, y)$ lies on the line $ax + by + c = 0$. This means that x and y satisfy the equation $ax + by + c = 0$.

That is,

$$a\left(x_1 + \frac{ar}{\sqrt{a^2 + b^2}}\right) + b\left(y_1 + \frac{br}{\sqrt{a^2 + b^2}}\right) + c = 0$$

$$\Rightarrow ax_1 + by_1 + c = -r\left(\frac{a^2 + b^2}{\sqrt{a^2 + b^2}}\right)$$

Cross multiplication to obtain

$$\Rightarrow ax_1 + by_1 + c = -r\sqrt{a^2 + b^2}$$

$$\Rightarrow -r = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow |-r| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

The absolute value sign is introduced since the distance always has a positive magnitude. Therefore, the formula for finding the perpendicular distance of a point $P(x_1, y_1)$ from the line $ax + by + c = 0$ is given by, $r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

Example 3.4

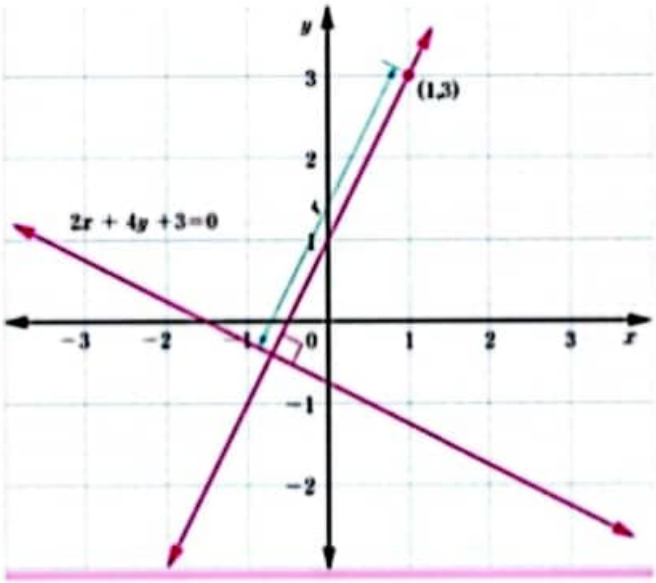
Find the perpendicular distance from the point (1, 3) to the line $2x + 4y + 3 = 0$.

Solution

Given the equation $2x + 4y + 3 = 0$.

Comparing with $ax + by + c = 0$, the constants are $a = 2$, $b = 4$ and $c = 3$.

Consider the following graph.



Let $(x_1, y_1) = (1, 3) \Rightarrow x_1 = 1$ and $y_1 = 3$.

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Using the perpendicular distance formula;

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow r = \frac{|2\times1 + 4\times3 + 3|}{\sqrt{2^2 + 4^2}} = \frac{17}{\sqrt{20}} = 3.8$$

$= 3.8$ units

Therefore, the perpendicular distance is 3.8 units.

Example 3.5

Find the length of the line segment drawn from the point (4, 6) and perpendicular to the line $3x + 4y - 10 = 0$.

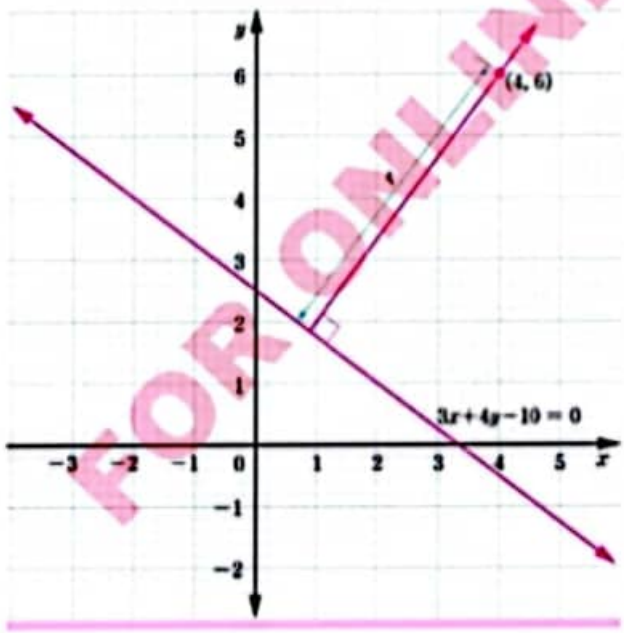
Solution

Given the equation, $3x + 4y - 10 = 0$.

Comparing with $ax + by + c = 0$,

the constants are, $a = 3$, $b = 4$, and $c = -10$.

Consider the following diagram.



Let $(x_1, y_1) = (4, 6) \Rightarrow x_1 = 4$ and $y_1 = 6$.

Using the perpendicular distance formula, it follows that

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow r = \frac{|3\times4 + 4\times6 - 10|}{\sqrt{3^2 + 4^2}} = \frac{26}{\sqrt{25}}$$

$= 5.2$ units

Therefore, the length of the perpendicular line segment is 5.2 units.

Exercise 3.2

1. Find the perpendicular distance:
   (a) From the point (3, 2) to the line $3x - 4y = -4$.
   (b) From the point $(0, -3)$ to the line $x = -5y - 2$.
   (c) From the point $(0, a)$ to the line $4x = 3y$.
   (d) From the point (h, k) to the line $3x + 4y - h = 0$.
   (e) From the point (0, 0) to the line $3x = 4y$, and hence comment on your results.
2. The perpendicular distance of the point $(p, 2)$ from the line $3x + 4y + 1 = 0$ is $\frac{\sqrt{2}}{5}$ units. Find the values of p.
3. If the lines $Ax + By + C = 0$ and $-Bx + Ay + Bs - At = 0$ are perpendicular at point Q, where A, B, C, s and t are constants, show that the coordinates of Q are given by,

$$\left(\frac{B^2s - ABt - AC}{A^2 + B^2}, \frac{-ABs + A2t - BC}{A^2 + B^2}\right).$$

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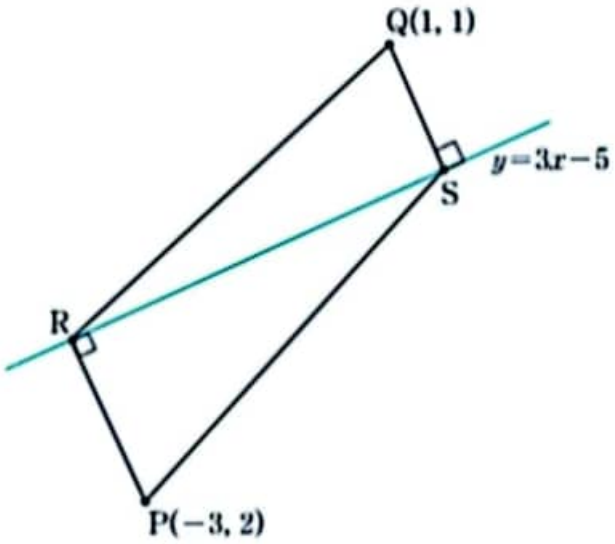
4. Given the following figure, find the distances $\overline{PR}$, $\overline{QS}$, and $\overline{RS}$.



5. If m and n are the lengths of the perpendicular lines from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$, and $x\sec\theta - y\operatorname{cosec}\theta = k$, respectively, prove that $m^2 + 4n^2 = k^2$.

6. If the perpendicular distance of the point $(3,-1)$ from the line $12x + ky = 7$ is $\frac{24}{13}$ units, find the value of k.

7. Given that the distance of the point $(n, 1)$ from the line $5y - 12x = 82$ is 5 units, determine the value of n.

8. Given a triangle with vertices P(5, 1), Q(3, 6), and R$(-1, a)$. If the length of the altitude from point R from $\overline{PQ}$ is $\frac{36\sqrt{29}}{29}$ units, find the value of a.

9. Prove that the perpendicular distances of the points $(-2,-1)$ and $(2, 2)$ from the line $3x - 4y + 5 = 0$ are equal.

10. The perpendicular distance of the point $(2,-5)$ from the line $3x + ry + 16 = 0$ is 1 unit. Verify that the values of r satisfies the equation $24r^2 - 220r + 475 = 0$.

Locus

When a point moves in such a way that it satisfies a given condition or a given set of conditions, the path it traces is called a locus. For example, if a point moves in the plane in such a way that it is always equidistant from a given point, then its locus is a circle. Also, if a set of points in three-dimensions are equidistant from a given point, its locus is a sphere. If a point (x, y) moves so that it satisfies any given condition, it will describe a locus or a curve.

The relationship between the coordinates (x, y) of any point will only be true if the point lies on the locus.

Example 3.6

Find the equation of the locus of a point which moves in such a way that it is equidistant from the point (2, 4) and the point (0, −3). What is the name of the equation of the locus formed?

Solution

Let P(x, y) be the point which is equidistant from the points (2, 4) and (0, −3) as shown in the following figure.

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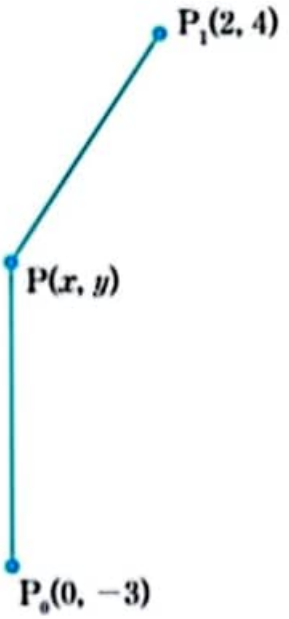


By using the distance formula, it implies that

$$\sqrt{(x-2)^2+(y-4)^2} = \sqrt{(x-0)^2+(y-(-3))^2}$$

Squaring both sides gives,

$$(x-2)^2+(y-4)^2 = (x-0)^2+(y+3)^2$$

$$\Rightarrow x^2-4x+4+y^2-8y+16 = x^2+y^2+6y+9$$

$$\Rightarrow -4x-14y+11 = 0$$

$$\Rightarrow 4x+14y-11 = 0$$

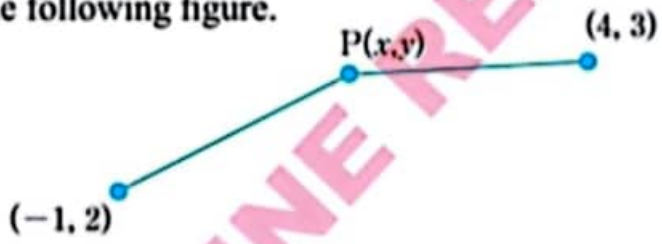
Therefore, the locus is a straight line whose equation is $4x+14y-11=0$.

Example 3.7

A point moves in such a way that its distance from the point $(-1, 2)$ is two times its distance from the point $(4, 3)$. Find the equation of the locus formed.

Solution

Consider the following figure.



Let $P(x, y)$ be the point from points $(-1, 2)$ and $(4, 3)$ that satisfies the relation. The distance from point $(-1, 2)$ is equal to 2 times the distance from the point $(4, 3)$.

Using the distance formula, it implies that

$$\sqrt{(x-(-1))^2+(y-2)^2} = 2\sqrt{(x-4)^2+(y-3)^2}$$

Squaring both sides gives,

$$(x+1)^2+(y-2)^2 = 4[(x-4)^2+(y-3)^2]$$

Expanding both sides gives,

$$x^2+2x+1+y^2-4y+4 = 4[x^2-8x+16+y^2-6y+9]$$

Simplification of the equation gives,

$$3x^2+3y^2-34x-20y+95=0$$

Therefore, the locus is $3x^2+3y^2-34x-20y+95=0$.

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Exercise 3.3

1. In each of the following, find the locus of a point which moves in such a way that:
   (a) It is 4 units to the right of the y-axis.
   (b) It is 5 units above the x-axis.
   (c) Its distance from the origin is 5.
   (d) Its distance from the point $(-2, -3)$ is 8.
   (e) The sum of the squares of its distances from the points $(2, 0)$ and $(6, 0)$ is 16 units.
   (f) Its distance from the point $(0, 0)$ is twice its distance from the line $y = 4$.
   (g) Its distance from the point $(0, 0)$ is half its distance from the line $x = 3$.
   (h) The difference between the squares of its distance from the points $(-2, 0)$ and $(2, 0)$ is 3 units.
2. Given the points A(4, 0) and B(0, 3). Find the equation of the locus of a point $P(x, y)$ which moves in such a way that $\overline{PA} = \overline{PB}$.
3. A point $P(x, y)$ moves in such a way that its distance from a line parallel to the y-axis through the point $(-a, 0)$ is equal to its distance from the point $(a, 0)$. Find the equation of the locus formed.
4. A point $P(x, y)$ is twice as far from the point A(3, 0) as it is from the line $x = 5$. Find the equation of the locus of P.
5. Find the locus of the point which moves in such a way that its perpendicular distances from the lines $4x - 3y = 0$ and $5x + 12y = 0$ are equal.
6. Find the locus of a point $Q(x, y)$ which moves in such a way that the sum of its distances from points C(2, 1) and D(−2, 1) is equal to 6.
7. If T is the point (3, −4) and S is the point (−2, −2), find the locus of a point R which moves such that $\overline{TR}^2 + 2\overline{SR}^2 = 16$.
8. Determine the locus of a point which moves such that its distance from the point $(-6, -4)$ is three times its distance from the point (12,12).
9. If G is the point (−4, 5) and H is the point (4,5), find the locus of a point P which moves such that $\overline{GP} + \overline{HP} = 10$.
10. A point B is restricted such that it is equidistant from the points M(2,4) and N(−6, −8). Determine the Cartesian equation of B.

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Ratio theorem

The ratio theorem is a fundamental concept in coordinate geometry. It deals with dividing a line segment into two parts based on a given ratio. A point can divide a line segment either internally or externally.

Internal division

Consider a line segment $\overline{AB}$ with two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be a point dividing the line $\overline{AB}$ internally in the ratio $m:n$ on rectangular Cartesian coordinates as shown in Figure 3.4.

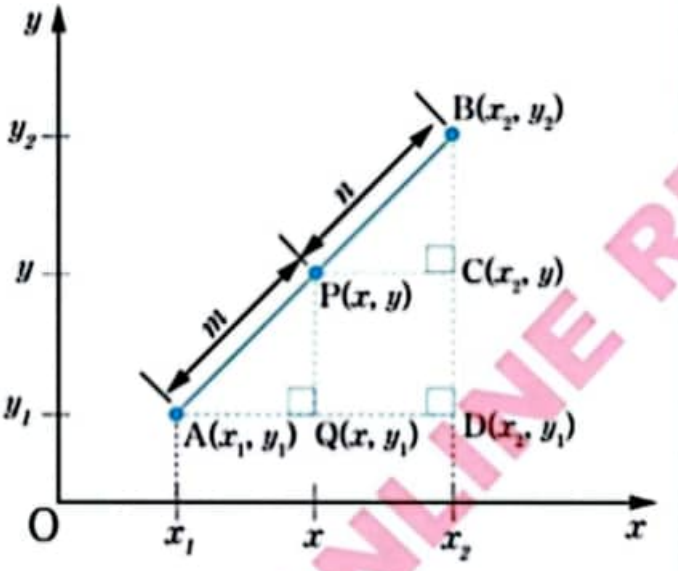

Figure 3.4: *Internal division of a line*

In Figure 3.4, it can be observed that ΔAPQ and ΔPBC are similar.

Hence, $\frac{\overline{AQ}}{\overline{PC}} = \frac{\overline{AP}}{\overline{PB}}$.

But $\overline{AQ} = x - x_1$ and $\overline{PC} = x_2 - x$.

Thus, $\frac{x - x_1}{x_2 - x} = \frac{m}{n}$.

Making x the subject of the formula gives,

$$x = \frac{mx_2 + nx_1}{m+n} \qquad (3.1)$$

Similarly, $\frac{\overline{QP}}{\overline{CB}} = \frac{\overline{AP}}{\overline{PB}}$.

But $\overline{QP} = y - y_1$ and $\overline{CB} = y_1 - y$.

Thus, $\frac{y - y_1}{y_2 - y} = \frac{m}{n}$.

Making y the subject of the formula gives,

$$y = \frac{my_2 + ny_1}{m+n} \qquad (3.2)$$

The expressions for x and y in equations (3.1) and (3.2) give the coordinates of the point $P(x, y)$ which divides the line joining the points, $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$, respectively.

That is,

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

Example 3.8

Find the coordinates of the point that divides internally the line joining the points, $A(-4, 10)$ and $B(8, 4)$ in the ratio of 1:3.

Solution

Given the ratio $1:3 \Rightarrow m = 1$ and $n = 3$.

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Also,

$(x_1, y_1) = (-4, 10)$ and $(x_2, y_2) = (8, 4)$.

From $P(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$,

$$\Rightarrow P(x, y) = \left(\frac{1(8)+3(-4)}{1+3}, \frac{1(4)+3(10)}{1+3}\right)$$

$$= \left(\frac{8-12}{4}, \frac{4+30}{4}\right)$$

$$= \left(-1, \frac{17}{2}\right)$$

Therefore, the point of internal division is

$P(x, y) = \left(-1, \frac{17}{2}\right)$.

Example 3.9

A point P divides internally the line segment joining the points A(4, 6) and B(−5, 2) in the ratio $\overline{AP} : \overline{PB} = 2:3$. Find the coordinates of P.

Solution

Given the ratio $2:3 \Rightarrow m = 2$ and $n = 3$.

Also, $(x_1, y_1) = (4, 6)$, and $(x_2, y_2) = (-5, 2)$.

From $P(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$,

$$\Rightarrow P(x, y) = \left(\frac{(2\times-5)+(3\times4)}{2+3}, \frac{(2\times2)+(3\times6)}{2+3}\right)$$

$$= \left(\frac{-10+12}{5}, \frac{4+18}{5}\right)$$

$$= \left(\frac{2}{5}, \frac{22}{5}\right)$$

Therefore, the point of internal division is $P(x, y) = \left(\frac{2}{5}, \frac{22}{5}\right)$.

External division

Let $P(x, y)$ be a point which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio m n. The formula for computing the coordinates of P can be derived by considering the sketch in Figure 3.5.

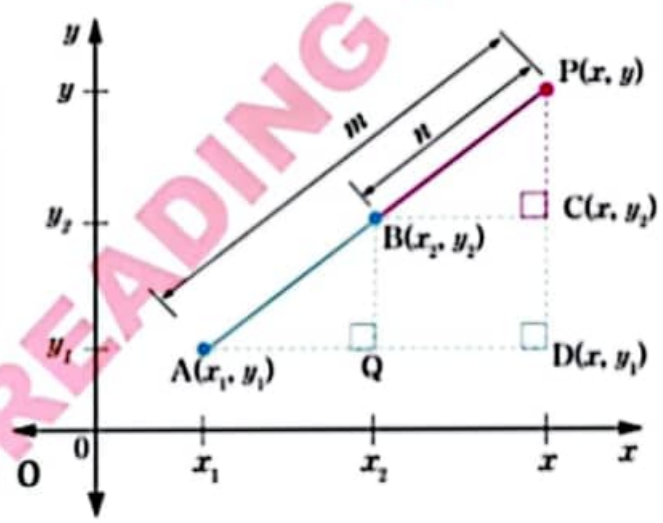

Figure 3.5: *External division of a line*

In Figure 3.5, ΔAPD is similar to ΔBPC.

Hence, $\frac{\overline{BC}}{\overline{AD}} = \frac{\overline{BP}}{\overline{AP}}$.

Thus, $\frac{x - x_2}{x - x_1} = \frac{n}{m}$.

Making x the subject of the formula gives,

$$x = \frac{nx_1 - mx_2}{n - m}. \qquad (3.3)$$

Similarly, $\frac{\overline{PC}}{\overline{PD}} = \frac{\overline{BP}}{\overline{AP}}$.

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Thus, $\frac{y-y_2}{y-y_1}=\frac{n}{m}$.

Making y the subject of the formula gives, $y=\frac{ny_1-my_2}{n-m}$. (3.4)

If equations (3.1) and (3.3) are compared, it can be observed that they are similar except that m in equation (3.1) has been replaced by $-m$ in equation (3.3). Similarly, m in equation (3.2) has been replaced by $-m$ in equation (3.4). The expressions for x and y in equations (3.3) and (3.4), respectively, are the coordinates of the point $P(x,y)$ which divides the line joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ externally in the ratio $m:n$.

Thus, $P(x,y)=\left(\frac{nx_1-mx_2}{n-m},\frac{ny_1-my_2}{n-m}\right)$.

Example 3.10

Find the coordinates of the point that externally divides the line joining the points, A(–2, 8) and B(7, 3) in the ratio of 2:3.

Solution

Let $P(x,y)$ be the point dividing the line in the ratio $2:3 \Rightarrow m=2$ and $n=3$.

Also, let $(x_1,y_1)=(-2,8)$ and $(x_2,y_2)=(7,3)$.

From $P(x,y)=\left(\frac{nx_1-mx_2}{n-m},\frac{ny_1-my_2}{n-m}\right)$,

$$P(x,y)=\left(\frac{3(-2)-2(7)}{3-2},\frac{3(8)-2(3)}{3-2}\right)$$

$$=(-20,18).$$

Therefore, the point of external division is $P(x,y)=(-20,18)$.

Example 3.11

If the point $P(x,6)$ divides externally the line segment joining the points, A (–1, 3) and B (4, y) in the ratio 3 : 2, find the values of x and y.

Solution

Given the ratio $3:2 \Rightarrow m=3$ and $n=2$.

Let $(x_1,y_1)=(-1,3)$ and $(x_2,y_2)=(4,y)$.

From $P(x,y)=\left(\frac{nx_1-mx_2}{n-m},\frac{ny_1-my_2}{n-m}\right)$.

$$\Rightarrow (x,6)=\left(\frac{2(-1)-(3)(4)}{2-(3)},\frac{2(3)-(3)y}{2-(3)}\right)$$

The value of x is given by;

$$x=\frac{-2-12}{-1} \Rightarrow x=14$$

Also, the value of y is given by;

$$6=\frac{2(3)-(3)y}{2-(3)} \Rightarrow 6=\frac{6-3y}{-1}$$

$$\Rightarrow -6-6=-3y \Rightarrow y=4$$

Therefore, the values of $x=14$ and $y=4$.

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Exercise 3.4

1. For each of the following pairs of points, find the points which divide the line segment both internally and externally:
   (a) A(3, 6) and B(−8,−2) in the ratio 2:3
   (b) C(1, 1) and D(−3,5) in the ratio 1:1
   (c) Q(−3,12) and R(−8,−2) in the ratio 2:3
   (d) S(3, 6) and T(7, 6) in the ratio 5:1

2. Points A and B have coordinates (7, 6) and (−6, −10), respectively. If the point C divides $\overline{AB}$ externally in the ratio of 1:2, find the coordinates of C.

3. A line joins the points P(−4,−3) and Q(3,2). Given that, point B divides $\overline{PQ}$ internally in the ratio 1:2 and point R divides $\overline{PQ}$ internally in the ratio 2:5, find the length of $\overline{BR}$.

4. A line $\overline{PQ}$ is divided internally in the ratio 2:1 by the point A(1, 1), and externally in the ratio 5:2 by the point B(4, 7). Find the coordinates of P and Q.

5. A line $\overline{PQ}$ is divided internally in the ratio $m:n$ by a point B. Given that P, Q, and B have the coordinates (−2, −4), (3, 11), and (1,5), respectively, find the values of m and n.

6. The line $2x+y-4=0$ divides the line segment joining the points A(2,−2) and B(3,7). Find the ratio of the line segment.

7. Verify that the points (2,7) and (−52,−11) divide the points K(8,9) and L(−7,4) internally in the ratio 4:3.

8. A point R(2,−3) divides externally the line segment joining the points P(−2,3) and Q(−4, 6) in the ratio $m:n$. Find the values of m and n.

9. Find the ratio in which the point (1,7) divides the line segment joining the points (4,3) and (10,−8) externally.

10. Determine the coordinates of the point that divides internally the line joining the points D(5,2) and E(7,9) in the ratio 4:9.

11. A line joins the points U(−8,−6) and V(4,5). Given that, point A(x,y) divides $\overline{UV}$ internally in the ratio 2:4 and the point B(x_1,y_1) divides $\overline{UV}$ internally in the ratio 6:8, find the coordinates of A and B.

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A circle

A circle is the locus of a point which moves such that its distance from a fixed point is constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle. The diameter of a circle is any line segment through the centre of the circle with end points at the circumference of the circle. Objects such as coins, wheels of a bicycle or vehicle, camera lenses, pizzas, Ferris wheels, rings, steering wheels, and bottle lids are all examples of circular objects. If the middle point of a circular object is located, then the distance from the middle point towards any point of the edge will always be the same.

Standard equation of a circle

Suppose $C(a,b)$ is the point at the centre of a circle, r is its radius, and $P(x, y)$ is any point on the edge of the circle as shown in Figure 3.6, then the standard equation of the circle can be obtained as follows.

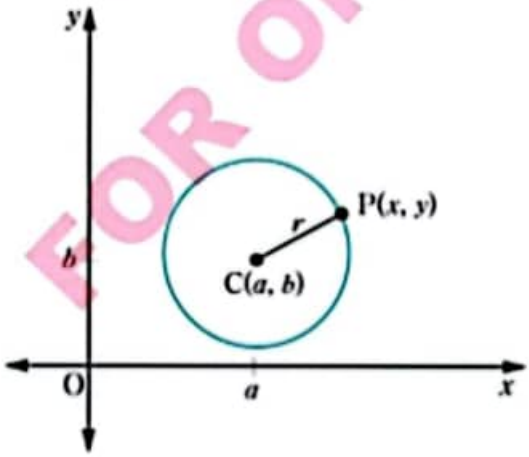

Figure 3.6: *A circle*

From Figure 3.6, the radius, $r = \overline{CP}$.

Using the distance formula gives,

$$\overline{CP} = \sqrt{(x-a)^2 + (y-b)^2}$$

But $\overline{CP} = r$.

Thus, $\sqrt{(x-a)^2 + (y-b)^2} = r$.

Squaring both sides gives,

$$(x-a)^2 + (y-b)^2 = r^2 \qquad (3.5)$$

Equation (3.5) is the standard equation of a circle.

Example 3.12

Write the standard equation of a circle, given the following:

(a) Centre (3, 2) and radius 4 units.

(b) Centre (0, 4) and diameter 6 units.

Solution

(a) Given the centre $(a, b) = (3, 2)$ and $r = 4$. Using the standard equation of the circle gives,

$$(x-a)^2 + (y-b)^2 = r^2$$
$$\Rightarrow (x-3)^2 + (y-2)^2 = 4^2$$

Therefore, the standard equation is $(x-3)^2 + (y-2)^2 = 16$.

(b) Given the centre $(a,b) = (0,4)$ and $d = 2r = 6$ units $\Rightarrow r = 3$ units. The standard equation of a circle is given by $(x-0)^2 + (y-4)^2 = 3^2$

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$\Rightarrow (x-0)^2 + (y-4)^2 = 9$

Therefore, the standard equation is
$x^2 + (y-4)^2 = 9$.

The general equation of a circle

The general equation of a circle is an expansion of the standard form of the equation of a circle of radius r and centre (a, b).

Thus, from equation (3.5), the general equation is obtained as follows:

$(x-a)^2 + (y-b)^2 = r^2$, expanding gives,

$x^2 - 2xa + a^2 + y^2 - 2yb + b^2 = r^2$

$\Rightarrow x^2 + y^2 - 2xa - 2yb + a^2 + b^2 = r^2$

$\Rightarrow x^2 + y^2 - 2xa - 2yb + a^2 + b^2 - r^2 = 0$

$\Rightarrow x^2 + y^2 + 2(-a)x + 2(-b)y + a^2 + b^2 - r^2 = 0$ (3.6)

Let $g = -a$, $f = -b$ and $c = a^2 + b^2 - r^2$.
Equation (3.6) becomes;

$x^2 + y^2 + 2gx + 2fy + c = 0$ (3.7)

Equation (3.7) is called the general equation of the circle.

Note that, when comparing the coordinates of the centre and radius of the circle, the coefficients of x^2 and y^2 in any form of the equation of a circle must be the same and positive. Otherwise, it will not conform to the equation of a circle.

Since $c = a^2 + b^2 - r^2$, then

$r^2 = a^2 + b^2 - c$.

$\Rightarrow r = \sqrt{a^2 + b^2 - c}$

Since $g = -a$ and $f = -b$ then
$a = -g$ and $b = -f$.

Hence, the centre and radius of a circle are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

Example 3.13

Find the centre and the radius of the circle whose equation is
$x^2 + y^2 - 4x + 2y + 1 = 0$.

Solution

Given the equation
$x^2 + y^2 - 4x + 2y + 1 = 0$.

Compare the given equation with the general equation of a circle.

The values are $g = -2$, $f = 1$, and $c = 1$. The radius, r and the centre of the circle are given by,

$r = \sqrt{g^2 + f^2 - c}$.

$= \sqrt{(-2)^2 + (1)^2 - 1} = \sqrt{4 + 1 - 1}$

$\Rightarrow r = 2$

Centre $= (-g, -f)$

$= (2, -1)$

Therefore, the centre of the circle is $(2, -1)$ and radius is 2 units.

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Alternatively, it is possible to find the centre and radius if the general equation of a circle is changed into the standard equation by completing the square. That is,

$x^2 + y^2 - 4x + 2y + 1 = 0$

$\Rightarrow x^2 - 4x + 4 + y^2 + 2y + 1 = 4$

$\Rightarrow (x-2)^2 + (y+1)^2 = 4$

$\Rightarrow (x-2)^2 + (y+1)^2 = 2^2$

Compare the resulting equation with the standard equation of a circle,

$(x-a)^2 + (y-b)^2 = r^2$

Thus, centre $(2,-1)$ and $r = 2$.

Therefore, the centre of the circle is $(2,-1)$ and its radius is 2 units.

Example 3.14

Find the general equation of the circle whose centre and radius are (1, 3) and 2 units, respectively.

Solution

From the general form of equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Centre = (1, 3), then $g = -1$ and $f = -3$.

The radius, $r = \sqrt{g^2 + f^2 - c}$.

$\Rightarrow 2 = \sqrt{(-1)^2 + (-3)^2 - c}$

$\Rightarrow 2 = \sqrt{10 - c} \Rightarrow c = 6$

Substituting all the values into the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ gives,

$x^2 + y^2 - 2x - 6y + 6 = 0$.

Therefore, the general equation of the circle is $x^2 + y^2 - 2x - 6y + 6 = 0$.

Exercise 3.5

1. Find the general equation of a circle using each of the following information:
   (a) Centre = (0, 2) and radius = 4 units
   (b) Centre = (–2, 2) and radius = 5 units
   (c) Centre = (0, 0) and diameter = 4 units
   (d) Centre $= \left(-\frac{1}{2}, 5\right)$ and radius = 10 units
   (e) Centre = (2, –3) and diameter $= \frac{\sqrt{2}}{2}$ units
2. Find the centre and radius of each of the circles given by the following equations:
   (a) $(x-1)^2 + (y+2)^2 = 45$
   (b) $x^2 + y^2 = 25$
   (c) $x^2 + y^2 - 4y = 0$
   (d) $(x+1)^2 + y^2 = 4$
   (e) $(2x-1)^2 + (2y+3)^2 = 4$
3. Find the equation of a circle which touches both axes at a distance 4 units from the origin.
4. Find the equation of the circle of radius 12 units whose centre is at the point at which the lines $y = 4x + 3$ and $y = 5x + 44$ meet.
5. Find the equation of the circle whose centre is at the point (5, 2) and passes through the centre of the circle $x^2 + y^2 - 3x - 4y - 1 = 0$.
6. Determine the radius and centre of the circle $3x^2 + 3y^2 - 12x + 15y + 21 = 0$.

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7. Show that the equation of the circle with centre $\left(-\frac{1}{8}, \frac{1}{16}\right)$ and radius $\frac{1}{4}$ units is $256x^2 + 256y^2 + 64x - 32y - 11 = 0$.

8. Determine the centre and diameter of the circle $(x-5)^2 + (y+6)^2 = 12$.

9. Given the points D $(3,-1)$ and E $(5,3)$,
   (a) Show that the locus of a point $P(x, y)$ which moves so that $\overline{PD}^2 + \overline{PE}^2 = 28$ is a circle whose equation is $2x^2 + 2y^2 - 16x - 4y + 7 = 0$.
   (b) Find the centre, radius, and diameter of the circle.

10. Determine the equation of the circle whose centre is at the point $(-6, -4)$ and diameter is 24 units.

Equation of a circle given the end points of its diameter

Let $\overline{AB}$ be a diameter of a circle with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$. If $P(x, y)$ is any point on the circle, then $A\hat{P}B = 90°$. Figure 3.7 illustrates the statement.

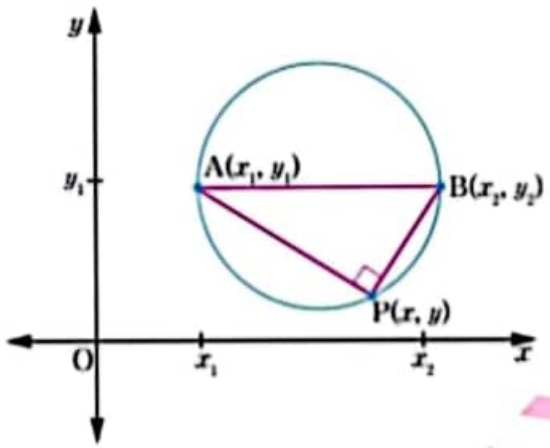

Figure 3.7: *Circle with end points of a diameter*

In Figure 3.7, the gradient of $\overline{AP} = \frac{y - y_1}{x - x_1}$

Thus,

$$m_1 = \frac{y - y_1}{x - x_1} \qquad (3.8)$$

Similarly, the gradient of $\overline{BP} = \frac{y - y_2}{x - x_2}$.

That is,

$$m_2 = \frac{y - y_2}{x - x_2} \qquad (3.9)$$

But $\overline{AP}$ is perpendicular to $\overline{BP}$, that is $m_1 m_2 = -1$. Substituting equations (3.8) and (3.9) gives,

$$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

This implies that,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0. \qquad (3.10)$$

Equation (3.10) can be used to find the equation of a circle when given the two end points of a diameter.

Example 3.15

Find the equation of the circle of diameter $\overline{AB}$ with end points $A(2, 1)$ and $B(-2, 2)$.

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Solution

Given the end points of a diameter, $A(x_1, y_1) = (2,1)$ and $B(x_2, y_2) = (-2,2)$.

Use the equation of the circle of the form

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Substituting the given values in the equation gives,

$$(x-2)(x-(-2))+(y-1)(y-2)=0$$

$$\Rightarrow (x-2)(x+2)+(y-1)(y-2)=0$$

$$\Rightarrow x^2-2x+2x-4+y^2-y-2y+2=0$$

$$\Rightarrow x^2+y^2-3y-2=0$$

Therefore, the equation of a circle is $x^2+y^2-3y-2=0$.

Equation of a circle passing through three given points

Suppose $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, and $A_3(x_3, y_3)$ are three points on a circle with the centre (a, b) and radius r. Substitution of each of the given point in the standard equation of the circle helps to determine the unknown values.

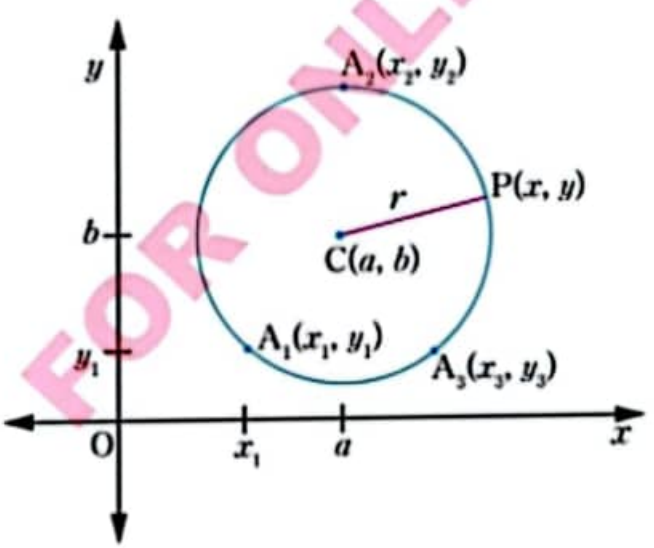

Figure 3.8: *Three points on a circle*

Using Figure 3.8, the standard equation of the circle is given by,

$$(x-a)^2+(y-b)^2=r^2.$$

Substitute the three given points lying on a circle:

For a point $A_1(x_1, y_1)$, the equation will be;

$$(x_1-a)^2+(y_1-b)^2=r^2 \qquad (3.11)$$

For a point $A_2(x_2, y_2)$, the equation will be;

$$(x_2-a)^2+(y_2-b)^2=r^2 \qquad (3.12)$$

For a point $A_3(x_3, y_3)$, the equation will be;

$$(x_3-a)^2+(y_3-b)^2=r^2 \qquad (3.13)$$

The three equations (3.11), (3.12), and (3.13) can be combined to form a system of simultaneous equations. That is,

$$\begin{cases}(x_1-a)^2+(y_1-b)^2=r^2\\(x_2-a)^2+(y_2-b)^2=r^2\\(x_3-a)^2+(y_3-b)^2=r^2\end{cases}$$

The system of equations can be solved simultaneously to obtain the values of a, b, and r which when substituted back, the standard equation of the circle is obtained. The same procedure can be used when the general equation of a circle is considered and gives the following system of simultaneous equations:

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$$\begin{cases} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \end{cases}$$

The system of equations can be solved simultaneously to obtain the values of g, f, and c which when substituted back, the general equation of the circle will be obtained.

Example 3.16

Find the equation of the circle which passes through the points, A(3, 1), B(2, 6), and C(3, 2).

Solution

The given points, $A(x_1, y_1) = (3, 1)$, $B(x_2, y_2) = (2, 6)$, and $C(x_3, y_3) = (3, 2)$ lie on a circle. Thus, they satisfy the system of general equations of a circle.

The system of general equations of a circle formed by the points is given by:

$$\begin{cases} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \end{cases}$$

Substitute the given points to get the values of g, f, and c:

$$\begin{cases} 3^2 + 1^2 + 2(3)g + 2(1)f + c = 0 \\ 2^2 + 6^2 + 2(2)g + 2(6)f + c = 0 \\ 3^2 + 2^2 + 2(3)g + 2(2)f + c = 0 \end{cases} \Rightarrow \begin{cases} 6g + 2f + c = -10 \\ 4g + 12f + c = -40 \\ 6g + 4f + c = -13 \end{cases}$$

Solving the system of equations simultaneously, the values of g, f, and c are $\frac{15}{2}$, $-\frac{3}{2}$, and -52, respectively.

Therefore, the equation of a circle is $x^2 + y^2 + 15x - 3y - 52 = 0$.

Alternatively: The three given points can be substituted in the standard equation of the circle.

$(x-a)^2 + (y-b)^2 = r^2$

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For the point $A(x_1, y_1) = (3, 1)$, the equation will be

$(3-a)^2 + (1-b)^2 = r^2$ ..... (i)

For the point $B(x_2, y_2) = (2, 6)$, the equation will be

$(2-a)^2 + (6-b)^2 = r^2$ ..... (ii)

For the point $C(x_3, y_3) = (3, 2)$, the equation will be

$(3-a)^2 + (2-b)^2 = r^2$ .....(iii)

Equations (i), (ii), and (iii) can be combined to obtain,

$$\begin{cases}(3-a)^2 + (1-b)^2 = r^2 \\ (2-a)^2 + (6-b)^2 = r^2 \\ (3-a)^2 + (2-b)^2 = r^2\end{cases}$$

Equating equations (i) and (ii) gives,

$(3-a)^2 + (1-b)^2 = (2-a)^2 + (6-b)^2$

$\Rightarrow a^2 - 6a + 9 + b^2 - 2b + 1 = a^2 - 4a + 4 + b^2 - 12b + 36$

$\Rightarrow -2a + 10b = 30$

$\Rightarrow a - 5b = -15$ .....(iv)

Equating equations (ii) and (iii) gives,

$(2-a)^2 + (6-b)^2 = (3-a)^2 + (2-b)^2$

$\Rightarrow a^2 - 4a + 4 + b^2 - 12b + 36 = a^2 - 6a + 9 + b^2 - 4b + 4$

$\Rightarrow 2a - 8b = -27$ .....(v)

Solve equations (iv) and (v) simultaneously to get, $a = -\frac{15}{2}$ and $b = \frac{3}{2}$.

Thus, the centre of the circle is $(a, b) = \left(-\frac{15}{2}, \frac{3}{2}\right)$. The radius of the circle is computed by substituting the values of a and b in one of the equations in the system of simultaneous equations. Using equation (i), it follows that

$$\left(3-\left(-\frac{15}{2}\right)^2\right)^2 + \left(1-\frac{3}{2}\right)^2 = r^2$$

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$$\left(3+\frac{15}{2}\right)^2+\left(3+\frac{3}{2}\right)^2=r^2$$

$$\Rightarrow r^2=\frac{221}{2}$$

$$\Rightarrow r=\sqrt{\frac{221}{2}}$$

Thus, the circle has centre at $\left(-\frac{15}{2},\frac{3}{2}\right)$ and radius, $r=\sqrt{\frac{221}{2}}$.

The equation of the circle is given by;

$$\left(x+\frac{15}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{221}{2}$$

$$\Rightarrow x^2+15x+\frac{225}{4}+y^2-3y+\frac{9}{4}=\frac{221}{2}$$

Thus, $x^2+y^2+15x-3y-52=0$.

Therefore, the equation of the circle is $x^2+y^2+15x-3y-52=0$.

Exercise 3.6

1. Find the equation of the circle with $\overline{AB}$ as a diameter when the coordinates of A and B are respectively:
   (a) $(-2,3)$ and $(-5,2)$
   (b) $(6,1)$ and $(-2,-5)$
   (c) $(-1,1)$ and $(0,-3)$
   (d) $(8,-3)$ and $(-7,-1)$
   (e) $(0,3)$ and $(-4,0)$
2. Find the equation of the circle which passes through each of the following points:
   (a) $(-1,-5)$, $(6,2)$, and $(-2,-2)$
   (b) $(1,2)$, $(0,-6)$, and $(4,0)$
   (c) $(-5,-10)$, $(-6,-5)$, and $(12,7)$
   (d) $(1,1)$, $(1,7)$, and $(8,8)$
   (e) $(1,-5)$, $(1,2)$, and $(0,-2)$
3. A triangle has vertices (0, 6), (4, 0), and (6, 0). Find the equation of the circle passing through the three points.
4. Show that the equation of the circle with the line joining the points F(−4, −3) and G(5,6) as its diameter is $x^2+y^2-x-3y-38=0$. Hence, find its centre and radius.
5. Find the equation of the circle which is circumscribed about the triangle whose vertices are H(6,−1), K(5,2), and J(−2,3).

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6. Show that the equation of the circle passing through the points, C(−2,−4) and D(4,−6) and the line segment $\overline{CD}$ as a diameter is $x^2 + y^2 - 2x + 10y + 16 = 0$.

7. The end points of the diameter of a circle are P(−16,4) and Q(8,12). Find;
   (a) the coordinates of the centre
   (b) the radius of the circle
   (c) the equation of the circle

8. Verify that the equation of the circle that is circumscribed about the triangle with vertices J(5,0), K(6,0), and L(8,6) is $x^2 + y^2 - 11x - 7y + 30 = 0$.

9. Assume you are a surveyor tasked with mapping out a circular garden in a park. Three trees are marked as reference points, located at the first tree (5, 8), the second tree (3, −4) and the third tree (−2, 6), respectively. Determine the equation of the circular boundary of the garden.

Equation of a tangent to a circle

A tangent to a circle is a straight line which touches the circle at only one point. The point where it touches the circle is called the point of tangency. The tangent line to a circle is always perpendicular to the line from the centre to the point of tangency as shown in Figure 3.9.

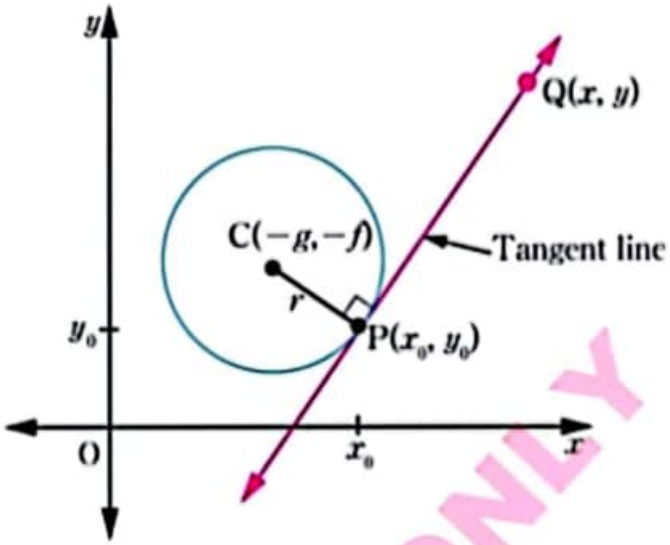

Figure 3.9: *Tangent line to a circle*

In Figure 3.9, C(−g, −f) is the centre of the circle, $P(x_0, y_0)$ is the point of tangency and Q(x, y) is any other point on the tangent line.

The gradient of the line $\overline{CP}$ from the centre to the point of tangency is given by, $\overline{CP} = \frac{y_0 + f}{x_0 + g}$.

The tangent line is always perpendicular to the line drawn from the point of tangency to the centre of the circle. The slope (gradient) of the tangent line equals the negative reciprocal of the slope of $\overline{CP}$. Therefore, the slope of the tangent line is $-\left(\frac{x_0 + g}{y_0 + f}\right)$.

Thus, the equation of the tangent line is given by $\frac{y - y_0}{x - x_0} = -\left(\frac{x_0 + g}{y_0 + f}\right)$.

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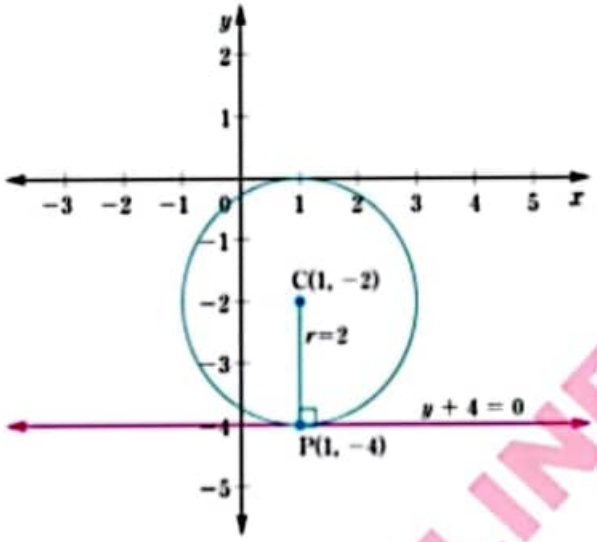
Example 3.17

Find the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ at the point P(1, – 4).

Solution

Comparing the given equation $x^2 + y^2 - 2x + 4y + 1 = 0$ with the general equation of the circle, the centre of the circle is $(-g, -f) = (1, -2)$ and the radius of the circle is $r = 2$. Consider the following figure.



From the Figure, the slope of $\overline{CP}$ is

$$m_1 = \frac{-4-(-2)}{1-1} = \text{undefined}.$$

Undefined slope denotes that the line is parallel to the y-axis. Thus, the tangent is parallel to the x-axis.

Thus, slope of the tangent,

$$m_2 = -\left(\frac{1-1}{-4-(-2)}\right) = 0.$$

If $Q(x, y)$ is any other point on the tangent line, then

$$m_2 = \frac{y-(-4)}{x-1} \Rightarrow 0 = \frac{y+4}{x-1}.$$

Therefore, the equation of the tangent line is $y + 4 = 0$.

Example 3.18

If $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, show that

$$a = \left|\frac{c}{\sqrt{1+m^2}}\right|.$$

Solution

Given $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$. The tangent line satisfies the equation of the circle.

Thus, substituting $y = mx + c$ in the circle gives,

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow x^2 + m^2x^2 + 2mcx + c^2 = a^2$$

$$\Rightarrow (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0,$$

which is a quadratic equation.

Recall the general formula for solving a quadratic equation, $Ax^2 + Bx + C = 0$.

That is, $x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

$\Rightarrow A = 1 + m^2$, $B = 2mc$, and $C = c^2 - a^2$

Substituting the values of A, B, and C into the general quadratic formula gives,

$$x = \frac{-2mc \pm \sqrt{(2mc)^2 - 4(1+m^2)(c^2-a^2)}}{2(1+m^2)}$$

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Since the line is tangent to the circle, then it touches the circle, and hence the discriminant is zero.

Thus, $(2mc)^2 - 4(1+m^2)(c^2-a^2) = 0.$

$\Rightarrow 4m^2c^2 - 4(1+m^2)(c^2-a^2) = 0$

$\Rightarrow m^2c^2 - (1+m^2)(c^2-a^2) = 0$

$\Rightarrow m^2c^2 - (c^2 + m^2c^2 - a^2 - a^2m^2) = 0$

$\Rightarrow -c^2 + a^2 + a^2m^2 = 0$

$\Rightarrow c^2 = a^2(1+m^2)$

Thus, $a^2 = \dfrac{c^2}{1+m^2}$.

Take square root on both sides to get, $a = \pm\dfrac{c}{\sqrt{(1+m^2)}}$.

Therefore, $a = \left|\dfrac{c}{\sqrt{1+m^2}}\right|$.

Example 3.19

Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the straight line $4x + 3y + 5 = 0$.

Solution

Any straight line parallel to the given equation is in the form of $4x + 3y + c = 0$, where c is a constant.

The equation of the circle is given as

$x^2 + y^2 - 6x + 4y = 12$

$\Rightarrow (x-3)^2 + (y+2)^2 = 5^2$

Thus, the centre of the circle is $(3,-2)$ and the radius is 5.

For the straight line to be tangent to the circle, its distance from the point $(3,-2)$ must be equal to 5.

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Hence, substituting $(x_1, y_1) = (3, -2)$, $a = 4$, $b = 3$, and $r = 5$ in the perpendicular distance formula,

$$r = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$\Rightarrow 5 = \left| \frac{12 - 6 + c}{\sqrt{4^2 + 3^2}} \right|$$

It follows that

$$25 = |6 + c|$$

$$\Rightarrow 25 = \pm(6 + c)$$

$$\Rightarrow c = 19 \text{ or } c = -31.$$

Therefore, the equations of the tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

Equation of a normal to a circle

The normal to a circle is a straight line drawn at 90° to the tangent at the point where the tangent touches the circle. On the other hand, the normal is referred to as a perpendicular line to the tangent at the point of contact to the circle as shown in Figure 3.10.

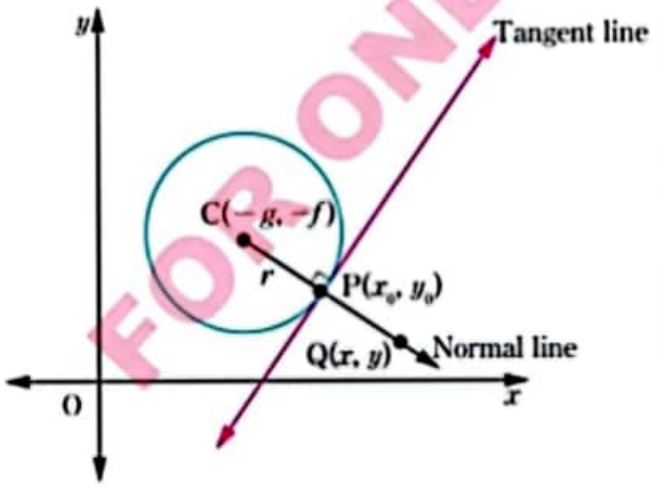

Figure 3.10: *Normal line to a circle*

From Figure 3.10, it can be observed that the slope of the normal is the same as the slope of $\overline{CP}$, that is, the slope of the line joining the centre of the circle and the given point of tangency. In Figure 3.10, $P(x_0, y_0)$ is the point of tangency on the circle and $Q(x, y)$ is any other point on the line normal to the circle.

Now, the slope of $\overline{CP} = \frac{y_0 + f}{x_0 + g}$.

Since the slope of the normal line PQ is equal to the slope of $\overline{CP}$, then the equation of the normal line PQ, is given as $\frac{y - y_0}{x - x_0} = \frac{y_0 + f}{x_0 + g}$.

Example 3.20

Find the equation of the normal to the circle $x^2 + y^2 - 6x + 4y = 12$ at the point $(-1, -5)$.

Solution

Compare the equation of a circle $x^2 + y^2 - 6x + 4y = 12$ with the general equation of the circle. The centre of the circle is $(-g, -f) = (3, -2)$ and radius, $r = 5$.

Slope of the normal equation is given by, $m = \frac{-5 + 2}{-1 - 3} = \frac{3}{4}$.

Suppose (x, y) is any other point on the normal, then

$$m = \frac{y + 5}{x + 1}$$

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Thus, $\frac{y+5}{x+1}=\frac{3}{4}$.

$\Rightarrow -3x+4y+17=0$

Therefore, the equation of the normal is $-3x+4y+17=0$.

Exercise 3.7

1. Find the equations of the tangent and the normal to each of the following circles at the points given:
   (a) $8x^2+8y^2=5, \left(\frac{3}{4},-\frac{1}{4}\right)$
   (b) $x^2+y^2=25, (-3,4)$
   (c) $(x-1)^2+(y+2)^2=5, (3,-3)$
   (d) $4x^2+4y^2-4x+8y-15=0, \left(1\frac{1}{2}, 1\right)$
   (e) $x^2+y^2-6x-2y-3=0, (5,4)$
   (f) $x^2+y^2+4x+6y=0, (0,0)$
2. A tangent to the circle $x^2+y^2=5$ at the point $(1, -2)$ also touches the circle $x^2+y^2-8x+6y+20=0$. Find the coordinates of the corresponding point of contact.
3. Find the values of k such that the line $3x+4y=k$ touches the circle $x^2+y^2=10x$.
4. Find the equation of the normal to the circle $2x^2+2y^2-2x-5y+3=0$ at the point $(1, 1)$.
5. Show that the tangents to the circle $x^2+y^2+6x-2y+5=0$ at the points $P(-1, 2)$ and $Q(-2, -1)$, respectively meet at the origin O.
6. Show that the tangent to the circle $x^2+y^2-6x+2y+5=0$ at the point $(1, 0)$ touches the circle $5x^2+5y^2=4$.
7. Find the equation of the diameter of the circle $x^2+y^2-4x-6y+12=0$ that is parallel to the line $x+y=7$.
8. Find the equations of the tangents to the circle $x^2+y^2+4x-2y-24=0$ at the points where the circle crosses the line $y-x=0$.
9. Find the condition for the line $y=mx+c$ to touch the circle $(x-a)^2+(y-b)^2=r^2$. Hence, find the point of contact.
10. Find the equations of the tangents to the circle $x^2+y^2=25$ which are parallel to the line $4x-3y-2=0$.
11. Find the equations of the tangents to the circle $x^2+y^2-6x+10y+29=0$ which are parallel to the line $2x+y+8=0$.
12. Show that the line $y=2x$ is tangent to the circle $x^2+y^2-8x-y+5=0$.
13. Find the equations of the tangents to the circle $x^2+y^2-6x+4y+8=0$ which are perpendicular to the line $2x-y-1=0$.

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14. The tangent to the circle $x^2+y^2-2x-6y+5=0$ at the point $(3, 4)$ meets the x-axis at M. Find the distance of M from the centre of the circle.

15. Find the equation of the circle centred at D$(-3,-5)$ and touches the line $12x+5y-4=0$.

Point of intersection of circles

There are three cases in which two circles can intersect. The first case is when two circles touch each other externally, the second case is when two circles of different sizes touch each other internally, and the third case is when two circles intersect at two distinct points.

(i) External point of intersection of circles

Two circles touch each other externally if and only if the distance between their centres is equal to the sum of their radii. Consider two circles touching each other externally as shown in Figure 3.11.

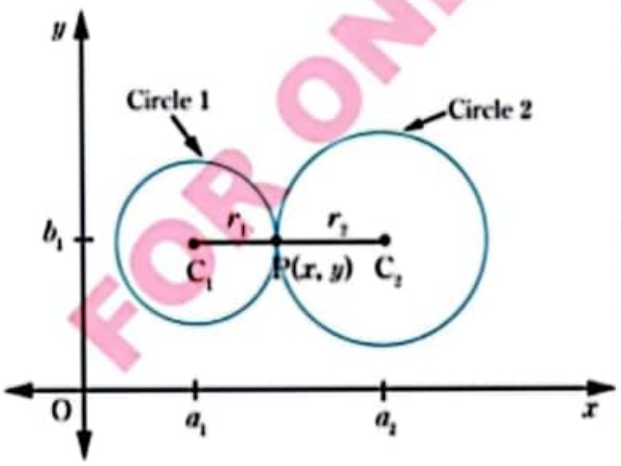

Figure 3.11: *Two circles intersecting externally*

Suppose $C_1(a_1,b_1)$ and $C_2(a_2,b_2)$ are the coordinates of the centres of the circles $C_1:(x-a_1)^2+(y-b_1)^2=r_1^2$ and $C_2:(x-a_2)^2+(y-b_2)^2=r_2^2$, where r_1 and r_2 are the radii of the circles.

From Figure 3.11, the distance between the centres of the circles C_1 and C_2 is given by;

$$\overline{C_1C_2}=\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}$$

The distance $\overline{C_1C_2}$ is equal to the sum of their radii which is given by $r=r_1+r_2$.

Thus, $\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}=r_1+r_2$.

Therefore, the condition for two circles to intersect each other externally at one point is given by;

$$\overline{C_1C_2}=r_1+r_2$$

Example 3.21

Show that the circles whose equations are $x^2+y^2-4x-6y-3=0$ and $x^2+y^2-12x-12y+71=0$ intersect externally at one point.

Solution

If the two circles intersect externally, then they must satisfy the equation

$\overline{C_1C_2}=r_1+r_2$.

Given that,

$C_1: x^2+y^2-4x-6y-3=0$

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Its centre is $C_1(2, 3)$ and radius is $r_1 = 4$ units.

$C_2 : x^2 + y^2 - 12x - 12y + 71 = 0$,

Its centre is $C_2(6, 6)$ and radius is $r_2 = 1$ unit.

Thus, the distance between the centres is given by,

$$\overline{C_1C_2} = \sqrt{(6-2)^2 + (6-3)^2}$$
$$= \sqrt{25}$$
$$= 5$$

Sum of the radii, $r_1 + r_2 = 4 + 1 = 5$.

Thus, $\overline{C_1C_2} = r_1 + r_2 = 5$.

Therefore, the two circles intersect externally at one point because the distance between their centres is equal to the sum of their radii.

(ii) Internal point of intersection of circles

Two circles touch each other internally if and only if the distance between their centres is equal to the difference between their radii. Consider two circles of different sizes intersecting each other internally as shown in Figure 3.12.

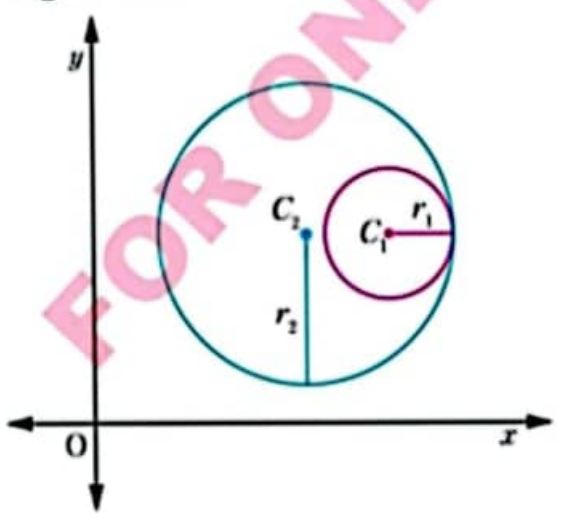

Figure 3.12: *Two circles intersecting internally*

In Figure 3.12, the distance between the centres of the circles, $C_1(a_1, b_1)$ and $C_2(a_2, b_2)$ is given by,

$$\overline{C_1C_2} = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} \qquad (3.14)$$

Since $r_2 > r_1$, then the difference between their radii is $r = r_2 - r_1$.

Thus,

$$\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} = r_2 - r_1 \qquad (3.15)$$

Equating equations (3.14) and (3.15) gives,

$\overline{C_1C_2} = r_2 - r_1$

Therefore, the condition for two circles to intersect each other internally at one point is $\overline{C_1C_2} = r_2 - r_1$.

Example 3.22

Show that the two circles whose equations are $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ intersect internally.

Solution

The two circles intersect internally if they satisfy the equation $\overline{C_1C_2} = r_2 - r_1$.

Given $C_1 : x^2 + y^2 + 2x - 8 = 0$, its centre is $C_1(-1, 0)$ and radius is $r_1 = 3$ units.

$C_2 : x^2 + y^2 - 6x + 6y - 46 = 0$, its centre is $C_2(3, -3)$ and radius is $r_2 = 8$ units.

The distance between the centres is

$$\overline{C_1C_2} = \sqrt{(3-(-1))^2 + ((-3)-0)^2}$$
$$= \sqrt{25}$$
$$= 5$$

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The difference of the radii is given by,
$r_2 - r_1 = 8 - 3 = 5$

Thus, $\overline{C_1C_2} = r_2 - r_1 = 5$.
Therefore, the circles intersect internally at one point because the distance between their centres is equal to the difference between their radii.

(iii) Intersection of two circles at two distinct points

The intersection of two circles at distinct points occurs when the circles overlap each other and meet at two distinct point, A and B. The points of intersection, A and B lie on the circles and satisfy the equations of both circles as shown in Figure 3.13.

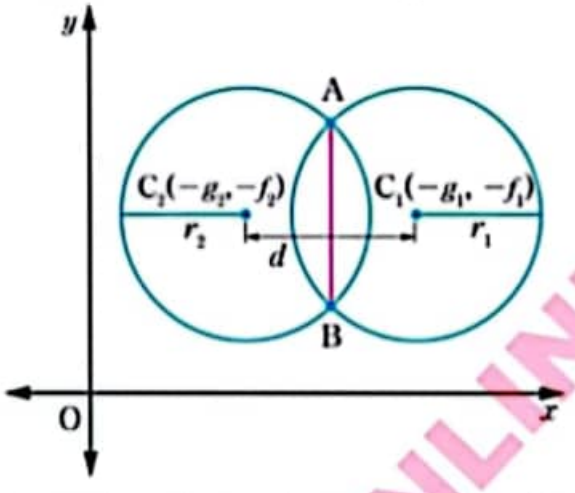

Figure 3.13: *Circles intersecting at two distinct points*

In Figure 3.13, the condition for the two circles to intersect at two distinct points is that, the distance between the two centres of the circles is less than the sum of their radii. That is, $\overline{C_1C_2} < r_1 + r_2$.

The line segment joining $\overline{AB}$ is called a common chord. The common chord is a straight line obtained by solving the two equations of the circles simultaneously. Thus, the equation of the common chord of the circles $C_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $C_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is obtain by solving the system,

$$-\begin{cases} x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \\ x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{cases}$$

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

Therefore, the equation of the common chord of the two circles is given by,
$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Example 3.23

Find the equation of the common chord of the two circles $x^2 + y^2 - 2y - 4 = 0$ and $x^2 + y^2 - x + y - 12 = 0$. Hence, find the two distinct points of intersection.

Solution

Given two circles $x^2 + y^2 - 2y - 4 = 0$ and $x^2 + y^2 - x + y - 12 = 0$.
Solve the two equations simultaneously as follows:

$$-\begin{cases} x^2 + y^2 + 0x - 2y - 4 = 0 \\ x^2 + y^2 - x + y - 12 = 0 \end{cases}$$

$$x - 3y + 8 = 0$$

Thus, the equation of the common chord is $x - 3y + 8 = 0$.

The points of intersection are obtained by solving simultaneously the equation of the common chord with one of the equations of the circles. In this case, take the equation

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$x^2 + y^2 - 2y - 4 = 0$, then substitute the equation of a common chord into the equation of the selected circle as follows:

$(3y-8)^2 + y^2 - 2y - 4 = 0$

$\Rightarrow y^2 - 5y + 6 = 0$

$\Rightarrow y = 2$ or $y = 3$.

Substitute back the values of y in the equation of the common chord to get the values of x. That is, when $y = 2 \Rightarrow x = -2$ and when $y = 3 \Rightarrow x = 1$.

Therefore, the equation of the common chord is $x - 3y + 8 = 0$ and the points of intersection of the circles are $(-2, 2)$ and $(1, 3)$.

Exercise 3.8

1. Show that the circles $x^2 + y^2 - 2x - 2y - 7 = 0$ and $x^2 + y^2 - 10x - 8y + 37 = 0$ intersect each other externally.
2. Show that the circles $x^2 + y^2 - 4x - 2y = 0$ and $x^2 + y^2 - 8x - 10y - 4 = 0$ touch each other internally.
3. Find the equation of the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$.
4. Find the coordinates of the point where the common chord of the circles $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 2x - 4y - 5 = 0$ meets the line joining their centres.
5. Find the points of intersection of the circles given by the equations $x^2 + y^2 - 4x - 6y = 0$ and $(x-1)^2 + (y+1)^2 = 16$.
6. Find the equation of the circle which passes through the points $(-2, 2)$, $(2, 4)$, and $(5, -5)$. Show that this circle touches the circle $2x^2 + 2y^2 - 17x + 16y + 65 = 0$, hence find the coordinates of the point of contact.

Orthogonal circles

Two circles are said to be orthogonal if and only if they intersect in such a way that the tangents at the points of intersection are perpendicular, as shown in Figure 3.14.

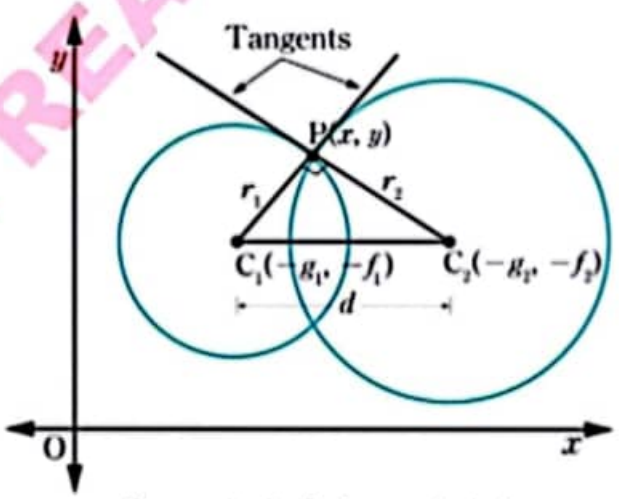

Figure 3.14: *Orthogonal circles*

In Figure 3.14, $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ are the centres of the circles and $P(x, y)$ is a point of intersection. Let r_1 and r_2 be the radii of the circles, respectively and d be the distance between the centres of the circles.

For the circles to be orthogonal, the triangle C_1PC_2 must be a right-angled triangle at P.

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Therefore, from Pythagoras' theorem, $d^2 = r_1^2 + r_2^2$.

That is, the square of the distance between the centres of the circles must be equal to the sum of the squares of their radii.

Example 3.24

Prove that the two circles $x^2 + y^2 - 8x - 14y + 56 = 0$ and $x^2 + y^2 - 4x - 8y + 16 = 0$ are orthogonal.

Solution

The two circles are orthogonal if they satisfy the equation $d^2 = r_1^2 + r_2^2$.

$C_1 : x^2 + y^2 - 8x - 14y + 56 = 0$

Its centre is $C_1(4, 7)$ and radius is $r_1 = 3$ units.

Also,

$C_2 : x^2 + y^2 - 4x - 8y + 16 = 0$

Its centre is $C_2(2, 4)$ and radius is $r_2 = 2$ units.

But $d^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$.

$\Rightarrow d^2 = (4-2)^2 + (7-4)^2$

$\Rightarrow d^2 = 2^2 + 3^2$

$= 13$

Thus, $d^2 = 13$.

But $r_1^2 + r_2^2 = 3^2 + 2^2 = 13$

Hence, $d^2 = r_1^2 + r_2^2 = 13$.

Therefore, the circles are orthogonal.

Exercise 3.9

1. Determine whether or not the following circles are orthogonal:
   (a) $2x^2 + 2y^2 - 12x + 8y + 4 = 0$ and $2x^2 + 2y^2 + 16x + 4y - 44 = 0$
   (b) $x^2 + y^2 - 8x + 6y - 23 = 0$ and $x^2 + y^2 - 2x - 5y - 16 = 0$
   (c) $x^2 + y^2 - 8x + 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$
   (d) $x^2 + y^2 - 2x + 8y + 1 = 0$ and $x^2 + y^2 + 2y - 9 = 0$
   (e) $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 + 4y - 3 = 0$
2. If two circles are orthogonal, show that $2g_1g_2 + 2f_1f_2 = c_1 + c_2$, where $(-g_1, -f_1)$ and $(-g_2, -f_2)$ are centres, and c_1 and c_2 are the constants of the two circles, respectively.
3. Find the equation of the circle which passes through the point $(1, 2)$ and cuts orthogonally each of the circles $x^2 + y^2 - 2x + 8y - 7 = 0$ and $x^2 + y^2 = 9$.
4. For each of the following, find the values of k if the circles:
   (a) $x^2 + y^2 - 2x + 22y + 5 = 0$ and $x^2 + y^2 + 14x + 6y + k = 0$ intersect at right angles.
   (b) $x^2 + y^2 + 4x + 6y + 4 = 0$ and $x^2 + y^2 - 2x + 14y + k = 0$ intersect orthogonally.

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5. Find the equation of the circle which intersects orthogonally the circles $x^2+y^2-4x-6y+11=0$ and $x^2+y^2-10x-4y+21=0$ and has its centre on $2x+3y-7=0$.

6. Determine the equation of the circle which passes through the point $(2,3)$, has its centre on the line $x+y=4$, and intersects orthogonally the circle $x^2+y^2-4x+2y-3=0$.

7. Prove that the equation of the circle with its centre on the y-axis, and crosses the circles $x^2+y^2+5x-2y-3=0$ and $x^2+y^2-2x+6y+7=0$ orthogonally is $2x^2+2y^2+5y+1=0$.

8. Given two fixed points, E$(3,0)$ and F$(0,2)$.
   (a) Find the equation of the locus of a point R such that $2\overline{RE}=3\overline{RF}$. Verify that the locus is a circle passing through the origin.
   (b) Show that the equation of the tangent to the circle in (a) from the origin is $2x-3y=0$.
   (c) Prove that the circle in (a) cuts the circle $x^2+y^2-3x-4y=0$ at right angles

9. Given that a circle C is orthogonal to $x^2+y^2-2x-4y-2=0$. If the circle C passes through the origin and the point N$(2,1)$, then verify that its equation is $(x-2)^2+\left(y+\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2$.

10. Find the equation of the circle which touches the line $3x+4y+5=0$ and crosses orthogonally the circles $x^2+y^2-6y+1=0$ and $x^2+y^2-4y+1=0$.

11. If the circles $(x-3)^2+(y+2)^2=1$ and $x^2+y^2+8x+2y+m=0$ cross orthogonally, find the value of m.

Length of a tangent from a point to a circle

Consider a circle whose equation is $x^2+y^2+2gx+2fy+c=0$. The circle is centred at $(-g,-f)$ and has a radius of $\sqrt{g^2+f^2-c}$. Let $P(x_1,y_1)$ be a point outside the circle as shown in Figure 3.15.

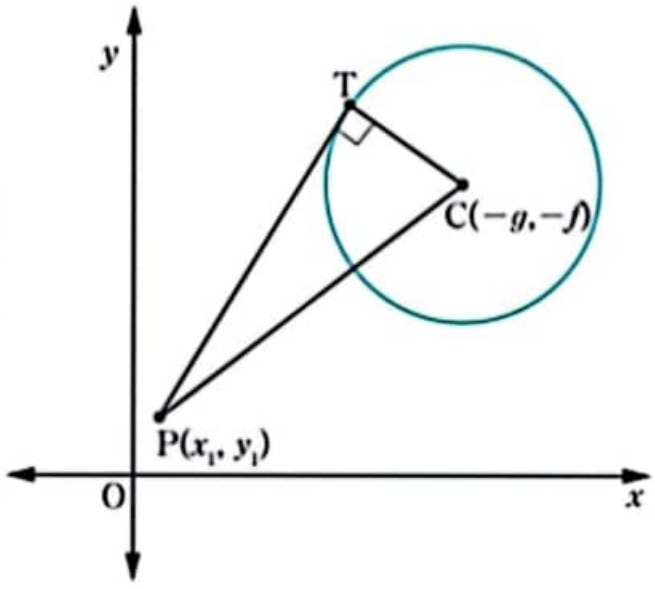

Figure 3.15: *Length of a tangent from a point to a circle*

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The tangent line meets at a right angle with the line passing through the centre at the point of contact.

Using the Pythagoras' theorem, it follows that $\overline{CP}^2 = \overline{CT}^2 + \overline{TP}^2$.

This implies that,

$$\overline{TP}^2 = \overline{CP}^2 - \overline{CT}^2 \qquad (3.16)$$

But $\overline{CT}$ is the radius of the circle, which is given by;

$$\overline{CT} = \sqrt{g^2 + f^2 - c}$$

$$\overline{CT}^2 = g^2 + f^2 - c \qquad (3.17)$$

Applying the distance formula, it follows that $\overline{CP} = \sqrt{(x_1 + g)^2 + (y_1 + f)^2}$

$$\Rightarrow \overline{CP}^2 = (x_1 + g)^2 + (y_1 + f)^2 \qquad (3.18)$$

Substituting the values in equations (3.17) and (3.18) into equation (3.16) gives,

$$\overline{TP}^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Thus, $\overline{TP} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

Therefore, the length of a tangent from point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by, $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ units.

Example 3.25

Find the length of a tangent from the point P(2,5) to the circle $x^2 + y^2 - 2x - 3y - 1 = 0$.

Solution

Given the equation of a circle with the corresponding values $P(x_1, y_1) = (2,5)$, $g = -1, f = -\frac{3}{2}$, and $c = -1$.

Length of tangent is given by;

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

Substituting the values gives,

$$\sqrt{2^2 + 5^2 + 2(-1)(2) + 2\left(-\frac{3}{2}\right)(5) - 1}$$

= 3 units

Therefore, the length of a tangent is 3 units.

Exercise 3.10

1. Find the length of the tangent from the point (12, –9) to the circle $3x^2 + 3y^2 - 7x + 22y + 9 = 0$.
2. A tangent is drawn to a circle of radius 5 cm from a point located 8 cm away from the end point of the radius on the circumference of the circle. Find the length of the tangent.
3. What is the length of a tangent from the point P(4, 2) to the circle with centre (0, 0) and radius 4 units?

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4. Find the length of a tangent to the circle $x^2+y^2-2x+4y-3=0$ from the centre of the circle $x^2+y^2+6x+8y-1=0$.
5. Compute the length of the tangent from the point P(9, 8) to the circle which passes through the points E(5, 7), F(−2, 6), and G(6, 0).
6. Show that the lengths of the tangents from the point (0, 5) to the circles $x^2+y^2+2x-4=0$ and $x^2+y^2-y+1=0$ are equal.
7. Find the equation of the circle such that the lengths of the tangents from the points (−1, 0), (0, 2), and (−2, 1) to the circle are 3, $\sqrt{10}$, and $3\sqrt{3}$ units respectively.
8. Show that the length of a tangent from the point (3,4) to the circle $x^2+y^2-4x+6y-1=0$ is 6 units.
9. Determine the length of a tangent from the point M(6,4) to the circle with centre (−2,−3) and radius 5 units.
10. Verify that the lengths of tangents from the point T(1,5) to the circles, $x^2+y^2+4x+2y-3=0$ and $x^2+y^2-6x+4y-3=0$ are equal.

Chapter summary

1. If L_1 and L_2 are two intersecting lines, and θ is one of the angles between the lines, then the other angle between the lines is $180°-\theta$, where $\theta=\tan^{-1}\left(\frac{m_2-m_1}{1+m_1m_2}\right)$.
2. The perpendicular distance of point $P(x_1, y_1)$ from a line $ax+by+c=0$ is given by, $\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$.
3. The coordinates of the point $P(x, y)$ which divides internally and externally the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$ are $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ and $\left(\frac{nx_1-mx_2}{n-m}, \frac{ny_1-my_2}{n-m}\right)$, respectively.
4. A circle is the locus of a point which moves so that its distance from a fixed point is always constant.
5. The equations of the tangent and normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point $P(x_0, y_0)$ are $\frac{y-y_0}{x-x_0}=-\left(\frac{x_0+g}{y_0+f}\right)$ and $\frac{y-y_0}{x-x_0}=\frac{y_0+f}{x_0+g}$, respectively.
6. Two circles are said to be orthogonal if and only if they intersect in such a way that the tangents at the points of intersection are perpendicular.
7. A circle whose equation is $x^2+y^2+2gx+2fy+c=0$ is centred at $(-g, -f)$ and has a radius of $\sqrt{g^2+f^2-c}$ units.

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8. If the circles,
$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and
$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
intersect orthogonally, then
$d^2 = r_1^2 + r_2^2$ or $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

9. The equation of the common chord to the two circles C_1 and C_2 is defined by $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$.

10. The common chord of two circles C_1 and C_2 is a line passing through the point of intersection of the circles.

Revision exercise 3

1. Find the value of k if the circles $x^2 + y^2 + 2x - 4y + k = 0$ and $x^2 + y^2 - 6x + 4y - 19 = 0$ intersect orthogonally.

2. A triangle has its vertices at points A(1, 7), B(5, 5), and C(8, −1). The point M is the midpoint of $\overline{AB}$, L is the point on $\overline{BC}$ such that $\overline{BL} : \overline{LC} = 1:2$, and N is the point on $\overline{AC}$ such that $\overline{AN} : \overline{NC} = 3:1$. Find the coordinates of L, M, and N.

3. Find the locus of a point P which moves so that $\overline{PA} + \overline{PB} = 1$ given that A (1, 0) and B (−1, 0).

4. Find the equations of the tangents to the circle $x^2 + y^2 - 2x - 4y = 0$ which are parallel to the line $x + 2y + 3 = 0$.

5. Find the equation of the circle which intersects orthogonally the circles $x^2 + y^2 + 2x - 2y - 2 = 0$ and $x^2 + y^2 + 6x - 4y + 4 = 0$ and has its centre on the line $4x - 3y + 3 = 0$.

6. If the perpendicular distance of the point $(m, 3)$ from the line $x - 2y + 1 = 0$ is $\frac{3\sqrt{5}}{5}$ units, find the value of m.

7. Find the coordinates of the point that divides internally and externally the line joining the points $(-2, 3)$ and $(4, 6)$ in the ratio 3:2.

8. Find the coordinates of the point that divides the line joining the points (3, 4) and (4, 2):
   (a) Internally in the ratio 3:2.
   (b) Externally in the ratio 3:2.

9. Find the length of the tangent from the centre of the circle $x^2 + y^2 + 2x - 6y + 4 = 0$ to the circle $x^2 + y^2 - 4x + 5y + 3 = 0$.

10. Show that the circles,
$x^2 + y^2 - 2x + 4y = 0$ and
$x^2 + y^2 - 16x + 16y + 96 = 0$
touch each other. Hence, find the equation of the common chord and the point of contact.

11. Find the points, L and M which divide the line joining (1, 2) and (10, 8) into three equal parts.

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12. Find the distance between the pair of straight lines $x+2y=3$ and $2x+4y+6=0$. (Leave your answer in surd form).

13. If the line $3x+4y=12$ is a tangent to the circle which touches both axes in the first quadrant, find the equations of the circles.

14. Find the locus of a moving point $P(x, y)$ which is equidistant from the point $A(0, b)$ and the line $y=-b$.

15. Find the distance of the point of intersection of the line $2x+3y=21$ and $3x-4y+11=0$ from the line $8x+6y+5=0$.

16. Find the angle between a pair of lines whose equation is $4x^2-24xy+11y^2=0$.

17. Find the equations of a line passing through the point (3, 2) and makes an angle of 45Y with the line $y+3x-7=0$.

18. Determine the equations of two orthogonal lines through the point (3, 2), which make an angle of 45° with the line $-2x+y-1=0$.

19. Find an acute angle between the straight lines in $2x^2+5xy-12y^2=0$.

20. Find the value of m for which $m(2x^2+y^2)+(x+y+2)(x-y)=4$ is a circle. Hence, find the centre of the resulting circle.

21. Find the equation of the circle which is circumscribed about the triangle whose vertices are (−2, 3), (5, 2), and (−4, −1).

22. Describe the locus of a point whose distance from the origin to the point (1, −2) is in the ratio 2:3.

23. Find the equation of the circle whose centre is (2, −3) and passes through the point (3, −5).

24. A line from point A(−4, 0) intersects the straight line $2y+3x-14=0$ at point B. If a perpendicular line is extended to point C in such a way that $\overline{BC}=\frac{1}{2}\overline{AB}$, determine the coordinates of point C.

25. If A = (9,13) and C = (4, −2), find the coordinates of point B given that $\overline{AB}=\frac{3}{2}\overline{BC}$.

26. Find the equation of the circle whose diameters are $x+y=2$ and $x-y=0$ and radius is 1 unit.

27. Find the acute angle between the lines $y=\sqrt{3}x-2$ and $x=4$.

28. If a point is equidistant from (2, 1) and (−7, −5), find its locus.

29. Find the locus of P such that it is equidistant from (3, 1) and the line $3x-4y+1=0$.

30. What is the locus of a point which moves in such a way that it is equidistant from point (2, 3) and the line $=4$?

31. Determine the equation of the circle which passes through the points A(1, 1), B(1, 7), and C(8, 8).

32. Show that the line $5x-2y=31$ touches the circle $x^2+y^2-8x-6y+15=0$.

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33. Find the equation of the circle, which has its centre on the line $x+y=0$ and passes through the points of intersection of the two circles $(x-1)^2+(y+5)^2=50$ and $(x+1)^2+(y+1)^2=10$.

34. Find the equation of the circle whose diameter joins the points A(0, 3) and B(4, 5).

35. For what values of a and b is the equation $ax^2+2bxy+2y^2+8x+12y=0$ a circle?

36. Show that the line $y=2x$ touches the circle $x^2+y^2-8x-y+5=0$. Hence, find the coordinates of the point of contact.

37. Determine the equation of a circle of radius 5 units which touches the x-axis and passes through the point (3, 1).

38. Find the centre and radius of the circle $x^2+y^2+4x-8y+4=0$. Find the length of the tangent to the circle from the point P(3, 8).

39. A circle centred at (2, 3) touches the line joining the points (0, 4) and (3, 1). Find the equation of the circle.

40. A line $2x+2y-3=0$ touches the circle $4x^2+4y^2+8x+4y-13=0$ at A. Find the equation of the line joining A to the origin.

41. Show that the tangents to the circle $x^2+y^2=169$ at (5, 12) and (12, −5) are perpendicular.

42. Find the value of c so that the circles $x^2+y^2-6x+4y+12=0$ and $x^2+y^2+8x+2y+c=0$ are orthogonal.

43. Find the equations of the lines that pass through the point (3, 1) and are tangent to the circle whose equation is $x^2+y^2-2=0$.

44. Find the equations of the tangent and normal to the circle $x^2+y^2-10x+4y-140=0$ at the point (−7, −7).

45. Find the equations of the tangents from the origin to the circle whose equation is $x^2+y^2-8x-y+5=0$.

46. What is the length of the tangent from point (8, 4) to the circle with centre at (3, 0) and radius 2 units?

47. Find the length of the tangents to the circle $2x^2+2y^2-4x+8y-6=0$ from the centre of the circle $3x^2+3y^2+18x+24y-3=0$.

48. Show that the line $y=x+1$ touches the circle $x^2+y^2-8x-2y+9=0$

49. Find the equation of the tangent to the circle $x^2+y^2-2x+y-5=0$ at the point (3, −2). If this tangent crosses the axes at A and B, calculate the area of the triangle OAB, where O is the origin.

50. Show that the line $y=px+q$ touches the circle $x^2+y^2=r^2$ if $q^2=r^2(1+p^2)$.

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Chapter Four

Functions

Introduction

In mathematics, a function is a relation between a set of inputs and a set of outputs satisfying a certain rule. Functions have the property that, each input is related to exactly one output. The set of inputs is called a domain, whereas the set of outputs is called the range. In this chapter, you will learn how to plot graphs of functions. The competencies developed will help you in designing machines, making predictions, studying growth relations, formulating mathematical models, developing computer programs, and in solving many other real life problems.

Think

Mathematics without the concept of functions.

Graphs of functions

A function can be represented graphically. A function has an independent variable and a dependent variable. Normally, x is the independent variable and y is the dependent variable. A common method used in drawing the graph of a function is to choose some values for the independent variable x and substitute them into the function to get a set of ordered pairs (x, y). Tabulate the values and locate the points (x, y) on the xy-plane, then connect the points to obtain the graph. Normally, using more points gives a better graph. This section, discusses how to plot graphs of polynomial functions up to 4th degree, rational functions, composite functions, exponential functions, and logarithmic functions.

Activity 4.1: Identifying a function from the drawn graph

Individually or in a group, perform the following tasks:

1. Draw each of the following figures in the xy-plane:
   (a) A parabola opening upwards or downwards.
   (b) A parabola opening to the left or to the right.
   (c) A circle of reasonable radius.

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2. In each of the figures drawn in task 1, draw a vertical line that intersects the figure at the middle.
3. What have you observed after performing task 2?
4. What conclusion can you make from the figures?

Polynomial functions

A function f: $\mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \quad + a_2 x^2 + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are constants, n is a positive integer, and $a_n \neq 0$ is called a polynomial function.

The degree of a polynomial function is the highest power of x appearing in the polynomial.

For example, $f_1(x) = x + 2$, $f_2(x) = x^2$, $f_3(x) = x^3 - 2x + 1$, $f_4(x) = 3x^4 - 2x^3 + 5$ are polynomial functions of degree 1, 2, 3, and 4, respectively.

Polynomial functions are classified depending on their degrees, that is:

(i) If the degree is 0, then the polynomial is a constant function.

(ii) If the degree is 1, then the polynomial is a linear function.

(iii) If the degree is 2, then the polynomial is a quadratic function.

(iv) If the degree is 3, then the polynomial is a cubic function.

(v) If the degree is 4, then the polynomial is a quartic function.

Graphs of linear functions

A function defined by $f(x) = y = mx + c$, where m is the gradient, c is the y-intercept, and x is the independent variable, is called a linear function. The graph of a linear function is always a straight line as shown in Figure 4.1. The x-intercept is denoted by a where $a = -\frac{c}{m}$.

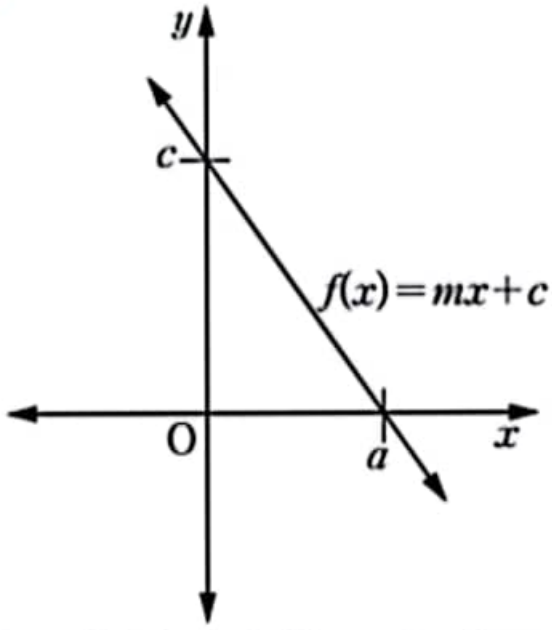

Figure 4.1: *Graph of* $f(x) = mx + c$, *for* $m < 0$

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Example 4.1

Draw the graph of $f(x) = 3x + 2$.

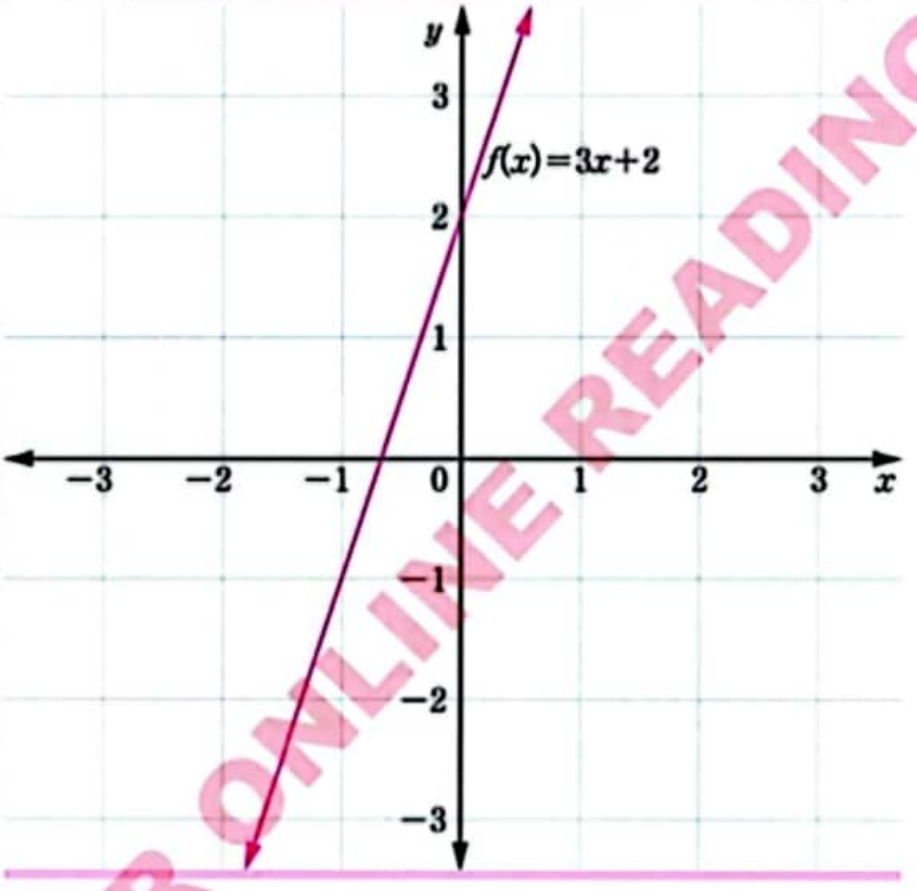
Solution

Given $f(x) = 3x + 2$.

Select values for graphing the function as shown in the following table.

x	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f(x)$	$-\frac{5}{2}$	-1	$\frac{1}{2}$	2	$\frac{7}{2}$

The graph of $f(x) = 3x + 2$ is shown in the following figure.



Alternatively,

The graph of $f(x) = 3x + 2$ can be sketched by using x and y-intercepts as follows.

x-intercept

When $y = 0$, $x = -\frac{2}{3}$,

y-intercept

When $x = 0$, $y = 2$

The graph of $f(x) = 3x + 2$ is shown in the following figure:

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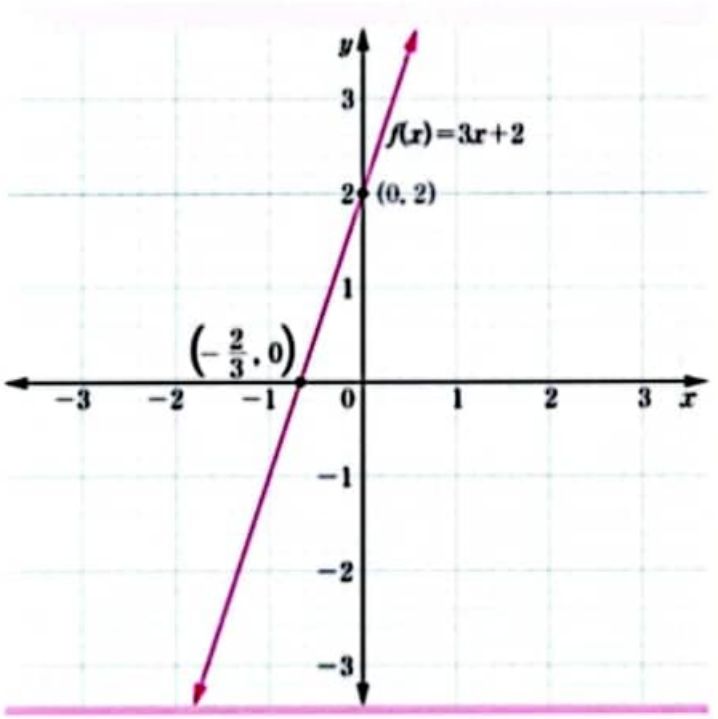


Example 4.2

Draw the graph of $f(x)=4x-1$.

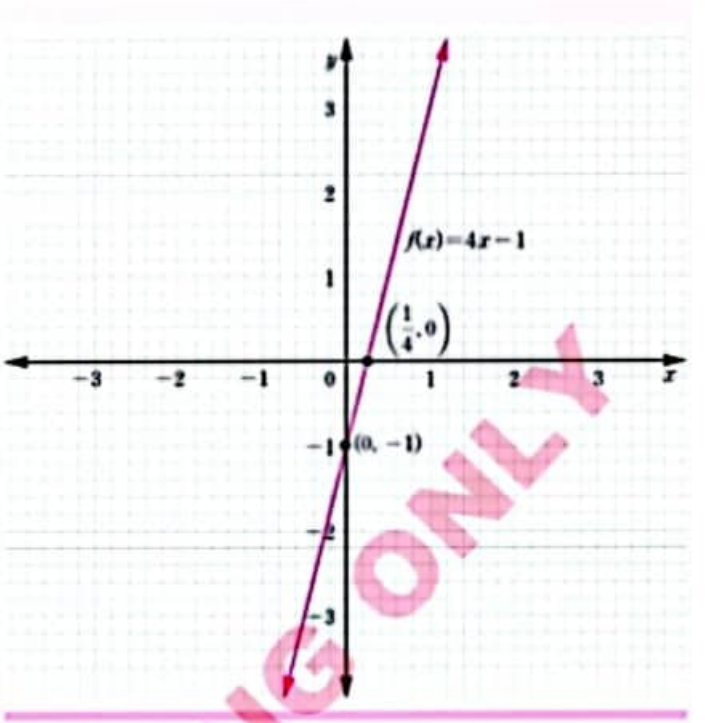
Solution

Given $f(x)=4x-1$.

The x and y-intercepts for graphing the function are tabulated as follows.

x	0	$\frac{1}{4}$
$f(x)$	−1	0

The graph of $f(x)=4x-1$ is shown in the following figure.



Exercise 4.1

Draw the graph of each of the following functions:

1. $f(x)=-2x+6$
2. $f(x)=\frac{1}{3}(4x+8)$
3. $f(x)=-3x+2$
4. $f(x)=4x+9$
5. $f(x)=-7x-3$
6. $f(x)=\frac{1}{2}x+8$
7. $f(x)=8x-\frac{1}{3}$
8. $f(x)=-9x$
9. $f(x)=\frac{2}{9}x+3$
10. $g(x)=\frac{4}{10}x$

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Graphs of quadratic functions

A polynomial function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants, and $a \neq 0$ is called a quadratic function. When $a > 0$, the graph opens upwards and when $a < 0$, it opens downwards as shown in Figure 4.2(a) and Figure 4.2 (b), respectively.

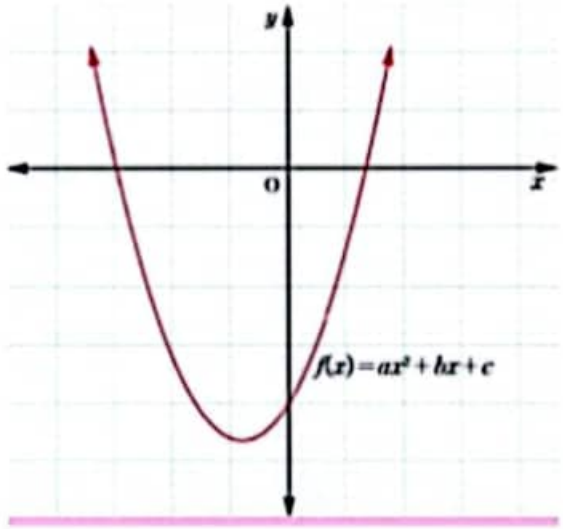


Figure 4.2(a) : *Graph of a quadratic function for $a > 0$*

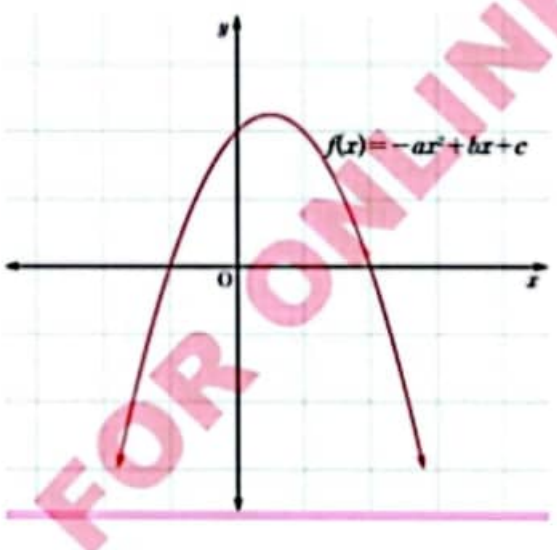


Figure 4.2(b) : *Graph of a quadratic function for $a < 0$*

Example 4.3

Draw the graph of $f(x) = x^2 + 4x + 2$.

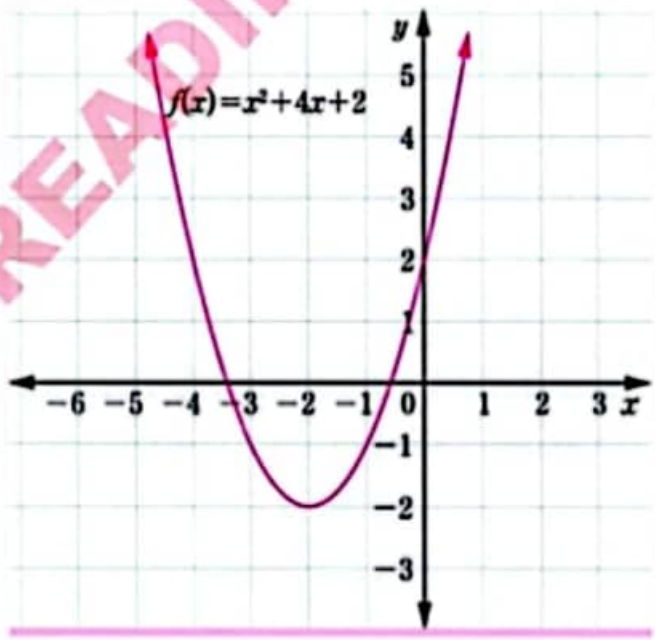
Solution

Given $f(x) = x^2 + 4x + 2$.

Select values for graphing the function as shown in the following table.

x	-4	-3	-2	-1	0
$f(x)$	2	-1	-2	-1	2

The graph of $f(x) = x^2 + 4x + 2$ is shown in the following figure.



Example 4.4

Draw the graph of $f(x) = -x^2 + 2x + 1$.

Solution

Given $f(x) = -x^2 + 2x + 1$.

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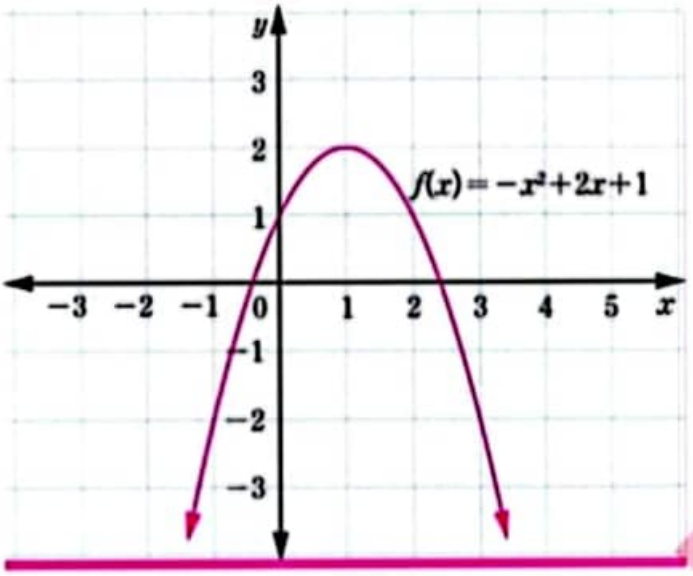
Selected values for graphing the function are shown in the following table.

x	-1	0	1	2	3
$f(x)$	-2	1	2	1	-2

The graph of $f(x)=-x^2+2x+1$ is shown in the following figure.



Example 4.5

Draw the graph of $f(x)=3x^2+5x+4$.

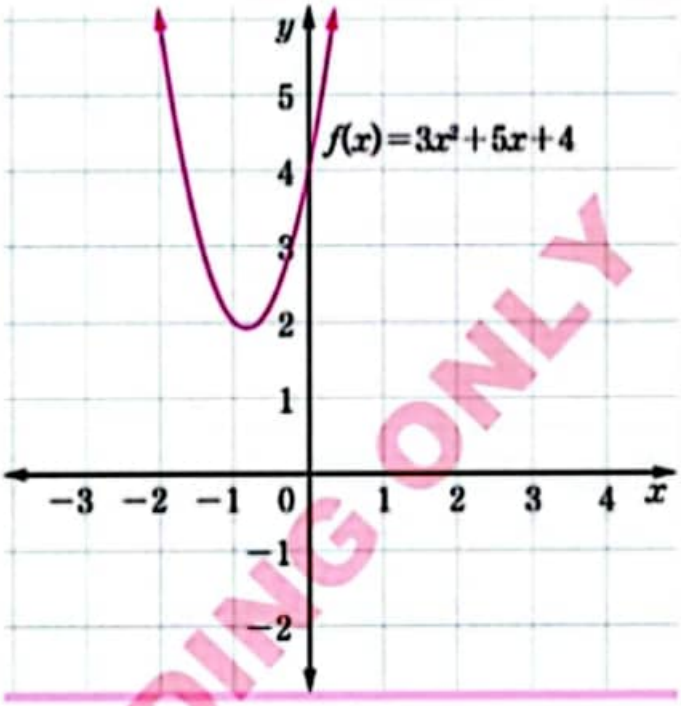
Solution

Given $f(x)=3x^2+5x+4$.

Select values for graphing the function as shown in the following table.

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
$f(x)$	6	$\frac{13}{4}$	2	$\frac{9}{4}$	4

The graph of $f(x)=3x^2+5x+4$ is shown in the following figure.



Exercise 4.2

Draw the graph of each of the following functions:

1. $f(x)=(x+3)^2$
2. $f(x)=3x^2+2x+1$
3. $f(x)=x^2+12x+3$
4. $f(x)=x^2-2x-3$, and $g(x)=-2x^2-5x+7$ on the same xy-plane.
5. $f(x)=3x^2-\frac{3}{4}x+1$
6. $f(x)=x^2$ and $g(x)=-2x+7$ on the same xy-plane.

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7. $f(x) = -(x^2 - 3x - 2) - 5$

8. $f(x) = \frac{3}{5}(x^2 + 2x) + 2$

9. $f(x) = \frac{1}{2}(\sqrt{4}x^2 + 5x + 1)$

10. $f(x) = -4x^2 - x - 3$

Graphs of cubic functions

A polynomial function of the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants and $a \neq 0$ is called a cubic function. The procedure for graphing a cubic function is the same as that used for graphing a quadratic function.

Example 4.6

Draw the graph of $f(x) = x^3 - 4x$ and state its domain and range.

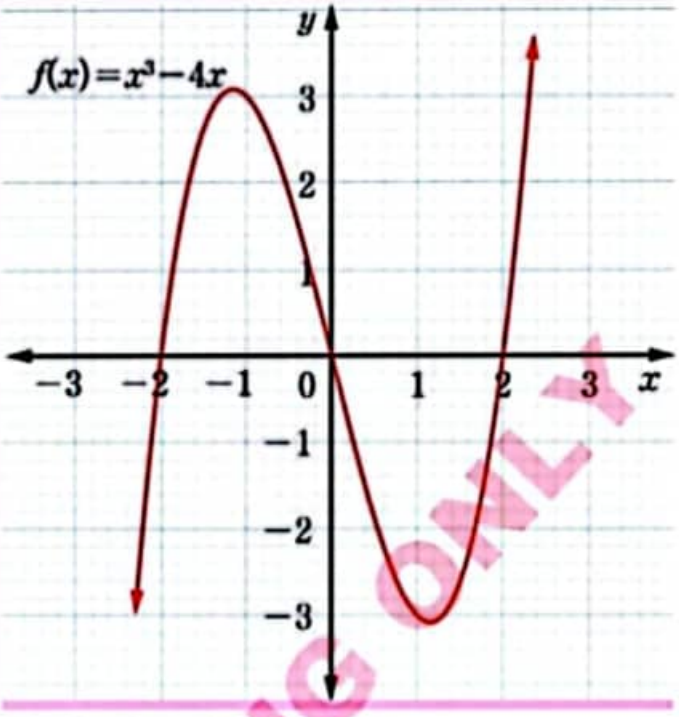
Solution

Given $f(x) = x^3 - 4x$.

Select a few values (x, y) for graphing the function as shown in the following table.

x	−2	−1	0	1	2
$f(x)$	0	3	0	−3	0

The graph of $f(x) = x^3 - 4x$ is shown in the following figure.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

Example 4.7

Draw the graph of $f(x) = x^3 + 3x^2 - 2x - 5$ and state how the graph behaves for large positive and negative values of x.

Solution

Given $f(x) = x^3 + 3x^2 - 2x - 5$.

Select few values for graphing the function as tabulated in the table.

x	−3	−2	−1	0	1
$f(x)$	1	3	−1	−5	−3

The graph of $f(x) = x^3 + 3x^2 - 2x - 5$ is shown in the following figure.

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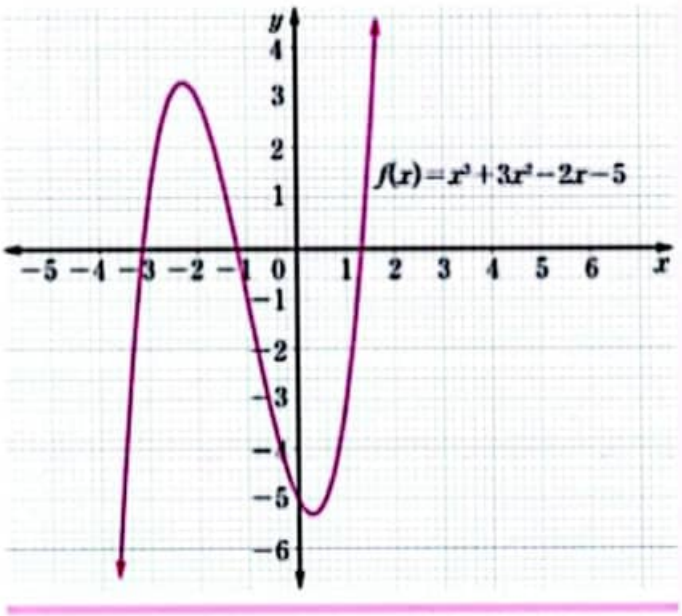


The graph shows that as x increases indefinitely, $f(x)$ increases and the graph of $f(x)$ is opening upwards. As x decreases indefinitely $f(x)$ decreases and the graph of $f(x)$ is opening downwards.

Example 4.8

Draw the graph of $f(x) = -x^3 - 2x^2 + x + 1$ and state its domain and range.

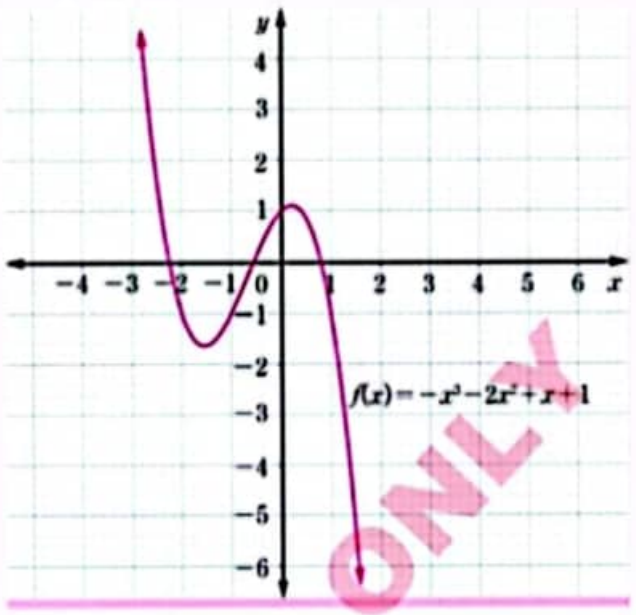
Solution

Given $f(x) = -x^3 - 2x^2 + x + 1$.

Select a few values (x, y) for graphing the function as tabulated in the table.

x	-2	-1	0	1
$f(x)$	-1	-1	1	-1

The graph of $f(x) = -x^3 - 2x^2 + x + 1$ is shown in the following figure.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

Exercise 4.3

Draw the graph of each of the following functions and hence state the domain and range:

1. $f(x) = x^3 - 3x$
2. $f(x) = 2x^3 + 2x^2 - 4x + 5$
3. $f(x) = 2x^3 + x^2 - x - 1$
4. $f(x) = x(x-2)(x+3)$
5. $f(x) = -(x^3 - 7x + 3)$
6. $f(x) = x(x^2 + 3x) + 4$
7. $f(x) = x^3 - 4x^2 + x$ and $g(x) = -\frac{1}{3}x^3 + \frac{3}{2}x^2 + x$ on the same xy-plane.
8. $f(x) = -x^3 - 5x^2 + 4$
9. $f(x) = (x-2)^3 - 4x$ and $h(x) = -x^2 + 4$ on the same xy-plane.
10. $g(x) = (2-x)(x^2 + 3x)$

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Graphs of quartic functions

A polynomial function of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are constants and $a \neq 0$ is called a quartic function. The procedure for graphing a quartic function is the same as that used for graphing a cubic function.

Example 4.9

Draw the graph of $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$.

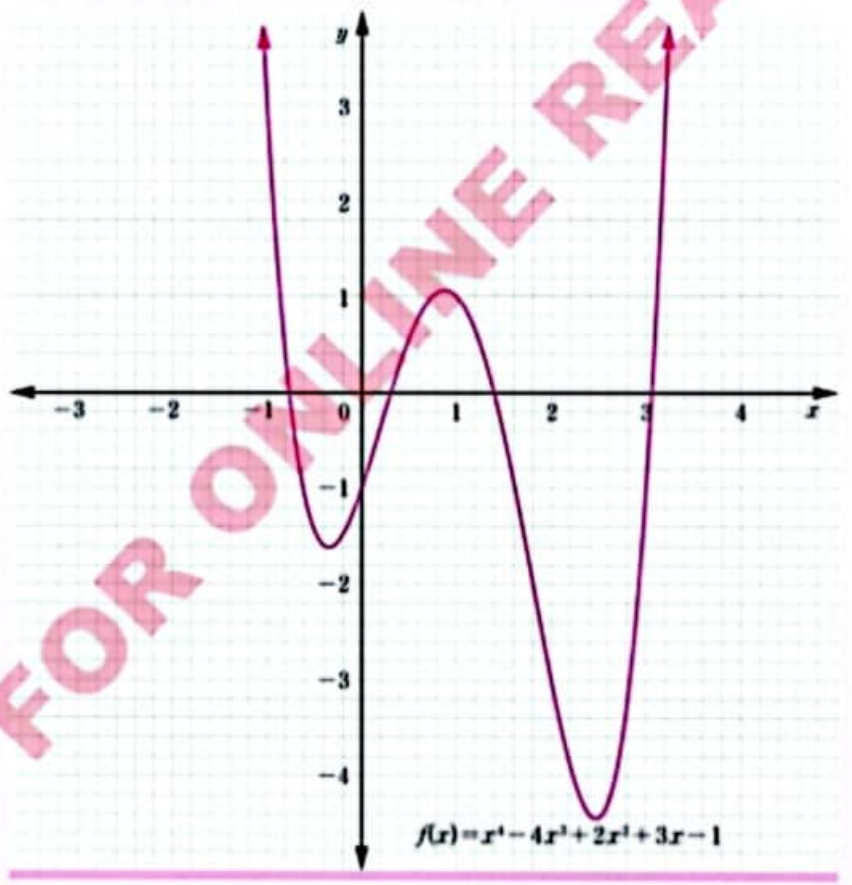
Solution

Given $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$.

Select a few values for graphing the function as tabulated in the table.

x	-1	0	1	2	3
$f(x)$	3	-1	1	-3	-1

The graph of $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$ is shown in the following figure.



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Example 4.10

Draw the graph of $f(x)=(x^2-3)(x^2-8)$ and give its domain and range.

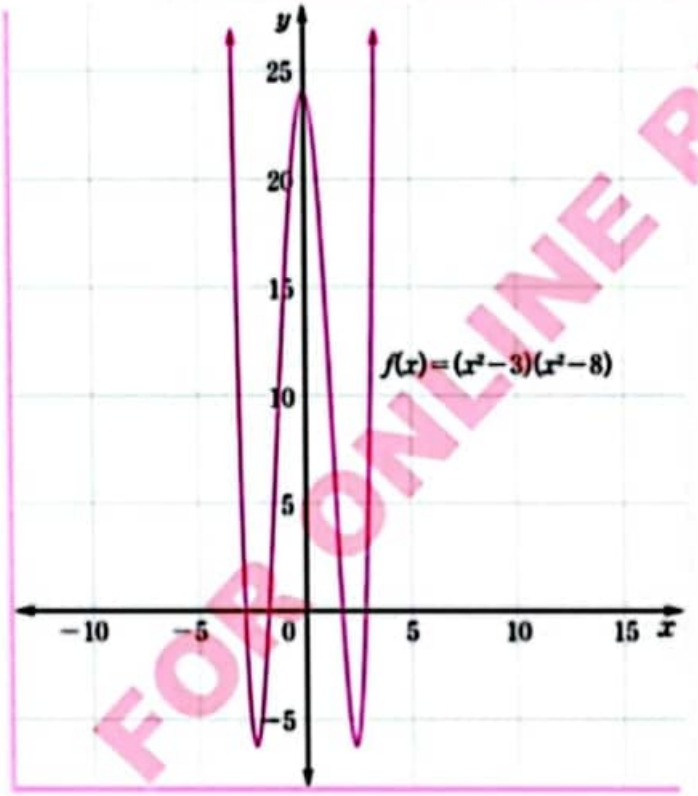
Solution

Given $f(x)=(x^2-3)(x^2-8)$.

Select a few values for graphing the function as tabulated in the table.

x	-3	-2	-1	0	1	2	3
$f(x)$	6	-4	14	24	14	-4	6

The graph of $f(x)=(x^2-3)(x^2-8)$ is shown in the following figure.



Domain $=\{x: x\in \mathbb{R}\}$

Range $=\left\{y: y\in \mathbb{R},\ y\geq -\frac{25}{4}\right\}$

Exercise 4.4

1. Given the function $f(x)=x^4-2x^3-x^2+2x$, find the zeros of $f(x)$.
2. Draw the graph of $f(x)=x^4-3x^3-4x^2+6x+4$ then find its domain and range.
3. Draw the graph of each of the following functions:
   (a) $f(x)=x^4-5x^2+10$.
   (b) $f(x)=-x^4-4x^3-x^2+8x$.
   (c) $f(x)=-(x-2)(x+2)^2(x+1)$
4. Draw the graph of $f(x)=4x^4-3x^2$.
5. Draw the graph of $f(x)=x^2\left(x^2-1\right)$ over the interval $-2\leq x\leq 2$ and find its range.
6. Draw the graph of $g(x)=\left(x^2-4\right)\left(x^2-16\right)$ and state how the graph behaves for large positives values.
7. Draw the graph of $h(x)=\left(x-2\right)\left(x^3-8\right)$ and state its domain and range.
8. Draw the graph of $f(x)=-4x^2\left(x^2-1\right)$ then identify its domain and range.
9. Describe the behavior for the graph of $g(x)=\left(\frac{1}{2}x^2-\frac{1}{4}\right)\left(\frac{1}{2}x^2-1\right)$ for large positive and negative values of x.
10. Draw the graph of $f(x)=(x-2)(x+1)(x-3)(x+3)$ in the interval of $-4\leq x\leq 3$. Hence, state its range.

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Graphs of rational functions

A function of the form $f(x)=\frac{p(x)}{q(x)}$, where the numerator $p(x)$ and denominator $q(x)$ are polynomial functions and $q(x)\neq 0$, is called a rational function. These types of functions may not be defined at some values. The value at which the rational function is not defined is known as a hole. As the values of x get closer to or far from a hole, the behaviour of the function is determined.

Asymptotes

An asymptote to a curve is a line towards which the curve approaches as x approaches a hole. There are three types of asymptotes that may exist in rational functions, namely; vertical asymptotes, horizontal asymptotes, and oblique asymptotes.

A vertical asymptote is a vertical line obtained by equating the denominator of a rational function to zero, that is $q(x)=0$. A horizontal asymptote is a horizontal line parallel to the axis of the independent variable. The horizontal asymptote of the function $f(x)$ is determined by reducing the term with the highest degree to a constant. This is done by dividing each term in the rational function by the variable with highest degree, then solving for y when the independent variable x approaches $\pm\infty$.

Generally, if $f(x)=\frac{p(x)}{q(x)}$ is a rational function, where p and q are polynomial functions with leading coefficients a and b, respectively, then the following hold:

(a) If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y=\frac{a}{b}$ is the horizontal asymptote of the graph $y=f(x)$.

(b) If the degree of $p(x)$ is less than the degree of $q(x)$, then $y=0$ is the horizontal asymptote of the graph of $y=f(x)$.

(c) If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y=f(x)$ has no horizontal asymptote.

An oblique asymptote is a straight line that is inclined at a certain angle with the x-axis. The equation of an oblique asymptote is a function of the form $y=mx+b$, where $m\neq 0$. This kind of asymptote exists when the degree of the numerator is higher than that of the denominator. Normally, the oblique asymptote is the quotient obtained by dividing the numerator by the denominator using long division method.

The graph of a rational function $f(x)=\frac{p(x)}{q(x)}$ has a hole at $x=a$ if:

both $p(x)$ and $q(x)$ have the common factor $(x-a)$.

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To find the y-coordinate of the hole in a rational function, substitute $x = a$ into a simplified expression of $f(x)$. In the graph, a hole (if any) is located using a small circle.

Steps for graphing a rational function

When graphing a rational function use the following steps:

1. Identify the holes, if any.
2. Find and locate the x and y-intercepts, if they exist.
3. Find the asymptotes (horizontal, vertical, or oblique) if any.
4. Draw the asymptotes on the graph, using dotted lines.
5. Sketch the graph by testing the behaviour of the graph.

Note that, the graph of a rational function may have many vertical asymptotes, but it will have at most one horizontal or oblique asymptote.

Example 4.11

Find the holes and vertical asymptotes of the function $f(x) = \dfrac{2x^2 - 4x}{x^2 + x - 6}$.

Solution

Given $f(x) = \dfrac{2x^2 - 4x}{x^2 + x - 6}$.

Factorization of the numerator and denominator gives,

$$f(x) = \frac{2x(x-2)}{(x+3)(x-2)} \quad \text{.....(i)}$$

It can be observed that $(x-2)$ is a factor in both the numerator and denominator. Setting this factor equals to zero, that is $x - 2 = 0$, gives $x = 2$.

Simplifying equation (i) gives,

$$f(x) = \frac{2x}{x+3}.$$

Substituting $x = 2$ gives,

$$f(2) = y = \frac{4}{5}.$$

Thus, a hole is at the point $\left(2, \frac{4}{5}\right)$.

The vertical asymptotes can be obtained by equating the denominator of the function to zero, that is $x^2 + x - 6 = 0$.

Solving for x gives $x = 2$ or $x = -3$.

Therefore, a hole is at a point $\left(2, \frac{4}{5}\right)$ and the vertical asymptotes are the lines $x = 2$ and $x = -3$.

Example 4.12

Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{x}{x-3}.$$

Solution

Let $y = f(x) = \dfrac{x}{x-3}$.

Finding the vertical asymptotes, equate the denominator of the function to zero, that is,

$x - 3 = 0 \Rightarrow x = 3$.

Thus, $x = 3$ is the vertical asymptote.

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Since the degrees of the numerator and denominator are equal, then the horizontal asymptote can be found as follows:

$$y = \frac{\frac{x}{x}}{\frac{x}{x} - \frac{3}{x}} = \frac{1}{1 - \frac{3}{x}}$$

As $x \to \pm\infty, y \to 1$

Therefore, the vertical asymptote is the line $x = 3$ and horizontal asymptote is the line $y = 1$.

Example 4.13

Find the oblique asymptote of the function $f(x) = \frac{4x^2 + 5}{x - 2}$.

Solution

Let $y = f(x) = \frac{4x^2 + 5}{x - 2}$.

Finding the oblique asymptote, divide the numerator by denominator using long division as follows.

$$\begin{array}{r} 4x + 8 \\ x - 2 \overline{)\,4x^2 + 5} \\ -(4x^2 - 8x) \\ \hline 8x + 5 \\ -(8x - 16) \\ \hline 21 \end{array}$$

Thus, $y = \frac{4x^2 + 5}{x - 2} = (4x + 8) + \frac{21}{x - 2}$.

Therefore, the oblique asymptote is the line $y = 4x + 8$.

Example 4.14

Sketch the graph of $f(x) = \frac{x}{x^2 - 9}$, and hence determine the domain and range.

Solution

Let $y = f(x) = \frac{x}{x^2 - 9}$.

Find the x and y-intercepts as follows.

The x-intercept is obtained when $y = 0$.

So, $0 = \frac{x}{x^2 - 9} \Rightarrow x = 0$

Thus, the x-intercept is at the point $(0, 0)$.

The y-intercept is obtained when $x = 0$.

That is, $y = \frac{0}{0 - 9}$

$\Rightarrow y = 0$

Thus, the y-intercept is at the point $(0, 0)$.

Find the vertical asymptotes by equating the denominator to zero. That is,

$x^2 - 9 = (x + 3)(x - 3) = 0$

$\Rightarrow x = -3$ or $x = 3$

Hence, the vertical asymptotes are the lines $x = -3$ and $x = 3$.

Draw $x = -3$ and $x = 3$ using dotted lines on the xy-plane.

Find the horizontal asymptote by dividing the expression by the term with the highest power.

Since the degree of the numerator is 1 and that of the denominator is 2, divide each term by x^2.

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That is, $f(x) = \dfrac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{9}{x^2}} = \dfrac{\frac{1}{x}}{1 - \frac{9}{x^2}}$

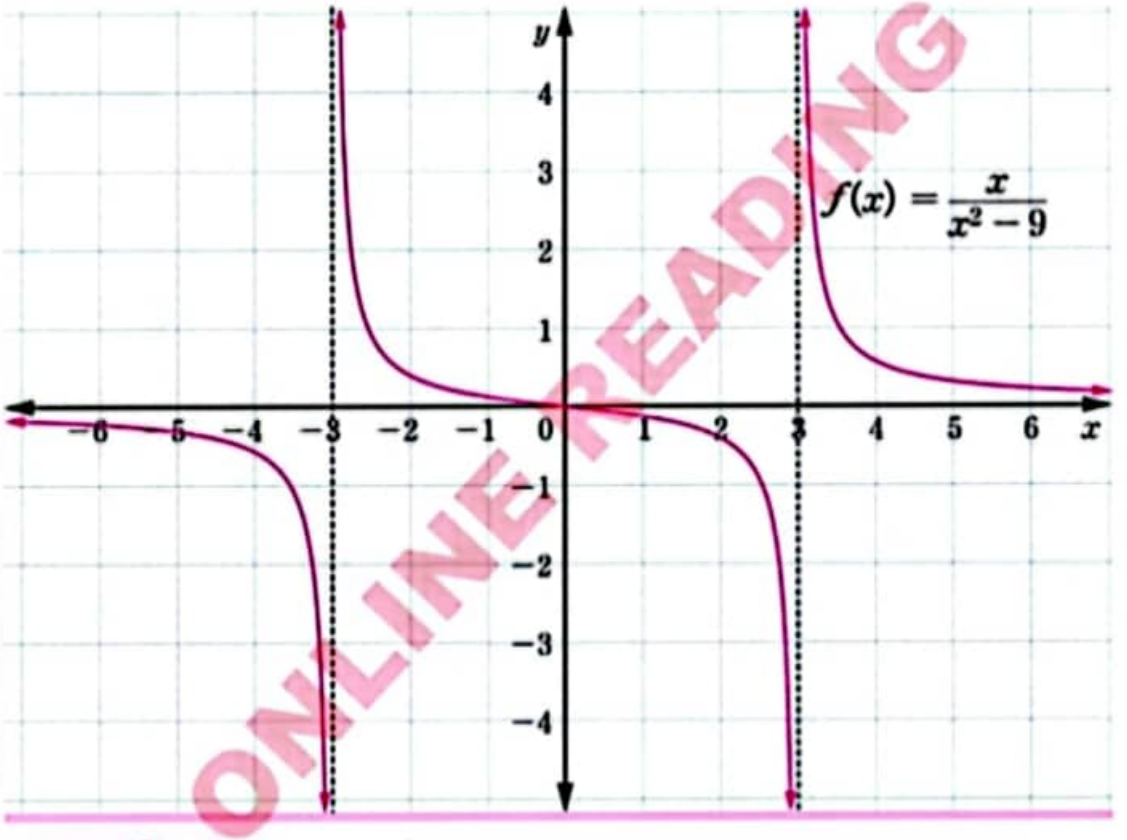
As $x \to \pm\infty, y \to 0$.

Thus, the horizontal asymptote is the line $y = 0$.

But the line $y = 0$ cannot be a dotted line as it define the x-axis.

Locate the points on either side of the vertical asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.

The graph of $f(x) = \dfrac{x}{x^2 - 9}$ is shown in the following figure.



Domain $= \{x : x \in \mathbb{R}, x \neq \pm 3\}$

Range $= \{y : y \in \mathbb{R}\}$

Example 4.15

Sketch the graph of $f(x) = \dfrac{x^3 - 3x}{3(x^2 - 7)}$, and hence determine the domain and range.

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Solution

Let $y = f(x) = \dfrac{x^3 - 3x}{3(x^2 - 7)}$.

Find the x and y-intercepts:

For x-intercept; $y = 0$.

So, $0 = \dfrac{x^3 - 3x}{3(x^2 - 7)}$

$x^3 - 3x = 0 \Rightarrow x(x^2 - 3) = 0$

$\Rightarrow x = 0$ or $x^2 - 3 = 0$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

Thus, the x-intercepts are at the points $(0, 0)$ and $(\pm\sqrt{3}, 0)$.

y-intercept; $x = 0$.

That is, $y = \dfrac{0 - 0}{3(0 - 7)}$.

$\Rightarrow y = 0$

Thus, the y-intercept is at the point $(0, 0)$.

Find the vertical asymptotes by equating the denominator to zero, that is,

$3(x^2 - 7) = 0$

$\Rightarrow x^2 = 7$

$\Rightarrow x = \pm\sqrt{7}$

Hence, the vertical asymptotes are the lines $x = -\sqrt{7}$ and $x = \sqrt{7}$.

Draw $x = -\sqrt{7}$ and $x = \sqrt{7}$ as dotted lines on the xy-plane.

Since the highest degree of the numerator is 3 and that of the denominator is 2, divide the numerator by the denominator to get the oblique asymptote.

That is,

$$\begin{array}{r} \frac{1}{3}x \\ 3x^2 - 21 \overline{)\, x^3 - 3x} \\ -\underline{(x^3 - 7x)} \\ 4x \end{array}$$

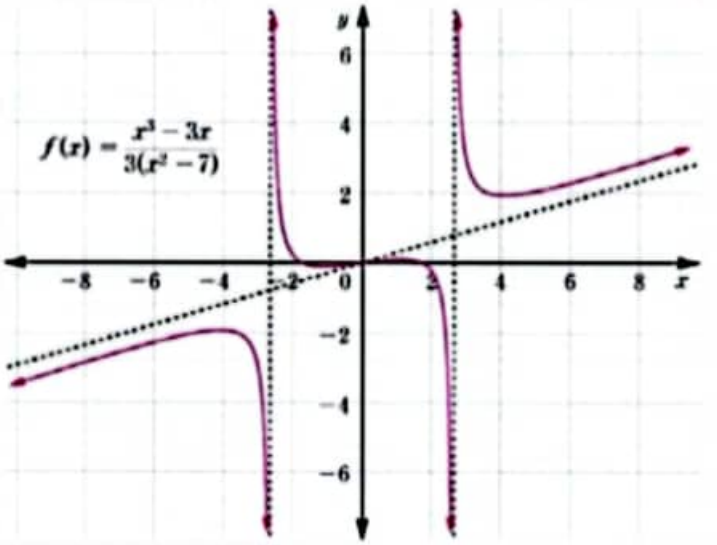
Thus,

$$f(x) = \frac{x^3 - 3x}{3(x^2 - 7)} = \frac{1}{3}x + \frac{4x}{3(x^2 - 7)}.$$

Hence, the oblique asymptote is the line $y = \dfrac{1}{3}x$.

Locate the points on either side of the vertical asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.

The graph of $f(x) = \dfrac{x^3 - 3x}{3(x^2 - 7)}$ is shown in the following figure.



üüüü $= \{x \; x \in \mathbb{R} \; x \neq \pm\sqrt{\;}\}$

Range $= \{y : y \in \mathbb{R}\}$

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Example 4.16

Sketch the graph of

$f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$ and hence determine the domain and range.

Solution

Let $y=f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$. Factorize the numerator and denominator as follows.

$$f(x)=\frac{(x-1)(x-1)}{x(x+2)(x-1)}$$

Thus, $f(x)=\dfrac{x-1}{x(x+2)}$.

Hence, the hole is at $x=1$. Substituting $x=1$ gives $f(1)=0$. This implies that the hole is at the point $(1,0)$.

For x-intercept; $y=0$.

$$\Rightarrow 0=\frac{x-1}{x(x+2)}$$

$$\Rightarrow x-1=0$$

$$\Rightarrow x=1$$

Thus, the x-intercept is at the point (1, 0).

For y-intercept; $x=0$.

$$\Rightarrow y=\frac{0-1}{0(0+2)}$$

$\Rightarrow y=\infty$ (undefined)

Thus, the y-intercept is at infinity (does not exist).

The vertical asymptotes are obtained by equating the denominator to zero, that is, $x(x+2)=0$.

$\Rightarrow x=0$ or $x=-2$

Hence, the vertical asymptotes are the lines $x=0$ and $x=-2$.

Draw the line $x=-2$ as dotted line on the graph.

But $x=0$ cannot be a dotted line as it defines the y-axis.

Since the degree of the numerator is 1 and the degree of the denominator is 2, the horizontal asymptotes is obtained as follows.

$$f(x)=\frac{\frac{x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+2\frac{x}{x^2}}=\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{2}{x}}.$$

As $x\to\pm\infty$, $y\to 0$.

Hence, the horizontal asymptote is the line $y=0$.

Locate the points on either side of the vertical asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.

The graph of $f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$ is shown in the following figure.

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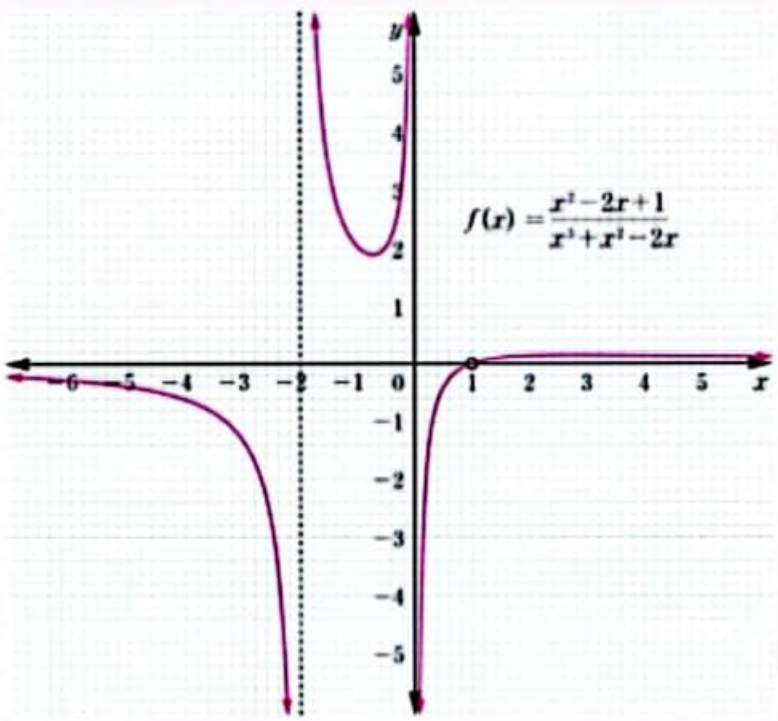

Domain $= \{x : x \in \mathbb{R},\ x \neq -2,\ x \neq 0,\ x \neq 1\}$

Range $= \{y : y \in \mathbb{R}, y \geq 1.866,\ y \leq 0.134,\ y \neq 0\}$

Example 4.17

Sketch the graph of the function $f(x) = \dfrac{3x+3}{x(3-x)}$, and use it to show that it has no real values between $\dfrac{1}{3} < y < 3$. Hence, state its domain and range.

Solution

Let $y = f(x) = \dfrac{3x+3}{x(3-x)}$.

Find the x and y-intercepts:

For x-intercept set $y = 0$.

$\Rightarrow 0 = \dfrac{3x+3}{x(3-x)}$

$\Rightarrow x = -1$

Thus, the x-intercept is at the point $(-1, 0)$.

For y-intercept set $x = 0$, that is,

$y = \dfrac{3(0)+3}{0(3-0)}$

$\Rightarrow y = \infty$

Thus, the y-intercept is at infinity (does not exist).

The vertical asymptotes are obtained by equating the denominator to zero, that is,

$x(3-x) = 0$

$\Rightarrow x = 0$ or $x - 3 = 0$. This gives $x = 0$ or $x = 3$.

Hence, the vertical asymptotes are the lines $x = 0$ and $x = 3$.

Draw $x = 3$ as a dotted line on the graph.

Determine the horizontal asymptote as follows.

Since the degree of the numerator is 1 and that of the denominator is 2, divide each term by x^2.

That is, $f(x) = \dfrac{3\frac{x}{x^2} + \frac{3}{x^2}}{3\frac{x}{x^2} - \frac{x^2}{x^2}}$

$= \dfrac{\frac{3}{x} + \frac{3}{x^2}}{\frac{3}{x} - 1}$

As $x \to \pm\infty$, $y \to 0$.

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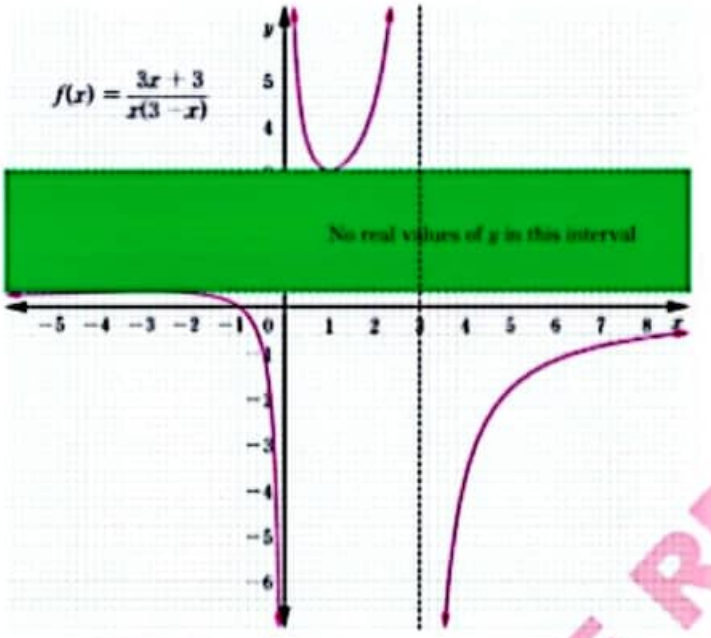
Hence, the horizontal asymptote is the line $y = 0$. Locate the points on either side of the vertical asymptotes, also select some value of x and determine the corresponding values of y. Use the selected values to sketch the graph.

The graph of $f(x) = \dfrac{3x+3}{x(3-x)}$ is shown in the following figure.



Domain $= \{x : x \in \mathbb{R}, x \neq 0, x \neq 3\}$

Range $= \left\{y : y \in \mathbb{R}, y \leq \dfrac{1}{3} \text{ and } y \geq 3\right\}$

Exercise 4.5

Sketch the graph of each of the following functions and state the domain and range:

1. $f(x) = \dfrac{x}{(x-3)(x+2)}$

2. $f(x) = \dfrac{x^3+4x^2+5x+2}{x^2+x-6}$

3. $f(x) = \dfrac{2x}{(x-3)(x+3)}$

4. $f(x) = \dfrac{x^2-4}{x^2-4x}$

5. $f(x) = \dfrac{x^3+x}{x^2-9}$

6. $f(x) = \dfrac{x^3+2x^2+1}{(x+1)(x-1)}$

7. $f(x) = \dfrac{(x-1)(x^3-2x^2-2x+1)}{x^3-x^2-x+1}$

8. $f(x) = \dfrac{2x^4+x^2-3}{x^2-9}$

9. $f(x) = \dfrac{x^2-2}{x^2-1}$

10. $f(x) = \dfrac{x^2-3x-2}{x^2-4}$

Composite functions

A composite function is a function which is obtained by substituting one function into another. Suppose $f(x)$ and $g(x)$ are two functions, then the composite of $f(x)$ and $g(x)$ is given as $f[g(x)]$ or $g[f(x)]$. The composite function $f[g(x)]$ is written as $(f \circ g)(x)$, and read as "f of g of x". The function $g(x)$ is the inner function and the function $f(x)$ is the outer function. Hence, $f[g(x)]$ can be read as "the function g is the inner function of the outer function f". The composite function is sometimes written as $f \circ g$.

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Activity 4.2: Verifying whether a composite of two linear functions is also a function

Individually or in a group, perform the following tasks:

1. Write down any two linear functions.
2. Substitute one function into another for the functions obtained in task 1.
3. Verify whether the result obtained in task 2 is a function or not.

Steps in forming a composite function

The following steps are used in forming a composite function:

1. Decide which is the inner and which is the outer function.
2. Substitute the inner function for the variable x of the outer function.
3. Simplify the resulting composite function.

Example 4.18

Given the functions $g(x) = 2x - 2$ and $f(x) = x^2 + 5$, find each of the following:

(a) $(g \circ f)(x)$

(b) $(f \circ g)(x)$

Solution

Given

$g(x) = 2x - 2$ and $f(x) = x^2 + 5$.

(a) To find $(g \circ f)(x)$, substitute $x^2 + 5$ for x into the function $g(x) = 2x - 2$.

That is, $(g \circ f)(x) = 2(x^2 + 5) - 2$

$= 2x^2 + 10 - 2$

$= 2x^2 + 8$

Therefore, $(g \circ f)(x) = 2x^2 + 8$.

(b) To find $(f \circ g)(x)$, substitute $g(x) = 2x - 2$ for x into the function $f(x) = x^2 + 5$. That is,

$(f \circ g)(x) = (2x - 2)^2 + 5$

$= 4x^2 - 8x + 9$.

Therefore,

$(f \circ g)(x) = 4x^2 - 8x + 9$.

Example 4.19

Given $f(x) = 2x + 7$, find $(f \circ f)(x)$.

Solution

Given $f(x) = 2x + 7$.

Substitute $2x + 7$ for x into the function $f(x) = 2x + 7$.

That is, $(f \circ f)(x) = 2(2x + 7) + 7$

$= 4x + 14 + 7$

$= 4x + 21$.

Therefore, $(f \circ f)(x) = 4x + 21$.

Example 4.20

Evaluate $f[g(6)]$ given that, $f(x) = 6x + 2$ and $g(x) = x - 2$.

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Solution

Given $f(x)=6x+2$ and $g(x)=x-2$.

Substitute $x-2$ for x into the function $f(x)=6x+2$

That is, $f[g(x)]=6(x-2)+2$

$=6x-12+2$

$=6x-10$

Hence,

$f[g(6)]=6(6)-10=36-10=26$

Therefore, $f[g(6)]=26$.

Example 4.21

Given $f(x)=\sqrt{x-3}$ and $g(x)=\ln(1+x^2)$, find $g(f(x))$.

Solution

Given that $f(x)=\sqrt{x-3}$ and $g(x)=\ln(1+x^2)$.

Substitute $\sqrt{x-3}$ for x into the function $g(x)=\ln(1+x^2)$ that is,

$g(f(x))=\ln\left[1+(f(x))^2\right]$

$=\ln\left[1+\left(\sqrt{x-3}\right)^2\right]$

$=\ln|1+(x-3)|$

$=\ln|x-2|$

Therefore, $g(f(x))=\ln|x-2|$.

Example 4.22

Given $f(x)=x^2+4$ and $g(x)=x-9$. Solve for x if $(f\circ g)(x)=(g\circ f)(x)$.

Solution

Given that $f(x)=x^2+4$ and $g(x)=x-9$.

Substitute $x-9$ for x into the function $f(x)=x^2+4$. That is,

$(f\circ g)(x)=(x-9)^2+4$

$=x^2-18x+85$

Substitute x^2+4 for x into the function $g(x)=x-9$. That is,

$(g\circ f)(x)=(x^2+4)-9$

$=x^2-5$

But $(f\circ g)(x)=(g\circ f)(x)$, thus

$x^2-18x+85=x^2-5$

$\Rightarrow -18x+85=-5$

$\Rightarrow -18x=-90$

Thus, x 5.

Therefore, $x=5$.

Example 4.23

Find $(f\circ g)(x)$ and $(g\circ f)(x)$ given that $f=\{(2,5),(5,7),(9,0)\}$ and $g=\{(3,2),(4,5),(0,7)\}$.

Solution

Given $f=\{(2,5),(5,7),(9,0)\}$ and $g=\{(3,2),(4,5),(0,7)\}$. The composite functions are given by:

$f[g(3)]=f[2]=5\rightarrow(3,5)$

$g[f(2)]=g[5]$ does not exist.

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$f[g(4)] = f[5] = 7 \rightarrow (4, 7)$
$g[f(5)] = g[7]$ does not exist.
$f[g(0)] = f[7]$ does not exist.
$g[f(9)] = g[0] = 7 \rightarrow (9, 7)$
Therefore, $(f \circ g)(x) = \{(3, 5), (4, 7)\}$ and $(g \circ f)(x) = \{(9,7)\}$.

Example 4.24

If $f(x) = 2x^2 + 4x + 4$, and $(f \circ g)(x) = 2x^2 - 8x + 10$, find $g(x)$.

Solution

Given $f(x) = 2x^2 + 4x + 4$ and $(f \circ g)(x) = 2x^2 - 8x + 10$.
But $(f \circ g)(x) = f[g(x)]$.

Substitute $g(x)$ for x into the function $f(x)$ to obtain,
$2[g(x)]^2 + 4g(x) + 4 = 2x^2 - 8x + 10$

Rearrangement gives,
$2[g(x)]^2 + 4g(x) - (2x^2 - 8x + 6) = 0$

This is a quadratic equation in $g(x)$.
Thus,

$$g(x) = \frac{-4 \pm \sqrt{16 + 8(2x^2 - 8x + 6)}}{4}$$

$$= -1 \pm \sqrt{1 + \frac{1}{2}(2x^2 - 8x + 6)}$$

$$= -1 \pm \sqrt{1 + x^2 - 4x + 3}$$

$$= -1 \pm \sqrt{x^2 - 4x + 4}$$

$$= -1 \pm \sqrt{(x-2)^2}$$

$$= -1 \pm (x - 2)$$

Therefore,
$g(x) = x - 3$ and $g(x) = 1 - x$.

Exercise 4.6

1. Given $f(x) = 3x + 5$ and $g(x) = 4 - x$, find $f(g(x))$ and $g(f(x))$ and use the results to verify whether or not the composition of the functions is commutative.

2. Find $f(x)$ and $f\left(\frac{1}{2}\right)$ if
   (a) $(f \circ g)(x) = \frac{x^4 + x^2}{1 + x^2}$ and $g(x) = 1 - x^2$.
   (b) $(f \circ g)(x) = 3x^2 - 6x + 11$ and $g(x) = x^2 - 2x + 3$.

3. Given the functions $f = \{(-1, 1), (0, 3), (4, 6)\}$ and $g = \{(1, 1), (3, 5), (7, 9)\}$. Find $(g \circ f)$ and determine its domain and range.

4. In each of the following pairs of functions, find $(f \circ g)(x)$:
   (a) $f(x) = x^2 - 2$, $g(x) = x + 3$
   (b) $f(x) = x + 4$, $g(x) = x^2 - 11$
   (c) $f(x) = e^x - 2$, $g(x) = x^3$
   (d) $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

5. Show that if $f : A \rightarrow B$ and $g : B \rightarrow C$ are one to one, then $g \circ f : A \rightarrow C$ is also one to one.

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6. Find $g(-4)$ if $f(x)=\dfrac{x+9}{3x}$ and $(f\circ g)(x)=\dfrac{2(x+5)}{3(2x+1)}$

7. If $f(x)=\sqrt{27-x^2}$ and $g(x)=x^2-2$, solve the equation $(g\circ f)(x)=0$.

8. Let $f(x)=3x-2$, $g(x)=4x$, and $h(x)=x^2+2$. Compute each of the following:
   (a) $f(g(3))$ (d) $f(g(h(3)))$
   (b) $f(h(6))$ (e) $h(g(f(5)))$
   (c) $(g\circ h)(12)$ (f) $h(x-1)$

9. Given that $h(x)=4x-13$ and $g(x)=2x^2-26x+9$, find:
   (a) $(g\circ h)(x)$
   (b) $(h\circ g)(x)$

10. If $f=\{(3,8),(2,5),(4,-5),(9,3)\}$ and g $=\{(7,2),(-5,3),(5,7),(8,10),(1,7)\}$, find $f\circ g$ and $g\circ f$.

11. If f and g are sets of ordered pairs such that $f=\{(-2,1),(0,3),(4,5)\}$ and $g=\{(1,1),(3,3),(7,9)\}$. Find $g\circ f$.

12. Find the composite function $f\circ g$ given that $f=\{(3,6),(5,7),(9,0)\}$ and $g=\{(2,3),(4,5),(6,7)\}$.

Graphs of composite functions

The procedures for drawing graphs of composite functions are similar to those applied to other functions.

Example 4.25

Given $f(x)=x+3$ and $g(x)=x^2+2x$, sketch the graphs of $(f\circ g)(x)$ and $(g\circ f)(x)$ on the same xy-plane.

Solution

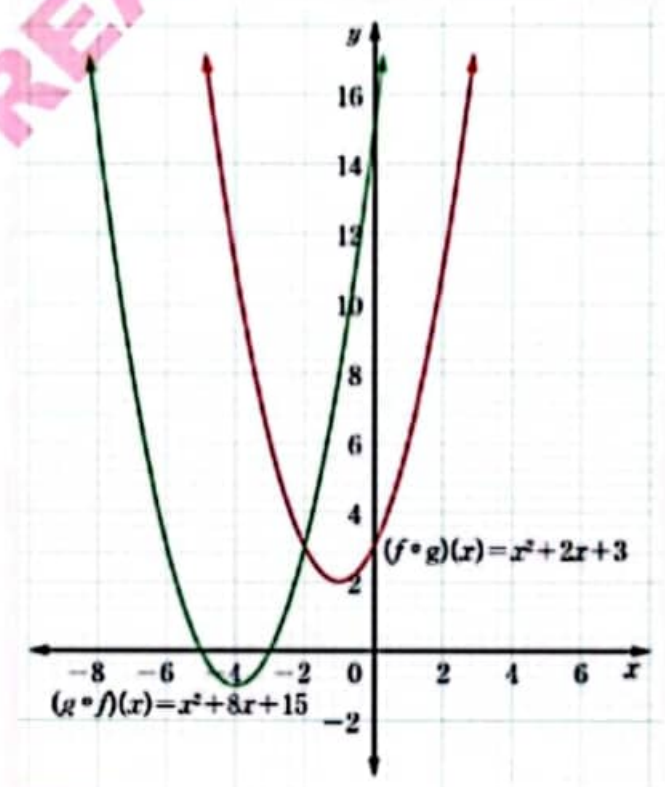
Given $f(x)=x+3$ and $g(x)=x^2+2x$.

Substitute x^2+2x for x into $f(x)=x+3$ and $x+3$ for x in $g(x)=x^2+2x$.

That is, $f(g(x))=x^2+2x+3$ and $(g(f(x))=(x+3)^2+2(x+3)=x^2+8x+15$.

Hence, $(f\circ g)(x)=x^2+2x+3$ and $(g\circ f)(x)=x^2+8x+15$.

The graphs of $(f\circ g)(x)$ and $(g\circ f)(x)$ are plotted in the following figure.



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Example 4.26

Given $f(x)=3x-1$ and $g(x)=\dfrac{4}{x-2}$, sketch separately the graphs of $(f\circ g)(x)$ and $(g\circ f)(x)$.

Solution

Given $f(x)=3x-1$ and $g(x)=\dfrac{4}{x-2}$. It follows that,

$$f(g(x))=3\left(\frac{4}{x-2}\right)-1$$

$$=\frac{12}{x-2}-\frac{1}{1}=\frac{12-x+2}{x-2}$$

$$=\frac{14-x}{x-2}$$

Thus, $(f\circ g)(x)=\dfrac{14-x}{x-2}$.

For x-intercept; $y=0$.

Thus, $0=\dfrac{14-x}{x-2}$.

$\Rightarrow 0=14-x$

$\Rightarrow x=14$

Thus, the x-intercept is at the point $(14, 0)$.

For y-intercept; $x=0$.

$\Rightarrow y=\dfrac{14-0}{0-2}$

$\Rightarrow y=-7$

Thus, the y-intercept is at the point $(0, -7)$.

To determine the vertical asymptote, equate the denominator to zero, that is $x-2=0$.

$\Rightarrow x=2$.

Hence, the vertical asymptote is the line $x=2$.

Sketch $x=2$ as a dotted line on the xy-plane.

Determine the horizontal asymptote as follows. Since the degree of numerator and denominator are equal, divide both numerator and denominator by x. It follows that,

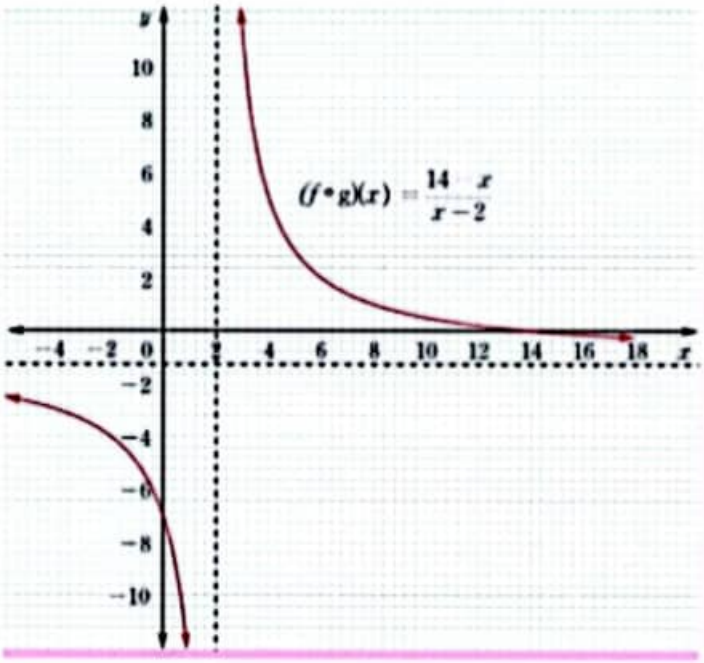
$$(f\circ g)(x)=\frac{\frac{14}{x}-\frac{x}{x}}{\frac{x}{x}-\frac{2}{x}}=\frac{\frac{14}{x}-1}{1-\frac{2}{x}}$$

Thus, as $x\to\pm\infty$, $y\to -1$.

Hence, the horizontal asymptote is the line $y=-1$.

Draw $y=-1$ as a dotted line on the xy-plane. Locate the points on either side of the vertical asymptote, also, select some values of x and determine the corresponding values of y. Use the values to sketch the graph.

Therefore, the graph of $(f\circ g)(x)=\dfrac{14-x}{x-2}$ is as shown in the following figure.



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For the composite function $g(f(x))$, substitute $3x-1$ for x into $g(x)=\frac{4}{x-2}$, so that

$g(f(x))=\frac{4}{(3x-1)-2}$.

$\Rightarrow g(f(x))\Rightarrow f(f(x))=\frac{4}{3x-3}$

Thus, $(g\circ f)(x)=\frac{4}{3x-3}$

For x-intercept; $y=0$.

Thus, $0=\frac{4}{3x-3}$, implies there is no x-intercept.

For y-intercept; $x=0$.

Thus, $y=\frac{4}{3(0)-3}$

$\Rightarrow y=-\frac{4}{3}$

Hence, y-intercept is at the point $\left(0,-\frac{4}{3}\right)$.

For vertical asymptote, equate the denominator to zero, that is,

$3x-3=0$

$\Rightarrow x=1$

Hence, the vertical asymptote is the line $x=1$.

Draw $x=1$ as a dotted line on the xy-plane.

Determine the horizontal asymptote as follows.

Since the degree of the numerator is 0, and the degree of the denominator is 1, divide both numerator and denominator by x. That is,

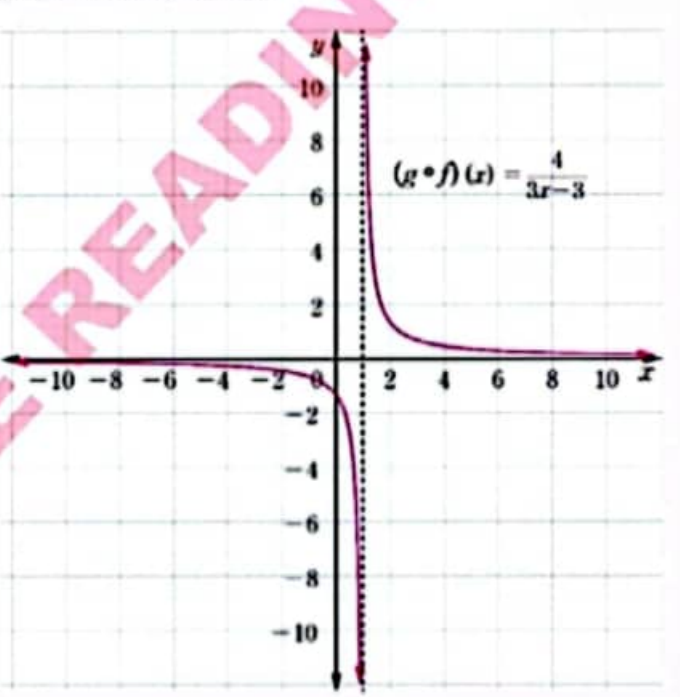
$$y=(g\circ f)(x)=\frac{\frac{4}{x}}{\frac{3x}{x}-\frac{3}{x}}=\frac{\frac{4}{x}}{3-\frac{3}{x}}$$

Thus, as $x\to\pm\infty$, $y=0$.

Hence, the horizontal asymptote is the line $y=0$.

Plot the points on either side of the vertical asymptote, also select some values of x and determine the corresponding values of y. Use these values to sketch the graph.

Therefore, the graph of $(g\circ f)(x)=\frac{4}{3x-3}$ is shown as follows.



Exercise 4.7

1. The functions g and h are defined by $g(x)=x^2+1$ and $h(x)=x+4$, where $x\in\mathbb{R}$. Sketch the graph of $(g\circ h)(x)$ and state its domain and range.

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2. If $f(x)=x^2+2x+3$, find the function $g(x)$, given that $(f\circ g)(x)=x^2-4x+6$. Hence, sketch the graph of $(f\circ g)(x)$.

3. Given $f(x)=2x^2+1$ and $g(x)=\frac{4x}{x^2-2}$, find:
   (a) $(f\circ g)(x)$
   (b) The asymptotes of $(f\circ g)(x)$.
   Hence, sketch the graph of $(f\circ g)(x)$ and state its domain and range.

4. Given $f(x)=6x^2-1$ and $g(x)=\frac{6}{x-4}$, sketch the graph of $(g\circ f)(x)$.

5. Given that $(f\circ g)(x)=3x^2+10x+15$ and $f(x)=x^2+2x+1$.
   (a) Find $g(x)$
   (b) Sketch the graph of $(f\circ g)(x)$.

6. If $(f\circ g)(x)=\frac{x^2+2}{x^4+4}$ and $g(x)=x^2+1$, find $f(x)$.

7. Given that $f(x)=4x+1$ and $g(x)=6x+k$, find the value of k if $f(g(x))=g(f(x))$. Hence, sketch the graph of $g(f(x))$.

8. A function is defined by $g:x\to x^2-10$ for $x\in\mathbb{R}$. Find the value of x for which $(g\circ g)(x)=26$. Hence, sketch the graph of $(g\circ g)(x)$.

9. If $f(x)=3x-2$ and $h(x)=\frac{1}{1+x}$, determine the intercepts and the asymptotes of $(f\circ h\circ h)(x)$. Hence, sketch the graph of the composite function.

10. Given the functions $f(x)=3x+4$ and $g(x)=x^2$. Find
    (a) $(f\circ g)(x)$
    (b) $(g\circ f)(x)$
    Hence, draw the graph of $(g\circ f)(x)$.

11. Given $f(x)=x^2$, $g(x)=x-1$, and $h(x)=\frac{x}{2}$, find:
    (a) $(f\circ g\circ h)(x)$ and sketch its graph.
    (b) $(h\circ g\circ f)(4)$

12. Find $(f\circ g)(x)$ and $(g\circ f)(x)$ in each of the following pairs of functions and sketch their graphs on the same xy-plane:
    (a) $f(x)=x+6$ and $g(x)=x+3$
    (b) $f(x)=x^2+2$ and $g(x)=x$
    (c) $f(x)=3x+1$ and $g(x)=6-3x$

Graphs of exponential functions

An exponential function $f(x)$ is a function of the form $g(x)=a^{f(x)}$ or $y=a^{f(x)}$, where the base $a>0$, $a\neq 1$, and $f(x)$ is any function of x. To draw the graph of an exponential function, choose some

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values of the independent variable x and determine the corresponding values of the dependent variable y. Tabulate the values and plot them on the xy-plane.

Example 4.27

Draw the graph of $g(x)=2^x$.

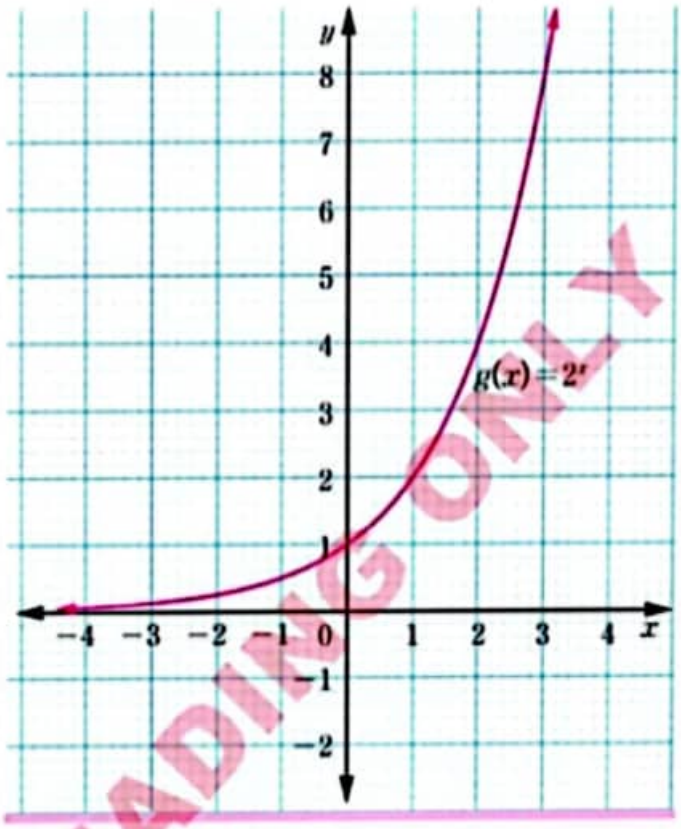
Solution

Given $g(x)=2^x$.

Select a few values for graphing the function as shown in the following table.

x	-3	-2	-1	0	1	2	3
$g(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

The graph of $g(x)=2^x$ is shown in the following figure:



Example 4.28

Draw the graph of $g(x)=\left(\frac{1}{2}\right)^x$, hence determine its domain and range.

Solution

Given $g(x)=\left(\frac{1}{2}\right)^x$.

Select a few values for graphing the function as shown in the following table.

x	-3	-2	-1	0	1	2	3
$g(x)=\left(\frac{1}{2}\right)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph of $g(x)=\left(\frac{1}{2}\right)^x$ is shown in the following figure.

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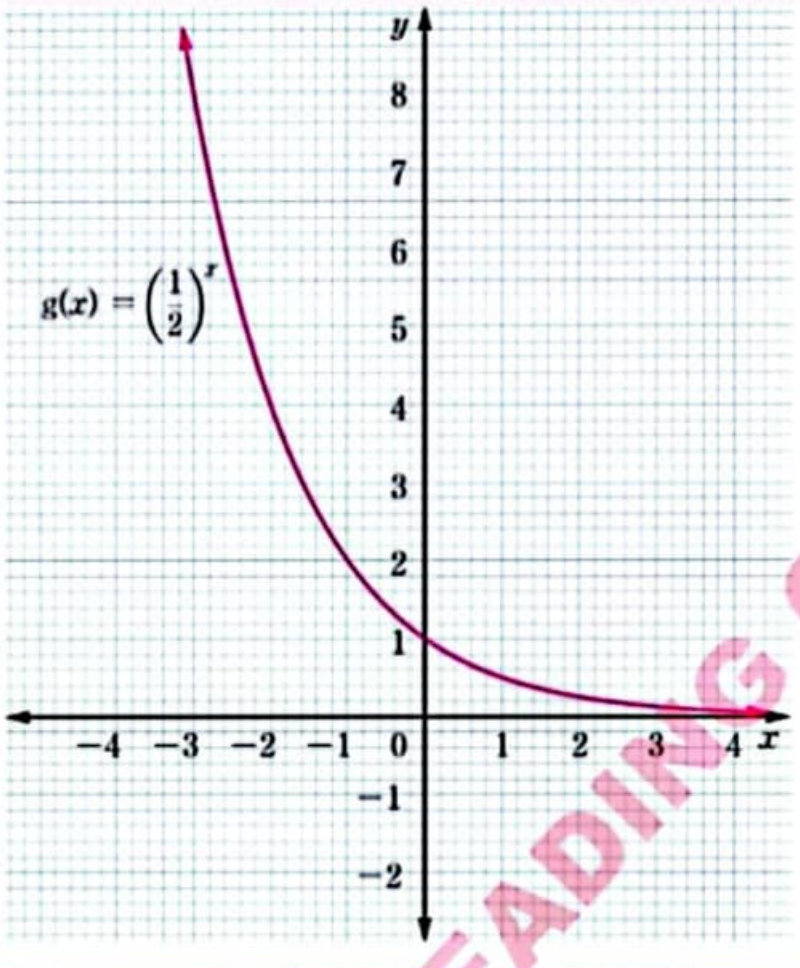

Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}, y > 0\}$

Example 4.29

Draw the graphs of $f(x) = e^x$, $g(x) = -e^x$, and $h(x) = e^{x-1}$ on the same xy-plane.

Solution

Select a few values for graphing the functions as shown in the following table.

x	−3	−2	−1	0	1	2
$f(x) = e^x$	0.0498	0.1353	0.3679	1	2.7183	7.3891
$g(x) = -e^x$,	−0.0498	−0.1353	−0.3679	−1	−2.7183	−7.3891
$h(x) = e^{x-1}$	0.0183	0.0498	0.1353	0.3679	1	2.7183

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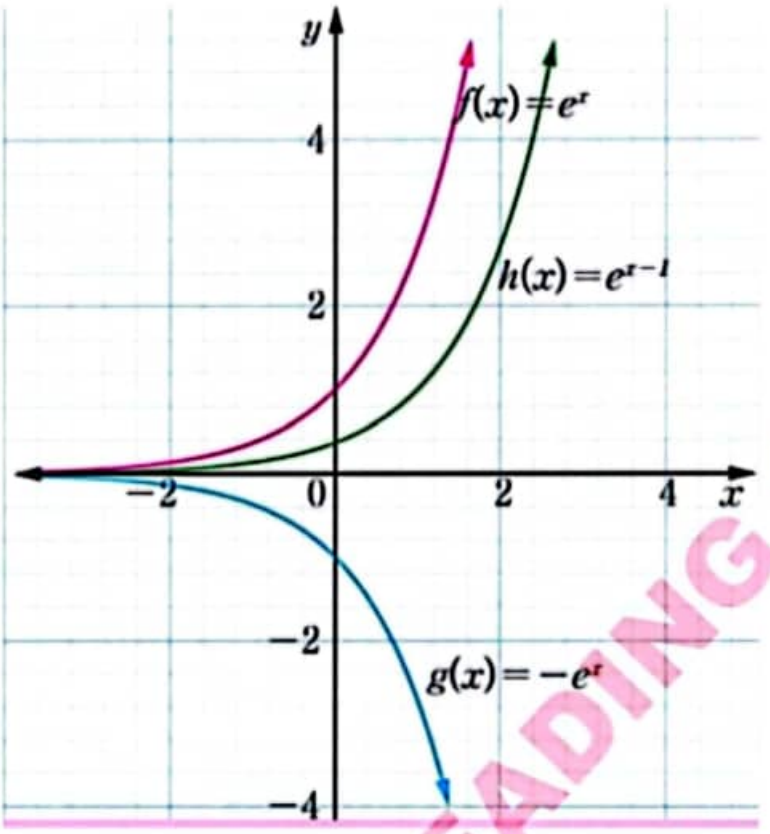
The graphs of $f(x)=e^x$, $g(x)=-e^x$, and $h(x)=e^{x-1}$, are shown in the following figure.



Exercise 4.8

1. Draw the graph of $f(x)=\left(\frac{1}{3}\right)^x$ and determine its domain and range.
2. Graph each of the following functions and state its domain and range.
   (a) $f(x)=2^{x+4}-4$
   (b) $f(x)=-e^{x+2}$
   (c) $f(x)=\left(\frac{1}{2}\right)^{x-2}-3$
   (d) $f(x)=-2^{x+3}+1$
   (e) $f(x)=e^{-x}+3$
3. Draw the graph of $y=4^x$, $y=5^x$, and $f(x)=\left(\frac{1}{3}\right)^x$ on the same xy-plane.
4. Sketch the graph of $f(x)=a^x$ for the following values of a, on the same xy-plane.
   (a) $a=3$ (b) $a=7$ (c) $a=2$
   (d) $a=6$ (e) $a=\frac{1}{6}$ (f) $a=\frac{1}{7}$
5. Graph each of the following exponential functions.
   (a) $f(x)=8^x$ (d) $f(x)=2^{x+4}$
   (b) $f(x)=e^{x-1}$ (e) $f(x)=2^{x-3}-1$
   (c) $f(x)=6^{x+1}$

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6. Graph each of the following functions and state the domain and range.

   (a) $f(x)=\left(\frac{1}{7}\right)^{x-1}+7$

   (b) $f(x)=\left(\frac{1}{4}\right)^{x+2}-1$

   (c) $f(x) = {}^{x+1}$

   (d) $f(x)=3^{x+2}+1$

7. Sketch the graph of $f(x)=e^{(x+3)\ln 2}$.

8. Draw the graphs of $f(x) = {}^{2x}$, $g(x)=-e^{2x}$, and $h(x)=e^{2x+1}$ on the same xy-plane.

9. Draw the graphs of $f(x)=4^{x+3}$ and $g(x)=-4^{x+3}$ on the same xy-plane.

10. Draw the graph of $g(x)=\frac{1}{2}e^{-1.5x}$ over a range $x=-2.5$ to $x=2.5$ and hence determine the value of $g(x)$ when $x=-0.5$ and the value of x when $g(x)=3.5$.

Graphs of logarithmic functions

If a is any positive real number not equal to 1, then the function defined as $f(x)=\log_a x$, $x>0$, is called a logarithmic function.

A logarithmic function is the inverse of an exponential function, that is, if $y=a^x$, then $x=\log_a y$, $a>0$, $a\neq 1$.

The procedure for drawing graphs of logarithmic functions is similar to the procedure used for drawing exponential functions.

Example 4.30

Draw the graph of $f(x)=\log_2 x$ and state its domain and range.

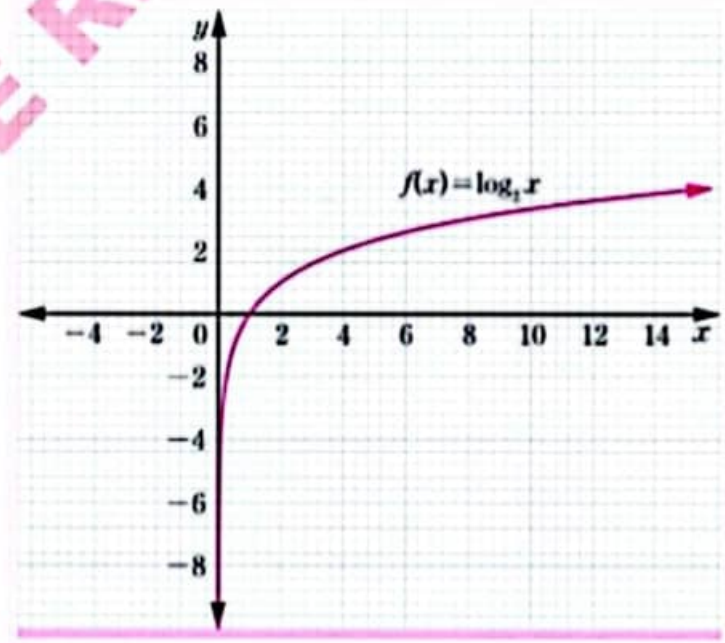
Solution

Given $f(x)=\log_2 x$.

Select a few values for graphing the function as shown in the following table.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x)=\log_2 x$	−2	−1	0	1	2

The graph of $f(x)=\log_2 x$ is shown in the following figure.



Domain $=\{x: x\in \mathbb{R}, x>0\}$,
Range $=\{y: y\in \mathbb{R}\}$.

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Example 4.31

Draw the graph of $f(x) = \ln x$.

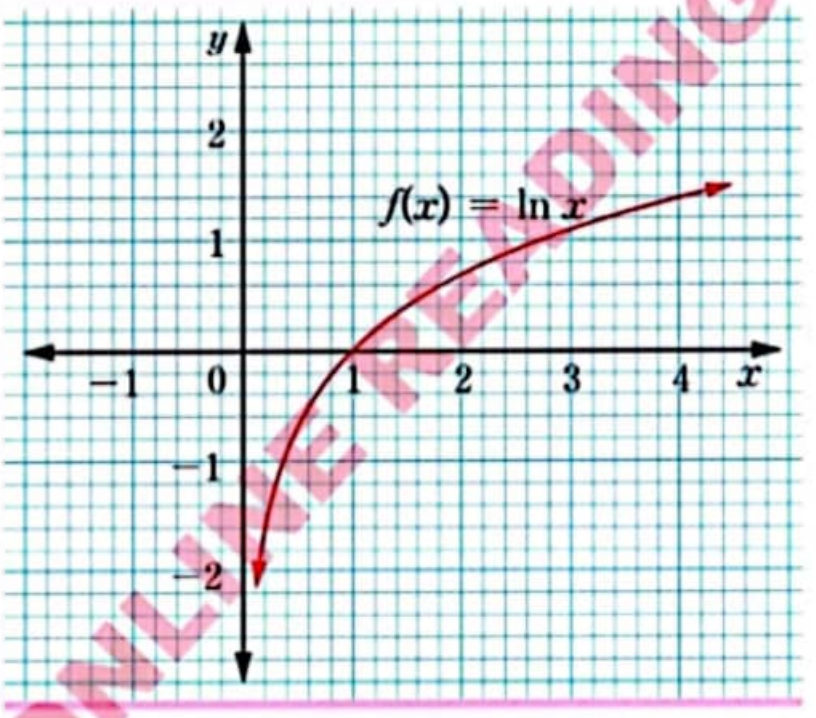
Solution

Given $f(x) = \ln x$.

Select a few values for graphing the function as shown in the following table.

	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \ln x$	−2.08	−1.39	−0.69	0	0.69	1.39

The graph of $f(x) = \ln x$ is shown in the following figure.



Example 4.32

Draw the graph of $f(x) = \log_{\frac{1}{2}} x$, hence state its domain and range.

Solution

Given $f(x) = \log_{\frac{1}{2}} x$.

Select a few values for graphing the function as shown in the following table.

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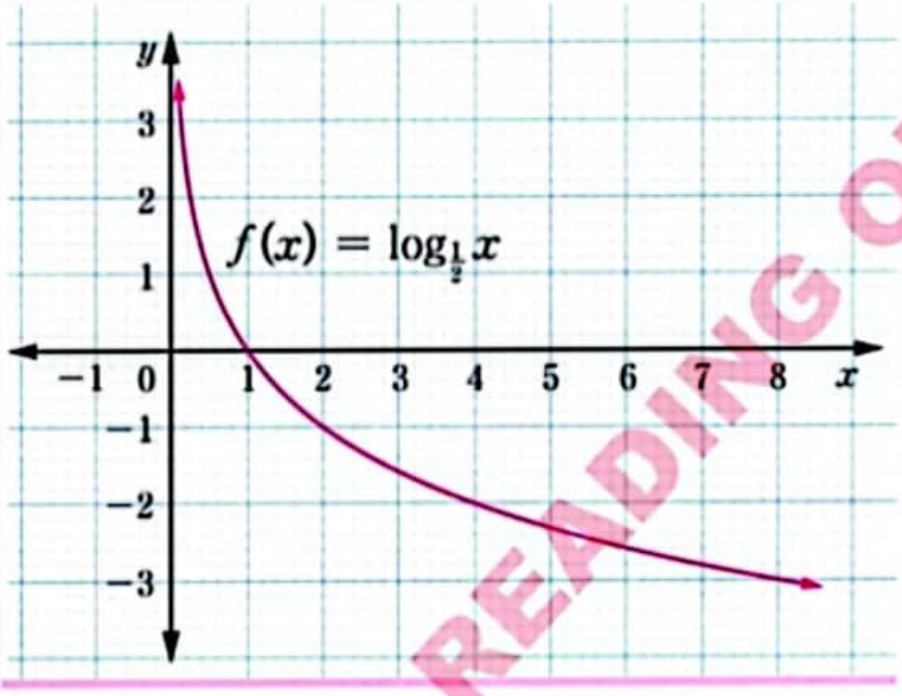
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x)=\log_{\frac{1}{2}} x$	3	2	1	0	−1	−2	−3

The graph of $f(x)=\log_{\frac{1}{2}} x$ is plotted in the following figure.



Therefore, the domain $=\{x : x \in \mathbb{R}, x>0\}$ and range $=\{y : y \in \mathbb{R}\}$.

Example 4.33

Draw the graph of $f(x)=\log_4(x+2)+1$, hence state its domain and range.

Solution

Given $f(x)=\log_4(x+2)+1$.

Select a few values for graphing the function as shown in the following table.

x	−1.8	−1.7	−1.6	−1	0	1	1.5	2
$f(x)=\log_4(x+2)+1$	−0.2	0.1	0.3	1	1.5	1.8	1.9	2

The graph of $f(x)=\log_4(x+2)+1$ is shown in the following figure.

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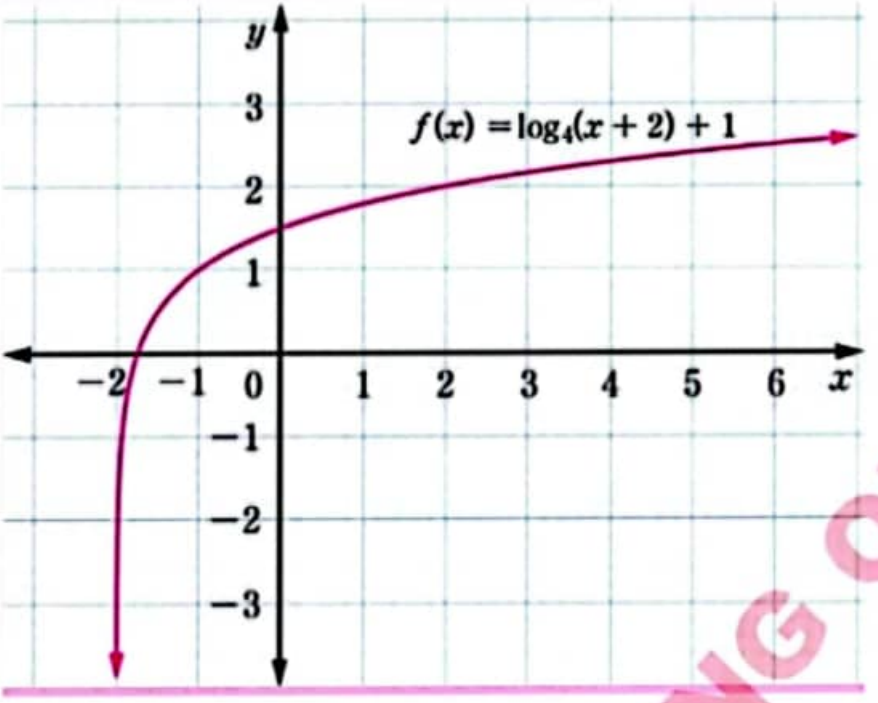

Therefore, the domain $=\{x : x \in \mathbb{R},\ x > -2\}$ and range $=\{y : y \in \mathbb{R}\}$.

Example 4.34

Draw the graphs of $f(x) = \log_3 x$, $g(x) = \log_6 x$, $h(x) = \log_{\frac{1}{7}} x$, and $k(x) = \log_{\frac{1}{4}} x$ on the same xy-plane.

Solution

Select a few values for graphing the functions as shown in the following table:

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$h(x) = \log_{\frac{1}{7}} x$	1.7	1.1	0.6	0	−0.6	−1.1	−1.7
$f(x) = \log_3 x$	−3	−2	−1	0	1	2	3
$g(x) = \log_6 x$	−1.8	−1.2	−0.6	0	0.6	1.2	1.8
$k(x) = \log_{\frac{1}{4}} x$	2.4	1.6	0.8	0	−0.8	−1.6	−2.4

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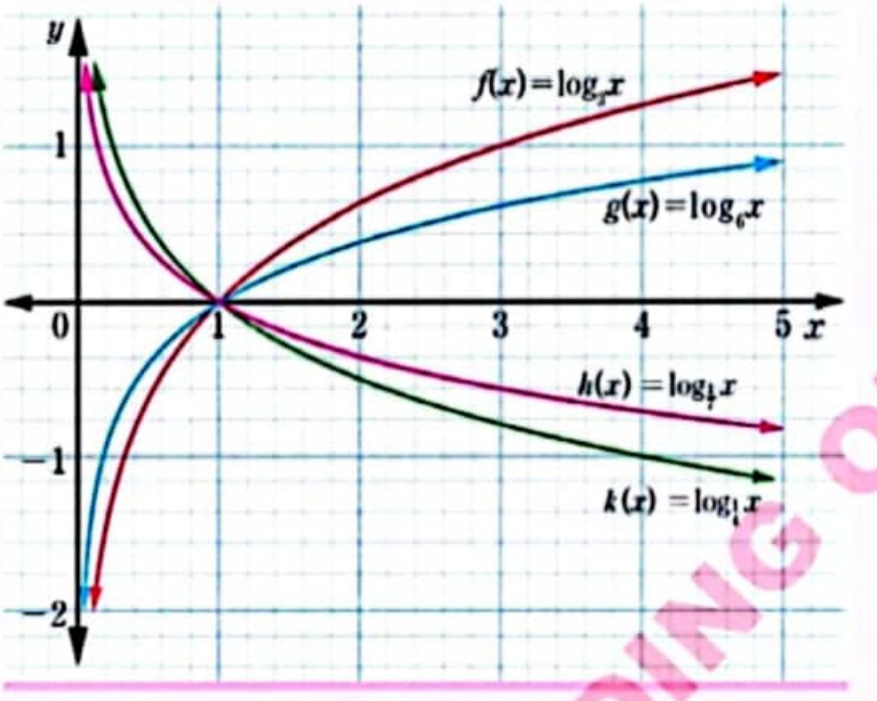
The graphs of $f(x)=\log_3 x$, $g(x)=\log_6 x$, $h(x)=\log_{\frac{1}{7}} x$, and $k(x)=\log_{\frac{1}{4}} x$ are shown in the following figure.



Exercise 4.9

1. Draw the graph and identify the domain and range of each of the following:
   (a) $y=\log_6(x-1)-4$
   (b) $y=\log_6(3x-14)+1$
   (c) $y=-\log_5(x-1)+4$
   (d) $y=\log_5(x+2)+1$
   (e) $y=\log_2(x-2)-3$
2. Draw the graph of $f(x)=-2\log(x-3)+1$.
3. Draw the graph of $f(x)=-2\log_5(-(x-3))+1$.
4. Draw the graph of $f(x)=\log_9\left(\frac{15}{7}x\right)$, find its domain and range.
5. Draw the graphs of $f(x)=e^{2x}$ and $g(x)=\ln 2x$ on the same xy-plane.
6. Draw the graph of $f(x)=1+\log_2 x$.
7. Draw the graph of $f(x)=4^{x+2}-6$. Hence, use it to find the
   (a) Domain and range of $f(x)$.
   (b) Intercepts of $f(x)$.
8. Draw the graphs of $f(x)=\log_2 x$, $g(x)=\log_4 x$, $h(x)=\log_{\frac{1}{2}} x$, and $k(x)=\log_{\frac{1}{4}} x$ on the same xy-plane. Hence,

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(a) determine their point of intersections

(b) describe their behaviours as x become very large.

9. Draw the graph of $g(x)=\log_3(x-2)+1$, hence state its domain and range.

10. Draw the graph of $f(x)=\ln(1+2x)$ and $g(x)=\ln(2x-1)$ on the same xy-plane.

Chapter summary

1. A function is a relation between a set of inputs and a set of outputs which satisfies to a certain rule.
2. A polynomial function has the form $y=f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are costants, $a_0 \neq 0$, and n $\mathbb{N}$ or $n=0$.
3. A rational function is any function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x)\neq 0$.
4. A composite function is a function obtained by substituting one function into the other. If $f(x)$ and $g(x)$ are two functions, then the composite function $f[g(x)]$ is abbreviated as $(f\circ g)(x)$.
5. An exponential function $g(x)$ is a function of the form $g(x)=a^{f(x)}$ or $y=a^{f(x)}$ where the base $a>0$, $a\neq 1$, and $x\in\mathbb{R}$.
6. A logarithmic function is the inverse of an exponential function.

Revision exercise 4

1. Draw the graph of $f(x)=2x-x^2$ and state its domain and range.
2. Draw the graph of each of the following functions:
   (a) $f(x)=-\dfrac{5}{3}x+8$, and label the x-intercept.
   (b) $f(x)=\dfrac{1}{3}x+1$ and $g(x)=4$ on the same xy-plane and determine the point where $f(x)=g(x)$.
   (c) $f(x)=\begin{cases}x+2, & x\in[-2,2)\\ 2x-3, & x\in[1,3]\end{cases}$
3. Draw the graph of each of the following functions:
   (a) $f(x)=-x^2+4x-3$.
   (b) $f(x)=x^2-4x+3$, and determine the domain and range.
   (c) $f(x)=-x^2$, $g(x)=-\dfrac{1}{3}x^2$, and $h(x)=-3x^2$ on the same xy-plane.
4. Find the domain in which the functions $f(x)=x^2-3$ and $g(x)=3x-5$ are equal.

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5. Draw the graph of each of the following functions and determine its domain and range:

(a) $f(x)=(x+2)(x+4)(x-2)$

(b) $f(x)=x^3-9x+8$ for $-4\le x\le 4$

(c) $f(x)=x^3+3x^2+x-2$

6. Draw the graph of each of the following functions:

(a) $f(x)=3x^4-7x^2$

(b) $f(x)=x^4-2x^2$

(c) $f(x)=x^4-2x^3-5x^2$

(d) $f(x)=-(x^4-4x^3-x^2+10x+2)$

7. For each of the following rational functions, find the vertical asymptotes, horizontal asymptotes, oblique asymptotes, and holes (if any), then, sketch the graph and describe its behaviour near the asymptotes. Hence, state its domain and range.

(a) $f(x)=\dfrac{x}{x^2+x-14}$

(b) $f(x)=\dfrac{x^3+2}{x^2-2}$

(c) $f(x)=\dfrac{2x^2+6x}{x^2+4x+3}$

(d) $f(x)=\dfrac{3}{3-x^2}$

(e) $f(x)=\dfrac{x}{(x+1)(x-2)}$

(f) $f(x)=\dfrac{5}{(x-1)^2(x+2)}$

(g) $f(x)=\dfrac{1}{(x-1)(x+3)}$

(h) $f(x)=\dfrac{x^3+2x^2-9x-18}{x^3+6x^2+5x-12}$

(i) $f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$

(j) $f(x)=\dfrac{12}{x^2+2x-3}$

(k) $f(x)=\dfrac{3x-9}{x^2-x-2}$

(l) $f(x)=\dfrac{2-3x}{x^2+3x-3}$

(m) $f(x)=\dfrac{2x^3}{x^2-9}$

8. Given that $x=\{1, 2, 3, 4, 5\}$, $g(x)=\{(1, 0),(2, 1),(3, 3),(4, 6),(5, 8)\}$ and $f(x)=2x-4$, find:

(a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$

9. Given that $f=\{(1, 7), (2, 9), (3, 12), (4, 15), (5, 19)\}$ and $g=\{(7,3), (-1,5), (9,4), (2,-3)\}$.

Find:

(a) $f\circ g$ (b) $g\circ f$

10. Given that $f(x)=x^2+2x+2$, $g(x)=x-4$, and $h(x)=\cos x$, find each of the following functions:

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(a) $(f \circ g)(x)$ (c) $(g \circ f \circ h)(x)$

(b) $(g \circ f)(x)$

11. Given that $(g \circ f)(x) = 18x^2 + 2$ and $g(x) = 2x^2 + 5x + 8$, find $f(x)$.

12. If $f(x) = 4 - x$ and $g(x) = \frac{4}{x}$, show that $f \circ g \neq g \circ f$.

13. Given that $f(x) = 2x - 2$ and $g(x) = \frac{5}{x-2}$, find:

(a) $(f \circ g)(x)$ (c) $(f \circ g)(3)$

(b) $(g \circ f)(x)$ (d) $(f \circ g)(4)$

(e) Sketch the graph of $(g \ f)(x)$ and state its domain and range.

14. Given that $f(x) = \sqrt{20 - x^2}$ and $g(x) = x^2 - 4$, solve for x if $(f \circ g)(x) = 0$.

15. If $f(x) = x^2 - x$ and $g(x) = \frac{x-1}{x+2}$, find each of the following functions:

(a) $(f \circ g)(x)$ (b) $(g \circ f)(-1)$

16. If $f(x) = \sqrt{x-4}$ and $g(x) = |x|$, find each of the following functions:

(a) $(f \circ g)(8)$ (b) $(g \circ f)(20)$

17. Draw the graph in each of the following functions and determine the domain and range:

(a) $f(x) = \left(\frac{5}{3}\right)^{-x}$

(b) $f(x) = 4^x$

(c) $f(x) = \left(\frac{1}{3}\right)^{x+1} - 1$

(d) $f(x) = 2^{x-1} + 2$

(e) $f(x) = \left(\frac{5}{3}\right)^x$

(f) $f(x) = \left(\frac{1}{4}\right)^{x+2}$

(g) $f(x) = \left(\frac{1}{4}\right)^{x-1} + 4$

(h) $f(x) = e^{x-1}$

(i) $f(x) = -\left(\frac{1}{6}\right)^{-x}$

18. Draw the graph of each of the following functions:

(a) $f(x) = -4\log(x-2) + 1$

(b) $f(x) = \log\left(\frac{x+2}{x-3}\right)^2$.

19. (a) If $f(x) = \log_3 x$ and $g(x) = x - 1$, graph $(f \circ g)(x)$ and find its domain and range.

(b) Draw the graph of $f(x) = (\log_6 x) - 2$.

20. Draw the graph of $f(x) = x^3 - 9x + 5$ and state the behaviour of the function for positive and negative large values of x.

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Chapter Five

Introduction

Algebra deals with the representation of numbers and quantities in formulae and equations by symbols and letters. In this chapter, you will learn about sequences and series, proofs by mathematical induction, roots of polynomial functions, remainder and factor theorems, operations on polynomials, inequalities, matrices, binomial theorem, and partial fractions. The competencies developed can be applied in various real-life situations such as in solving problems related to landscape designing, computer programming, real estate planning, business and finance management, geometry, budgeting, making a schedule of activities, cooking, shopping, and in many other fields.

Think

The World without algebra.

Sequences and series

A sequence is an itemized collection of elements in which repetitions are allowed, while a series is the sum of all elements in a sequence.

Sequence

A set of numbers or algebraic expressions for which a member can be obtained from a preceding member by a definite rule defines a sequence. The numbers that make up the sequence are called terms of the sequence. The terms of a sequence are named by using their positions.

In general, a sequence is written as; $a_1, a_2, a_3, ..., a_n$, where a_1 is the first term, a_2 is the second term, a_3 is the third term, and a_n is the n^{th} term.

The n^{th} term is also called the general term of the sequence. In this case, the subscript of each term represents the position of the term. A sequence is either finite or infinite depending on the number of terms in a sequence. If the number of terms in the sequence is known, then the sequence is called a finite sequence, otherwise it is an infinite

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sequence. Generally, the sequence $a_1, a_2, a_3, \ldots$ is denoted by a_n. Other defined notation for a sequence can be used.

For example, in the sequence 1, 3, 5, 7, 9, 11, … the rule is that, 2 is added to obtain the next term.

By using symbols, it is written as $a_1 = 1$, $a_2 = a_1 + 2$, $a_3 = a_2 + 2, \ldots$

In general, $a_{n+1} = a_n + 2$, where $n = 1, 2, 3, \ldots$

Activity 5.1: Identifying a formula for a sequence

Individually or in a group, study the pattern of the given objects and perform the tasks that follows.

1. Find the rule of the given pattern of the objects.
2. Suggest the number of objects in columns, 6 and 7 in task 1.
3. Write the formula for terms of the pattern.
4. Share your findings with your fellow students for more inputs.

Example 5.1

Find the first three terms of the following sequences:

(a) $a_n = \dfrac{1}{n+1}$ (b) $a_n = \dfrac{1}{2}n(n+1)$

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Solution

(a) Given $a_r = \frac{1}{r+1}$.

The first three terms can be obtained as follows;

$$a_1 = \frac{1}{1+1} = \frac{1}{2},$$

$$a_2 = \frac{1}{2+1} = \frac{1}{3}, \text{ and}$$

$$a_3 = \frac{1}{3+1} = \frac{1}{4}.$$

Therefore, the first three terms are $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$.

(b) Given $a_r = \frac{1}{2}r(r+1)$.

The first three terms can be obtained as follows;

$$a_1 = \frac{1}{2} \times 1(1+1) = 1,$$

$$a_2 = \frac{1}{2} \times 2(2+1) = 3, \text{ and}$$

$$a_3 = \frac{1}{2} \times 3(3+1) = 6.$$

Therefore, the first three terms are 1, 3, and 6.

Series

When the terms of a sequence are linked together with signs of addition or subtraction, the resulting expression is known as a series. A series is said to be finite if it is the sum of a finite number of terms. Otherwise, it is infinite. For instance, the series, $a_n = 1 - 2n + 3n^2 - 4n^3$ is finite , while the series, $1 + 2 + 3 + 4 + 5 + \cdots$ is infinite.

Example 5.2

Find the next two terms of the series, $1 + 1 + 20 + 210 + 2300 + \cdots$

Solution

By observing the given series, it can be seen that the next term is ten times the sum of the previous two consecutive terms.

The 6^{th} term $= 10(210 + 2300) = 25{,}100$

Similarly,

the 7^{th} term $= 10(2300 + 25100) = 274{,}000$

Therefore, the next two terms are 25,100 and 274,000.

Sigma notation

Consider the sum of the first 6 terms of the series $3 + 6 + 9 + 12 + 15 + 18$. This series can also be written as $3\times1 + 3\times2 + 3\times3 + 3\times4 + 3\times5 + 3\times6$. All terms of this series are of the form $3n$. Thus, $3n$ is the general term. The series can be defined as the sum of terms of the form $3n$, where n takes all integral values in order from 1 to 6 inclusive. The Greek letter $\sum$ called sigma is used to denote "the sum of terms". Therefore, the series can be defined more precisely as $\sum_{k=1}^{6} 3k$.

This means that "the sum of all terms of the form $3k$, where k takes all integer values from 1 to 6 inclusive".

The symbol can be used to indicate the finite and infinite series.

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For example,

$\sum_{k=1}^{6} 3k = 3+6+9+12+15+18$

is a finite series and

$\sum_{k=1}^{\infty} 3k = 3+6+9+12+15+18+\cdots$

is an infinite series.

The sigma notation shortens the expression of a series and enables one to select a particular term of a series without writing down all the preceding terms. For instance, given the series $\sum_{k=2}^{19} 3(k-1)$, then:

The first term is obtained when
$k = 2 \Rightarrow 3(2-1) = 3$

The second term is obtained when
$k = 3 \Rightarrow 3(3-1) = 6$

The tenth term is obtained when
$k = 11 \Rightarrow 3(11-1) = 30$

The last term is obtained when
$k = 19 \Rightarrow 3(19-1) = 54.$

Example 5.3

Write $\sum_{k=1}^{4} (-1)^{k+1} \cdot 3^{k-1}$ in expanded form.

Solution

$$\sum_{k=1}^{4} (-1)^{k+1} \cdot 3^{k-1} = (-1)^2 3^0 + (-1)^3 3^1 + (-1)^4 3^2 + (-1)^5 3^3$$
$$= (1)(1) + (-1)(3) + (1)(9) + (-1)(27)$$
$$= 1 - 3 + 9 - 27$$

Therefore, $\sum_{k=1}^{4} (-1)^{k+1} \cdot 3^{k-1} = 1 - 3 + 9 - 27.$

Example 5.4

Evaluate $\sum_{k=5}^{12} 2^{4-k}$.

Solution

$$\sum_{k=5}^{12} 2^{4-k} = 2^{4-5} + 2^{4-6} + 2^{4-7} + 2^{4-8} + 2^{4-9} + 2^{4-10} + 2^{4-11} + 2^{4-12}$$
$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-7} + 2^{-8}$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$$

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$$= \frac{255}{256}$$

Therefore, $\sum_{k=5}^{12} 2^{4-k} = \frac{255}{256}$.

Properties of the sigma notation

If c is any constant that does not depend on summation of index, then:

1. $\sum_{k=n_0}^{n} ca_k = c\sum_{k=n_0}^{n} a_k$ 2. $\sum_{k=n_0}^{n} (ca_k \pm cb_k) = c\sum_{k=n_0}^{n} a_k \pm c\sum_{k=n_0}^{n} b_k$ 3. $\sum_{k=n_0}^{n} c = cn$

Example 5.5

Evaluate $\sum_{k=1}^{8} [4+(k-1)]$.

Solution

$$\begin{aligned}\sum_{k=1}^{8} [4+(k-1)] &= (4+0)+(4+1)+(4+2)+(4+3)+(4+4)+(4+5)\\ &\quad +(4+6)+(4+7)\\ &= 4+5+6+7+8+9+10+11\\ &= 60.\end{aligned}$$

Therefore, $\sum_{k=1}^{8} [4+(k-1)] = 60$.

Example 5.6

Determine $\sum_{k=1}^{n} 1$.

Solution

$$\sum_{k=1}^{n} 1 = 1 + 1 + 1 + 1 + \cdots + 1.$$

(This process is repeated "n" times as 1 does not change with the change in k). Since the terms are not changing, we can take one term and multiply by the number of times.

Thus, $\sum_{k=1}^{n} 1 = 1 \times n = n$.

Therefore, $\sum_{k=1}^{n} 1 = n$.

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Activity 5.2: Deducing a formula for the sum of the first n natural numbers

Individually or in a group, perform the following tasks:

1. Write the formula for the sum of the first n terms of an arithmetic progression.
2. Use the formula in task 1 to deduce the formula for the first n natural numbers.
3. What have you observed from tasks 1 and 2? Give comments.
4. Share your results with your fellow students for more inputs.

The sum of the first n natural numbers is $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Example 5.7

Prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Solution

Let $S_n = \sum_{k=1}^{n} k$.

Expanding S_n gives,

$S_n = 1+2+3+\cdots+(n-2)+(n-1)+n$ .....(i)

The same sum can also be obtained by reversing the order of the series, that is:

S_n ℵ⋈ℵ(ℵ 1) $(n$ 2) 3 2 1 .....(ii)

Arrange equations (i) and (ii) so that corresponding terms come together:

$$+\begin{cases} S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \end{cases} \quad \text{.....(iii)}$$

Add the two equations in the system (iii) to get;

$2S_n = (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1)$

$= (n+1)n$

Thus, $S_n = \frac{1}{2}(n+1)n$.

Therefore, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

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Sum of squares of the first n natural numbers

Suppose k is a natural number and k^2 is its square. If n consecutive natural numbers are considered, then the sum of their squares in expanded form is $1^2+2^2+3^2+4^2+5^2+6^2+\cdots+n^2$. In sigma notation, this is written as $\sum_{k=1}^{n} k^2$.

Now suppose S_n is the required sum, it follows that

$$S_n=\sum_{k=1}^{n} k^2=1^2+2^2+3^2+4^2+5^2+6^2+\cdots+n^2.$$

The value of k^2 can be obtained from the expansion of $(k-1)^3$ as follows;

$$(k-1)^3=k^3-3k^2+3k-1$$

$$\Rightarrow k^3-(k-1)^3=3k^2-3k+1 \quad \text{.....(i)}$$

Substituting $k = 1, 2, 3, \ldots, (n-2), (n-1), n$ in equation (i), it implies that:

When $k=1$; $1^3-0^3=3(1)^2-3(1)+1$

When $k=2$; $2^3-1^3=3(2)^2-3(2)+1$

$\vdots \quad \vdots \quad \vdots \qquad \vdots$

When $k=n-1$; $(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1$

When $k=n$; $n^3-(n-1)^3=3n^2-3n+1$

Adding all the terms from both sides gives,

$$(1^3-0^3)+(2^3-1^3)+\ldots+[(n-1)^3-(n-2)^3]+[n^3-(n-1)^3]$$

$$=3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+(1+1+1+\cdots+1;\ n \text{ times})$$

$$\Rightarrow n^3-0^3=3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+(1+1+1+\cdots+1;\ n \text{ times})$$

$$\Rightarrow n^3=3S_n-3\left(\frac{n(n+1)}{2}\right)+n$$

Thus, $3S_n=n^3+\frac{3}{2}n(n+1)-n$

$$=n(n+1)(n-1)+\frac{3}{2}n(n+1)$$

$$=n(n+1)\left[(n-1)+\frac{3}{2}\right].$$

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$$\Rightarrow 3S_n = \frac{n(n+1)(2n+1)}{2}$$

Thus, $S_n = \frac{n(n+1)(2n+1)}{6}$.

Therefore, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Example 5.8

Prove that $\sum_{k=1}^{n}(6k^2-8k+2) = n(n-1)(2n+1)$. Hence, evaluate $\sum_{k=1}^{64}(6k^2-8k+2)$.

Solution

$$\sum_{k=1}^{n}(6k^2-8k+2) = 6\sum_{k=1}^{n}k^2 - 8\sum_{k=1}^{n}k + 2\sum_{k=1}^{n}1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

But $\sum_{k=1}^{n}k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^{n}k^2 = \frac{1}{6}n(n+1)(2n+1)$.

Substituting these values into equation (i) gives,

$$\sum_{k=1}^{n}(6k^2-8k+2) = 6\left[\frac{1}{6}n(n+1)(2n+1)\right] - 8\left[\frac{1}{2}n(n+1)\right] + 2n$$

$$= n(n+1)(2n+1) - 4n(n+1) + 2n$$

$$= n\left[2n^2+3n+1-4n-4+2\right]$$

$$= n\left[2n^2-n-1\right]$$

$$= n(n-1)(2n+1)$$

Thus, $\sum_{k=1}^{n}(6k^2-8k+2) = n(n-1)(2n+1)$.

$$\Rightarrow \sum_{k=1}^{64}(6k^2-8k+2) = 64(64-1)(2\times 64+1)$$

$$= 64(63)(129)$$

$$= 520,128$$

Therefore, $\sum_{k=1}^{n}(6k^2-8k+2) = n(n-1)(2n+1)$ and $\sum_{k=1}^{64}(6k^2-8k+2) = 520,128$

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Activity 5.3: Deriving a formula for the sum of cubes of the first n natural numbers

Individually or in a group, perform the following tasks:

1. Write down the formula for $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} 1$, and $\sum_{k=1}^{n} k^2$.
2. Find the expansion of $(k-1)^4$.
3. Use tasks 1 and 2 to deduce that the sum of the cubes of the first n natural numbers is given by the formula $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$.
4. Have you observed any challenge from this activity? Give reasons.
5. Share your findings with other students for more inputs.

Example 5.9

Simplify $\sum_{k=1}^{n} k(k-1)(2k+3)$. Hence, evaluate $\sum_{k=1}^{100} k(k-1)(2k+3)$.

Solution

$$\sum_{k=1}^{n} k(k-1)(2k+3) = \sum_{k=1}^{n} (2k^3 + k^2 - 3k)$$

$$= 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k$$

$$= 2\left[\frac{1}{4}n^2(n+1)^2\right] + \frac{1}{6}n(n+1)(2n+1) - 3\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{2}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)(3n^2+5n-8)$$

$$= \frac{1}{6}n(n+1)(n-1)(3n+8)$$

Thus, $\sum_{k=1}^{n} k(k-1)(2k+3) = \frac{1}{6}n(n+1)(n-1)(3n+8)$ and

$$\sum_{k=1}^{100} k(k-1)(2k+3) = \frac{1}{6}(100)(101)(99)(308)$$

$$= 51,328,200.$$

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Therefore, $\sum_{k=1}^{n} k(k-1)(2k+3) = \frac{1}{6}n(n+1)(n-1)(3n+8)$

and $\sum_{k=1}^{100} k(k-1)(2k+3) = 51,328,200.$

Exercise 5.1

1. Write each of the following series by using the sigma notation:
   (a) $8 + 11 + 14 + 17 + 20$
   (b) $2 + 4 + 6 + 8 + 10 + \cdots + 28$
   (c) $32 + 16 + 8 + 4 + 2 + 1 + \frac{1}{2}$
   (d) $1 + 4 + 9 + 16 + \cdots + 121$
   (e) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$
   (f) $-1 + 4 - 9 + 16 - 25 + \cdots + n$

2. Write each of the following in expanded form:
   (a) $\sum_{k=1}^{6}(k+3)$
   (b) $\sum_{k=2}^{4}(2k^3 - k^2 + 8k + 1)$
   (c) $\sum_{k=1}^{7}(k - 3k^2)$
   (d) $\sum_{k=2}^{8}(1 - 2^k)$
   (e) $\sum_{k=0}^{9} 2^{k-2}$

3. Evaluate each of the following series:
   (a) $\sum_{k=0}^{5}(11 + 6k)$
   (c) $\sum_{k=1}^{6}\left(\frac{2}{5}\right)k$
   (e) $\sum_{k=0}^{18}(-1)^k$
   (b) $\sum_{k=0}^{5} 2^{k-1}$
   (d) $\sum_{k=6}^{13}(57 - k)$

4. Evaluate each of the following series:
   (a) $\sum_{k=1}^{n} 4k$
   (b) $\sum_{k=1}^{n}(k^2 + 3k)$
   (c) $\sum_{k=1}^{n} k(k+2)(k+3)$

5. Write each of the following series in sigma notation:
   (a) $1\times1 + 2\times3 + 3\times5 + 4\times7$
   (b) $1 - x + x^2 - x^3 + x^4$
   (c) $y + 2y^2 + 3y^3 + 4y^4 + \cdots.$
   (d) $-4 - 1 + 2 + 5 + \cdots + 17$
   (e) $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$
   (f) $1\times4 + 4\times7 + 7\times10 + 10\times13$

6. Write down the first three terms of each of the following:
   (a) $\sum_{k=2}^{n} 2^k$
   (b) $\sum_{k=2}^{n} k(k-1)$
   (c) $\sum_{k=0}^{20} \frac{k+2}{(k+1)(2k+1)}$
   (d) $\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$
   (e) $\sum_{k=1}^{n}(-1)^{k+1}k^2$
   (f) $\sum_{k=1}^{n} 6k(4k-3)(2k+1)$

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7. Simplify $\sum_{k=1}^{n} 2k(k-4)(k+6)$. Hence, evaluate $\sum_{k=20}^{40} 2k(k-4)(k+6)$.

8. Prove that $\sum_{k=1}^{n}(16k^2-10k+7)=\frac{n}{3}(16n^2+9n+14)$. Hence, evaluate $\sum_{k=1}^{208}(16k^2-10k+7)$.

9. Prove that $\sum_{k=1}^{n}(80k^3-220k^2+150)=\frac{1}{3}n(n+1)(60n^2-160n-85)$. Hence, evaluate $\sum_{k=6}^{18}(80k^3-220k^2+150k)$.

10. Show that the sum to n terms of the series $(1\times3\times6)+(2\times5\times9)+(3\times7\times12)+\cdots$ is $\frac{3}{2}n(n^3+4n^2+5n+2)$.

Proof by mathematical induction

Mathematical induction is a method that is used to prove a given statement which is formulated in terms of n where n is a natural number. Mathematical induction establishes that the first statement in an infinite sequence of objects is true, and prove that, if any one statement in the sequence is true, then so is the next one. This leads to the conclusion that the statement is true for all given values of n.

In general, proof by mathematical induction is an indirect method of proof which is used in case a direct method is either not possible or not convenient.

Activity 5.4: Verifying that the method of induction holds true on passing information through various people from the first to the last person

In a group, perform the following tasks:

1. Form a group of a convenient number of students.
2. Members of each group should stand in a straight line.
3. Let the first student in each line tell his/her secret to the one standing next. This process has to continue by each student to telling the same secret to the next student in the line and stops at the last student.
4. Give your observations in each of the following:
   (a) Does the first student in your group know the secret?
   (b) Does everybody in the line know the secret?
   (c) Does the last student know the secret (if it was not altered on the way)? In each case give reasons.
5. What have you observed during this activity? Give reasons.
6. Share your findings with other groups for more inputs.

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From Activity 5.4, the following observations can be made:

Let P_n be the statement that "S_n students know the secret" and S_n be the n^{th} student in the line.

S_2 knows the secret because it was told by S_1. So, P_2 is true.

S_3 knows the secret because it was told by S_2. So, P_3 is true, etc.

S_n knows the secret because it was told by S_{n-1}. So, P_n is true as well.

Remark: If the secret is unaltered (remained the same), then the secret is induced from the first student to the last. At the end, all students in the line know the secret.

The method used is to establish that each student got the secret correctly and conclude that the secret was induced from one student to the last, hence proved by induction.

Proof by mathematical induction is done in three steps as follows:

1. Show that the statement is true for an arbitrary natural number, say $n = 1$. (or true for any fixed natural number)
2. Assume the statement is true for $n = k$.
3. Using the assumption made in step 2, prove that the statement is also true for $n = k+1$.

Generally, the principle of mathematical induction is stated as follows:

If n is a natural number and a statement P_n is true for $n = 1$ and also true for $n = k+1$, then the statement P_n is true for all values of n.

Example 5.10

Prove by mathematical induction that if n is a natural number, then $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Proof

Let P_n be the statement, $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. That is,

$$P_n : 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

For $n=1$, check if P_1 is true. Consider the left-hand side of P_n;

$$P_1 : \sum_{r=1}^{1} r = 1$$

Consider the right-hand side of P_n;

$$P_1 : \frac{1}{2}(1)(1+1) = 1$$

Since P_1 is the same in both sides, then P_n is true, for $n=1$.

Suppose P_n is true for $n=k$.

$$\Rightarrow P_k : 1+2+3\cdots+k = \frac{1}{2}k(k+1)$$

Now show that P_n is true for $n=k+1$.

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$$P_{k+1}: 1+2+3+\ldots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{1}{2}(k+1)(k+2)$$

Thus, P_{k+1} is true. Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers.

Example 5.11

Use mathematical induction to prove that, $n^3 - n$ is divisible by 3 for all natural numbers.

Proof

Let P_n be the statement: $n^3 - n$ is divisible by 3.

For $n=1$, check if P_1 is true:

$P_1: \frac{n^3-n}{3}=\frac{1^3-1}{3}=0$, which is divisible by 3.

Thus, P_1 is true.

Suppose P_n is true for $n=k$, that is,

$P: \frac{k^3-k}{3}=m \Rightarrow k^3-k=3m$, where m is an integer.

$\Rightarrow k^3=k+3m$ .....(i)

Now show that P_n is true for $n=k+1$. That is,

$P_{k+1}: (k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1$

$=k^3+3k^2+2k$

$=(3m+k)+3k^2+2k$, substituting the value of k^3 from equation (i)

$=3k^2+3k+3m$

$=3\left(k^2+k+m\right)$ is divisible by 3 since 3 is its factor.

Thus, P_{k+1} is true. Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers.

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Example 5.12

Prove by mathematical induction that, if n is a natural number greater than 1, then

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}.$$

Proof

Let P_n be the statement,

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}$$

For $n=2$, check if P_2 is true.

Consider the left-hand side of P_n;

$$P_2:\ \frac{2+3}{(1)(2)(3)}=\frac{5}{6}$$

Consider the right-hand side of P_n;

$$P_2:\ \frac{3}{2}-\frac{2+2}{2(3)}=\frac{5}{6}$$

Since P_2 is the same in both sides, then P_n is true for $n=2$.

Suppose P_n is true for $n=k$.

$$\Rightarrow P_k:\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{k+3}{(k-1)k(k+1)}=\frac{3}{2}-\frac{k+2}{k(k+1)}.$$

Now show that P_n is also true for $n=k+1$. That is,

$$P_{k+1}:\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{k+3}{(k-1)k(k+1)}+\frac{k+4}{k(k+1)(k+2)}$$

$$=\frac{3}{2}-\frac{k+2}{k(k+1)}+\frac{k+4}{k(k+1)(k+2)}$$

$$=\frac{3}{2}-\left[\frac{k+2}{k(k+1)}-\frac{k+4}{k(k+1)(k+2)}\right]$$

$$=\frac{3}{2}-\frac{(k+2)(k+2)-(k+4)}{k(k+1)(k+2)}$$

$$=\frac{3}{2}-\frac{k^2+3k}{k(k+1)(k+2)}$$

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$$= \frac{3}{2} - \frac{k(k+3)}{k(k+1)(k+2)}$$

$$= \frac{3}{2} - \frac{k+3}{(k+1)(k+2)}$$

Thus, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers greater than one.

Exercise 5.2

Use mathematical induction to prove each of the following statements for every natural number n:

1. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
2. $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$
3. $\sum_{k=1}^{n} (k+1) = \frac{n(n+3)}{2}$
4. $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$
5. $2+6+10+14+\cdots+(4n-2) = 2n^2$
6. $1\times3+3\times5+5\times7+\cdots+(2n-1)(2n+1) = \frac{1}{3}n(4n^2+6n-1)$
7. $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
8. $\frac{1}{1\times3}+\frac{1}{2\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{n(n+2)} = \frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$
9. $\sum_{k=0}^{n} am^k = a\left(\frac{m^{n+1}-1}{m-1}\right)$
10. $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$

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11. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

12. $\sum_{k=1}^{n} k^3 = \left(\frac{1}{2}n(n+1)\right)^2$

13. $2^{3n} - 1$ is divisible by 7.

14. $n(n-1)(2n-1)$ is divisible by 6, for $n > 1$.

15. $7^{2n} + 2^{3n-3} \times 3^{n-1}$ is divisible by 25.

Roots of polynomial functions

Given a polynomial function, $p(x)$, the values of the independent variable x for which the polynomial has zero value can be determined. This section discusses the general form of a polynomial function and how its roots can be obtained.

Polynomial functions

A polynomial function $p(x)$ is an algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers and n is a natural number. The degree of a polynomial is the highest positive integer exponent of the variable x with non-zero coefficient. Polynomials are named basing on their degrees. Thus, a polynomial of degree 1 is called linear, a degree 2 polynomial is called a quadratic, a degree 3 polynomial is called a cubic, a degree 4 polynomial is called a quartic, a degree 5 polynomial is called a quintic, and so on.

Roots of polynomials

The values of x for which the value of the polynomial $p(x)$ is zero are called roots or zeros of $p(x)$. The roots are called zeros because they are the values of x at which the polynomial has value zero. The highest exponent of the variable gives the number of roots the polynomial has. If the highest exponent of a polynomial is 2, the polynomial has two roots. If the highest exponent is 3, then the polynomial has three roots, and so on.

Example 5.13

Given that $p(x) = (x-3)(x^2-1)$, find the roots of $p(x)$.

Solution

The roots of $p(x)$ are obtained when $p(x) = 0$.

$\Rightarrow (x-3)(x^2-1) = 0$

$\Rightarrow (x-3)(x+1)(x-1) = 0$

Thus, either

$x-3=0$, $x+1=0$ or $x-1=0$

$\Rightarrow x = 3$, $x = -1$ or $x = 1$

Therefore, the roots of $p(x)$ are -1, 1, and 3.

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Relationships between roots and coefficients of a quadratic equation

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, such that $a \neq 0$. The roots of a quadratic equation are obtained by solving the equation. Alternatively, the roots can be obtained from the coefficients of the equation. Therefore, there are relationships between coefficients of a quadratic equation and its roots.

Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.
Thus, either $x - \alpha = 0$ or $x - \beta = 0$.

This suggests that, the equation $ax^2 + bx + c = 0$ is obtained by multiplying its factors, $(x - \alpha)$ and $(x - \beta)$. That is,

$$(x-\alpha)(x-\beta) = 0.$$

$$\Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta = 0 \quad \text{..... (i)}$$

Now consider the equation, $ax^2 + bx + c = 0$:

Divide it by a throughout to obtain,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{..... (ii)}$$

Comparing coefficients of corresponding terms of equations (i) and (ii) gives,

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Hence, the sum of the roots of a quadratic equation is given by $\alpha + \beta = -\frac{b}{a}$ and the product of the roots is given by $\alpha\beta = \frac{c}{a}$.

Generally, if the roots of a quadratic equation are α and β, then the equation of the quadratic equation can be expressed in the form $x^2 - (\alpha+\beta)x + \alpha\beta = 0$. That is,

x^2 – (sum of roots)x + (product of roots) = 0.

Example 5.14

If the roots of the equation $2x^2 + 3x - 2 = 0$ are α and β, find the values of the following:

(a) $\alpha + \beta$ (b) $\alpha\beta$

Solution

(a) Given $2x^2 + 3x - 2 = 0$. Compare it with the equation $ax^2 + bx + c = 0$.

$a = 2$, $b = 3$, $c = -2$, $\alpha + \beta = -\frac{b}{a}$.

Thus, $\alpha + \beta = -\frac{3}{2}$.

Therefore, $\alpha + \beta = -\frac{3}{2}$.

(b) $\alpha\beta = \frac{c}{a}$

$$\Rightarrow \alpha\beta = -\frac{2}{2} = -1$$

Therefore, $\alpha\beta = -1$.

Example 5.15

If α and β are the roots of the equation $2x^2 - 3x - 8 = 0$;

(a) Find the values of each of the following:

(i) $\alpha^2 + \beta^2$ (iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

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(b) Write a quadratic equation with the roots $\frac{1}{\beta}+\frac{1}{\alpha}$ and $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

Solution

(a) Given $2x^2-3x-8=0$, then $\alpha+\beta=\frac{3}{2}$ and $\alpha\beta=-\frac{8}{2}=-4$

(i) Since $(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$,

then $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(\frac{3}{2}\right)^2-2(-4)$$

$$=\frac{41}{4}.$$

Therefore, $\alpha^2+\beta^2=\frac{41}{4}$.

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}$

$$=\frac{3}{2}\div(-4)$$

$$=-\frac{3}{8}$$

Therefore, $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{3}{8}$.

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{41}{4}\div(-4)$$

$$=-\frac{41}{16}$$

Therefore, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-\frac{41}{16}$.

(b) The quadratic equation with the roots $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$ and $\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)$ is given by;

$x^2-(\text{sum of roots})x+(\text{product of roots})=0$

$$\Rightarrow x^2-\left[\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\right]x+\left[\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\right]=0$$

$$\Rightarrow x^2-\left(-\frac{3}{8}+\left(-\frac{41}{16}\right)\right)x+\left(-\frac{3}{8}\right)\left(-\frac{41}{16}\right)=0$$

$$\Rightarrow x^2+\frac{47}{16}x+\frac{123}{128}=0$$

$$\Rightarrow 128x^2+376x+123=0$$

Therefore, the required equation is $128x^2+376x+123=0$.

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Example 5.16

Find the values of m in the equation $x^2 + mx + 45 = 0$, given that the sum of the square of the roots is 54.

Solution

Given $x^2 + mx + 45 = 0$, then $\alpha + \beta = -m$ and $\alpha\beta = 45$.

But $\alpha^2 + \beta^2 = 54$.

$\Rightarrow (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\Rightarrow (-m)^2 = \alpha^2 + \beta^2 + 2(45)$

$\Rightarrow m^2 = \alpha^2 + \beta^2 + 90$

$\Rightarrow m^2 = 54 + 90$

$\Rightarrow m^2 = 144$

$\Rightarrow m = 12$ or -12

Therefore, $m = 12$ or $m = -12$.

Relationships between roots and coefficients of a cubic equation

Suppose α, β, and γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, then $(x-\alpha)$, $(x-\beta)$, and $(x-\gamma)$ are its factors. The equation can be obtained by multiplying its factors, that is, $(x-\alpha)(x-\beta)(x-\gamma) = 0$.
On expanding, it gives;

$$x^3 - (\gamma + \alpha + \beta)x^2 + (\beta\gamma + \gamma\alpha + \alpha\beta)x - \alpha\gamma\beta = 0 \quad \text{.....(i)}$$

Dividing equation $ax^3 + bx^2 + cx + d = 0$ by a throughout gives,

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \quad \text{.....(ii)}$$

Comparing coefficients of corresponding terms of equations (i) and (ii) gives the following relationships:

$\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, and $\alpha\beta\gamma = -\frac{d}{a}$.

Therefore, the sum of the roots of a cubic equation is $\alpha + \beta + \gamma = -\frac{b}{a}$, the sum of the products of the roots in pairs is given by $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, and the product of the roots is $\alpha\beta\gamma = -\frac{d}{a}$.

Generally, if the roots of a cubic equation are α, β, and γ, then the equation of the cubic equation can be expressed in the form

x^3 – (sum of roots)x^2 + (sum of product of roots in pairs)x – (product of roots) = 0.

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Example 5.17

If α, β, and γ are the roots of the equation $2x^3 - 5x^2 + 4x - 1 = 0$, find the values in each of the following:

(a) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ (b) $\frac{1}{\alpha\beta\gamma}$ (c) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}$

Hence, find a cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$.

Solution

(a) Given $2x^3 - 5x^2 + 4x - 1 = 0$, then:

$$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{5}{2},\ \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = 2, \text{ and } \alpha\beta\gamma = -\frac{d}{a} = \frac{1}{2}.$$

Thus, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$$= 2 \div \frac{1}{2}$$

$$= 4.$$

Therefore, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = 4.$

(b) $\frac{1}{\alpha\beta\gamma} = 1 \div \frac{1}{2} = 2.$

Therefore, $\frac{1}{\alpha\beta\gamma} = 2.$

(c) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$

$$= \frac{5}{2} \div \frac{1}{2}$$

$$= 5$$

Therefore, $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma} = 5.$

The cubic equation is $x^3 - \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)x^2 + \left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}\right)x - \left(\frac{1}{\alpha\beta\gamma}\right) = 0$

$\Rightarrow x^3 - 4x^2 + 5x - 2 = 0$

Therefore, the cubic equation is $x^3 - 4x^2 + 5x - 2 = 0$.

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Example 5.18

The roots of the equation $x^3+tx^2+ux+30=0$ are in the ratio 2:3:5. Find the values of t and u.

Solution

Let k be the factor of the roots, then $\alpha=2k$, $\beta=3k$, and $\gamma=5k$.

Given $x^3+tx^2+ux+30=0$, then

$\alpha+\beta+\gamma=-t \Rightarrow 2k+3k+5k=-t$

$\Rightarrow 10k=-t$ .....(i)

Also,

$\alpha\beta+\alpha\gamma+\beta\gamma=u \Rightarrow 6k^2+10k^2+15k^2=u$

$\Rightarrow 31k^2=u$ .....(ii)

Also, $\alpha\beta\gamma=-30 \Rightarrow 30k^3=-30$

$\Rightarrow k=-1$

On substituting $k=-1$ in equation (i) gives,

$10(-1)=-t \Rightarrow t=10$.

From equation (ii)

$u=31(-1)^2=31$

Therefore, $t=10$ and $u=31$.

Exercise 5.3

1. If α and β are the roots of the equation $(1-q)x^2-2x-1=0$, find the values of $\alpha+\beta$ and $\alpha\beta$.
2. If α and β are the roots of the equation $2x^2-3x+8=0$, find the values of
   (a) $\alpha^2\beta+\alpha\beta^2$ (b) $\alpha^3+\beta^3$
3. If α and β are the roots of the equation $3x^2+5x-1=0$, find the equations whose roots are:
   (a) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$
   (b) $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$
4. If $x^2+mx+n=0$ and $px^2+2mx-3n=0$ have common roots, show that $n(p+3)^2=5m^2(p-2)$.
5. If α and β are the roots of the equation $17x^2-34x=18$, find the value of $\alpha-\beta$.
6. If α and β are the roots of the equation $x^2+3x-3=0$, find the quadratic equation whose roots are $2(\alpha+\beta)$ and $\frac{1}{3\alpha\beta}$.
7. If α and β are the roots of the equation $2x^2+5x-3=0$, find the equation whose roots are given by the following pairs of the roots:
   (a) $(3\alpha+\beta)$ and $(3\beta+\alpha)$
   (b) α^2 and β^2
   (c) $\frac{3}{\alpha}$ and $\frac{3}{\beta}$
   (d) $\frac{2\alpha}{\beta}$ and $\frac{2\beta}{\alpha}$
   (e) $(\alpha+3)$ and $(\beta+3)$
   (f) $(\alpha-5)$ and $(\beta-5)$
8. If the sum of the squares of the roots of the equation $kx^2+tx+s=0$ is 1, show that $t^2=2ks+k^2$.

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9. If the roots of the equation $ax^2+bx+c=0$ are in the ratio $p:q$, prove that $ac(p+q)^2=b^2pq$.
10. If α, β, and γ are the roots of the equation $2x^3+3x^2-x-8=0$, find the equation whose roots are α^2, β^2, and γ^2.
11. The equation $x^3+2x^2-5x+1=0$ has roots α, β, and γ. Determine the equations with the following roots:
    (a) 2α, 2β, and 2γ
    (b) $\alpha+\beta$, $\alpha+\gamma$, and $\beta+\gamma$
12. The roots of the equation $\ddot{u}^3-5\ ^2+\ +12=0$ are α, β, and γ. Calculate the values of each of the following expressions:
    (a) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$
    (b) $\alpha^2+\beta^2+\gamma^2$
    (c) $a^{\ddot{u}}+\beta\ +\gamma$
    (d) $\frac{1}{\alpha\beta\gamma}$
    (e) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\gamma\beta}$
    (f) $(\alpha+2)(\beta+2)(\gamma+2)$

Remainder theorem

Sometimes it is not easy to find factors of polynomials of degree three or higher or to find the remainder when polynomials are divided by an expression.

In arithmetic, numbers can be divided using the long division method. By this method, the dividend equals the product of the divisor and quotient plus a remainder.

For instance, dividing 627 by 50 gives 12 with a remainder 27. Thus, $627=50\times12+27$. This relationship is also applicable to polynomials.

Consider the polynomial $p(x)=2x^2+5x-4$. It can be divided by any polynomial of degree less than or equal to 2. Choosing to divide $p(x)$ by $x-2$, the long division method gives,

$$\begin{array}{r} 2x+9 \\ \text{Divisor} \rightarrow x-2\overline{)\,2x^2+5x-4} \leftarrow \text{Dividend} \\ \underline{-(2x^2-4x)} \\ 9x-4 \\ \underline{-(9x-18)} \\ 14 \leftarrow \text{Remainder} \end{array}$$

From the long division, $2x^2+5x-4$ is the dividend, $x-2$ is the divisor, $2x+9$ is the quotient, and 14 is a remainder. Thus, $2x^2+5x-4=(2x+9)(x-2)+14$.

Generally, the remainder theorem states that, if $p(x)$ is a dividend, $q(x)$ is a quotient, $x-a$ is a divisor, and $r(x)$ is a remainder, then $p(x)=q(x)\times(x-a)+r(x)$. If $x=a$, then $p(a)=r(a)$. Therefore, the remainder theorem enables us to find the remainder when a polynomial is divided by an expression without actually carrying out the steps of the long division.

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Example 5.19

Find the remainder when $p(x) = 3x^3 - 5x^2 + 2$ is divided by $x + 4$.

Solution

The divisor $x + 4$ can be rewritten as $x - (-4)$. Thus, the remainder is $p(-4)$.

But $p(-4) = 3(-4)^3 - 5(-4)^2 + 2$

$= 3 \times (-64) - 5 \times 16 + 2$

$= -270$

Therefore, when $p(x) = 3x^3 - 5x^2 + 2$ is divided by $x + 4$, the remainder is -270.

Example 5.20

If $hx^3 - 11x^2 + 4x + 12$ is divided by $x - 5$, the remainder is 132. Find the value of the constant h.

Solution

Let $p(x) = hx^3 - 11x^2 + 4x + 12$.

The divisor is $x - 5 \Rightarrow x - 5 = 0$.

$\Rightarrow p(5) = h(5)^3 - 11(5)^2 + 4(5) + 12$

$= 125h - 275 + 20 + 12$

$\Rightarrow p(5) = 125h - 243$. But $p(5) = 132$.

$\Rightarrow 125h - 243 = 132$

$\Rightarrow 125h = 375$

Thus, $h = 3$.

Therefore, the value of h is 3.

Example 5.21

When the expression $x^5 + 4x^2 + cx + d$ is divided by $x^2 - 1$, the remainder is $2x + 3$. Find the values of c and d.

Solution

Let $p(x) = x^5 + 4x^2 + cx + d$.

The divisor is $x^2 - 1 \Rightarrow x^2 - 1 = 0$.

$\Rightarrow x = -1$ or $x = 1$.

For $x = 1$, $p(1) = 1 + 4 + c + d$

But $p(1) = 2(1) + 3 = 5$

$\Rightarrow 5 = 5 + c + d$

$\Rightarrow c + d = 0$ .....(i)

For $x = -1$, $p(-1) = -1 + 4 - c + d$

$\Rightarrow p(-1) = 3 - c + d$. But $p(-1) = 2(-1) + 3 = 1$.

$\Rightarrow 1 = 3 - c + d$

$\Rightarrow c - d = 2$ .....(ii)

Solving equations (i) and (ii) simultaneously gives $c = 1$ and $d = -1$.

Therefore, the values of c and d are 1 and -1, respectively.

Factor theorem

The factor theorem states that, if a polynomial $p(x)$ is such that the remainder $p(a) = 0$, then $x - a$ is a factor of $p(x)$. This implies that $p(x) = (x - a) \times q(x)$. In other words, if $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$. Thus, the factor theorem is a special case of the remainder theorem in the sense that the factor theorem is deduced from the remainder theorem when $r(x) = 0$.

Example 5.22

Determine whether or not $x + 1$ is a factor of the polynomial $p(x) = x^4 - 3x^3 + 4x^2 - 8$.

Solution

Given $p(x) = x^4 - 3x^3 + 4x^2 - 8$.

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Rewrite $x+1$ as $x-(-1)$. This gives, $p(-1)=(-1)^4-3(-1)^3+4(-1)^2-8=0$
Since $p(-1)=0$, then $x+1$ is a factor of $p(x)$.

Therefore, $x+1$ is a factor of the polynomial $p(x)$.

Example 5.23

Find the factors of x^3+2x^2-x-2.

Solution

Let $p(x)=x^3+2x^2-x-2$.
From the factor theorem, if $x-a$ is a factor of $p(x)$, then $p(a)=0$.
Choose values of a and substitute into $p(x)$ to check or inspect whether $p(a)=0$.
The factors of the constant term, -2 are $1, 2, -1, -2$. So, the values to choose must belong to the set $\{\pm 1, \pm 2\}$.

When $x=1$: $(1)=1^3+2(1)^2-1-2=0$

Thus, $x-1$ is a factor of $p(x)$.

When $x=-1$: $p(-1)=(-1)^3+2(-1)^2-(-1)-2=0 \Rightarrow p(-1)=0$

Thus, $x+1$ is a factor of $p(x)$.

When $x=2$: $p(2)=2^3+2(2)^2-2-2=12 \Rightarrow p(2)\neq 0$.

Thus, $x-2$ is not a factor of $p(x)$.

When $x=-2$: $p(-2)=(-2)^3+2(-2)^2+2-2=0 \Rightarrow p(-2)=0$.

Thus, $x+2$ is a factor of $p(x)$.

Therefore, $p(x)=x^3+2x^2-x-2=(x+1)(x-1)(x+2)$.

Operations on polynomials

Algebraic operations such as addition, subtraction, multiplication, and division can be applied on polynomial functions. Activity 5.5 highlights some operations on polynomials.

Activity 5.5: Recognizing the use of polynomials to model blood flow

Individually or in a group, read the following information, then perform the tasks that follow.

Cardiac output is the amount of blood pumped through the heart. The output is measured by a technique called dye dilution. A doctor injects dye into a vein near the heart and measures the amount of dye in the arteries over time.

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The cardiac output of a particular patient can be approximated by the function $f(t) = 0.0056t^3 - 0.22t^2 + 2.33t$, where t represents time in seconds after injection, $0 \le t \le 23$ and $f(t)$ represents the concentration of dye (in milligrams per litre).

1. Evaluate $f(t)$ for $t = 0$ and $t = 3$.
2. Describe what do the values in task 1 represent.
3. Share your findings with other students.

Addition and subtraction of polynomials

Addition and subtraction of polynomials is simply done by adding or subtracting the terms of the same power. The exponent of the variable in a polynomial is always a non-negative integer.

Example 5.24

Given $p(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15$ and $q(x) = x^4 + x^3 - 5x^2 - 6x - 10$, determine the expression $p(x) + q(x)$.

Solution

Given $p(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15$ and $q(x) = x^4 + x^3 - 5x^2 - 6x - 10$. It follows that, $p(x) + q(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15 + x^4 + x^3 - 5x^2 - 6x - 10$

Combine like terms as follows.

$$p(x) + q(x) = (3x^4 + x^4) + (2x^3 + x^3) + (-10x^2 - 5x^2) + (8x - 6x) + (15 - 10)$$
$$= 4x^4 + 3x^3 - 15x^2 + 2x + 5$$

Therefore, $p(x) + q(x) = 4x^4 + 3x^3 - 15x^2 + 2x + 5$.

Example 5.25

Simplify $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$ and write your answer in descending powers of x.

Solution

Given $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$.

Combine like terms to get,

$14x^3 + (3x^2 + x^2) + (x - x) + (7 + 2) = 14x^3 + 4x^2 + 9$

Therefore, $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x) = 14x^3 + 4x^2 + 9$.

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Example 5.26

Find $(10x^5-4x^4+3x^3+x^2-10x+9)-(6x^5-10x^3+12x+19)$.

Solution

Insert the signs of the second polynomial, combine like terms, then simplify as follows.

$(10x^5-4x^4+3x^3+x^2-10x+9)-(6x^5-10x^3+12x+19)$

$=(10x^5-4x^4+3x^3+x^2-10x+9)+(-6x^5+10x^3-12x-19)$

$=(10x^5-6x^5)+(-4x^4+0x^4)+(3x^3+10x^3)+(x^2+0x^2)+(-10x-12x)+(9-19)$

$=4x^5-4x^4+13x^3+x^2-22x-10.$

Therefore, $(10x^5-4x^4+3x^3+x^2-10x+9)-(6x^5-10x^3+12x+19)=$
$4x^5-4x^4+13x^3+x^2-22x-10.$

Example 5.27

If $px^4-qx^3+rx^2+sx+t+2x^4-3x^3+2x-18=24x^4-16x^3+8x^2+10x-24$, find the values of p, q, r, s, and t.

Solution

Given $px^4-qx^3+rx^2+sx+t+2x^4-3x^3+2x-18=24x^4-16x^3+8x^2+10x-24.$

$\Rightarrow px^4-qx^3+rx^2+sx+t=(24x^4-16x^3+8x^2+10x-24)-(2x^4-3x^3+2x-18)$

$=(24x^4-16x^3+8x^2+10x-24)+(-2x^4+3x^3-2x+18)$

$=(24x^4-2x^4)+(-16x^3+3x^3)+(8x^2)+(10x-2x)+(-24+18)$

$=22x^4-13x^3+8x^2+8x-6$

$\Rightarrow px^4-qx^3+rx^2+sx+t=22x^4-13x^3+8x^2+8x-6$

Comparing the coefficients of like terms gives,
$p=22$, $q=13$, $r=8$, $s=8$, and $t=-6$.

Therefore, $p=22$, $q=13$, $r=8$, $s=8$, and $t=-6$.

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Multiplication of polynomials

Multiplication of polynomial is the process of multiplying together two or more polynomials. This can be performed by applying the distributive property for multiplication to polynomials. To multiply two polynomials by each other, take each term of the first polynomial and multiply to the second polynomial.

Example 5.28

Find $(2x^2+x-3)\times(x^2-2x+5)$.

Solution

Multiply each term of the first polynomial by the second polynomial and then combine like terms as follows.

$$\begin{aligned}(2x^2+x-3)\times(x^2-2x+5) &= 2x^2(x^2-2x+5)+x(x^2-2x+5)+(-3)(x^2-2x+5)\\ &= 2x^4-4x^3+10x^2+x^3-2x^2+5x-3x^2+6x-15\\ &= 2x^4-4x^3+x^3+10x^2-2x^2-3x^2+5x+6x-15\\ &= 2x^4-3x^3+5x^2+11x-15.\end{aligned}$$

Therefore, $(2x^2+x-3)\times(x^2-2x+5)=2x^4-3x^3+5x^2+11x-15$.

Division of polynomials

Division of polynomials is an arithmetic operation where a polynomial is divided by another polynomial with the same or lower degree as compared to the dividend. Division of polynomials can be done in two ways, namely long division method and synthetic division method.

Long division method

Division of polynomials can be done by using a method similar to long division of whole numbers. The following are steps to follow when dividing polynomials by using long division method:

1. Rewrite the problem in long division form. Make sure that both polynomials are written in decreasing powers, filling in any missing term with a term whose coefficient is zero.
2. Divide the first term of the dividend by the first term of the divisor. Place the quotient to its corresponding like term. Multiply the quotient by the divisor, place the product below its like terms, then subtract.

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3. Bring down the next term of the dividend.
4. Repeat steps 2 and 3 until the last term is obtained.
5. Write the final answer. If there is a remainder, place the remainder over the divisor and add it to the quotient.

Example 5.29

Divide $2x^4-3x^3+5x-36$ by x^2+x+2.

Solution

Rewrite the problem in long division form and follow the steps discussed.

$$
\begin{array}{r}
2x^2-5x+1 \\
x^2+x+2\overline{)\,2x^4-3x^3+0x^2+5x-36} \\
-\underline{(2x^4+2x^3+4x^2)} \\
-5x^3-4x^2+5x \\
-\underline{(-5x^3-5x^2-10x)} \\
x^2+15x-36 \\
-\underline{(x^2+x+2)} \\
14x-38
\end{array}
$$

Thus, $2x^4-3x^3+5x-36$ divided by x^2+x+2 gives $2x^2-5x+1$ with a remainder $14\ -38$.

Therefore, $\dfrac{2x^4-3x^3+5x-36}{x^2+x+2}=2x^2-5x+1+\dfrac{14x-38}{x^2+x+2}$.

Synthetic division method

Synthetic division is a simplified method for dividing a polynomial by another polynomial of degree one. It is performed by writing down only coefficients of several powers of the variable and changing sign of the constant term in the divisor so as to replace the usual subtraction by addition.

Example 5.30

Use synthetic division to divide $6x^3+x-1$ by $x+2$.

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Solution

Since the divisor is of the form $x-(-2)$, use the additive inverse of 2 which is -2.

Step 1: Write down all the coefficients of the dividend in a row, from left to right, and then place -2 in that same row, to the left of the leading coefficient of the dividend.

$$\begin{array}{l|llll} -2 & 6 & 0 & 1 & -1 \\ \hline \end{array}$$

Step 2: Bring down the leading coefficient of the dividend which is 6, and then multiply it by the divisor -2 to get -12. Write -12 to the corresponding coefficient 0.

$$\begin{array}{l|llll} -2 & 6 & 0 & 1 & -1 \\ & & -12 & & \\ \hline & 6 & & & \end{array}$$

Step 3: Add 0 and -12 to get -12, then write it to its corresponding coefficient -12.

$$\begin{array}{l|llll} -2 & 6 & 0 & 1 & -1 \\ & & -12 & & \\ \hline & 6 & -12 & & \end{array}$$

Step 4: Apply Steps 2 and 3 to the result. That is, multiply -12 by -2 to get 24. Write 24 to the corresponding coefficient 1, then add 1 and 24 to get 25. Write 25 to the corresponding coefficient 24.

$$\begin{array}{l|llll} -2 & 6 & 0 & 1 & -1 \\ & & -12 & +24 & \\ \hline & 6 & -12 & +25 & \end{array}$$

Step 5: Apply Steps 3 and 4 to the result. That is, multiply 25 by -2 to get -50. Write -50 to the corresponding cefficcient -1, then add -50 and -1 to get -51. Write -51 to the corresponding coefficient -50.

$$\begin{array}{l|llll} -2 & 6 & 0 & 1 & -1 \\ & & -12 & +12 & -50 \\ \hline & 6 & -12 & +25 & \boxed{-51} \end{array}$$

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Step 6: Read the coefficients of the quotient from the bottom row and write the quotient. That is,
$q(x) = 6x^2 - 12x + 25$.

The remainder, $r(x)$ is -51, which is the last number in the bottom row.

Thus, $6x^3 + x - 1$ divided by $x+2$ gives $6x^2 - 12x + 25$ with a remainder -51.

Therefore, $\dfrac{6x^3 + x - 1}{x+2} = 6x^2 - 12x + 25 + \left(-\dfrac{51}{x+2}\right)$.

Example 5.31

By using synthetic division, find the remainder when $p(x) = x^4 - 2x^3 - 15x + 2$ is divided by $x - 2$.

Solution

2	1	−2	0	−15	2
		+2	+0	+0	−30
	1	+0	+0	−15	−28

Therefore, the remainder is -28.

Exercise 5.4

1. Given that $p(x) = -18x^5 + 14x^4 - 24x^3 - 24x + 70$ and $q(x) = -25x^5 + 72x^4 - 32x^3 + 12x^2 - 50$, find:
   (a) $p(x) + q(x)$ (b) $p(x) - q(x)$
2. If $x - 4$ and $x + 6$ are factors of the expression $x^3 + ax^2 + bx + c$ and it leaves a remainder of -18 when divided by $x + 3$, find the values of a, b, and c.
3. Use synthetic division to find the remainder when $2x^4 + 3x^3 - 2x + 5$ is divided by $x + 5$.
4. Find the constant e, f, and g such that when $y^5 - 7y^3 + 4y - 2$ is divided by $(y-1)(y+1)(y-3)$ the remainder is $ey^2 + fy + g$.
5. Expand each of the following polynomials:
   (a) $(x^2 - 2x - 3)^2$ (b) $(3x^2 - x + 1)(-x^2 + 2x - 1)$

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6. Find the values of m, n, and p if $m+n(x-1)+p(x-1)(x-2)+2(x-1)(x-2)(x-3)$ is divided by $(x-1)$, $(x-2)$, and $(x-3)$, gives the remainders 10, 3, and 3, respectively.

7. By using synthetic division method, find the remainder and the quotient in each of the following:

(a) $\dfrac{x^3-3x+1}{x-2}$ (d) $\dfrac{x^2-8x-1}{x+5}$

(b) $\dfrac{x^4+x^3-8x^2+2x-1}{x+1}$ (e) $\dfrac{x^4-4x^2+2x+6}{x-4}$

(c) $\dfrac{x^6-4x^4+2x-1}{x+2}$ (f) $\dfrac{2x^4+x^3+1}{x+1}$

8. Use the long division method to find the quotient and remainder in each of the following:

(a) $\dfrac{2x^5+8x^2-3x-2}{x^2-1}$ (c) $\dfrac{2x^4+x^3+1}{x+1}$

(b) $\dfrac{1-x^2+x^4}{x^2+x+1}$ (d) $\dfrac{x^4-a^4}{x-a}$

9. Find the remainder in each of the following expressions:

(a) $\dfrac{x^3+x^2+4x-4}{x-1}$ (c) $\dfrac{x^5+x^4-x-2}{x-3}$ (e) $\dfrac{16x^3+x^2-1}{x-2}$

(b) $\dfrac{x^3-3x+5}{x+1}$ (d) $\dfrac{8x^4+32x^2-x-7}{x-4}$

10. If the remainder of $\dfrac{x^2-8x-t}{x+2}$ is -10, find the value of t and hence factorize the polynomial.

11. Show that $x+1$ is a factor of $x^3-x^2-10x-8$.

Inequalities

It is common to compare two objects that are not of the same size. For instance, people say, "She must be older than you" or "No way, we cannot be equal" or "*a* goat is smaller than a cow". These statements compare two objects of different sizes or unequal sizes. In mathematics, a statement describing relationships like; "greater than, greater than or equal, less than, or less than or equal" between two numbers or two algebraic expressions is called inequality.

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In general, inequalities are statements that compare two values or sizes. The inequality symbols include: $\leq$ (less than or equal), $\geq$ (greater than or equal), $<$ (strictly less than), and $>$ (strictly greater than).

The inequality that involves variables can be solved and its solution is not a single value, but a set of values that satisfy the inequality. The method of solving inequalities is very much like the method used for solving equations. The only difference is on multiplying or dividing both sides by a negative number. That is, when both sides of a given inequality are either multiplied or divided by a negative quantity, the sense of inequality must change (that is $>$ changes to $<$, $<$ changes to $>$, $\geq$ changes to $\leq$, and $\leq$ changes to $\geq$).

Activity 5.6: Recognizing the solution of a quadratic inequality

Individually or in a group, perform the following tasks:

1. List down the steps of drawing the graph of the quadratic function.
2. Use the steps in task 1 to draw the graph of $f(x) = x^2 + 2x - 8$ on the interval $-6 \leq x \leq 4$.
3. Identify the solution of $f(x) \geq 0$ using the graph in task 2.
4. Identify the solution of $f(x) \leq 0$ using the graph in task 2.
5. What have you observed in tasks 3 and 4? Give reasons.
6. Share your findings with other students for more inputs.

Quadratic inequalities

A quadratic inequality is a polynomial of degree two that uses an inequality sign instead of an equal sign. There are several ways of solving quadratic inequalities. They include graphical and algebraic methods. Quadratic inequalities can be solved graphically by first rewriting the inequality in standard form, with zero on one side. Sketch the quadratic function and determine where it is above or below the x-axis. Also, the inequality can be solved algebraically by making use of a sign chart. To make a sign chart, use the function and test values in each region bounded by the roots. The aim is to check if the function is positive or negative.

Example 5.32

Solve $x^2 + 2x \leq 0$ graphically.

Solution

Given $x^2 + 2x \leq 0$.

$\Rightarrow x(x+2) \leq 0$

Let $f(x) \leq 0$, where $f(x) = x(x+2)$

The graph of $f(x)$ is shown in the following figure.

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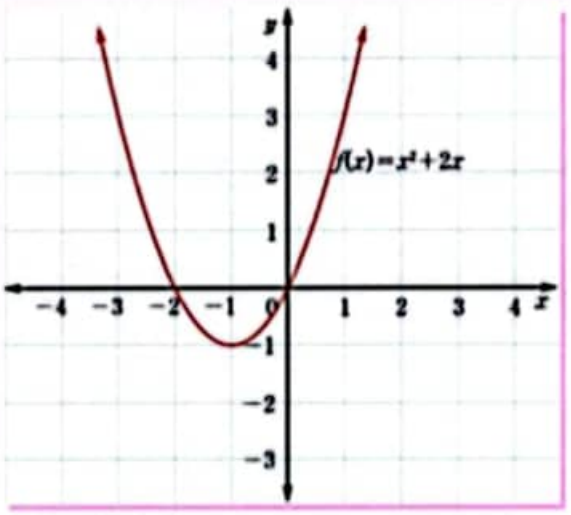


From the graph of $f(x)$, the region where $f(x) \leq 0$ is below the x-axis. The values of x corresponding to the portion of the graph satisfy the inequality $f(x) \leq 0$. The values of x where $f(x) = 0$, that is $x = 0$ and $x = -2$ are part of the solution since they satisfy the inequality.

Thus, the set of values of x which satisfy the given inequality is $-2 \leq x \leq 0$.

Therefore, the solution is $\{x : -2 \leq x \leq 0\}$.

Example 5.33

Solve $x^2 + x - 6 > 0$ algebraically.

Solution

Given $x^2 + x - 6 > 0$.

$\Rightarrow (x-2)(x+3) > 0$

$\Rightarrow$ Either $x - 2 > 0$ and $x + 3 > 0$ or $x - 2 < 0$ and $x + 3 < 0$.

$\Rightarrow$ Either $x > 2$ and $x > -3$ or $x < 2$ and $x < -3$.

Consider the signs of the factors in the following table of intervals.

	$x < -3$	$-3 < x < 2$	$x > 2$
$x - 2$	−ve	−ve	+ve
$x + 3$	−ve	+ve	+ve
$(x-2)(x+3)$	+ve	−ve	+ve

The solution for which $(x-2)(x+3) > 0$ is $x < -3$ or $x > 2$.

Therefore, the solution is $\{x : x < -3 \text{ or } x > 2\}$.

Example 5.34

Solve $-x^2 - x + 6 > 0$ graphically.

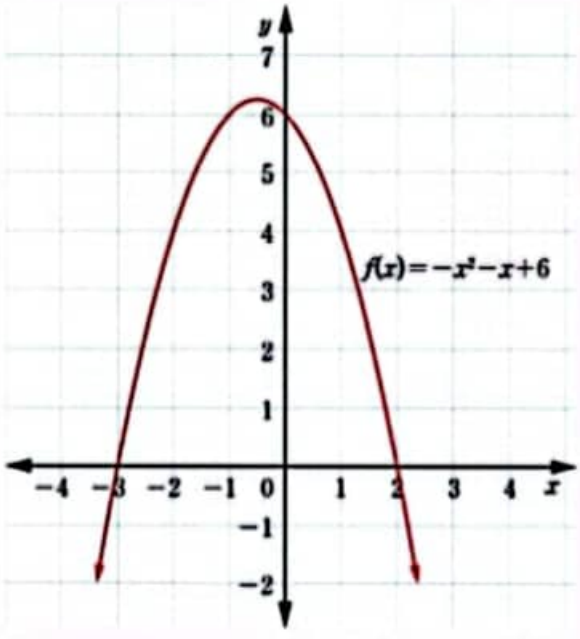
Solution

Given $-x^2 - x + 6 > 0$.

$\Rightarrow (x-2)(-x-3) > 0$

Let $f(x) > 0$, where $f(x) = (x-2)(-x-3)$.

The graph of $f(x)$ is shown in the following figure.



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From the graph of $f(x)$, the portion where $f(x) > 0$ is above the x-axis. The values of x corresponding to this portion of the graph satisfy the inequality $f(x) > 0$. The values of x where $f(x) = 0$ do not satisfy the inequality, hence $x = -3$ and $x = 2$ are not part of the solution.

Thus, the set of values of x which satisfy the given inequality is $-3 < x < 2$.

Therefore, the solution is $\{x : -3 < x < 2\}$.

Rational inequalities

Rational inequalities are fraction inequalities that have a polynomial expression in the numerator, denominator or both. Generally, a rational inequality compares two expressions in the form $\frac{p(x)}{q(x)} < 0$; $\frac{p(x)}{q(x)} > 0$, $\frac{p(x)}{q(x)} \leq 0$ or $\frac{p(x)}{q(x)} \geq 0$, where $q(x) \neq 0$.

Note that, the properties of fractions can be applied to simplify rational inequalities, such as cancelling common factors from the numerator and the denominator.

The following are steps in solving rational inequalities:

1. Rewrite the inequality to contain a zero on the right-hand side. Create a single fraction on the left-hand side.
2. Find the critical values of x that make the numerator equal to zero and that make the denominator equal to zero.
3. Construct a table of intervals in which the columns are separated by the critical values.
4. Select a test point in each interval and check to see if it satisfies the original inequality or the inequality set is less than 0.
5. Test the roots of the numerator for their possibility of inclusion to the overall solution. Include the roots to get a true statement if they are part of the overall solution.
6. Exclude the roots of the denominator in the overall solution since they make the rational expression undefined.

Example 5.35

Find the values of x for which $\frac{2x+1}{x} > 1$.

Solution

Given $\frac{2x+1}{x} > 1$.

Make the right-hand side equal to zero.

That is, $\frac{2x+1}{x} - 1 > 0$.

$$\Rightarrow \frac{2x+1-x}{x} > 0$$

$$\Rightarrow \frac{x+1}{x} > 0$$

$$\Rightarrow f(x) = \frac{x+1}{x} > 0$$

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The numerator of $f(x)$ is zero when $x=-1$, and the denominator is zero when $x=0$. So the critical values are $x=-1$ and $x=0$.
Construct a table of intervals in which the columns are separated by the critical values as follows.

	$x<-1$	$-1<x<0$	$x>0$
$x+1$	−ve	+ve	+ve
x	−ve	−ve	+ve
$f(x)=\frac{x+1}{x}$	+ve	−ve	ü

The solution for which $\frac{2x+1}{x}>1$ is $x<-1$ or $x>0$.

Therefore, the solution is $\{x: x<-1 \text{ or } x>0\}$.

Example 5.36

Find the possible values of x for which $f(x)=\frac{(x+1)(x-3)(x+4)}{x-2}\leq 0$.

Solution

Given $f(x)=\frac{(x+1)(x-3)(x+4)}{x-2}\leq 0$.

The numerator of $f(x)$ is zero when $(x+1)(x-3)(x+4)=0$, thus $x=-1$, $x=3$, and $x=-4$. The denominator $f(x)$ is zero when $x-2=0\Rightarrow x=2$. So, the critical values are $x=-1$, $x=3$, $x=-4$, and $x=2$.

Construct a table of intervals in which the columns are separated by the critical values as follows.

	$x<-4$	$-4<x<-1$	$-1<x<2$	$2<x<3$	$x>3$
$x+4$	−ve	+ve	+ve	+ve	+ve
$x+$	−ve	−ve	+ve	+ve	+ve
$x-2$	−ve	−ve	−ve	+ve	+ve
$x-3$	−ve	−ve	−ve	−ve	+ve
$f(x)$	+ve	−ve	+ve	−ve	+ve

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The solution for which $f(x) < 0$ is $-4 < x < -1$ or $2 < x < 3$. The critical values $x = -4$, $x = -1$, and $x = 3$ satisfy the overall solution and the critical value of the denominator does not satisfy the solution as it makes the rational expression undefined.

Therefore, the solution is $\{x : -4 \leq x \leq -1 \text{ or } 2 < x \leq 3\}$.

Absolute value inequalities

The symbol $| \, |$ is the sign of the absolute value. The absolute value of a real number x is written as $|x|$, where

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

The solution set of $|x| = a$ is the union of the solution sets of the two equations. Similarly, an equation that has the sign of an absolute value is equivalent to two equations without the sign of the absolute value. An absolute value inequality is an expression with absolute functions as well as inequality signs. For instance, the expression $|p| < c$ is an absolute value inequality containing "a less than" symbol. This expression is defined as follows.

Definition 1 : If $|p| < c$, then $-c < p < c$. If $|p| \leq c$, then $-c \leq p \leq c$, where p is any algebraic expression.

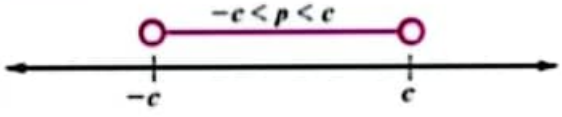
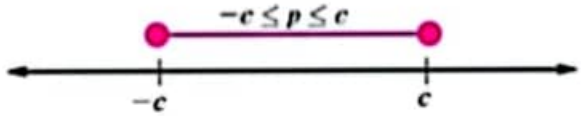
Note that, the value of c cannot be negative because if $c < 0$, then $|p| < c$ becomes $|p| < 0$, which is impossible. Thus, the definition of $|p|$ requires that $|p|$ is not negative.

Definition 2:

If $|p| > c$, then $p > c$ or $p < -c$.

If $|p| \geq c$, then $p \geq c$ or $p \leq -c$, where p is any algebraic expression.

Solutions of absolute value inequalities can be presented algebraically or graphically. The general solutions of such graphs are as follows.

Type of inequality	Equivalent inequality to be solved	Nature of the graph
$\|p\| < c$	$-c < p < c$	
$\|p\| \leq c$	$-c \leq p \leq c$	

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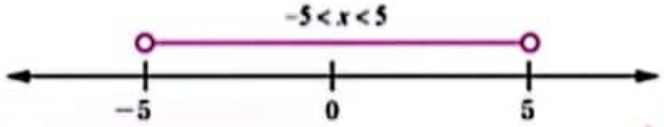
$\lvert p\rvert > c$	$p > c$ or $p < -c$	
$\lvert p\rvert \geq c$	$p \geq c$ or $p \leq -c$	

Example 5.37

Solve $|2x| < 10$ and plot its solution on the number line.

Solution

By definition, $|p| < c$ is equivalent to $-c < p < c$.
This implies that $-10 < 2x < 10$.
Dividing by 2 each term gives $-5 < x < 5$.
This solution is plotted on the number line as follows.



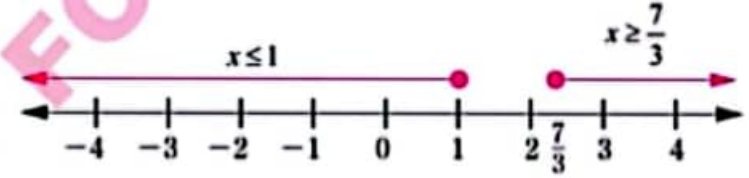
Therefore, the solution is $\{x \in \mathbb{R} : -5 < x < 5\}$.

Example 5.38

Solve $|3x - 5| \geq 2$ and plot its solution set on the number line.

Solution

Generally, the solution of $|p| \geq c$ is given by $p \geq c$ or $p \leq -c$.
Thus, the solution of $|3x - 5| \geq 2$ is given by $3x - 5 \geq 2$ or $3x - 5 \leq -2$.
$\Rightarrow 3x \geq 7$ or $3x \leq 3$
Dividing by 3 both sides gives; $x \geq \frac{7}{3}$ or $x \leq 1$
This solution is plotted on the number line as follows:



Therefore, the solution is $\left\{x \in \mathbb{R} : x \leq 1,\ x \geq \frac{7}{3}\right\}$.

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Example 5.39

Solve $\left|\frac{2-3x}{2}\right| \leq 6$ and plot its solution set on the number line.

Solution

Since $|p| \leq c$ is equivalent to $-c \leq p \leq c$.

This implies that the solution of $\left|\frac{2-3x}{2}\right| \leq 6$ is given by,

$$-6 \leq \frac{2-3x}{2} \leq 6$$

Multiplying by 2 in each term gives,

$$-12 \leq 2-3x \leq 12$$

Subtracting 2 from each term gives,

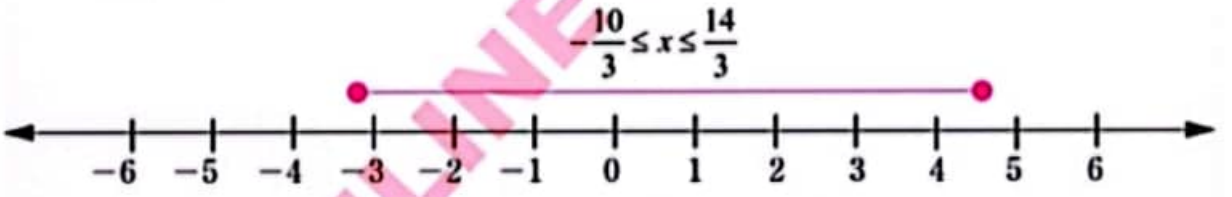
$$-14 \leq -3x \leq 10$$

Divide by -3 each term and change the sense of inequality signs:

$$\Rightarrow \frac{14}{3} \geq x \geq -\frac{10}{3}$$

$$\Rightarrow -\frac{10}{3} \leq x \leq \frac{14}{3}$$

This solution is shown on the number line as follows.



Therefore, the solution set is $\left\{x \in \mathbb{R} : -\frac{10}{3} \leq x \leq \frac{14}{3}\right\}$.

Exercise 5.5

Solve questions 1 – 19.

1. $|3-2x| < |4+x|$
2. $\left|\frac{x+3}{x-4}\right| > 2$
3. $|2x+2| > 5$
4. $|2x-1|+4 < 13$
5. $|3x-1|-x \geq 0$
6. $|x-3|+|x+1| > 4$

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7. $\frac{3x-1}{x-1}<2$ 8. $\frac{x}{x^2-1}$ 0 9. $\frac{(x+1)(x-1)}{x}\leq 0$

10. $x^2-8x+15\geq 0$ 11. $x^2-5x\leq 0$ 12. $\frac{(x+1)(x+5)}{(x+2)(2x+3)}>0$

13. $\frac{2x^2+x+5}{2x^2+x-3}\leq 1$ 14. $\log_x\left(\frac{x+3}{x-1}\right)>1$ 15. $\frac{x^2+2}{x^2-1}<-2$

16. $\frac{x-2}{2x-3}<\frac{x}{3x-2}$ 17. $\frac{x+3}{x-2}<\frac{4}{3x-3}$

18. $\log\left(\frac{1-x}{1+2x}\right)<1$ 19. $|8x-4|+10>4$

20. Show that $0\leq\frac{(x+1)^2}{x^2+x+1}\leq\frac{4}{3}$ for all real values of x.

Matrices

A set of numbers arranged in a rectangular array having m rows and n columns and enclosed by a square bracket [ ] or round bracket ( ) is called a matrix. Matrices are denoted by capital letters such as A, B, C or any other letter, and described by using rows and columns with rows appearing first. A matrix $m\times n$ (read as m by n) having m rows and n columns is written as follows.

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The entry a_{ij} is the element in the i^{th} row and j^{th} column of the matrix. For instances, a_{12} is the element in the first row and in the second column. The order or dimension of a matrix is given by $m\times n$, where m is the number of rows and n is the number of columns. If a matrix has 3 rows and 2 columns, then its order is written as 3×2.

If the number of rows and columns are equal, the matrix is called a square matrix. If the number of rows and columns are not equal, the matrix is just called a rectangular matrix.

For instances, $A=\begin{pmatrix} 1 & 1 & 3 \\ 2 & 5 & 6 \\ 7 & 7 & 5 \end{pmatrix}$ is a 3×3 square matrix and $B=\begin{bmatrix} -4 & -2 \\ -8 & -1 \\ -7 & -9 \end{bmatrix}$ is a 3×2 rectangular matrix.

Activity 5.7 provides a fundamental insight on the dimension of a matrix.

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Activity 5.7: Recognizing the number of rows and columns in the matrix

Individually or in a group, read carefully the following paragraph and then perform the tasks that follow.

An electronic company produces laptops, mobile phones, and desktop computers. Each laptop requires 10 units of gold, 8 units of copper, and 12 units of iron. Each mobile phone requires 9 units of gold, 7 units of copper, and 5 units of iron. Each desktop computer requires 15, 11, and 14 units of gold, copper, and iron, respectively.

1. Present this information in matrix form.
2. Identify the number of rows and columns of the matrix in task 1.
3. Identify the order of the matrix in task 1.
4. What are the similarities and differences between the number of rows and columns identified in task 2?
5. Share your findings with your fellow students for more inputs.

Types of matrices

There are several types of matrices. The following are some types of matrices:

Row and column matrices

A matrix consisting of a single row and any number of columns is called a row matrix or row vector, whereas a matrix having a single column and any number of rows is called a column matrix or a column vector. For instances, $D=\begin{bmatrix}1 & 4 & 8\end{bmatrix}$ and $E=\begin{bmatrix}-6\\-8\\-10\end{bmatrix}$ are row and column matrices, respectively.

Null or zero matrix

A matrix in which all the elements are 0 is called a null or zero matrix. For instance, the matrix $C=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ is a zero matrix of order 3×3.

Square matrix

A matrix having the same number of rows and columns is called a square matrix. For instance,

$$C=\begin{bmatrix}2 & 6 & 10\\-2 & -5 & 12\\-4 & -8 & -15\end{bmatrix}$$

is a square matrix. Any matrix of order $n\times n$ is a square matrix.

Scalar matrix

A square matrix whose elements in the leading diagonal are equal or the same, and the rest of the elements are zero is called a scalar matrix.

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For instance,

$$K = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$ is a scalar matrix of order 3×3.

Diagonal matrix

A square matrix in which all elements are zero except those in the main or leading diagonal is called a diagonal matrix.

For instance, $G = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ is a diagonal matrix of order 3×3.

Identity or unit matrix

A square matrix whose elements in the leading diagonal are one and the rest of the elements are zero is called an identity or unit matrix. For instance,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is an identity matrix of order 3×3.

Multiplication of 3×3 matrices

Two matrices can be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. The dimension of the resulting matrix will have the number of rows of the first matrix and the number of columns of the second matrix. That is, if matrix A is of order $m\times p$ and matrix B is of order $q\times n$, then multiplication of the two matrices, AB is possible if and only if $p = q$, and the resulting matrix will be of order $m\times n$.

The steps for multiplying a matrix by another matrix are as follows:

Let matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

and matrix $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$.

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To obtain the matrix AB, use the following steps:

1. Multiply the elements of the first row in A by the elements of the first column in B, that is $(a_{11} \quad a_{12} \quad a_{13}) \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$ ..... 1st element is obtained.

2. Multiply the elements of the first row in A by the elements of the second column in B, that is $(a_{11} \quad a_{12} \quad a_{13}) \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$ ..... 2nd element is obtained.

3. Multiply the elements of the first row in A by the elements of the third column in B, that is $(a_{11} \quad a_{12} \quad a_{13}) \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$ ..... 3rd element is obtained.

4. Multiply the elements of the second row in A by the elements of the first column in B, that is $(a_{21} \quad a_{22} \quad a_{23}) \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$ .....4th element is obtained.

5. Multiply the elements of the second row in A by the elements of the second column in B, that is $(a_{21} \quad a_{22} \quad a_{23}) \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$ .....5th element is obtained.

6. Multiply the elements of the second row in A by the elements of the third column in B, that is $(a_{21} \quad a_{22} \quad a_{23}) \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$ .....6th element is obtained.

7. Multiply the elements of the third row in A by the elements of the first column in B, that is $(a_{31} \quad a_{32} \quad a_{33}) \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$ .....7th element is obtained.

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8. Multiply the elements of the third row in A by the elements of the second column in B, that is $\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}$ ..... 8^{th} element is obtained.

9. Multiply the elements of the third row in A by the elements of the third column in B, that is $\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$ ..... 9^{th} element is obtained.

Thus, $AB = \begin{pmatrix} 1^{st} \text{ element} & 2^{nd} \text{ element} & 3^{rd} \text{ element} \\ 4^{th} \text{ element} & 5^{th} \text{ element} & 6^{th} \text{ element} \\ 7^{th} \text{ element} & 8^{th} \text{ element} & 9^{th} \text{ element} \end{pmatrix}$.

Therefore,

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Example 5.40

If $G = \begin{pmatrix} -2 & -8 & 9 \\ -6 & -5 & 7 \\ 12 & 10 & -3 \end{pmatrix}$ and $H = \begin{pmatrix} 4 & -4 & -10 \\ 13 & -9 & -11 \\ 3 & 6 & -15 \end{pmatrix}$, find GH.

Solution

$$GH = \begin{pmatrix} -2 & -8 & 9 \\ -6 & -5 & 7 \\ 12 & 10 & -3 \end{pmatrix} \times \begin{pmatrix} 4 & -4 & -10 \\ 13 & -9 & -11 \\ 3 & 6 & -15 \end{pmatrix} =$$

$$\begin{pmatrix} (-2\times 4)+(-8\times 13)+(9\times 3) & (-2\times -4)+(-8\times -9)+(9\times 6) & (-2\times -10)+(-8\times -11)+(9\times -15) \\ (-6\times 4)+(-5\times 13)+(7\times 3) & (-6\times -4)+(-5\times -9)+(7\times 6) & (-6\times -10)+(-5\times -11)+(7\times -15) \\ (12\times 4)+(10\times 13)+(-3\times 3) & (12\times -4)+(10\times -9)+(-3\times 6) & (12\times -10)+(10\times -11)+(-3\times -15) \end{pmatrix}$$

$$= \begin{pmatrix} -85 & 134 & -27 \\ -68 & 111 & 10 \\ 169 & -156 & -185 \end{pmatrix}$$

Therefore, $GH = \begin{pmatrix} -85 & 134 & -27 \\ -68 & 111 & 10 \\ 169 & -156 & -185 \end{pmatrix}$.

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Example 5.41

Given that $E=\begin{pmatrix}4 & a & -6\\ -8 & c & -4\\ 10 & 3 & b\end{pmatrix}$, $F=\begin{pmatrix}12\\ -15\\ -10\end{pmatrix}$, and $EF=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}$. Find the values of a, b, and c.

Solution

$$EF=\begin{pmatrix}4 & a & -6\\ -8 & c & -4\\ 10 & 3 & b\end{pmatrix}\times\begin{pmatrix}12\\ -15\\ -10\end{pmatrix}=\begin{pmatrix}4\times12+a\times(-15)+(-6)\times(-10)\\ -8\times12+c\times(-15)+(-4)\times(-10)\\ 10\times12+3\times(-15)+b\times(-10)\end{pmatrix}$$

$$=\begin{pmatrix}108-15a\\ -56-15c\\ 75-10b\end{pmatrix}$$ since, $EF=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}$, then

$$\begin{pmatrix}108-15a\\ -56-15c\\ 75-10b\end{pmatrix}=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}$$

Equating the corresponding elements gives,

$108-15a=-72\Rightarrow a=12$,

$-56-15c=109\Rightarrow c=11$,

$75-10b=-80\Rightarrow b=15.5$.

Therefore, $a=12$, $b=15.5$, and $c=-11$.

Example5.42

Given $M=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}$, verify that MI = IM = M, where I is the 3×3 multiplicative identity matrix.

Solution

$$MI=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}\times\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}$$

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$$\text{Also, } IM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -12 & 20 & 40 \\ 14 & -8 & -2 \\ 26 & 15 & 6 \end{pmatrix} = \begin{pmatrix} -12 & 20 & 40 \\ 14 & -8 & -2 \\ 26 & 15 & 6 \end{pmatrix}$$

Therefore, $MI = IM = M$.

Transpose of a matrix

Let A be an $m \times n$ matrix, then the $n \times m$ matrix obtained after interchanging the rows and columns of matrix A is called the transpose of matrix A and it is denoted by A^T. For example, if $A = \begin{pmatrix} 2 & 8 & -7 \\ 4 & -2 & 9 \\ 6 & 1 & 5 \end{pmatrix}$, then $A^T = \begin{pmatrix} 2 & 4 & 6 \\ 8 & -2 & 1 \\ -7 & 9 & 5 \end{pmatrix}$.

Example 5.43

Given $L = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix}$ and $M = \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$, verify that

$(LM)^T = M^T L^T$.

Solution

Consider the left-hand side

$$LM = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix} \times \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} -40+30-12 & -10-45+0 & -70-30+108 \\ 32-8-3 & 8+12+0 & 56+8+27 \\ 24-20+1 & 6+30+0 & 42+20-9 \end{pmatrix}$$

$$= \begin{pmatrix} -22 & -55 & 8 \\ 21 & 20 & 91 \\ 5 & 36 & 53 \end{pmatrix}$$

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Thus, $(LM)^T = \begin{pmatrix} -22 & 21 & 5 \\ -55 & 20 & 36 \\ 8 & 91 & 53 \end{pmatrix}$

Similarly, consider the right-hand side:

Since $M = \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$, then $M^T = \begin{pmatrix} 4 & -2 & -1 \\ 1 & 3 & 0 \\ 7 & 2 & 9 \end{pmatrix}$

Also, $L = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix}$, $L^T = \begin{pmatrix} -10 & 8 & 6 \\ -15 & 4 & 10 \\ 12 & 3 & -1 \end{pmatrix}$.

$\Rightarrow M^T L^T = \begin{pmatrix} 4 & -2 & -1 \\ 1 & 3 & 0 \\ 7 & 2 & 9 \end{pmatrix} \times \begin{pmatrix} -10 & 8 & 6 \\ -15 & 4 & 10 \\ 12 & 3 & -1 \end{pmatrix}$

Thus, $M^T L^T = \begin{pmatrix} -22 & 21 & 5 \\ -55 & 20 & 36 \\ 8 & 91 & 53 \end{pmatrix}$

Therefore, $(ML)^T = M^T L^T$.

Exercise 5.6

1. If $N = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, verify that $N^2 - 4N - 5I = M$, where I is an identity matrix and M is a null matrix.

2. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 9 \\ 1 & -3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 5 & 8 \\ -6 & 10 & 12 \\ 13 & 4 & 7 \end{pmatrix}$, find AB.

3. Given that $J = \begin{bmatrix} 8 & 7 & 2 \\ 10 & 6 & 4 \\ 12 & 14 & 21 \end{bmatrix}$ and $K = \begin{bmatrix} -4 & 5 & -10 \\ -6 & 4 & -12 \\ -8 & 10 & -18 \end{bmatrix}$, show that $JK \neq KJ$.

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4. If $L=\begin{pmatrix}80 & 50 & 40\\ 20 & 30 & -50\\ -90 & -70 & -95\end{pmatrix}$ and $M=\begin{pmatrix}-19\\ -57\\ -38\end{pmatrix}$, evaluate each of the following:

   (a) LM (b) ML (c) M^2

5. Given that $P=\begin{pmatrix}c & 15 & -20\\ -18 & 12 & d\\ 14 & 20 & 25\end{pmatrix}$, $Q=\begin{pmatrix}-4\\ -6\\ e\end{pmatrix}$, and $PQ=\begin{pmatrix}-70\\ 36\\ 49\end{pmatrix}$. Find the values of c, d, and e.

6. Find $\begin{pmatrix}-2 & 12 & 0\\ -4 & 8 & -2\\ -5 & 4 & -6\end{pmatrix}\times\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 8 & 1 & -2\end{pmatrix}\times\begin{pmatrix}10\\ -10\\ 20\end{pmatrix}$.

7. If $R=\begin{pmatrix}-14 & 40 & -8\\ -16 & 20 & -6\\ -21 & 30 & -4\end{pmatrix}$, show that $RI = IR = R$, where I is the 3×3 identity matrix.

8. If $P=\begin{pmatrix}1 & 3 & 2\\ 2 & 2 & 3\\ 3 & 1 & 1\end{pmatrix}$ and $Q=\begin{pmatrix}-2 & 1 & -4\\ -1 & 2 & -5\\ -3 & 3 & -6\end{pmatrix}$, find:

   (a) $(PQ)^T$

   (b) Q^TP^T. Hence, show that $(PQ)^T = Q^TP^T$.

9. The distribution of number of grades A, B, and C, and grade points from the departments of Geography, Chemistry, and Biology are represented by matrices M and N, respectively as follows.

$$M=\begin{array}{r}\\ \text{Geography}\\ \text{Chemistry}\\ \text{Biology}\end{array}\begin{array}{c}\begin{matrix}A & B & C\end{matrix}\\ \begin{bmatrix}45 & 60 & 50\\ 35 & 55 & 40\\ 48 & 50 & 36\end{bmatrix}\end{array} \quad \text{and} \quad N=\begin{array}{c}\\ A\\ B\\ C\end{array}\begin{bmatrix}20\\ 18\\ 12\end{bmatrix}$$

   Find the matrix MN and interpret it.

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10. Three students buy soft drinks of type D_1, D_2, and D_3. The first student buys 8, 12, and 16 bottles of D_1, D_2, and D_3, respectively. The second student buys 5, 9, and 10 bottles of D_1, D_2, and D_3, respectively. The third student buys 15, 18, and 12 bottles of D_1, D_2, and D_3, respectively.
    (a) Represent the information in matrix form.
    (b) If the cost of each bottle of D_1, D_2, and D_3 is 500 Tanzanian shillings, use matrix operations to find the total amount of money spent by each student individually.

Minors of a 3×3 matrix

The minor of an element in a matrix is the determinant of a sub-matrix formed by deleting the rows and columns for a given element. Each element in a 3×3 matrix has its own minor. The minors may be denoted by M_{ij}, where i and j are the deleted rows and columns, respectively.

The steps for obtaining the minors are as follows:

Let matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and M_{ij} be the minors of A.

For M_{11}, delete the first row and first column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{11}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=(a_{22}\times a_{33})-(a_{23}\times a_{32})$$

For M_{12}, delete the first row and second column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{12}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=(a_{21}\times a_{33})-(a_{23}\times a_{31})$$

For M_{13}, delete the first row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{13}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=(a_{21}\times a_{32})-(a_{22}\times a_{31})$$

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For M_{21}, delete the second row and the first column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{21}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=(a_{12}\times a_{33})-(a_{13}\times a_{32})$$

For M_{22}, delete the second row and the second column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{22}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}=(a_{11}\times a_{33})-(a_{13}\times a_{31})$$

For M_{23}, delete the second row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{23}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}=(a_{11}\times a_{32})-(a_{12}\times a_{31})$$

For M_{31}, delete the third row and the first column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{31}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}=(a_{12}\times a_{23})-(a_{13}\times a_{22})$$

For M_{32}, delete the third row and the second column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{32}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}=(a_{11}\times a_{23})-(a_{13}\times a_{21})$$

For M_{33}, delete the third row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows.

$$M_{33}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=(a_{11}\times a_{22})-(a_{12}\times a_{21})$$

Therefore, $M_{ij}=\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{33} & M_{33} \end{pmatrix}$ is a matrix of minors.

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Cofactors of a 3 × 3 matrix

The cofactor of an element a_{ij} is obtained by multiplying by $(-1)^{i+j}$ the minor M_{ij}. The matrix of cofactors of a matrix A is abbreviated as cof (A).

Generally, if $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$\text{cof}(a_{ij}) = (-1)^{i+j} M_{ij}$.

The cofactors of a matrix A are obtained as follows:

The cofactor of $a_{11} = (-1)^{1+1} \times M_{11}$

The cofactor of $a_{12} = (-1)^{1+2} \times M_{12}$

The cofactor of $a_{13} = (-1)^{1+3} \times M_{13}$

The cofactor of $a_{21} = (-1)^{2+1} \times M_{21}$

The cofactor of $a_{22} = (-1)^{2+2} \times M_{22}$

The cofactor of $a_{23} = (-1)^{2+3} \times M_{23}$

The cofactor of $a_{31} = (-1)^{3+1} \times M_{31}$

The cofactor of $a_{32} = (-1)^{3+2} \times M_{32}$

The cofactor of $a_{33} = (-1)^{3+3} \times M_{33}$

The pattern of signs may be useful in calculating cofactors of a 3×3 matrix.

That is, $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$.

Generally, the matrix of cofactors of a matrix A is given by,

$\text{cof}(A) = \left[(-1)^{i+j} M_{ij}\right]$.

For a 3×3 matrix A, the matrix of cofactors is,

$$\text{cof}(A) = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}.$$

Example 5.44

Given $M=\begin{pmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{pmatrix}$, find:

(a) The minors, M_{13} and M_{22}.

(b) The cofactors of the elements a_{13} and a_{22}.

Solution

(a) $M_{13} = \begin{vmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{vmatrix} = \begin{vmatrix} -12 & 28 \\ 15 & -39 \end{vmatrix}$

$= (-12 \times -39) - (28 \times 15)$

$= 48$

$M_{22} = \begin{vmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{vmatrix} = \begin{vmatrix} -8 & -72 \\ 15 & 95 \end{vmatrix}$

$= (-8 \times 95) - (-72 \times 15)$

$= 320$

Therefore, the minors, M_{13} and M_{22} are 48 and 320, respectively.

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(b) Cofactor of $a_{13} = (-1)^{1+3} \times M_{13}$

$= 1 \times 48$

$= 48$

cofactor of $a_{22} = (-1)^{2+2} \times M_{22}$

$= (1) \times 320$

$= 320$

Therefore, the cofactors of the elements a_{13} and $_{22}$ are 48 and 320, respectively.

Determinant of a 3 × 3 matrix

A real number that represents the magnitude of square matrix is called the determinant of the matrix. The determinant of a matrix A is denoted by det(A) or $|A|$. It can be calculated provided that A is a square matrix. If $|A| = 0$, then the matrix A is called a singular matrix, and if $|A| \neq 0$, then is called a non-singular matrix.

Finding determinants of 3×3 matrices

The following steps can be used in determining the determinant of a 3×3 matrix:

1. Choose any row or any column in the given matrix.
2. Compute the cofactors of each element in the chosen row or column.
3. The determinant is equal to the sum of the products of the elements and their respective cofactors.

Consider the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Choose the first row with elements a_{11}, a_{12}, and a_{13}. According to the patterns of signs (+ or −) for calculating cofactors, it follows that

cofactor of $a_{11} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, cofactor of $a_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, and cofactor of $a_{13} = +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Therefore, $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

Example 5.45

Given $D = \begin{pmatrix} 19 & -20 & -24 \\ 36 & 18 & -16 \\ -10 & 15 & 14 \end{pmatrix}$, evaluate $|D|$.

Solution

Using the first row of the matrix D:

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$$|D| = \begin{vmatrix} 19 & -20 & -24 \\ 36 & 18 & -16 \\ -10 & 15 & 14 \end{vmatrix}$$

$$= 19\begin{vmatrix} 18 & -16 \\ 15 & 14 \end{vmatrix} - (-20)\begin{vmatrix} 36 & -16 \\ -10 & 14 \end{vmatrix} + (-24)\begin{vmatrix} 36 & 18 \\ -10 & 15 \end{vmatrix}$$

$$= 9348 + 6880 - 17280$$

$$= -1052$$

Therefore, $|D| = -1052$.

Example 5.46

Given $W = \begin{pmatrix} 2 & 4 & -3 \\ -1 & -2 & 1 \\ 5 & 3 & -4 \end{pmatrix}$ and $V = \begin{pmatrix} -5 & -2 & 6 \\ -6 & 5 & 1 \\ 2 & 3 & -3 \end{pmatrix}$, evaluate:

(a) $\det\left((WV)^T\right)$ (b) $\det\left(V^T W^T\right)$.

Hence, show that $\det\left((WV)^T\right) = \det\left(V^T W^T\right)$.

Solution

(a) $WV = \begin{pmatrix} 2 & 4 & -3 \\ -1 & -2 & 1 \\ 5 & 3 & -4 \end{pmatrix} \times \begin{pmatrix} -5 & -2 & 6 \\ -6 & 5 & 1 \\ 2 & 3 & -3 \end{pmatrix}$

$$= \begin{pmatrix} -40 & 7 & 25 \\ 19 & -5 & -11 \\ -51 & -7 & 45 \end{pmatrix}$$

$$\Rightarrow (WV)^T = \begin{pmatrix} -40 & 19 & -51 \\ 7 & -5 & -7 \\ 25 & -11 & 45 \end{pmatrix}$$

Using the first column of the matrix $(WV)^T$, it follows that

$$\det\left((WV)^T\right) = -40\begin{vmatrix} -5 & -7 \\ -11 & 45 \end{vmatrix} - 7\begin{vmatrix} 19 & -51 \\ -11 & 45 \end{vmatrix} + 25\begin{vmatrix} 19 & -51 \\ -5 & -7 \end{vmatrix}$$

$$= 12080 - 2068 - 9700$$

$$= 322$$

Therefore, $\det\left((WV)^T\right) = 322$.

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(b) $V^T = \begin{pmatrix} -5 & -6 & 2 \\ -2 & 5 & 3 \\ 6 & 1 & -3 \end{pmatrix}$ and $W^T = \begin{pmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \\ -3 & 1 & -4 \end{pmatrix}$

$$\Rightarrow V^T W^T = \begin{pmatrix} -5 & -6 & 2 \\ -2 & 5 & 3 \\ 6 & 1 & -3 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \\ -3 & 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -40 & 19 & -51 \\ 7 & -5 & -7 \\ 25 & -11 & 45 \end{pmatrix}$$

Using the third column of matrix $W^T V^T$, it follows that

$$\det\left(V^T W^T\right) = +(-51)\begin{vmatrix} 7 & -5 \\ 25 & -11 \end{vmatrix} - (-7)\begin{vmatrix} -40 & 19 \\ 25 & -11 \end{vmatrix} + 45\begin{vmatrix} -40 & 19 \\ 7 & -5 \end{vmatrix}$$

$$= -2448 - 245 + 3015$$

$$= 322$$

üüü(ü ^{T} ^{T}) =

Therefore, $\det\left((WV)^T\right) = 322 = \det\left(V^T W^T\right)$.

Exercise 5.7

1. Calculate each of the following:

(a) $\begin{vmatrix} 5 & 9 & 6 \\ 3 & 7 & 8 \\ 21 & 55 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} -2 & 4 & 1 \\ 5 & 10 & 3 \\ 3 & -6 & -11 \end{vmatrix}$

2. Find the matrix of the minors of each of the following matrices:

(a) $D = \begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$ (b) $E = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 5 \\ 3 & 1 & 3 \end{pmatrix}$

3. Find the matrix of cofactors of each of the following matrices:

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 6 \\ 1 & 5 & 0 \\ 4 & -1 & 3 \end{pmatrix}$

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4. Use cofactors to evaluate each of the following:

(a) $\begin{vmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -2 & 6 \\ 2 & 1 & -3 \\ 3 & 0 & 5 \end{vmatrix}$

5. If $J = \begin{pmatrix} 3 & -3 & 0 \\ 6 & 3 & 9 \\ 12 & 3 & 24 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 3 & 0 \\ 6 & -9 & 3 \\ 3 & 3 & -3 \end{pmatrix}$, show that $\left|(JK)^T\right| = \left|J^T K^T\right|$.

6. Verify each of the following:

(a) $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

7. In each of the following, show that:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(b) $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

(c) $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

8. Given $K = \begin{pmatrix} -18 & -16 & 39 \\ -42 & 15 & 41 \\ 32 & 28 & 60 \end{pmatrix}$ and $L = \begin{pmatrix} -5 & -10 & 14 \\ -6 & -11 & 7 \\ -4 & 15 & 2 \end{pmatrix}$, show that

(a) $\det(KL) = \det(LK)$ (b) $\det\left((KL)^T\right) = \det\left(L^T K^T\right)$

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9. Find the value of the unknown element in each of the following matrices:

(a) $D=\begin{pmatrix}-2 & 1 & 1\\ 3 & 2 & 2\\ 1 & m & 4\end{pmatrix}$ if D is a singular matrix

(b) $E=\begin{pmatrix}8 & -2 & r\\ 2 & -3 & -2\\ 6 & 3 & 8\end{pmatrix}$ if $|E|=-328$

10. Given that matrix $C=\begin{pmatrix}-1 & 2 & 3\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$, find:

(a) The matrix from the expression C^2-2C+I, where I is an identity matrix.

(b) det(C2 – 2C + I)

Solution of systems of linear simultaneous equations by Cramer's rule

Consider the following system of linear equations:

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=d_1\\ a_{21}x+a_{22}y+a_{23}z=d_2\\ a_{31}x+a_{32}y+a_{33}z=d_3\end{cases}$$

Write the system of equations in matrix form as follows:

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}$$

Let $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ be the coefficient matrix of the system of linear equations.

By Cramer's rule, the values of x, y, and z are obtained by using the following formulae:

$x=\frac{|A_x|}{|A|}$, $y=\frac{|A_y|}{|A|}$, and $z=\frac{|A_z|}{|A|}$, where $|A|$ is the determinant of the coefficient matrix.

That is, $|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$,

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$|A_x|$ is the determinant of matrix A, with the first column entries replaced by the values d_1, d_2, and d_3. That is,

$$|A_x| = \begin{vmatrix} d_1 & a_{12} & a_{13} \\ d_2 & a_{22} & a_{23} \\ d_3 & a_{32} & a_{33} \end{vmatrix},$$

$|A_y|$ is the determinant of matrix A, with the second column entries replaced by the values d_1, d_2, and d_3. That is,

$$|A_y| = \begin{vmatrix} a_{11} & d_1 & a_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & d_3 & a_{33} \end{vmatrix}, \text{ and}$$

$|A_z|$ is the determinant of matrix A, where the third column entries are replaced by the values d_1, d_2, and d_3. That is,

$$|A_z| = \begin{vmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \\ a_{31} & a_{32} & d_3 \end{vmatrix}.$$

Example 5.47

Solve the following system of linear equations by using Cramer's rule.

$$\begin{cases} x + y + z = 3 \\ x - 3y + 2z = 0 \\ 5x - y - 3z = 1 \end{cases}$$

Solution

The matrix form of the given system of equations is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{pmatrix}$.

$$\Rightarrow |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} -3 & 2 \\ -1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix}$$

$$= 11 + 13 + 14$$

$$= 38$$

$$\Rightarrow |A_x| = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= 3\begin{vmatrix} -3 & 2 \\ -1 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix}$$

$$= 33 + 2 + 3$$

$$= 38$$

$$\Rightarrow |A_y| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix}$$

$$= -2 + 39 + 1$$

$$= 38$$

$$\Rightarrow |A_z| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -3 & 0 \\ 5 & -1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix}$$

$$= -3 - 1 + 42$$

$$= 38$$

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By using Cramer's rule;

$$\Rightarrow x = \frac{|A_x|}{|A|} = \frac{38}{38}$$

$$\Rightarrow x = 1$$

$$\Rightarrow y = \frac{|A_y|}{|A|} = \frac{38}{38}$$

$$\Rightarrow y = 1$$

$$\Rightarrow z = \frac{|A_z|}{|A|} = \frac{38}{38}$$

$$\Rightarrow z = 1$$

Therefore, the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Adjoint of a matrix

The adjoint matrix of a matrix A is the transpose of the matrix of cofactors of A.

Thus, adjoint matrix of A is given by:

$\text{Adj}(A) = (\text{cof}(A))^T$.

But $\text{cof}(A) = \left((-1)^{i+j} M_{ij}\right)$, where M_{ij} are the minors of A.

Therefore, $\text{Adj}(A) = \left((-1)^{i+j} M_{ij}\right)^T$.

Inverse of a 3×3 matrix

The inverse of a non-singular matrix A is denoted by A^{-1}. The inverse of a matrix A exists if and only if A is non-singular.

Properties of inverse of matrices

Let A and B be matrices whose inverses are A^{-1} and B^{-1}, respectively.

The following properties hold:

(a) $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

(b) $\left(A^{-1}\right)^{-1} = A$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) When $|A| = 0$, then A is singular matrix. Hence, $\frac{1}{|A|}$ is undefined and therefore singular matrices have no inverse.

If A is a 3×3 matrix, then A^{-1} is defined as $A^{-1} = \frac{\text{Adj}(A)}{|A|}$, where Adj(A) and $|A|$ are the adjoint and determinant of matrix A, respectively.

Example 5.48

Given $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$, find each of the following:

(a) det (A)
(b) Minors of A
(c) Cofactors of A
(d) Adjoint of A
(e) A^{-1}

Solution

(a) Given $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$.

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$$\Rightarrow \det(A) = 1\begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= 1(-2) - 2(1) + 1(3)$$

$$= -2 - 2 + 3$$

$$= -1$$

Therefore, $\det(A) = -1$.

(b) Let $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$ be a matrix of minors of A.

Given $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$, then the following are the minors of A:

$$M_{11} = \begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} = -2,\ M_{12} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1,\ M_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = 3,$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1,\ M_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1,\ M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1,$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1,\ M_{32} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,\ M_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1.$$

Therefore, the matrix of the minors of $A = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(c) The matrix of cofactors is obtained by multiplying the matrix of minors by the signs $(-1)^{i+j}$, as shown in the following pattern.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Thus, the matrix of cofactors is $\text{cof}(A) = \left((-1)^{i+j} M_{ij}\right) = \begin{pmatrix} -2 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

Therefore, $\text{cof}(A) = \begin{pmatrix} -2 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

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(d) $\text{Adj}(A)=[\text{Cof}(A)]^T$

$$\Rightarrow \text{Adj}(A)=\begin{pmatrix}-2 & -1 & 3\\ -1 & 1 & -1\\ 1 & 0 & -1\end{pmatrix}^T$$

$$=\begin{pmatrix}-2 & -1 & 1\\ -1 & 1 & 0\\ 3 & -1 & -1\end{pmatrix}$$

Therefore, adjoint of $A=\begin{pmatrix}-2 & -1 & 1\\ -1 & 1 & 0\\ 3 & -1 & -1\end{pmatrix}$.

(e) The inverse of matrix A is determined by dividing each element in the adjoint matrix by the determinant of A. That is,

$$A^{-1}=\frac{\text{Adj}(A)}{|A|}=-\frac{1}{1}\begin{pmatrix}-2 & -1 & 1\\ -1 & 1 & 0\\ 3 & -1 & -1\end{pmatrix}=\begin{pmatrix}2 & 1 & -1\\ 1 & -1 & 0\\ -3 & 1 & 1\end{pmatrix}$$

Therefore, $A^{-1}=\begin{pmatrix}2 & 1 & -1\\ 1 & -1 & \\ -3 & 1 & 1\end{pmatrix}$.

Solution of systems of linear equations using inverse matrix method

Consider the following system of linear equations

$$\begin{cases} a_{11}x+a_{12}y+a_{13}z=d_1\\ a_{21}x+a_{22}y+a_{23}z=d_2\\ a_{31}x+a_{32}y+a_{33}z=d_3\end{cases} \qquad (5.1)$$

The system of linear equations (5.1) can be written in matrix form as;

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix} \qquad (5.2)$$

Thus, the equations (5.2) becomes;

$$A\underline{x}=\underline{d} \qquad (5.3)$$

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where $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\underline{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

and $\underline{d}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

Pre-multiply the equations (5.3) by A^{-1} on both sides to obtain;

$A^{-1}A\underline{x}=A^{-1}\underline{d}$.

But $A^{-1}A=I$, where I is an identity matrix.

Thus, $I\underline{x}=A^{-1}\underline{d}$.

$\Rightarrow \underline{x}=A^{-1}\underline{d}$

Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

Example 5.49

Solve the following system of linear equations by using the inverse matrix method.

$$\begin{cases} x+y+z=3 \\ x+2y+3z=4 \\ x+4y+9z=6 \end{cases}$$

Solution

The matrix form of this equation is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Let $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, then

$$|A|=1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}-\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}+\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$=6-6+2$

$=2$

The cofactors of A are as follows:

$$C_{11}=\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}=6$$

$$C_{12}=-\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}=-6$$

$$C_{13}=\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}=2$$

$$C_{21}=-\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix}=-5$$

$$C_{22}=\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}=8$$

$$C_{23}=-\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}=-3$$

$$C_{31}=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1$$

$$C_{32}=-\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}=-2$$

$$C_{33}=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1$$

Thus, $\text{cof}(A)=\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

But $\text{Adj}(A)=(\text{cof}(A))^T$.

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$$\Rightarrow \text{Adj}(A) = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Now from the equation $\underline{x} = A^{-1}\underline{d}$.

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, $x = 2$, $y = 1$, and $z = 0$.

Exercise 5.8

1. Let R be the matrix,

$$R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

(a) Find the inverse of R

(b) Use the inverse obtained in (a) to solve the following system of equations.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 2z = 14 + 3y \\ 3x + y = z - 2 \end{cases}$$

2. Solve each of the following system of equations by using the determinant method:

(a) $\begin{cases} 2x + y + z = 11 \\ 3y - z = -1 \\ 2z = 8 \end{cases}$

(b) $\begin{cases} 4x + 6y + 4z = 28 \\ 4x + 2y - z = 15 \\ x + y + 3z = 8 \end{cases}$

(c) $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ 2x + 2y - 3z = 4 \end{cases}$

(d) $\begin{cases} x + y + z = 4 \\ 2x + y - z = -3 \\ -x + 3y + 4z = 19 \end{cases}$

(e) $\begin{cases} 3x + 2y + z = 5 \\ 2x + y - z = 2 \\ 2x + 2y + 2z = 0 \end{cases}$

3. (a) Find the adjoint of the matrix,

$$\begin{pmatrix} 3 & -2 & -2 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

(b) Use the adjoint obtained in (a) to solve the following system of linear equations.

$$\begin{cases} 3x = 1 + 2y + 2z \\ 2x + 3y = z + 13 \\ x + 3z = y - 8 \end{cases}$$

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4. If $D=\begin{pmatrix}2 & 1 & 0\\ 1 & 5 & 2\\ 1 & -1 & 1\end{pmatrix}$ and $E=\begin{pmatrix}-1 & 2 & 0\\ 1 & 3 & 2\\ 2 & 0 & 1\end{pmatrix}$, find $D^{-1}E$.

5. Use the inverse method to solve the following system of equations.

$$\begin{cases} 4a-b+5c=8\\ 5a+7b-3c=42\\ 3a+4b+c=27 \end{cases}$$

6. The following system of linear equations represents the currents flowing in a unbalanced three-plane, star-connected electrical network. Use Crammer's rule to determine the values of I_1, I_2, and I_3.

$$\begin{cases} 2I_1-5I_2+3I_3=14\\ 9I_1+3I_2-4I_3=13\\ 7I_1+3I_2-2I_3=3 \end{cases}$$

7. The height, h of an object thrown into the air is known to be given by a quadratic function of time (t) of the form, $h=kt^2+mt+n$. If the object is at $h=\frac{23}{4}$ metres when $t=\frac{1}{2}$ sec; at $h=7$ metres when $t=1$ sec; and $h=2$ metres when $t=2$ sec, find the values of k, m, and n by using Cramer's rule.

8. The daily cost of hospital, H is a linear function of the number of in-patients, I and out-patients, P plus a fixed cost q, given by $H=q+rP+sI$. The following data represents the cost for in-patients and out-patients for three days.

Day	Cost in Tshs, H	Number of in-patients, I	Number of out-patients, P
1	55,600	320	80
2	53,800	280	72
3	56,800	320	96

Set up a linear system of equations, hence use the inverse matrix method to find the values of q, r, and s.

9. A manufacturer produces three types of products: simple, medium, and complex. Each product is first processed in machine M_1, then in machine M_2, and lastly sent to machine M_3 for finishing. The following table shows the number of minutes and the total time available.

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Machine / Product	M_1	M_2	M_3
Simple	180	90	270
Medium	135	180	90
Complex	90	270	135
Total time	2,700	4,050	3,150

Calculate the number of units of the three types of products produced by constructing a matrix equation and then solving it by the inverse matrix method.

10. In a market survey, three commodities, namely groundnuts, beans, and rice were considered. In finding out the index number some fixed weights were assigned to three varieties V_1, V_2, and V_3 of each of the commodities. The following table provides the information regarding the consumption of three commodities according to three varieties and the total weight received by the commodity.

Varieties / Commodities	V_1	V_2	V_3	Total weight
Groundnuts	324	540	648	2,916
Beans	216	432	540	2,268
Rice	108	216	324	1,188

Find the weights assigned to the three varieties by using inverse matrix method, given that the weights assigned to a commodity are equal to the sum of weights of the various varieties multiplied by the corresponding consumption.

Binomial theorem

The binomial expression $(a + b)$ can be raised to different powers. When $(a + b)$ is raised to any power different from 1, the need of having its expanded form arises. These expansions range from simple expansion when the power is small to complex expansion when the power is large, fraction, or negative. The pattern

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in the expansion may give a rule for obtaining the coefficients of the terms in the expansion as it was discovered by a French Mathematician called Blaise Pascal. Activity 5.8 gives an overview of the pattern of expansion of the binomial expression.

Activity 5.8: Identifying coefficients of a binomial expression

Individually or in a group, perform the following tasks:

1. Write the expansion of $(a+b)^1$, $(a+b)^2$, and $(a+b)^3$.
2. Expand $(a+b)^n$ for $n = 4$, $n = 5$, and $n = 6$.
3. Study how every coefficient in the following table is obtained and then fill in the exponent in the expanded form and the binomial coefficients of the last three rows.

Binomial expression	Exponent	Expanded form	Binomial coefficients
$(a+b)^0$	0	1	1
$(a+b)^1$	1	$a+b$	1 1
$(a+b)^2$	2	$a^2+2ab+b^2$	1 2 1
$(a+b)^3$	3	$a^3+3a^2b+3ab^2+b^3$	1 3 3 1
$(a+b)^4$	4		
$(a+b)^5$	5		
$(a+b)^6$	6		

4. Is it possible to identify the binomial coefficients for $(a+b)^{20}$? Give reasons.
5. What did you observe in tasks 1, 2, and 3?
6. Share your results with other students for more inputs.

The following can be deduced from Activity 5.8.

1. The exponent of a is higher on the first term from left and decreases by 1 as you move to the next terms.
2. The exponent of b is zero on the first term from left and increases by 1 as you move to the next terms.
3. The sum of the exponents for a and b in each term is n, where n is the power.
4. The coefficients are symmetric since they increase from the beginning to the line of symmetry and then decrease to the end.
5. If the coefficients are detached from the expansions, then a triangle is formed. The resulting triangle is called Pascal's triangle.

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From Activity 5.8, consider the coefficients of the terms. Writing these as a triangular array for the binomial expression $(a+b)^n$, where $n = 0,1,2,3,$ and 4 gives,

$(a+b)^0$: 1

$(a+b)^1$: 1 1

$(a+b)^2$: 1 2 1

$(a+b)^3$: 1 3 3 1

$(a+b)^4$: 1 4 6 4 1

The previous triangular array is called Pascal's triangle and clearly it has a pattern. Two things can be observed from Pascal's triangle:

(i) Each row starts and end with 1

(ii) Each coefficient of a term, except the first and the last is obtained by adding the two adjacent coefficients immediately above it in the previous row.

Example 5.50

Expand $(2x+3)^3$ in descending powers of x.

Solution

Given $(2x+3)^3$, where $a = 2x$, $b = 3$, and $n = 3$.

Applying the Pascal's triangle to expand: gives,

$$(2x+3)^3 = 1(2x)^3(3)^0 + 3(2x)^2(3)^1 + 3(2x)^1(3)^2 + 1(2x)^0(3)^3$$

$$= 8x^3 + 36x^2 + 54x + 27.$$

Therefore, $(2x+3)^3 = 8x^3 + 36x^2 + 54x + 27$.

Example 5.51

Simplify $(2+\sqrt{3})^4 - (2-\sqrt{3})^4$ leaving your answer in surd form.

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Solution

Apply Pascal's triangle to expand:

$$\Rightarrow (2+\sqrt{3})^4 = 1(2)^4(\sqrt{3})^0 + 4(2)^3(\sqrt{3})^1 + 6(2)^2(\sqrt{3})^2 + 4(2)^1(\sqrt{3})^3 + 1(2)^0(\sqrt{3})^4$$

$$= 16 + 32\sqrt{3} + 72 + 24\sqrt{3} + 9$$

$$= 97 + 56\sqrt{3}$$

and

$$(2-\sqrt{3})^4 = 1(2)^4(-\sqrt{3})^0 + 4(2)^3(-\sqrt{3})^1 + 6(2)^2(-\sqrt{3})^2 + 4(2)^1(-\sqrt{3})^3 + 1(2)^0(-\sqrt{3})^4$$

$$= 16 - 32\sqrt{3} + 72 - 24\sqrt{3} + 9$$

$$= 97 - 56\sqrt{3}$$

$$\Rightarrow (2+\sqrt{3})^4 - (2-\sqrt{3})^4 = (97+56\sqrt{3}) - (97-56\sqrt{3})$$

$$= 112\sqrt{3}$$

Therefore, $(2+\sqrt{3})^4 - (2-\sqrt{3})^4 = 112\sqrt{3}$.

Example 5.52

Find the expansion of $\left(1+\frac{1}{4}x\right)^4$. If $x = 0.1$, evaluate the value of $(1.025)^4$, correct to four decimal places.

Solution

Apply Pascal's triangle as follows.

$$\left(1+\frac{1}{4}x\right)^4 = 1(1)^4\left(\frac{1}{4}x\right)^0 + 4(1)^3\left(\frac{1}{4}x\right)^1 + 6(1)^2\left(\frac{1}{4}x\right)^2 + 4(1)^1\left(\frac{1}{4}x\right)^3 + 1(1)^0\left(\frac{1}{4}x\right)^4$$

$$= 1 + x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{256}x^4$$

Thus, $\left(1+\frac{1}{4}x\right)^4 = 1 + x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{256}x^4$.

Substituting $x = 0.1$ in the equation gives,

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$$\left(1+\frac{1}{4}(0.1)\right)^4 = 1+0.1+\frac{3}{8}(0.1)^2+\frac{1}{16}(0.1)^3+\frac{1}{256}(0.1)^4$$

$$\Rightarrow (1.025)^4 = 1.103812891$$

Therefore, $\left(1+\frac{1}{4}x\right)^4 = 1+x+\frac{3}{8}x^2+\frac{1}{16}x^3+\frac{1}{256}x^4$ and $(1.025)^4 \approx 1.1038$.

Binomial expansion

Pascal's triangle is tedious when the power is large. An easy way to expand the expression for higher powers is to apply the binomial theorem. Actually, the binomial theorem is obtained after studying the expansions of the individual expressions. It is the general expansion that can be used to obtain all expansions. The binomial theorem states that,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n, \quad (5.4)$$

where n is a positive integer.

Sometimes, the binomial theorem is presented with factorials. To understand how factorials are involved in the expression, consider the expansion of $(x+3)^4$ by using the binomial theorem (5.4) as shown in example 5.53.

Example 5.53

Expand $(x+3)^4$ in descending powers of x.

Solution

Apply the binomial theorem to obtain,

$$(x+3)^4 = \frac{x^4\times 3^0}{1} + \frac{4}{1}x^3\times 3^1 + \frac{4\times 3}{1\times 2}x^2\times 3^2 + \frac{4\times 3\times 2}{1\times 2\times 3}x^1\times 3^3 + \frac{4\times 3\times 2\times 1}{1\times 2\times 3\times 4}x^0\times 3^4$$

$$= x^4+12x^3+54x^2+108x+81$$

Therefore, $(x+3)^4 = x^4+12x^3+54x^2+108x+81$.

From Example 5.53, it can be noted that, the denominators of the terms are in a particular order. The denominators of the 5th, 4th, and 3rd terms are the factorials of 4, 3, and 2, respectively. This is due to the definition of the factorial which states that; if n is any positive integer, then $n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$, where $n!$ is the factorial of the positive integer n. In other words, $n!$ is a mathematical expression for number of ways to arrange a data set with n elements in it.

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The 1st and 2nd terms have denominators equal to 1. By following the trend from the 3rd term to the 5th term, it is possible to suggest the denominators of the 1st and 2nd terms in factorial. Since it has been observed that from the 5th term towards the 3rd term, there is a decrease of the denominator by 1. Hence, the denominators of the 2nd and 1st terms are 1! and 0!, respectively.

Note that, $0! = 1$ because there is only one way to arrange a data set with no elements in it.

To incorporate the factorial, the binomial theorem can now be modified to,

$$(a+b)^n = \frac{n!}{0!(n-0)!}a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \cdots + \frac{n!}{n!(n-n)!}b^n.$$

The expression can further be simplified to;

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n,$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^nC_r$.

Generally,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n \qquad (5.5)$$

Example 5.54

Expand $(x+1)^3$ by using binomial theorem.

Solution

From binomial theorem:

$$(x+1)^3 = \frac{3!}{0!3!}x^3 + \frac{3!}{1!2!}x^2(1) + \frac{3!}{2!1!}x^1(1)^2 + \frac{3!}{3(0)!}(1)^3$$

$$= x^3 + 3x^2 + \frac{3\times 2!}{2!}x + 1$$

$$= x^3 + 3x^2 + 3x + 1$$

Therefore, $(x+1)^3 = x^3 + 3x^2 + 3x + 1$.

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Example 5.55

Obtain the expansion of $(1+x-3x^2)^{10}$, as far as the term in x^3.

Solution

The expression $(1+x-3x^2)^{10}$ can be written as $[1+(x-3x^2)]^{10}$, which may be expanded by using the binomial theorem as follows.

$$[1+(x-3x^2)]^{10} = 1+10(x-3x^2)+\frac{10\times 9}{2!}(x-3x^2)^2+\frac{10\times 9\times 8}{3!}(x-3x^2)^3+\cdots$$
$$= 1+10x-30x^2+45x^2-270x^3+120x^3-\cdots$$
$$= 1+10x+15x^2-150x^3+\cdots$$

Therefore, $(1+x-3x^2)^{10} = 1+10x+15x^2-150x^3+\cdots$

Example 5.56

If x^3 and higher powers can be neglected, show that

$$(1-x)^5\left(2+\frac{1}{2}x\right)^{10} \approx 5(205+511x+578x^2).$$

Solution

Consider the binomial expansion of $(1-x)^5$ and $\left(2+\frac{1}{2}x\right)^{10}$.

$$\Rightarrow (1-x)^5 = 1+5(-x)^1+\frac{5\times 4}{2\times 1}(-x)^2+\cdots$$
$$= 1-5x+10x^2+\cdots$$

$$\text{Also, } \left(2+\frac{1}{2}x\right)^{10} = 2^{10}+10(2)^9\left(\frac{1}{2}x\right)+\frac{10(9)(2)^8\left(\frac{1}{2}x\right)^2}{1\times 2}+\cdots$$
$$= 2^{10}+5(2)^9x+90(2)^5x^2+\cdots$$

$$\Rightarrow (1-x)^5\left(2+\frac{1}{2}x\right)^{10} = 1-5x+10x^2+2^{10}+5(2)^9x+90(2)^5x^2+\cdots$$
$$= 1025+2555x+2890x^2+\cdots$$
$$= 5(205+511x+578x^2+\cdots)$$

Therefore, $(1-x)^5\left(2+\frac{1}{2}x\right)^{10} \approx 5(205+511x+578x^2)$.

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Exercise 5.9

1. Use Pascal's triangle to expand each of the following:

   (a) $(2x+3z)^5$ (b) $(a-b)^7$ (c) $\left(2x+\frac{2}{x}\right)^5$

2. Evaluate each of the following, leave the answer in surd form where appropriate:

   (a) $\left(\sqrt{2}+1\right)^6-\left(\sqrt{2}-1\right)^6$ (c) $\left(\sqrt{6}+\sqrt{2}\right)^3-\left(\sqrt{6}-\sqrt{2}\right)^3$

   (b) $\left(2+\sqrt{5}\right)^7+\left(2-\sqrt{5}\right)^7$ (d) $\left(\sqrt{2}+\sqrt{3}\right)^4+\left(\sqrt{2}-\sqrt{3}\right)^4$

3. Expand $\left(1+x+x^2\right)^{10}$ as a series in ascending powers of x up to and including the term in x^3. Hence, evaluate $(1.0101)^{10}$ correct to three decimal places.

4. Use the binomial theorem to evaluate each of the following correct to five significant figures:

   (a) $(1.009)^8$ (b) $(2.045)^{10}$ (c) $(2.098)^{12}$

5. The expansion of $(1+y)^{10}\left(1+my+ny^2\right)$ in ascending powers of y begins with the terms $1+y+y^2$. Find the numerical values of m and n.

6. In the expansion of $\left(1-2a+ka^2\right)^4$ as a series of powers of a, the coefficient of a^3 is zero. Verify that:

   (a) $k=-1\frac{1}{3}$ (b) The coefficient of a^4 is $-37\frac{1}{3}$.

7. Find the expansion of $(2+y)^5$ in ascending powers of y. Taking the first three terms of the expansion, put $y=0.001$ and find the value of $(2.001)^5$ correct to five decimal places.

8. Find the first four terms in the expansion of $\left(2+\frac{1}{4}y\right)^{10}$ in ascending powers of y. Hence, find the value of $(2.025)^{10}$ correctly to four significant figures.

9. Obtain the expansion of $(c-2)^2(1-c)^6$ in ascending powers of c as far as the term in c^4.

10. Find the expansion of $\left(3a-\frac{1}{2}b\right)^4$ by using the binomial theorem. Hence, obtain the value of $(29.5)^4$ correct to six significance figures.

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Binomial expansion for fractional and negative powers

The binomial expansion is applicable in the case when n is a positive integer. For the case when n is negative or fraction, the modified binomial theorem becomes:

$$(1+x)^n = \frac{1}{0!} + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots (5.6)$$

The modified binomial theorem is used in the case when n is negative or a fraction. The series of equation (5.6) does not terminate and it is only convergent with the limit of its sum as $(1+x)^n$ when x is between -1 and 1. This interval is also known as limits of x for which the expansion is valid.

Note that, when n is a positive integer, the series of equation (5.6) terminates at the term in x^n and its sum is $(1+x)^n$ for all values of x.

Example 5.57

Find the first four terms in the expansion of each of the following binomial expressions. State the range of values of x for which the expansion is valid.

(a) $(1+2x)^{-1}$ (b) $\sqrt[3]{2+x}$

Solution

(a) $(1+2x)^{-1} = 1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\frac{(-1)(-2)(-3)}{3!}(2x)^3+\cdots$

$$= 1-2x+\frac{8}{2\times 1}x^2-\frac{(6)8}{3\times 2\times 1}x^3+\cdots$$

$$= 1-2x+4x^2-8x^3+\cdots$$

Thus, $(1+2x)^{-1} = 1-2x+4x^2-8x^3+\cdots$

This expansion converges in the interval,

$|2x|<1 \Rightarrow -1<2x<1$

$\Rightarrow -\frac{1}{2}<x<\frac{1}{2}$ or $|x|<\frac{1}{2}$

Therefore, $(1+2x)^{-1} = 1-2x+4x^2-8x^3+\cdots$, it is valid in the interval $-\frac{1}{2}<x<\frac{1}{2}$ or $|x|<\frac{1}{2}$.

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(b) $\sqrt[3]{2+x} = 2^{\frac{1}{3}}\left[1+\frac{x}{2}\right]^{\frac{1}{3}}$

$$= \sqrt[3]{2}\left[1+\frac{1}{3}\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{x}{2}\right)^2}{2!}+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{x}{2}\right)^3}{3!}+\cdots\right]$$

$$= \sqrt[3]{2}\left[1+\frac{x}{6}-\frac{1}{36}x^2+\frac{5}{648}x^3+\cdots\right]$$

Thus, $\sqrt[3]{2+x} = \sqrt[3]{2}\left(1+\frac{x}{6}-\frac{1}{36}x^2+\frac{5}{648}x^3+\cdots\right)$.

This expansion converges in the range,

$$\left|\frac{x}{2}\right|<1 \Rightarrow -1<\frac{x}{2}<1$$

$$\Rightarrow -2<x<2 \text{ or } |x|<2.$$

Therefore, $\sqrt[3]{2+x} = \sqrt[3]{2}\left(1+\frac{1}{6}x-\frac{1}{36}x^2+\frac{5}{648}x^3+\cdots\right)$, it is valid in the interval $-2<x<2$ or $|x|<2$.

Example 5.58

If x is small compared to unity such that x^3 and higher powers can be neglected, show that $\frac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}} \approx 1-\frac{3}{2}x-\frac{33}{8}x^2$.

Solution

Apply the binomial expansion:

$$\Rightarrow \frac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}} = (1-4x)^{\frac{1}{2}}(1+3x)^{\frac{1}{3}}(1+x)^{-\frac{1}{2}}$$

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$$= \left[1+\frac{1}{2}(-4x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-4x)^2}{2!}+\cdots\right] \times \left[1+\frac{1}{3}(3x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(3x)^2}{2!}+\cdots\right] \times$$

$$\left[1-\frac{1}{2}(x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^2}{2!}+\cdots\right]$$

$$= \left(1-2x-2x^2+\cdots\right)\left(1+x-x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$= \left(1-x-5x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$= 1-\frac{3}{2}x-\frac{33}{8}x \ -\cdots$$

Therefore, $\dfrac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}} \approx 1-\frac{3}{2}x-\frac{33}{8}x^2.$

Example 5.59

Expand $\sqrt{1+x}$ in ascending powers of x as far as the term in x^2. Hence, estimate $\sqrt{30}$ correct to four significant figures.

Solution

Given $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$.

From $(1+x)^n = 1+nx+n\left(\frac{n-1}{2!}\right)x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

$$\Rightarrow (1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\cdots$$

$$= 1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

Thus, $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots.$

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Let $\sqrt{30}=\sqrt{25+5}$

$$\Rightarrow \sqrt{30}=\sqrt{25\left(1+\frac{1}{5}\right)}$$

$$=\sqrt{25}\sqrt{\left(1+\frac{1}{5}\right)}$$

$$=5\left(1+\frac{1}{5}\right)^{\frac{1}{2}}$$

Let $x=\frac{1}{5}$, then $\left(1+\frac{1}{5}\right)^{\frac{1}{2}}=(1+x)^{\frac{1}{2}}$.

$$\Rightarrow \sqrt{30}=5\left[1+\frac{1}{2}\left(\frac{1}{5}\right)-\frac{1}{8}\left(\frac{1}{25}\right)+\cdots\right]$$

$$=5\left[1+(0.5)(0.2)-(0.125)(0.04)+\cdots\right]$$

$$\approx 5[1.095]$$

$$\approx 5.475$$

Therefore,

$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$ and $\sqrt{30}\approx 5.475$.

Example 5.60

If x is small such that its cube and higher powers can be neglected, show that $\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$. Also, if $x=\frac{1}{8}$, show that $\sqrt{7}\approx 2\frac{83}{128}$.

Solution

Apply the binomial expansion:

$$\Rightarrow \sqrt{\frac{1-x}{1+x}}=(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$=\left[1+\frac{1}{2}(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-x)^2}{2!}+\cdots\right]\left[1+\left(-\frac{1}{2}\right)(x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^2}{2!}+\cdots\right]$$

$$=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots\right)$$

$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2+\cdots$$

$$=1-x+\frac{1}{2}x^2+\cdots$$

Hence, $\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$.

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Given $x=\frac{1}{8}$, then $\sqrt{\frac{1-\frac{1}{8}}{1+\frac{1}{8}}} \approx 1-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{8}\right)^2$.

$\Rightarrow \sqrt{\frac{7}{9}} \approx 1-\frac{1}{8}+\frac{1}{128}$

$\Rightarrow \frac{\sqrt{7}}{3} \approx \frac{113}{128}$

$\Rightarrow \sqrt{7} \approx \frac{113}{128} \times 3 \approx \frac{339}{128}$

Thus, $\sqrt{7} \approx 2\frac{83}{128}$.

Therefore, $\sqrt{\frac{1-x}{1+x}} = 1-x+\frac{1}{2}x^2$ and $\sqrt{7} \approx 2\frac{83}{128}$.

Exercise 5.10

1. Use the binomial theorem to show that
$$\frac{1}{(2+3y)^4} = \frac{1}{16}\left(1-6y+\frac{45}{2}y^2-\frac{135}{2}y^3+\cdots\right)$$

2. If x is very small compared to unity such that x^2 and higher powers can be neglected, show that $\frac{(\sqrt{1+x})\sqrt[3]{(1-x)^2}}{(1+x)\sqrt{(1+x)}} \approx 1-\frac{5}{3}x$

3. Express $\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$ as a power series, as far as the term in x^2. State the range of values of x for which the series is convergent.

4. Using the binomial theorem, show that $\frac{1}{(2+3y)^6} = \frac{1}{64}\left(1-9y+\frac{189}{4}y^2-\cdots\right)$
For what values of y is the expansion valid?

5. Use the binomial theorem to expand $\frac{(1+a)\sqrt[3]{(1-3a)^2}}{\sqrt{1+a^2}}$ in ascending powers

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of a as far as the term in a^2. State the interval of a for which the series is valid.

6. Obtain the first four terms in the expansion of $(1-16y)^{\frac{1}{4}}$. By substituting $y=\frac{1}{10000}$, evaluate $\sqrt[4]{39}$ correct to three decimal places.

7. When t is small, use the binomial theorem to show that $\frac{1}{(1+t)^2\sqrt{1-t}}=1-\frac{3}{2}t+\cdots$

8. Expand $\sqrt[3]{1+t}$ up to the term in t^2. Hence, by putting $t=\frac{1}{8}$, calculate $\sqrt[3]{9}$ correct to three decimal places.

9. If x is small such that x^3 and higher powers of x can be neglected, show that $(2x+3)(1-2x)^{10}\approx 3-58x+500x^2$.

10. Use the binomial theorem to expand $\frac{1+2n}{(n-1)(n+2)^2}$ neglecting n^3 and all higher powers. State the interval of values of n for which the expansion is valid.

The general term in the binomial expansion

Consider the binomial expansion (5.5), that is

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\cdots+b^n.$$

This equation can be written as:

$$(a+b)^n={}^nC_0a^nb^0+{}^nC_1a^{n-1}b+{}^nC_2a^{n-2}b^2+\cdots+{}^nC_ra^{n-r}b^r+\cdots+{}^nC_na^0b^n \quad (5.7)$$

From the expansion (5.7), it can be observed that:

1st term $={}^nC_0a^nb^0$, the 2nd term $={}^nC_1a^{n-1}b$, the 3rd term $={}^nC_2a^{n-2}b^2$, the 4th term $={}^nC_3a^{n-3}b^3$.

Hence, the $(r+1)^{th}$ term, is given by ${}^nC_ra^{n-r}b^r$.

Therefore, if T_{r+1} denotes the $(r+1)^{th}$ term, then $T_{r+1}={}^nC_ra^{n-r}b^r$.

Note that, the formulae for the general term of binomial expansion are:

For $(a+b)^n \Rightarrow T_{r+1}={}^nC_ra^{n-r}b^r$

For $(a-b)^n \Rightarrow T_{r+1}=(-1)^r\,{}^nC_ra^{n-r}b^r$

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For $(1+x)^n \Rightarrow T_{r+1} = {}^nC_r x^r$

For $(1-x)^n \Rightarrow T_{r+1} = (-1)^r {}^nC_r x^r$.

Example 5.61

Find the sixth term in the expansion of $\left(\frac{2}{x^2} - \frac{x^3}{4}\right)^8$.

Solution

From $\left(\frac{2}{x^2} - \frac{x^3}{4}\right)^8$, $n = 8$, $a = \frac{2}{x^2}$, $b = -\frac{x^3}{4}$, and $r = 5$.

$\Rightarrow T_{r+1} = {}^nC_r a^{n-r} b^r$

$\Rightarrow T_6 = {}^8C_5 \left(\frac{2}{x^2}\right)^{8-5} \left(-\frac{x^3}{4}\right)^5 = {}^8C_5 \left(\frac{2}{x^2}\right)^3 \left(-\frac{x^3}{4}\right)^5$

$= -\frac{7}{16} x^9$

Therefore, the sixth term is $-\frac{7}{16} x^9$.

Example 5.62

Find the term independent of x in each of the following:

(a) $\left(x^2 - \frac{1}{x^2}\right)^{14}$ (b) $\left(\frac{5}{2}x^2 - \frac{1}{5x}\right)^6$

Solution

(a) Let $(r+1)^{th}$ term be independent of x in the expansion.

Now, $T_{r+1} = {}^{14}C_r \left(x^2\right)^{14-r} \left(-\frac{1}{x^2}\right)^r = {}^{14}C_r x^{28-4r} (-1)^r$.

For this term to be independent of x, set $28 - 4r = 0 \Rightarrow r = 7$.

So, $(7+1)^{th} = 8^{th}$ term is independent of x.

$\Rightarrow T_{7+1} = T_8 = {}^{14}C_7 x^{28-28} (-1)^7 = -{}^{14}C_7$

$= -3432$

Therefore, the term independent of x is -3432.

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(b) Let $(r+1)^{th}$ term be independent of x in the given expression.

Now, $T_{r+1} = {}^6C_r\left(\frac{5}{2}x^2\right)^{6-r}\left(-\frac{1}{5x}\right)^r = {}^6C_r\left(\frac{5}{2}\right)^{6-r}(x^2)^{6-r}(-1)^r\frac{1}{5^r}x^{-r}$

$= {}^6C_r\left(\frac{5}{2}\right)^{6-r}(x)^{12-2r}(-1)^r\frac{1}{5^r}x^{-r}$

$= {}^6C_r\left(\frac{5}{2}\right)^{6-r}(x)^{12-3r}(-1)^r\frac{1}{5^r}$

For this term to be independent of x, set $12-3r=0$. That is, $12-3r=0 \Rightarrow r=4$. Thus, the 5^{th} term is independent of x.

$\Rightarrow T_5 = {}^6C_4\left(\frac{5}{2}\right)^2 x^0(-1)^4\frac{1}{5^4} = \frac{375}{4\times 625} = \frac{3}{20}.$

Therefore, the term independent of x is $\frac{3}{20}$.

Example 5.63

Find the coefficient of the term involving x^{10} in the expansion of $\left(x^6+\frac{1}{x^4}\right)^{15}$.

Solution

$T_{r+1} = {}^{15}C_r\left(x^6\right)^{15-r}\left(\frac{1}{x^4}\right)^r = {}^{15}C_r x^{90-10r}$

Find the value of r where the term containing x^{10} occurs.

$\Rightarrow x^{90-10r} = x^{10}$

Thus, $90-10r=10 \Rightarrow r=8$.

$\Rightarrow T_9 = {}^{15}C_8 x^{10} = 6435x^{10}$

Therefore, the coefficient of the term involving x^{10} is 6,435.

Middle terms in a binomial expansion

The binomial expansion of $(a+b)^n$ has $(n+1)$ terms. Therefore, if n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term, and if n is odd, the middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$.

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Example 5.64

Find the middle term in the expansion of $\left(4+\frac{1}{3}x\right)^{10}$.

Solution

Here, $n=10$, which is an even number. So, $\left(\frac{10}{2}+1\right)^{th}$ term = 6^{th} term is the middle term.

Hence, $T_6 = T_{5+1} = {}^{10}C_2(4)^5\left(\frac{x}{3}\right)^5 = \frac{258048}{243}x^5$

Therefore, the middle term is $\frac{258048}{243}x^5$.

Example 5.65

Find the middle terms in the expansion of $\left(2-\frac{x^4}{2}\right)^5$.

Solution

From $\left(2-\frac{x^4}{2}\right)^5$, $n=5$, which is an odd number.

So, $\left(\frac{5+1}{2}\right)^{th}$ and $\left(\frac{5+3}{2}\right)^{th}$, that is 3^{rd} and 4^{th} terms are middle terms.

Now, $T_3 = T_{2+1} = {}^5C_2(2)^2\left(-\frac{x^4}{2}\right)^3$

$$= \frac{5\times 4\times 3!}{2!3!}\times 2^2\times\left(-\frac{x^{12}}{8}\right)$$

$$= -5x^{12}$$

and $T_4 = T_{3+1} = {}^5C_3(2)^3\left(-\frac{x^4}{2}\right)^2$

$$= \frac{5\times 4\times 3!}{3!2!}\times 8\times\left(\frac{x^8}{4}\right)$$

$$= 20x^8$$

Therefore, the middle terms are $-5x^{12}$ and $20x^8$.

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Exercise 5.11

1. Show that there is no term independent of x in the expansion of $\left(2x^2-\frac{1}{4x}\right)^{11}$, $x\neq 0$.
2. If the coefficients of y^7 and y^8 in the expansion of $\left(3+\frac{1}{2}y\right)^n$ are equal, find the value of n.
3. Find the coefficient of z^5 in the expansion of $(1-z)^3(1+3z)^4$.
4. In the expansion of $(1+t)^{20}$, the coefficient of the r^{th} term and that of the $(r+1)^{th}$ term are in the ratio 1:2. Find the value of r.
5. Find the general term in the binomial expansion of each of the following:
   (a) $\left(\frac{4y}{5}-\frac{5}{2y}\right)^9$, $y\neq 0$
   (b) $\left(2y+\frac{1}{y}\right)^5$, $y\neq 0$
   (c) $\left(y^2-\frac{1}{y}\right)^{12}$, $y\neq 0$
   (d) $(t^2-s)^6$
6. Show that the 13^{th} term in the expansion of $\left(9y-\frac{1}{3}\right)^{-\frac{1}{2}}$, $y\neq 0$ is $18,564$.
7. Verify that the middle terms in the expansion of $\left(3-\frac{y^3}{6}\right)^7$ are $-\frac{105}{8}y^9$ and $\frac{35}{48}y^{12}$.
8. Find the values of p, q, and n in the expansion of $(p+q)^n$ if the first three terms of the expansion are 729, 7290, and 30375, respectively.
9. Find the term independent of y in the expansion of each of the following:
   (a) $\left(\frac{3y^2}{2}-\frac{1}{3y}\right)^{15}$
   (b) $\left(\sqrt{y}-\frac{3}{y^2}\right)^{10}$
   (c) $\left(1+y+2y^3\right)\left(\frac{3}{2}y^2-\frac{1}{3y}\right)^9$
   (d) $\left(y-\frac{1}{y}\right)^{14}$
10. Find the coefficient of the terms indicated in the expansions of the following:
    (a) $\left(2m+\frac{1}{m}\right)^7$, term in m^{-5}
    (b) $\left(p-\frac{2}{p}\right)^8$, term in p^6

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Partial fractions

In arithmetic, it is possible to decompose a fraction like $\frac{3}{4}$ into a sum of smaller fractions like $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$. Similarly, a rational expression can be decomposed into simpler parts known as partial fractions. For instance, $\frac{2x+1}{(x-1)(x+2)}$ can be decomposed as $\frac{2x+1}{(x-1)(x+2)}=\frac{1}{x-1}+\frac{1}{x+2}$. The fractions $\frac{1}{x-1}$ and $\frac{1}{x+2}$ are called partial fractions of $\frac{2x+1}{(x-1)(x+2)}$. The process of taking a rational expression and decomposing it into simpler rational expressions that can be added or subtracted to get the original rational expression is called partial fraction decomposition.

Decomposition of fractions

The process of splitting a fraction into its partial fractions is done by the method of comparing undetermined coefficients or the substitution method. The decomposition techniques differ depending on the nature of the expression on the denominator.

Decomposition of fractions whose denominators consist of non-repeated linear factors

If the denominator of a fraction is the product of linear factors or it is an expression which can be split into linear factors, then the fraction is split into terms of the form, $\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\frac{C}{a_3x+b_3}+\cdots$

Observe that, the number of partial fractions to be determined is normally shown by the factors in the denominator. The denominators of the partial fractions are the factors. However, their corresponding numerators are not known. In this case, the assumption is that, one constant for each denominator will be the numerator, then the values of constants can be computed.

Example 5.66

Find the partial fractions of $\frac{6x}{(x-1)(x+2)}$.

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Solution

There will be two partial fractions, thus choose constants A and B to be the numerators. The expression is rewritten as:

$$\frac{6x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Writing under common denominators gives,

$$\frac{6x}{(x-1)(x+2)} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

Since the denominators of left-hand side and right-hand side are equal, then the numerators are also equal.

That is,

$6x = A(x+2) + B(x-1)$.

To solve for the values of A or B, substitute any value of x.

However, for simplicity choose x in such a way that either A or B cancels.

Setting $x = 1$,

$$6\times 1 = A(1+2) + B(1-1)$$
$$6 = 3A + 0$$
$$\Rightarrow A = 2$$

Setting $x = -2$,

$$6\times -2 = A(-2+2) + B(-2-1)$$
$$\Rightarrow -12 = 0 - 3B$$
$$\Rightarrow B = 4$$

Thus, $\frac{6}{(x-1)(x+2)} = \frac{2}{(x-1)} + \frac{4}{(x+2)}$.

Therefore,

$$\frac{6x}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{4}{x+2}.$$

Decomposition of fractions whose denominators consist of non-repeated irreducible quadratic factors

If the denominator is a quadratic term which cannot be factorized, then the fraction is split into $\frac{Ax+B}{ax^2+bx+c}$.

However, the method is useful when the denominator is a product of either two quadratic terms which cannot be split further or it is a product of linear factors and a quadratic term which cannot be split into linear factors.

Example 5.67

Decompose $\frac{x+2}{(x-1)(x^2+1)}$ into its partial fractions.

Solution

Let $\frac{x+2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$.

Writing under common denominators gives,

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$$\frac{x+2}{(x-1)(x^2+1)} = \frac{A(x^2+1)+(x-1)(Bx+C)}{(x-1)(x^2+1)}$$

Equating numerators gives,

$$x+2 = A(x^2+1)+(x-1)(Bx+C)$$

$$\Rightarrow x+2 = (A+B)x^2+(-B+C)x+A-C$$

Equating the coefficients of x^2 terms gives,

$0 = A+B$ .....(i)

Equating the coefficients of x terms gives,

$1 = -B+C$ .....(ii)

Equating the constant terms gives,

$2 = A-C$ .....(iii)

Subtracting equation (i) from equation (iii) gives,

$2 = -B-C$ .....(iv)

Adding equations (ii) and (iv) gives, $3 = -2B \Rightarrow B = -\frac{3}{2}$

Subtracting equation (ii) from equation (iv) gives, $1 = -2C \Rightarrow C = -\frac{1}{2}$.

Substituting $C = -\frac{1}{2}$ into equation (iii) gives, $2 = A + \frac{1}{2} \Rightarrow A = \frac{3}{2}$.

$$\Rightarrow \frac{x+2}{(x-1)(x^2+1)} = \frac{\frac{3}{2}}{x-1} + \frac{-\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

Therefore, $\frac{x+2}{(x-1)(x^2+1)} = \frac{3}{2(x-1)} - \frac{3x+1}{2(x^2+1)}$.

Decomposition of fractions whose denominators consist of a repeated linear or quadratic factors

If the denominator is of the form $(cx + a)^n$, the number of fractions will be equal to n in the following order: $\frac{B_1}{cx+a}, \frac{B_2}{(cx+a)^2}, \frac{B_3}{(cx+a)^3}, \ldots, \frac{B_n}{(cx+a)^n}$

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If the denominator is of the form $(ax^2+bx+c)^n$ the number of fractions will be in the following order:

$$\frac{A_1x+B_1}{ax^2+bx+c}, \frac{A_2x+B_2}{(ax^2+bx+c)^2}, \frac{A_3x+B_3}{(ax^2+bx+c)^3}, \ldots, \frac{A_4x+B_n}{(ax^2+bx+c)^n}$$

where $A_1, A_2, A_3, \ldots, A_n, B_1, B_2, B_3, \ldots, B_n$ are constants.

Example 5.68

Decompose $\frac{2x+8}{(x-1)^3}$ into partial fractions.

Solution

Let $\frac{2x+8}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$.

Writing under common denominators gives,

$$\frac{2x+8}{(x-1)^3} = \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

Equating numerators gives,

$2x+8 = A(x-1)^2+B(x-1)+C$

Setting $x=1$, $10=0+0+C \Rightarrow C=10$

Setting $x=2$, $12=A+B+10 \Rightarrow A+B=2$ .....(i)

Setting $x=0$, $8=A-B+10 \Rightarrow A-B=-2$ .....(ii)

Solving equations (i) and (ii) simultaneously gives,

$A=0$ and $B=2$.

Therefore, $\frac{2x+8}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{10}{(x-1)^3}$.

Decomposition of fractions when the degree of the numerator is equal or higher than the degree of the denominator

When the degree of the numerator is equal or greater than the degree of the denominator, divide the numerator by the denominator and then proceed depending on the type of fraction to be decomposed.

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Example 5.69

Decompose $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}$ into partial fractions.

Solution

Since the numerator has degree greater than the denominator, divide the numerator by the denominator:

$$x^3-2x^2+3x-6\overline{)2x^4-4x^3-42}\quad 2x$$

$$-(2x^4-4x^3+6x^2-12x)$$

$$-6x^2+12x-42$$

Thus, $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}=2x+\dfrac{-6x^2+12x-42}{(x-2)(x^2+3)}$

$$\Rightarrow \frac{2x^4-4x^3-42}{(x-2)(x^2+3)}=2x+\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$$

$$\Rightarrow \frac{-6x^2+12x-42}{(x-2)(x^2+3)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$$

Writing under common denominators gives,

$$\frac{-6x^2+12x-42}{(x-2)(x^2+3)}=\frac{A(x^2+3)+(x-2)(Bx+C)}{(x-2)(x^2+3)}$$

Equating numerators gives,

$$-6x^2+12x-42=A(x^2+3)+(x-2)(Bx+C)$$

Setting $x=2\Rightarrow -42=7A\Leftrightarrow A=-6$

Setting $x=0\Rightarrow -42=3A-2C$ .....(i)

Substituting $A=-6$ into equation (i) gives, $C=12$.

Setting $x=1\Rightarrow -36=4A-B-C$ .....(ii)

Substitute $A=-6$ and $C=12$ into equation (ii) to obtain,

$-36=4(-6)-B-12$

$\Rightarrow -36=-B-36$

$\Rightarrow B=0$

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Thus, $\frac{-6x^2+12x-42}{(x-2)(x^2+3)}=-\frac{6}{x-2}+\frac{12}{x^2+3}$.

Therefore, $\frac{2x^4-4x^3-42}{(x-2)(x^2+3)}=2x-\frac{6}{x-2}+\frac{12}{x^2+3}$.

Example 5.70

Decompose $\frac{2x^3-x+4}{x^3+4x}$ into partial fractions.

Solution

Since the numerator and the denominator have the same degree, divide the numerator by the denominator as follows.

$$\begin{array}{r} 2 \\ x^3+4x\overline{)\ 2x^3+0x^2-x+4} \\ \underline{-(2x^3+0x^2+8x+0)} \\ -9x+4 \end{array}$$

Thus, $\frac{2x^3-x+4}{x^3+4x}=2+\frac{-9x+4}{x(x^2+4)}$.

The fraction $\frac{-9x+4}{x(x^2+4)}$ can be split further as follows:

$$\frac{-9x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

Now write the equation under common denominators as follows.

$$\frac{-9x+4}{x(x^2+4)}=\frac{A(x^2+4)+x(Bx+C)}{x(x^2+4)}$$

It follows that, $-9x+4=A(x^2+4)+x(Bx+C)$.

Setting $x=0$, $4=4A \Leftrightarrow A$ 1

Setting $x=1$, $-5=5+B+C \Leftrightarrow B+C=-10$ .....(i)

Setting $x=-1$, $13=5+B-C \Leftrightarrow B-C=8$.....(ii)

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Solving equations (i) and (ii) simultaneously gives,
$B=-1$ and $C=-9$.

Thus, $\dfrac{-9x+4}{x\left(x^2+4\right)}=\dfrac{1}{x}-\dfrac{x+9}{x^2+4}$.

Therefore, $\dfrac{2x^3-x+4}{x^3+4x}=2+\dfrac{1}{x}-\dfrac{x+9}{x^2+4}$.

Exercise 5.12

Decompose each of the following expressions into partial fractions:

1. $\dfrac{2x}{(x+1)(x+2)}$
2. $\dfrac{10-2x}{(x-3)(x-1)}$
3. $\dfrac{x^2-14x-10}{x^3-4x^2+x+6}$
4. $\dfrac{(x-1)(x-2)}{(2x-1)\left(x^2+3\right)}$
5. $\dfrac{\left(x^2-x+7\right)(x-2)}{x\left(x^2-x+1\right)}$
6. $\dfrac{3x^2+7x+5}{\left(x^2+2\right)(x+1)}$
7. $\dfrac{y^4+y^3-19y^2-44y-21}{(y+3)(y+2)(y+1)}$
8. $\dfrac{2x^2-10x+10}{(x-1)^2(x-2)(x-3)}$
9. $\dfrac{(x-3)^2}{x^3+1}$
10. $\dfrac{4t^2-28}{t^4-t^2-6}$

Summation of series using partial fractions

Some series may be summed using partial fractions. The series $\sum u_r$ is called a telescopic series if u_r can be expressed as a sum of terms which appear in the expression for u_{r+k}, where $k>0$, such that some of the terms of the series cancel on addition, leaving only a few terms which can be easily added. The following two examples explain on how to use partial fractions in finding the summation of the series.

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Example 5.71

Express $\frac{n+3}{(n-1)n(n+1)}$ in partial fractions and deduce that

$$\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}.$$

Hence, find the sum to infinity of this series.

Solution

Let $\frac{n+3}{(n-1)n(n+1)}=\frac{A}{n-1}+\frac{B}{n}+\frac{C}{n+1}$.

$$\Rightarrow \frac{n+3}{(n-1)n(n+1)}=\frac{An(n+1)+B(n-1)(n+1)+Cn(n-1)}{(n-1)n(n+1)}$$

Equating numerators gives;

$n+3=An(n+1)+B(n-1)(n+1)+Cn(n-1)$

Setting $n=1\Rightarrow 4=2A \Leftrightarrow A=2$

Setting $n=0\Rightarrow 3=-B\Leftrightarrow B=-3$

Setting $n=-1\Rightarrow 2=2C\Leftrightarrow C=1$

Thus, $\frac{n+3}{(n-1)n(n+1)}=\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$.

Deducing the sum of the series is easily done by setting the working in columns form:

Setting $n=2$; $\frac{5}{1\times 2\times 3}=2-\frac{3}{2}+\frac{1}{3}$

Setting $n=3$; $\frac{6}{2\times 3\times 4}=1-\frac{1}{1}+\frac{1}{4}$

Setting $n=4$; $\frac{7}{3\times 4\times 5}=\frac{2}{3}-\frac{3}{4}+\frac{1}{5}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Setting $n=n-2$; $\frac{n+1}{(n-3)(n-2)(n-1)}=\frac{2}{n-3}-\frac{3}{n-2}+\frac{1}{n-1}$

Setting $n=n-1$; $\frac{n+2}{(n-2)(n-1)(n)}=\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}$

Setting $n=n$; $\frac{n+3}{(n-1)n(n+1)}=\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$

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On adding, it implies that

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=2-\frac{3}{2}+1+\frac{1}{n}-\frac{3}{n}+\frac{1}{n+1}$$

$$=\frac{3}{2}+\frac{n+1-3(n+1)+n}{n(n+1)}$$

$$=\frac{3}{2}+\frac{-n-2}{n(n+1)}$$

$$=\frac{3}{2}-\frac{n+2}{n(n+1)}$$

Thus, $\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}$.

Note that, as $n\to\infty$, $\frac{n+2}{n(n+1)}\to0$. This means, $\frac{3}{2}-\frac{n+2}{n(n+1)}\to\frac{3}{2}$ as $n\to\infty$.

Thus, the infinite series $\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots$ is convergent and its sum to infinity is $\frac{3}{2}$.

Therefore, $\frac{n+3}{(n-1)n(n+1)}=\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$,

$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}$, and the sum to infinity of the series is $\frac{3}{2}$.

Example 5.72

Use partial fractions to find

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(4n^2-1)(2n+3)}.$$

Hence, determine the sum of this series when n tends to infinity.

Solution

Let $\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{A}{2n-1}+\frac{B}{2n+1}+\frac{C}{2n+3}$.

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$$\Rightarrow \frac{n+1}{(2n-1)(2n+1)(2n+3)} = \frac{A(2n+1)(2n+3)+B(2n-1)(2n+3)+C(2n-1)(2n+1)}{(2n-1)(2n+1)(2n+3)}$$

Equating numerators gives,

$$n+1 = A(2n+1)(2n+3)+B(2n-1)(2n+3)+C(2n-1)(2n+1)$$

Setting $n=\frac{1}{2} \Rightarrow \frac{3}{2} = 8A \Leftrightarrow A = \frac{3}{16}$

Setting $n=-\frac{1}{2} \Rightarrow \frac{1}{2} = -4B \Leftrightarrow B = -\frac{1}{8}$

Setting $n=-\frac{3}{2} \Rightarrow -\frac{1}{2} = 8C \Leftrightarrow C = -\frac{1}{16}$

Thus, $\frac{n+1}{(2n-1)(2n+1)(2n+3)} = \frac{3}{16(2n-1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$.

Setting $n=1$, $\frac{2}{1\times 3\times 5} = \frac{3}{16} - \frac{1}{24} - \frac{1}{80}$

Setting $n=2$, $\frac{3}{3\times 5\times 7} = \frac{3}{48} - \frac{1}{40} - \frac{1}{112}$

Setting $n=3$, $\frac{4}{5\times 7\times 9} = \frac{3}{80} - \frac{1}{56} - \frac{1}{144}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

Setting $n = n-2$, $\frac{n-1}{(2n-5)(2n-3)(2n-1)} = \frac{3}{16(2n-5)} - \frac{1}{8(2n-3)} - \frac{1}{16(2n-1)}$

Setting $n = n-1$, $\frac{n}{(2n-3)(2n-1)(2n+1)} = \frac{3}{16(2n-3)} - \frac{1}{8(2n-1)} - \frac{1}{16(2n+1)}$

Setting $n = n$, $\frac{n+1}{(2n-1)(2n+1)(2n+3)} = \frac{3}{16(2n-1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$

On adding, $\frac{2}{1\times 3\times 5} + \frac{3}{3\times 5\times 7} + \ldots = \frac{3}{16} - \frac{1}{24} + \frac{3}{48} - \frac{1}{16(2n+1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$

$$= \frac{5}{24} - \frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right)$$

Thus, $\frac{2}{1\times 3\times 5} + \frac{3}{3\times 5\times 7} + \frac{4}{5\times 7\times 9} + \cdots + \frac{n+1}{(4n^2-1)(2n+3)} = \frac{5}{24} - \frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right)$.

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Note that, as $n \to \infty$, $\frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right) \to 0$.

Thus, the infinity series $\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots$ is convergent, and its sum to infinity is $\frac{5}{24}$.

Therefore,

$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(4n^2-1)(2n+3)}=\frac{5}{24}-\frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right)$ and the sum to infinity of the series is $\frac{5}{24}$.

Exercise 5.13

1. Express $\frac{2}{n(n+1)(n+2)}$ in partial fractions and deduce that $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$. Hence, find the sum to infinity.
2. Use partial fractions to show that $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.
3. Find the sum of the series $\sum_{k=1}^{n}\frac{3k+2}{k(k+1)(k+2)}$, then evaluate the sum to infinity.
4. Decompose $\frac{2}{4n^2-1}$ into partial fractions, hence find $\sum_{k=1}^{n}\frac{2}{4k^2-1}$.
5. Evaluate $\sum_{k=1}^{n}\frac{4k}{(2k-1)(2k+1)(2k+3)}$ by using partial fractions.
6. Express $\frac{2}{n(n+2)}$ in partial fractions and deduce that $\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\cdots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$.
7. Express $\frac{1}{(2k-1)(2k+3)}$ into partial fractions. By multiplying by $\frac{1}{2r+1}$, or otherwise prove that $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)}=\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$.

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8. Use partial fractions to show that

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

9. Find the sum of the series $\frac{1}{2\times4}+\frac{1}{3\times5}+\frac{1}{4\times6}+\cdots+\frac{1}{(n+1)(n+3)}$.

10. Find the sum of the first n terms in each of the following series:

(a) $\frac{1}{3\times4\times5}+\frac{2}{4\times5\times6}+\frac{3}{5\times6\times7}+\cdots$ (b) $\frac{1}{1\times4}+\frac{1}{2\times5}+\frac{1}{3\times6}+\cdots$

(c) $\frac{1}{3\times6}+\frac{1}{6\times9}+\frac{1}{9\times12}+\cdots$

Chapter summary

1. An expression that links together the terms of a sequence with signs of addition or subtraction is called a series.
2. The principal of mathematical induction states that, if n is a natural number and the statement P_n is true for $n = 1$ (or any fixed natural number) and also true for $n = k + 1$, then the statement P_n is true for all values of n.
3. A polynomial function $p(x)$ has the form
$p(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0$
4. If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then
$(x-\alpha)(x-\beta)=0$
$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$
$\Rightarrow \alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$
5. If α, β, and γ are roots of a cubic equation $ax^3+bx^2+cx+d=0$, then
$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0$
$\Rightarrow \alpha+\beta+\gamma=-\frac{b}{a}$, $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$, and $\alpha\beta\gamma=-\frac{d}{a}$.
6. The remainder theorem states that, if $p(x)$ is a dividend, $q(x)$ is a quotient, $x-a$ is a divisor, and $r(x)$ is a remainder, then $p(x)=q(x)\times(x-a)+r(x)$. Setting $x = a$, gives $p(a) = r(a)$. This theorem is used to calculate the

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remainder when any polynomial is divided by a linear polynomial without carrying out the steps of the division process.

7. The formulae for the sum of the first n, n^2, and n^3 natural numbers are:

   (a) $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

   (b) $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

   (c) $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$

   (d) $\sum_{k=1}^{n} 1 = n$

8. If A is a 3×3 matrix, then:

   (a) $\text{Adj}(A) = [\text{cof}(A)]^T$

   (b) $A^{-1} = \frac{\text{Adj}(A)}{|A|}$ if $|A| \neq 0$.

9. If A is a 3×3 matrix and $|A| \neq 0$, then solving a system of equations by Cramer's rule is given by, $x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$, $z = \frac{|A_z|}{|A|}$ and by inverse matrix is given by, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

10. The formula for the general term of a binomial expansion $(a+b)^n$ is $T_{r+1} = {}^nC_r a^{n-r}b^r$.

11. The binomial theorem for fractional and negative powers is given by $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$, where $|x| < 1$.

12. The absolute value of a real number x is written as $|x|$, and it is defined by;

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

13. The following definitions apply for absolute value inequality expressions:

   (a) If $|p| < c$, then $-c < p < c$, where, $c \geq 0$.

   (b) If $|p| \leq c$, then $-c \leq p \leq c$.

   (c) If $|p| > c$, then $p > c$ or $p < -c$.

   (d) If $|p| \geq c$, then $p \geq c$ or $p \leq -c$,

   where p is any algebraic expression.

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Revision exercise 5

1. The roots of the equation $ax^2+bx+c=0$ are α and β. The roots of the equation $a^2x^2+b^2x+c^2=0$ are 2α and 2β. Show that the equation whose roots are $n\alpha$ and $n\beta$ is $x^2+2nx+4n^2=0$.

2. If one root of the equation $mx^2+nx+t=0$ is twice the other, show that $\frac{2n^2}{9t}=m$.

3. The roots of the equation $x^3+3x^2+5x+7=0$ are α, β, and γ. Find the equation whose roots are:
(a) 3α, 3β, and 3γ (b) α^2, β^2, and γ^2 (c) $\alpha+3$, $\beta+3$, and $\gamma+3$.

4. Use the remainder theorem to find the three factors of x^4+3x^2-4 and hence, decompose $\frac{2x^3-x^2-7x-14}{x^4+3x^2-4}$ into partial fractions.

5. The equations $x^2+kx+3=0$ and $x^2+x-1=0$ have a common root. If $k>0$, find the actual value of k.

6. Decompose $\frac{7+x}{1+x+x^2+x^3}$ into partial fractions. Assuming that $-1<x<1$, obtain an expression for $\frac{7+x}{1+x+x^2+x^3}$, give your answer in the form $a+bx+cx^2+dx^3+\cdots$ Hence, find the value of the coefficients as far as the term in x^5 inclusive.

7. Find n in the expansion $(a+y)^n$ if the 4^{th} and 13^{th} terms are equal.

8. Express $\frac{2x^2-3x+2}{(x-2)^8}$ in partial fractions.

9. Use the principal of mathematical induction to prove that the sum of cubes of any three consecutive natural numbers is divisible by 9.

10. Use mathematical induction to prove that x^n-y^n is divisible by $x-y$ for all positive integers.

11. Solve the inequality $\frac{2-x-3x^2}{8-2x-x^2}\geq 0$.

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12. Given that $y(x-1) = x^2 + 3$, where x is real show that y cannot take any value between -2 and 6.

13. Express the following as partial fractions.

(a) $\frac{1}{x^4-9}$ (b) $\frac{x^2+2x+18}{x(x^2+3)^2}$

14. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 10 & 5 & 1 \end{bmatrix}$.

15. Solve the following system of equations by using the matrix method:

$$\begin{cases} x+2y+z=10 \\ 2x+y+z=20 \\ x+3y+z=30 \end{cases}$$

16. Solve the following system of equations by using the matrix method:

$$\begin{cases} x+y+z=3 \\ 5x+4y+3z=11 \\ 10x+5y+z=11\frac{1}{2} \end{cases}$$

17. Write down the first four terms of the series expansion of each of the following and state the values of x where the expansion is valid:

(a) $(1+x)^{-3}$ (b) $(1-x^2)^{\frac{2}{3}}$

18. Find the 5^{th} term in the expansion of $(1-5x)^{\frac{7}{2}}$.

19. If $|x|<1$, show that $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots$

20. Express $\frac{7x+3}{(3x-1)(x+1)^2}$ in partial fractions and then use the binomial theorem to find the coefficient of x^n in ascending powers of x.

21. Use the binomial expansion to evaluate each of the following:

(a) $\sqrt{1.01}$ (b) $\sqrt[3]{0.98}$

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22. In the binomial expansion of $(1+x)^n$, where n is a positive integer, the coefficient of x^4 is $\frac{3}{2}$ times the sum of the coefficients of x^2 and x^3. Find the value of n and determine these three coefficients.

23. Prove that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$

24. Expand each of the following:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$ (b) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

25. Use the binomial theorem to expand $\sqrt{4+x}$ in ascending power of x up to four terms. Give the limits of x for which the expansion is valid.

26. Show that $3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = (n+1)^3 - 1$.

27. Prove each of the following statements by mathematical induction:

(a) $2+6+12+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

(b) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots+\frac{1}{n(n+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}$

28. Prove that $\begin{vmatrix} x^2 & 2xy & y^2 \\ y^2 & x^2 & 2xy \\ 2xy & y^2 & x^2 \end{vmatrix} = (x^3+y^3)^2$.

29. Write down the first four terms in the expansion of $(1-x)^{-2}$ in ascending powers of x, hence deduce that $\sum_{n=0}^{\infty}(a+bn)x^n = \frac{a+(b-a)x}{(1-x)^2}$.

30. Use Cramer's rule to solve the following system of equations:
$2x-5y+2z=14$, $9x+3y-4z=13$, $7x+3y-2z=3$.

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31. Evaluate each of the following:

(a) $\begin{pmatrix}2 & 3 & 1\end{pmatrix}\times\begin{pmatrix}3 & 2\\ 4 & 7\\ 1 & 6\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 2\\ 1 & -3 & -1\end{pmatrix}\times\begin{pmatrix}-1 & -2 & 1\\ -1 & 2 & 3\\ -1 & -2 & 2\end{pmatrix}$

32. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, verify that $A^2-4A-5I=0$.

33. Write the first three terms in the expression of $\frac{2+x}{(3-2x)^2}$ in ascending powers of x. Also, state the condition under which the expression is valid.

34. Use partial fractions to show that

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{5n^2+7n}{6(4n^2+8n+3)}$$

35. Prove by mathematical induction that:

$1\times3+3\times5+5\times7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$.

36. Find the values of k for which the equation $f(x)=0$, if $f(x)=3x^2-kx+3$ has:

(a) Repeated roots (b) Distinct real roots

37. Solve each of the following inequalities:

(a) $|3x+2|>|2x-3|$ (b) $\frac{(x+2)(x-5)}{(x-3)(x-2)}>1$

38. Find $\sum_{k=1}^{n}3(k^2+3k+2)$, hence evaluate $\sum_{k=15}^{65}3(k^2+3k+2)$.

39. A botanist requires 12, 6, and 7 units of copper, zinc, and calcium, respectively for his secondary school. A concentrated product contains 3, 1, and 1 units of copper, zinc, and calcium per carton, respectively. A diluted product contains 1, 0, and 2 units of copper, zinc, and calcium per carton, respectively. A dry product contains 1, 1, and 1 units of copper, zinc, and calcium per carton, respectively.

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(a) Formulate this information as a system of equations in matrix form.
(b) By using the inverse matrix method, find how many of each should be purchased in order to meet the requirements.

40. Prove that the sum of n terms of the series $1\times 3^2+4\times 4^2+7\times 5^2+10\times 6^2+\cdots$ is $\frac{1}{12}n(n+1)(9n^2+49n+44)-8n$.

41. (a) Given $f(x)=2x^4-3x^3+2x^2+8x-5$, $g(x)=2x^3-8x^2+15x+9$, and $h(x)=10x^4-16x^3+19x^2-8x+16$. Find
(i) $f(x)+2g(x)-3h(x)$ (ii) $h(x)-f(x)+3g(x)$

(b) Given $f(x)=mx^5-nx^4+px^3+qx^2+rx+t$,
$g(x)=8x^5-4x^4-7x^3+6x^2+10x-11$, and
$h(x)=25x^5-19x^4+12x^3-10x^2+16x-15$. If $f(x)+g(x)=h(x)$, find the values of m, n, p, q, r, and t.

42. If $g(x)=10x^2-4x$ and $h(x)=x^3-8x+2$, find
(a) $g(x)\times h(x)$ (b) $h(x)+g(x)$
(c) $h(x)-g(x)$ (d) $[g(x)]^2+2[h(x)]^2$

43. Given that $f(x)=x^6-4x^4+2x^2-8x+3$ and $g(x)=x-4$.
(a) Use long division method to find the quotient and remainder when $f(x)$ is divided by $g(x)$.
(b) Use the synthetic division method to find the quotient and remainder when $f(x)$ is divided by $g(x)$.

44. (a) Find the truth values of $f(x)\geq 0$ if $f(x)=x^2+2x-8$.
(b) Solve for y, if $|y-2|<1$.
(c) Given that $h(y)=\frac{(y+1)(y+5)}{(y+2)(2y+3)}$ find the solution of $h(y)$ when $h(y)>3$.

45. (a) A given function of x can be expressed in the form $p+q(x-3)+r(x-3)(x-4)+s(x-3)(x-4)(x-5)$ where p, q, r, and s are constants. When the function is divided inturn by $x-3$, $x-4$, $x-5$, and $x-6$ the remainders are 12, 21, 40, and 169. Determine the values of p, q, r, and s.
(b) Determine whether or not $x-1$ is a factor of $P(x)=x^4-2x^3+x^2-4x+4$.

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46. Given matrix $N = \begin{bmatrix} 9 & -12 & 6 \\ -2 & 15 & -3 \\ 7 & 8 & 5 \end{bmatrix}$. Find;

(a) The minors of N
(b) The cofactors of N
(c) The adjoint of N
(d) The determinant of N
(e) The inverse of N

47. A direct current circuit comprises three closed loops. Applying Kirchoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$\begin{cases} 4I_1 - 6I_2 + 3I_3 = 14 \\ 2I_1 + 5I_2 - 4I_3 = 6 \\ -3I_1 - 2I_2 + 10I_3 = 37 \end{cases}$$

Use Cramer's rule to find the values of I_1, I_2, and I_3.

48. The height h of an object thrown into the air is known to be given by a quadratic function of time (t) of the form, $h = mt^2 + nt + p$. If the object is at $h = 15$ metres when $t = \frac{1}{2}$ seconds, $h = 16$ metres when $t = 1$ second, and $h = 12$ metres when $t = 2$ seconds, then use the inverse matrix method to find the values of $m, n,$ and p.

49. (a) Show that the 13^{th} term in the expansion of $\left(9t - \frac{\sqrt{t}}{3t}\right)^{18}$ is 18,564.

(b) Determine the middle terms in the expansion of $\left(2 - \frac{x^4}{3}\right)^9$.

50. Given $f(n) = \frac{1}{(2n+1)(2n+3)}$.

(a) Resolve $f(n)$ into partial fractions.

(b) Evaluate $\sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)}$, using the result obtained in part (a).

(c) Evaluate $\sum_{k=1}^{100} \frac{1}{(2k+1)(2k+3)}$

(d) Deduce the sum to infinity in part (b).

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Chapter Six

Introduction

The word trigonometry comes from the Greek words "trigonon" which means triangle and "metron" which means to measure. Trigonometry deals with determination of measures of lengths of sides and angles of triangles by means of relevant trigonometric functions. The trigonometric functions are sine, cosine, tangent, secant, cosecant, and cotangent of an angle. In this chapter you will learn about trigonometric ratios, trigonometric identities, compound angle formulae, double angle formulae, trigonometric equations, factor formulae, radians and small angles, trigonometric functions, and inverse of trigonometric functions. The competencies developed can be applied in various real life situations such as in solving problems related to astronomy, navigation, architecture, oceanography, and in creation of maps. Generally, trigonometry has great practical importance to builders, architects, surveyors, engineers, and users in many other fields.

Think

Communication without satellites in space.

Trigonometric ratios

Trigonometric ratios are values of trigonometric functions determined from the ratio of sides of a right-angled triangle.

Activity 6.1: Deducing the formulae of trigonometric ratios

Individually or in a group, perform the following tasks:

1. Construct a right-angled triangle in the xy-plane.
2. Deduce the formulae for trigonometric ratios of sine, cosine, and tangent of an acute angle α.
3. Use the derived formulae in task 2 to deduce the formulae for secant, cosecant, and cotangent of an angle α.
4. Give a suggestion of the formulae deduced in tasks 2 and 3.
5. What have you observed in task 4? Give comments.
6. Share your findings with your fellow students for more inputs.

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The ratios of lengths of sides of a right-angled triangle with respect to any of its acute angles are known as the trigonometric ratios of that particular angle. The three sides of the right-angled triangle are hypotenuse (the longest side), opposite (side across from the angle), and adjacent (base of the angle). Consider the right-angled triangle OAB in Figure 6.1.

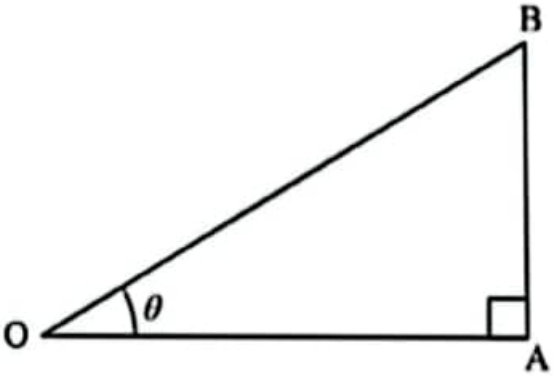


Figure 6.1: *A right-angled triangle OAB*

In Figure 6.1, $\overline{OB}$ is the hypotenuse of the right-angled triangle, $\overline{OA}$ is the adjacent side to angle θ, and $\overline{AB}$ is the opposite side to angle θ. The three basic trigonometric ratios with their abbreviations are defined as follows:

$$\sin\theta = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}} = \frac{\overline{AB}}{\overline{OB}}$$

$$\cos\theta = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}} = \frac{\overline{OA}}{\overline{OB}}$$

$$\tan\theta = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}} = \frac{\overline{AB}}{\overline{OA}}$$

Now, using Pythagoras' theorem, the following relation is obtained;

$$\left(\overline{OA}\right)^2 + \left(\overline{AB}\right)^2 = \left(\overline{OB}\right)^2.$$

The reciprocal of the trigonometric functions of sine, cosine, and tangent are cosecant, secant, and cotangent, respectively. These are defined from Figure 6.1 as follows:

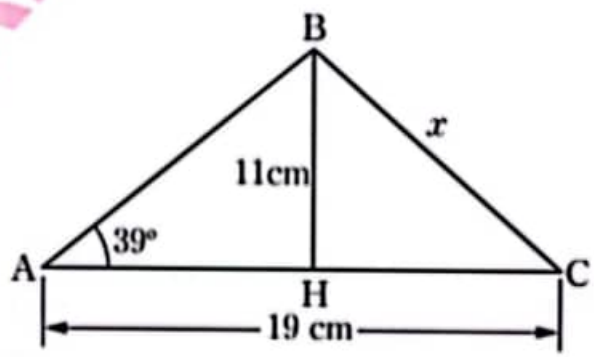
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\overline{OB}}{\overline{AB}},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\overline{OB}}{\overline{OA}}, \text{ and}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\overline{OA}}{\overline{AB}}.$$

Example 6.1

In the following figure, if $\overline{BH}$ is perpendicular to $\overline{AC}$, find the value of x correct to three significant figures.



Solution

From the figure, $\tan 39° = \frac{\overline{BH}}{\overline{AH}}$.

Thus, $\tan 39° = \frac{11}{\overline{AH}} \Rightarrow \overline{AH} = \frac{11}{\tan 39°}$

$\Rightarrow \overline{HC} = 19 - \overline{AH}$

$= 19 - \frac{11}{\tan 39°}$

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Using the Pythagoras' theorem for the right-angled triangle HBC, it gives;

$$(\overline{BH})^2 + (\overline{HC})^2 = (\overline{BC})^2$$

$$\Rightarrow 11^2 + (\overline{HC})^2 = x^2$$

Substitute $\overline{HC}$ to find the value of x, so that;

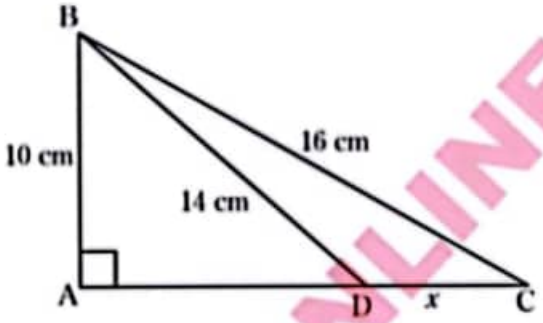
$$x = \sqrt{11^2 + \left(19 - \frac{11}{\tan 39°}\right)^2}$$

$= 12.3.$

Therefore, $x = 12.3$ cm.

Example 6.2

In the following figure, A is a right angle. Find the value of x correct to two decimal places.



Solution

Since A is a right angle, both triangles ABC and ABD in the figure are right angled-triangles.

Using the Pythagoras' theorem the value of x can be found as follows.

From a triangle BAD;

$$14^2 = 10^2 + (\overline{AD})^2 \Rightarrow \overline{AD} = \sqrt{14^2 - 10^2}$$

Also, from a triangle ABC;

$$16^2 = 10^2 + (\overline{AC})^2 \Rightarrow \overline{AC} = \sqrt{16^2 - 10^2}$$

Thus, $x = \overline{AC} - \overline{AD}$

$$= \sqrt{16^2 - 10^2} - \sqrt{14^2 - 10^2}$$

$$= \sqrt{156} - \sqrt{96}$$

$$= 2.69$$

Therefore, $x = 2.69$ cm.

Example 6.3

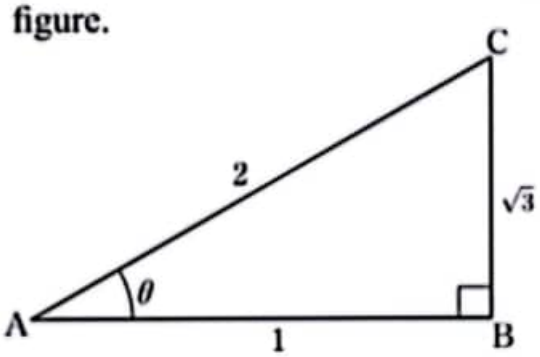
If $\tan\theta = \sqrt{3}$, where θ is an acute angle, find the value in each of the following, leaving your answer in surd form.

(a) $\sec\theta$ (b) $\cot\theta$ (c) $\text{cosec}\,\theta$

Solution

Given $\tan\theta = \sqrt{3}$. It implies that $\tan\theta = \frac{\sqrt{3}}{1}$.

The lengths of the sides of the corresponding right-angled triangle ABC are as shown in following figure.



Using the Pythagoras' theorem,

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$$

$$\Rightarrow 1^2 + \sqrt{3}^2 = \overline{AC}^2$$

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$\Rightarrow 4 = \overline{AC}^2$

$\Rightarrow \overline{AC} = 2$ units.

Thus, it follows that;

(a) $\sec\theta = \dfrac{1}{\cos\theta}$

But $\cos\theta = \dfrac{1}{2} \Rightarrow \sec\theta = 2$.

Therefore, $\sec\theta = 2$.

(b) $\cot\theta = \dfrac{1}{\tan\theta}$

But $\tan\theta = \sqrt{3}$.

$\Rightarrow \cot\theta = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Therefore, $\cot\theta = \dfrac{\sqrt{3}}{3}$.

(c) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

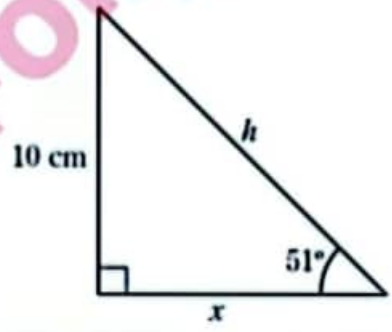
But $\sin\theta = \dfrac{\sqrt{3}}{2}$.

$\Rightarrow \text{cosec}\,\theta = \dfrac{2\sqrt{3}}{3}$.

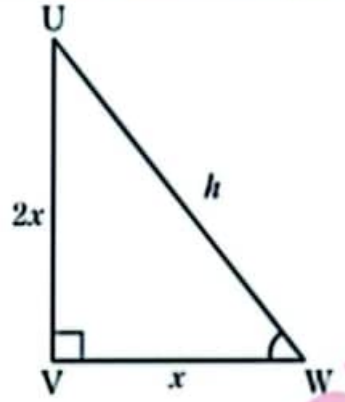
Therefore, $\text{cosec}\,\theta = \dfrac{2\sqrt{3}}{3}$.

Exercise 6.1

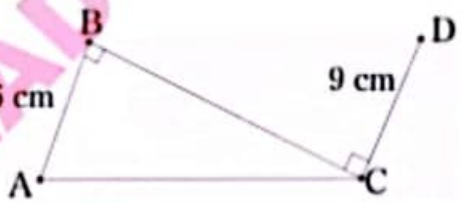
1. Find the values of x and h in the following figure.



2. Find the lengths of all sides of the following right-angled triangle with an area of 400 square units.



3. In the following figure, $\overline{AB}$ and $\overline{CD}$ are both perpendicular to $\overline{BC}$ and the size of angle ACB is 31°. Find the length of the line segment $\overline{BD}$, correct to 3 significant figures.



4. The area of a right-angled triangle JKL is 50 square centimetres. One of its angles is 45°. Find the lengths of the sides of the triangle.

5. In a right-angled triangle DEF, $\tan D = \dfrac{119}{120}$. Find the values of cosec D and sec D.

6. If $\sec B = \dfrac{13}{5}$, where B is an acute angle, find the value in each of the following;
   (a) cosec B (b) tan B

7. In a right-angled triangle LMN with angle L equal to 90°,

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find angles M and N so that $\sin M = \cos M$.

8. A rectangle has dimensions 10 cm by 5 cm. Determine the measures of the angles at the point where the diagonals intersect and find the length of the diagonal.
9. The lengths of sides PQ and QR of a scalene triangle PQR are 12 cm and 8 cm, respectively. If the size of angle R is 59°, find the length of side PR.
10. Maria is riding vertically in a hot air balloon directly over a point P on the ground. Maria spots a parked car on the ground at an angle of depression of 30°, then the balloon rises 50 metres. If the angle of depression to the car is 35°. How far is the car from point P?
11. If the shadow of a building increases by 10 metres when the angle of elevation of the sun rays decreases from 70° to 60°, what is the height of the building?

Trigonometric identities

Consider the right-angled triangle shown in Figure 6.2.

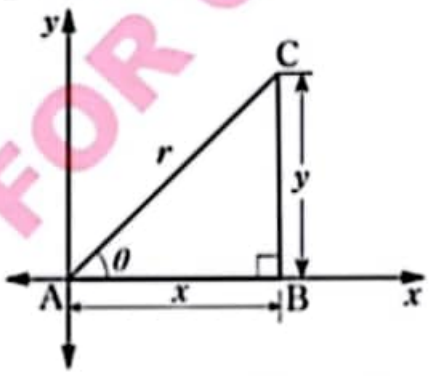

Figure 6.2: *A right-angled triangle in the xy-plane*

Using the Pythagoras' theorem, it follows that

$$\left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2 = \left(\overline{AC}\right)^2$$

Since $\overline{AC} = r$, $\overline{AB} = x$, and $\overline{BC} = y$, then

$$x^2 + y^2 = r^2$$

Divide throughout by r^2.

$$\Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

But $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$.

Substituting the values for $\cos\theta$ and $\sin\theta$ into $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ gives,

$$\cos^2\theta + \sin^2\theta = 1 \qquad (6.1)$$

Two similar identities can be deduced from equation (6.1) as follows.

Divide by $\sin^2\theta$ both sides of equation (6.1) as follows.

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

But $\frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$.

$$\Rightarrow \cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad (6.2)$$

Again, divide by $\cos^2\theta$ both sides of equation (6.1) as follows.

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

But $\frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$.

$$\Rightarrow 1 + \tan^2\theta = \sec^2\theta \qquad (6.3)$$

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Equations (6.1), (6.2), and (6.3) are also known as Pythagorean identities since they express the Pythagorean theorem in terms of trigonometric functions.

Example 6.4

Simplify $\sin\theta + \cos\theta\cot\theta$.

Solution

$$\sin\theta + \cos\theta\cot\theta = \sin\theta + \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}, \text{ but } \sin^2\theta + \cos^2\theta = 1$$

$$= \frac{1}{\sin\theta}$$

$$= \text{cosec}\theta$$

Therefore, $\sin\theta + \cos\theta\cot\theta = \text{cosec}\theta$.

Example 6.5

If $x = \sin\theta$, then show that $\dfrac{x}{\sqrt{1-x^2}} = \tan\theta$.

Solution

Given $x = \sin\theta$.

Thus, $\dfrac{x}{\sqrt{1-x^2}} = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

$= \dfrac{\sin\theta}{\sqrt{\cos^2\theta}}$, since $1-\sin^2\theta = \cos^2\theta$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$

Therefore, if $x = \sin\theta$, then $\dfrac{x}{\sqrt{1-x^2}} = \tan\theta$.

Example 6.6

Solve the equation $2\sin^2\theta - 1 = \cos\theta$, for values of θ between $0°$ and $360°$.

Solution

Given $2\sin^2\theta - 1 = \cos\theta$.

$\Rightarrow 2\sin^2\theta - \cos\theta - 1 = 0$

But $\sin^2\theta = 1 - \cos^2\theta$.

$\Rightarrow 2(1-\cos^2\theta) - \cos\theta - 1 = 0$

$\Rightarrow 2 - 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow -2\cos^2\theta - \cos\theta + 1 = 0$

Factorizing the resulted quadratic equation gives,

$(-2\cos\theta + 1)(\cos\theta + 1) = 0$

Hence, $\cos\theta = \dfrac{1}{2}$ or $\cos\theta = -1$.

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If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$.

If $\cos\theta = -1$, $\theta = 180°$.

Therefore, the values of θ between $0°$ and $360°$ are $60°, 180°$, and $300°$.

Example 6.7

Solve the equation
$4\cos\theta - 3\sec\theta = 2\tan\theta$, for
$-180° \le \theta \le 180°$.

Solution

Given $4\cos\theta - 3\sec\theta = 2\tan\theta$.

$\Rightarrow 2\tan\theta - 4\cos\theta + 3\sec\theta = 0$

$\Rightarrow 2\left(\frac{\sin\theta}{\cos\theta}\right) - 4\cos\theta + \frac{3}{\cos\theta} = 0$

Multiplying both sides by $\cos\theta$ gives,

$2\sin\theta - 4\cos^2\theta + 3 = 0$

But $\cos^2\theta = 1 - \sin^2\theta$.

$\Rightarrow 2\sin\theta - 4(1 - \sin^2\theta) + 3 = 0$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0.$

The general formula of solving a quadratic equation in $\sin\theta$ is

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 4$, $b = 2$, and $c = -1$.

Hence,

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$

So that, $\sin\theta = 0.3090$ or $\sin\theta = -0.8090$.

If $\sin\theta = 0.3090$, then

$\theta = \sin^{-1}(0.3090) = 18°$ or $162°$.

Also, if $\sin\theta = -0.8090$, then

$\theta = \sin^{-1}(-0.8090) = -54°$ or $-126°$.

Therefore, the values of θ are

$-126°, -54°, 18°$, and $162°$.

Example 6.8

Prove the identity
$\operatorname{cosec}^4 x - \operatorname{cosec}^2 x = \cot^4 x + \cot^2 x$.

Solution

Given $\operatorname{cosec}^4 x - \operatorname{cosec}^2 x = \cot^4 x + \cot^2 x$.

Proving from the right-hand side,

$\cot^4 x + \cot^2 x = \cot^2 x(\cot^2 x + 1)$

But $\cot x^2 = \operatorname{cosec}^2 x - 1$. That is,

$\cot^4 x + \cot^2 x = (\operatorname{cosec}^2 x - 1)(\operatorname{cosec}^2 x - 1 + 1)$

$= (\operatorname{cosec}^2 x - 1)(\operatorname{cosec}^2 x)$

$= \operatorname{cosec}^4 x - \operatorname{cosec}^2 x$

Therefore,

$\cot^4 x + \cot^2 x = \operatorname{cosec}^4 x - \operatorname{cosec}^2 x$.

Example 6.9

Verify the identity
$\tan x + \cot x = \sec x \operatorname{cosec} x$.

Solution

Given $\tan x + \cot x = \sec x \operatorname{cosec} x$.

From the left hand-side, it follows that

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$\Rightarrow \tan x + \cot x = \frac{1}{\cos x} \times \frac{1}{\sin x}$$

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$= \sec x \operatorname{cosec} x$

Therefore,

$\tan x + \cot x = \sec x \operatorname{cosec} x.$

Example 6.10

Eliminate θ from the following parametric equations.

$x = 2 + 4\cos\theta$

$y = 3 + 5\sin\theta$

Solution

Given

$x = 2 + 4\cos\theta$ and $y = 3 + 5\sin\theta.$

Write the given equations in terms of $\cos\theta$ and $\sin\theta$. That is,

$\cos\theta = \frac{x-2}{4}$ and $\sin\theta = \frac{y-3}{5}$

Square on both sides of each equation to obtain:

$\cos^2\theta = \left(\frac{x-2}{4}\right)^2$ and

$\sin^2\theta = \left(\frac{y-3}{5}\right)^2.$

Since $\sin^2\theta + \cos^2\theta = 1$, then

$\left(\frac{y-3}{5}\right)^2 + \left(\frac{x-2}{4}\right)^2 = 1.$

Therefore, $\left(\frac{y-3}{5}\right)^2 + \left(\frac{x-2}{4}\right)^2 = 1.$

Exercise 6.2

1. Simplify each of the following expressions:
   (a) $\sqrt{(1-\sin y)(1+\sin y)}$
   (b) $\sec\theta - \sec\theta\sin^2\theta$
   (c) $\frac{1}{\sec^2 x} + \frac{1}{\operatorname{cosec}^2 x}$
   (d) $\tan\theta + \frac{\cos\theta}{1+\sin\theta}$
2. Verify each of the following trigonometric identities:
   (a) $\frac{\sec x}{\cot x + \tan x} = \sin x$
   (b) $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
   (c) $\sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}} = \frac{1}{\sec x + \tan x}$
3. Solve the equation $32\cos^2\theta + 4\sin\theta - 31 = 0$, for $-180° \le \theta \le 180°$.
4. Eliminate θ in each of the following pair of equations:
   (a) $x = \sin\theta,\ y = \cos\theta$
   (b) $x = 2 + \sin\theta,\ \cos\theta = 1 + y$
   (c) $x = 3\sin\theta,\ y = \operatorname{cosec}\theta$
5. Solve each of the following equations for all values of θ between $0°$ and $360°$:
   (a) $3\cot\theta = \tan\theta$
   (b) $2\tan\theta = 5\operatorname{cosec}\theta + \cot\theta.$
   (c) $\operatorname{cosec}^2\theta = 3\cot\theta - 1.$
6. Prove each of the following identities:
   (a) $\cos^4 x - \sin^4 x + 1 = 2\cos^2 x$
   (b) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$
   (c) $\frac{\tan x - \sin x}{\sin^3 x} = \frac{\sec x}{1+\cos x}$

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(d) $\dfrac{\tan y+\sec y-1}{\tan y-\sec y+1}=\tan y+\sec y$

(e) $\sec A(1-\sin A)(\sec A+\tan A)=1$.

7. If $\cot^2\theta+3\text{cosec}^2\theta=7$, show that $\tan\theta=\pm1$.

8. Show that $\sqrt{\dfrac{1-\cos x}{1+\cos x}}=\text{cosec}\,x-\cot x$.

9. (a) If $m=4\sec 2\theta$ and $n=8\tan 2\theta$, show that $4m^2=64+n^2$.

(b) If $p=2\sin x$ and $3q=\cos x$, show that $p^2+36q^2=4$.

10. If $x=a(1-\text{cosec}\,\theta)$ and $y=a(\sec\theta+\tan\theta)$, prove that $xy^2+a^2(2a-x)=0$.

Compound angle formulae

The compound angle formula can be used to find the sine, cosine, and tangent of the sum and difference of angles. The basic operations on sums and differences of trigonometric functions can be computed using the concept of compound angles.

Consider the triangles in Figure 6.3.

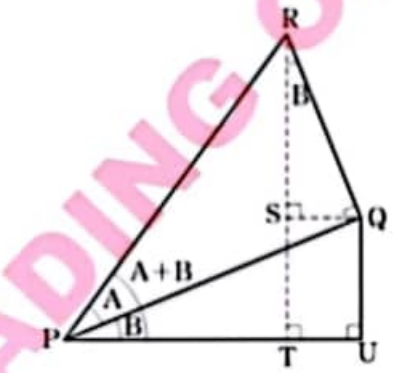

Figure 6.3: *Illustrative sketch for deriving compound angle formulae*

From Figure 6.3, angles A and B are acute such that $A+B<90°$.
Consider the right-angled triangle PRT. It follows that

$$\sin(A+B)=\frac{\overline{RT}}{\overline{PR}}=\frac{\overline{RS}+\overline{ST}}{\overline{PR}}=\frac{\overline{RS}+\overline{QU}}{\overline{PR}}.$$

but $\overline{ST}=\overline{QU}$. Thus,

$$\sin(A+B)=\frac{\overline{RS}}{\overline{PR}}+\frac{\overline{QU}}{\overline{PR}}$$

$$=\frac{\overline{QR}}{\overline{QR}}\times\frac{\overline{RS}}{\overline{PR}}+\frac{\overline{PQ}}{\overline{PQ}}\times\frac{\overline{QU}}{\overline{PR}}$$

$$=\frac{\overline{QR}}{\overline{PR}}\times\frac{\overline{RS}}{\overline{QR}}+\frac{\overline{PQ}}{\overline{PR}}\times\frac{\overline{QU}}{\overline{PQ}}$$

But $\dfrac{\overline{QR}}{\overline{PR}}=\sin A$, $\dfrac{\overline{RS}}{\overline{QR}}=\cos B$, $\dfrac{\overline{PQ}}{\overline{PR}}=\cos A$, and $\dfrac{\overline{QU}}{\overline{PQ}}=\sin B$.

Thus, $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

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Therefore, the compound angle formula for finding the sine of the sum of two angles is given by;

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad (6.4)$$

Suppose B is replaced by –B. Substituting it into equation (6.4), gives,

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B).$$

Note that, $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, and $\tan(-x) = -\tan x$, because $\cos x$ is an even function while $\sin x$ and $\tan x$ are odd functions.

Thus, $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Therefore, the compound angle formula for finding the sine of the difference of two angles is given by;

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

From Figure 6.3, considering the right-angled triangle PRT.

$$\cos(A+B) = \frac{\overline{PT}}{\overline{PR}} = \frac{\overline{PU} - \overline{TU}}{\overline{PR}}, \text{ but } \overline{PU} = \overline{PT} + \overline{TU}$$

$$\cos(A+B) = \frac{\overline{PU} - \overline{SQ}}{\overline{PR}} = \frac{\overline{PU}}{\overline{PR}} - \frac{\overline{SQ}}{\overline{PR}}, \text{ but } \overline{TU} = \overline{SQ}$$

$$= \frac{\overline{PQ}}{\overline{PQ}} \times \frac{\overline{PU}}{\overline{PR}} - \frac{\overline{QR}}{\overline{QR}} \times \frac{\overline{SQ}}{\overline{PR}}$$

$$= \frac{\overline{PQ}}{\overline{PR}} \times \frac{\overline{PU}}{\overline{PQ}} - \frac{\overline{QR}}{\overline{PR}} \times \frac{\overline{SQ}}{\overline{QR}}$$

But $\frac{\overline{PQ}}{\overline{PR}} = \cos A$, $\frac{\overline{PU}}{\overline{PQ}} = \cos B$, $\frac{\overline{QR}}{\overline{PR}} = \sin A$, and $\frac{\overline{SQ}}{\overline{QR}} = \sin B$.

Therefore, the compound angle formula for finding the cosine of the sum of two angles is given by;

$$\cos(A+B) = \cos A \cos B - \sin A \sin B. \qquad (6.5)$$

If B is replaced by –B, then equation (6.5), becomes,

$$\cos(A + (-B)) = \cos A \cos(-B) - \sin(-B).$$

Thus, $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

Therefore, the compound angle formula for finding the cosine of the difference of two angles is given by; $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

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Derivation of the compound angle formula for $\tan(A+B)$ can be done as follows.

From the trigonometric identities,

$\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$. Substituting equations (6.4) and (6.5) for $\sin(A+B)$ and $\cos(A+B)$ gives,

$$\tan(A+B)=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

Divide by $\cos A\cos B$ both the numerator and denominator to get;

$$\tan(A+B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{1-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\left(\dfrac{\sin A}{\cos A}\right)\left(\dfrac{\sin B}{\cos B}\right)}$$

Thus,

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Therefore, the compound angle formula for finding the tangent of the sum of two angles is written as;

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\qquad(6.6)$$

If B is replaced by $-B$, then equation (6.6) results to;

$$\tan\big(A+(-B)\big)=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$$

$$\Rightarrow\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Therefore, the compound angle formula for the tangent of the difference of two angles is written as,

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

In summary, the compound angle formulae are given by,

(i) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

(ii) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

(iii) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

Example 6.11

Prove that

$$\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y.$$

Solution

Given that

$$\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y.$$

Consider the left-hand side of the given identity. That is, $\cos(x+y)\cos(x-y)$

$=(\cos x\cos y-\sin x\sin y)\times$
$(\cos x\cos y+\sin x\sin y)$

$=\cos^2x\cos^2y+\cos x\cos y\sin x\sin y-$
$\cos x\cos y\sin x\sin y-\sin^2x\sin^2y$

$=\cos^2x\cos^2y-\sin^2x\sin^2y$

$=\cos^2x(1-\sin^2y)-(1-\cos^2x)\sin^2y$

$=\cos^2x-\cos^2x\sin^2y-\sin^2y$
$+\cos^2x\sin^2y$

$=\cos^2x-\sin^2y$

Therefore,

$$\cos(x+y)\cos(x-y)=\cos^2x-\sin^2y.$$

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Example 6.12

Veritfy that $\dfrac{\tan(A+B)-\tan A}{1+\tan(A+B)\tan A}=\tan B$.

Solution

By using the compound angle formula,

$$\frac{\tan(A+B)-\tan A}{1+\tan(A+B)\tan A}=\tan((A+B)-A)$$

$$=\tan(A-A+B)$$

$$=\tan B$$

Therefore, $\dfrac{\tan(A+B)-\tan A}{1+\tan(A+B)\tan A}=\tan B$.

Example 6.13

Simplify $(\sin x\cos y-\cos x\sin y)^2$
$+(\cos x\cos y+\sin x\sin y)^2$.

Solution

$(\sin x\cos y-\cos x\sin y)^2$
$+(\cos x\cos y+\sin x\sin y)^2$
$=\sin^2(x-y)+\cos^2(x-y)=1$

Therefore, $(\sin x\cos y-\cos x\sin y)^2$
$+(\cos x\cos y+\sin x\sin y)^2=1$.

Example 6.14

If $A+B+C=180°$, show that
$\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Solution

Given $A+B+C=180°$.

$\Rightarrow A+B=180°-C$

Applying tangent on both sides gives;
$\tan(A+B)=\tan(180°-C)$

By using the compound angle formula, it follows that

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\tan 180°-\tan C}{1+\tan 180°\tan C}$$

$$\Rightarrow\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{0-\tan C}{1+0\times\tan C}$$

since $\tan 180°=0$, it implies that

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

Cross multiplication gives,
$\tan A+\tan B=(1-\tan A\tan B)(-\tan C)$.

Thus,
$\tan A+\tan B=-\tan C+\tan A\tan B\tan C$

Therefore,
$\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Example 6.15

Solve the equation
$2\sin x=\cos(x+60°)$ for values of x between $0°$ and $360°$.

Solution

Given $2\sin x=\cos(x+60°)$.

$\Rightarrow 2\sin x=\cos x\cos 60°-\sin x\sin 60°$

$$\Rightarrow 2\sin x=\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$$

Divide by $\cos x$ on both sides to get;

$$2\tan x=\frac{1}{2}-\frac{\sqrt{3}}{2}\tan x$$

Collecting like terms gives;

$$2\tan x+\frac{\sqrt{3}}{2}\tan x=\frac{1}{2}$$

$$\Rightarrow\left(2+\frac{\sqrt{3}}{2}\right)\tan x=\frac{1}{2}$$

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$$\Rightarrow \left(\frac{4+\sqrt{3}}{2}\right)\tan x = \frac{1}{2}$$

$$\Rightarrow \left(4+\sqrt{3}\right)\tan x = 1$$

$$\Rightarrow \tan x = \frac{1}{4+\sqrt{3}}$$

$$\Rightarrow x = \tan^{-1}\left(\frac{1}{4+\sqrt{3}}\right)$$

$$\Rightarrow x = 9.9°, 189.9°$$

Therefore,
the values of x are $9.9°$ and $189.9°$.

Example 6.16

If $\tan(x+45°) = 2$, find the value of $\tan x$.

Solution

Given $\tan\left(x+45°\right) = 2$.

$$\Rightarrow \frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$$

$$\Rightarrow \frac{\tan x + 1}{1 - \tan x} = 2$$

$$\Rightarrow \tan x + 1 = 2(1 - \tan x)$$

$$\Rightarrow \tan x + 1 = 2 - 2\tan x$$

$$\Rightarrow 3\tan x = 1$$

$$\Rightarrow \tan x = \frac{1}{3}.$$

Therefore, $\tan x = \frac{1}{3}$.

Exercise 6.3

1. Find the values of $\sin\left(x+y\right)$, $\cos\left(x+y\right)$, and $\tan\left(x+y\right)$ in each of the following:

   (a) $\sin x = \frac{3}{5}$ and $\sin y = \frac{5}{13}$, where x and y lie in the first quadrant.

   (b) $\cos x = -\frac{12}{13}$ and $\cot y = \frac{24}{7}$, where x lies in the second quadrant and y lies in the third quadrant.

2. Find the values of $\sin\left(A-B\right)$, $\cos\left(A-B\right)$, and $\tan\left(A-B\right)$, given:

   (a) $\cos A = -\frac{12}{13}$ and $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B lies in the first quadrant.

   (b) $\sin A = \frac{3}{5}$ and $\sin B = -\frac{5}{13}$, where A lies in the first quadrant and B lies in the fourth quadrant.

3. If $\tan x - \tan y = m$ and $\cot y - \cot x = n$, prove that $\cot(x-y) = \frac{1}{m} + \frac{1}{n}$.

4. Prove each of the following identities:

   (a) $\tan\left(45° - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$

   (b) $\frac{\sin\left(x+y\right)}{\cos\left(x-y\right)} = \frac{\tan x + \tan y}{1+\tan x \tan y}$

   (c) $\frac{\tan\left(\alpha-\beta\right)}{\cot\left(\alpha+\beta\right)} = \frac{\tan^2\alpha + \tan^2\beta}{1-\tan^2\alpha\tan^2\beta}$

   (d) $\cot\left(\alpha+\beta\right) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

5. Given that A, B, and C are angles of a triangle, prove that:

   (a) $\cot\left(\frac{B+C}{2}\right) = \tan\frac{A}{2}$

   (b) $\tan B \tan C = 2$, if $\cos A = \cos B \cos C$

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(c) $\sin A \sin(A+2C)+\sin B \sin(B+2A)+\sin C \sin(C+2B)=0$

6. Solve each of the following equations for values of x between $0°$ and $360°$:
   (a) $\cos(x+45°)=\cos x$
   (b) $\sin(x-30°)=\frac{1}{2}\cos x$
   (c) $3\sin(x+10°)=4\cos(x-10°)$

7. Given that $\sin(x+\beta)=2\cos(x-\beta)$. Prove that $\tan x=\frac{2-\tan\beta}{1-2\tan\beta}$.

8. Evaluate each of the following expressions (Leave your answers in surd form).
   (a) $\sin 47° \cos 13° + \cos 47° \sin 13°$
   (b) $\frac{\tan 75° - \tan 15°}{1+\tan 75° \tan 15°}$
   (c) $\sin 75° \cos 300° + \cos 1470° \sin(-1020°)$.

9. If $\cot A \cot B = 2$, show that $\frac{\cos(A-B)}{\cos(A-B)}=3$.

10. Given that $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A \tan B}$. If $B=45°$, show that,
    $\tan \tan(A-B)=\frac{\tan^2 A-2\tan A+1}{\tan^2 A-1}$.

Double angle formulae

Double angle formulae are trigonometric identities which express trigonometric functions of 2θ in terms of trigonometric functions of θ. They are special cases of compound angle formulae.

The compound angle formulae under addition of two angles can be used to deduce the double angle formula.

Considering equation (6.4), that is $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

Replacing B by A, the double angle formula for sine of an angle is obtained as follows.

$\sin(A+A)=\sin A\cos A+\cos A\sin A$

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Thus, $\sin 2A = 2\sin A\cos A$

Therefore, the double angle formula for sine of an angle A is
$\sin 2A = 2\sin A\cos A$.

From equation (6.5), $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Replacing B by A gives the double angle formula for cosine of an angle, that is,

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\cos 2A = \cos^2 A - \sin^2 A$, where $\sin^2 A = 1 - \cos^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

Therefore, the double angle formula for cosine is,

$\cos 2A = 2\cos^2 A - 1$.

But $\cos^2 A = 1 - \sin^2 A$, thus,

$\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

Therefore, the double angle formula for cosine involving sine is,
$\cos 2A = 1 - 2\sin^2 A$.

From $\cos 2A = 2\cos^2 A - 1$

$\Rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$

and $\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$

Generally:

$\cos^2(mx) = \frac{1}{2}(1 + \cos 2mx)$ and

$\sin^2(mx) = \frac{1}{2}(1 - \cos 2mx)$, where

m is an integer

Considering equation (6.6), it follows that,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Replacing B by A gives the double angle formula for tangent of an angle, that is

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

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Therefore, the double angle formula for tangent of an angle is,

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

The double angle formulae can be summarized as follows:

(a) $\sin 2A = 2\sin A\cos A$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Example 6.17

Express $\sin 3\alpha$ in terms of $\sin\alpha$.

Solution

Let $\sin 3\alpha = \sin(2\alpha + \alpha)$. It follows that

$$\begin{aligned}\sin(2\alpha+\alpha) &= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\ &= (2\sin\alpha\cos\alpha)\cos\alpha + (1-2\sin^2\alpha)\sin\alpha \\ &= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha \\ &= 2\sin\alpha(1-\sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \\ &= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha \\ &= 3\sin\alpha - 4\sin^3\alpha\end{aligned}$$

Therefore, $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$.

Example 6.18

Prove that $\cos 2x = \cos^4 x - \sin^4 x$.

Solution

Proving from the right-hand side, it follows that

$$\begin{aligned}\cos^4 x - \sin^4 x &= (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x), \\ &= \cos^2 x - \sin^2 x, \text{ because } \cos^2 x + \sin^2 x = 1 \\ &= \cos 2x\end{aligned}$$

Therefore, $\cos 2x = \cos^4 x - \sin^4 x$.

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Example 6.19

Prove that $2\tan 2x = \dfrac{\cos x+\sin x}{\cos x-\sin x} - \dfrac{\cos x-\sin x}{\cos x+\sin x}$.

Solution

Consider the right-hand side of the given equation.

$$\frac{\cos x+\sin x}{\cos x-\sin x} - \frac{\cos x-\sin x}{\cos x+\sin x} = \frac{(\cos x+\sin x)^2-(\cos x-\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}$$

$$= \frac{(\cos^2 x+2\cos x\sin x+\sin^2 x)-(\cos^2 x-2\cos x\sin x+\sin^2 x)}{\cos^2 x-\sin^2 x}$$

$$= \frac{4\cos x\sin x}{\cos^2 x-\sin^2 x}$$

$$= \frac{2(2\cos x\sin x)}{\cos 2x} = \frac{2\sin 2x}{\cos 2x}$$

$$= 2\tan 2x$$

Therefore, $2\tan 2x = \dfrac{\cos x+\sin x}{\cos x-\sin x} - \dfrac{\cos x-\sin x}{\cos x+\sin x}$.

Example 6.20

Eliminate λ from the following equations:

$x = 3+4\sin\lambda$ and $y = 5\cos 4\lambda$

Solution

Given,

$x = 3+4\sin\lambda$ and $y = 5\cos 4\lambda$.

From $x = 3+4\sin\lambda$, make $\sin\lambda$ the subject to obtain, $\sin\lambda = \dfrac{x-3}{4}$

From $y = 5\cos 4\lambda$, write $\cos 4\lambda$ in terms of $\sin\lambda$, that is,

$$\cos 4\lambda = 1-2\sin^2 2\lambda$$
$$= 1-2(2\sin\lambda\cos\lambda)^2$$
$$= 1-2(4\sin^2\lambda\cos^2\lambda)$$
$$= 1-8\sin^2\lambda(1-\sin^2\lambda)$$

Hence, $y = 5(1-8\sin^2\lambda(1-\sin^2\lambda))$.

But $\sin\lambda = \dfrac{x-3}{4}$. Thus,

$$y = 5\left(1-8\left(\frac{x-3}{4}\right)^2\left(1-\left(\frac{x-3}{4}\right)^2\right)\right).$$

Therefore,

$$y = 5-40\left(\frac{x-3}{4}\right)^2\left(1-\left(\frac{x-3}{4}\right)^2\right).$$

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Example 6.21

Simplify $\dfrac{\sin 2\theta}{1+\cos 2\theta}$.

Solution

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+\left(2\cos^2\theta-1\right)}$$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

Therefore, $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

Example 6.22

Solve the equation

$2\sin 2x + 2\sin x + 2\cos x + 1 = 0$

for $-2\pi \le x \le 2\pi$.

Solution

Given,

$2\sin 2x + 2\sin x + 2\cos x + 1 = 0.$

But $\sin 2x = 2\sin x\cos x$

$\Rightarrow 2(2\sin x\cos x) + 2\sin x +$

$2\cos x + 1 = 0$

$\Rightarrow 4\sin x\cos x + 2\sin x +$

$2\cos x + 1 = 0$

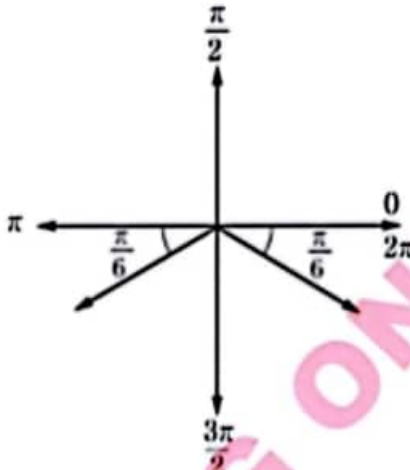
$\Rightarrow 2\sin x(2\cos x + 1) + 2\cos x + 1 = 0$

$\Rightarrow (2\cos x + 1)(2\sin x + 1) = 0$

$\Rightarrow 2\cos x + 1 = 0$ or $2\sin x + 1 = 0$

$\Rightarrow \cos x = -\dfrac{1}{2}$ or $\sin x = -\dfrac{1}{2}$

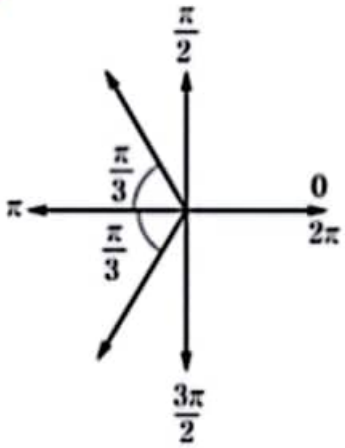
The following sketch shows the values of x satisfying the equation $\sin x = -\dfrac{1}{2}$ in the third and fourth quadrants.



From the sketch,

$$x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$$

The following sketch shows the values of x satisfying the equation $\cos x = -\dfrac{1}{2}$ in the second and third quadrants.



From the sketch,

$$x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$$

Therefore, the values of x are

$-\dfrac{4\pi}{3}, -\dfrac{5\pi}{6}, -\dfrac{2\pi}{3}, -\dfrac{\pi}{6}, \dfrac{2\pi}{3}, \dfrac{7\pi}{6}, \dfrac{4\pi}{3},$

and $\dfrac{11\pi}{6}$, for $-2\pi \le x \le 2\pi$.

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Half angle formulae

Half angle formulae are special trigonometric identities which are derived from the double angle formulae. These are useful in evaluating trigonometrical ratios of half angles and deriving other identities which involve half angles.

Consider the double angle formulae for sine and cosine.

$\sin 2A = 2\sin A\cos A$ and

$\cos 2A = 1-2\sin^2 A$

$= 2\cos^2 A - 1$

$= \cos^2 A - \sin^2 A$

The half angle formulae can be obtained by dividing the angle A by 2. That is,

(a) $\sin A = \sin\left(\frac{A}{2}+\frac{A}{2}\right)$

$= \sin\frac{A}{2}\cos\frac{A}{2}+\cos\frac{A}{2}\sin\frac{A}{2}$

$= 2\sin\frac{A}{2}\cos\frac{A}{2}$

Therefore, $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.

(b) $\cos A = \cos\left(\frac{A}{2}+\frac{A}{2}\right)$

$= \cos\frac{A}{2}\cos\frac{A}{2} - \sin\frac{A}{2}\sin\frac{A}{2}$

$= \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

Thus, $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$.

From the relation $\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$, it implies that $\cos^2\frac{A}{2} = 1-\sin^2\frac{A}{2}$.

Thus, the equation

$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$ becomes,

$\cos A = 1-\sin^2\frac{A}{2} - \sin^2\frac{A}{2}$

Thus, $\cos A = 1-2\sin^2\frac{A}{2}$.

Similary, the relation

$\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$ gives,

$\sin^2\frac{A}{2} = 1-\cos^2\frac{A}{2}$.

Thus, the equation

$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$ can be written as,

$\cos A = \cos^2\frac{A}{2} - \left(1-\cos^2\frac{A}{2}\right)$

$= 2\cos^2\frac{A}{2} - 1$

Thus, $\cos A = 2\cos^2\frac{A}{2} - 1$.

Therefore, $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$.

$= 1-2\sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1$.

(c) $\tan A = \frac{\sin A}{\cos A}$

But $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ and

$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

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$$\Rightarrow \tan A = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

Divide the numerator and denominator of the right-hand side of the equation by $\cos^2\frac{A}{2}$.

$$\Rightarrow \tan A = \frac{\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}}}{\frac{\cos^2\frac{A}{2}}{\cos^2\frac{A}{2}} - \frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}}$$

$$= \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

Therefore, $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$.

Example 6.23

Simplify $\frac{\cos^2\left(\frac{45°}{2}\right) - \sin^2\left(\frac{45°}{2}\right)}{\cos^2\left(\frac{135°}{2}\right) - \sin^2\left(\frac{135°}{2}\right)}$.

Solution

From $\cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \cos A$.

It follows that,

$$\frac{\cos^2\left(\frac{45°}{2}\right) - \sin^2\left(\frac{45°}{2}\right)}{\cos^2\left(\frac{135°}{2}\right) - \sin^2\left(\frac{135°}{2}\right)} = \frac{\cos 45°}{\cos 135°}$$

$$= \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$= -1$$

Therefore,

$$\frac{\cos^2\left(\frac{45°}{2}\right) - \sin^2\left(\frac{45°}{2}\right)}{\cos^2\left(\frac{135°}{2}\right) - \sin^2\left(\frac{135°}{2}\right)} = -1.$$

Example 6.24

Prove that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$.

Solution

Given that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$.

Consider the right-hand side of the equation.

$$\Rightarrow \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

Therefore, $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$.

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Example 6.25

Show that $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Solution

Given that $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Consider the left-hand side of the equation.

$$\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\frac{(1+\cos A)+\sin A}{(1-\cos A)+\sin A}$$

$$=\frac{2\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}$$

$$=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}$$

Therefore, $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Exercise 6.4

1. Prove each of the following identities:

   (a) $\cot\theta=\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}$

   (b) $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

   (c) $\cos 6\theta=1-2\sin^2 3\theta$

   (d) $\sin^2\frac{1}{2}\theta=\frac{1}{2}(1-\cos\theta)$

   (e) $\tan 4\theta=\frac{4\tan\theta(1-\tan\theta)(1+\tan\theta)}{1-6\tan^2\theta+\tan^4\theta}$

   (f) $\sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

   (g) $\cot 15^\circ=\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$

2. Prove each of the following identities:

   (a) $1-\frac{1}{2}\sin 2x=\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

   (b) $\cos x=\frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}$

   (c) $\frac{1+\sin C+\cos C}{1+\sin C-\cos C}=\cot\frac{C}{2}$

   (d) $\frac{1-\sin 2A}{\cos 2A}=\frac{1-\tan A}{1+\tan A}$

3. Simplify each of the following expressions:

   (a) $\frac{\sin 3A}{\sin A}-\frac{\cos 3A}{\cos A}$

   (b) $\frac{1+\cos 2\theta}{\sin 2\theta}$

   (c) $\left(\sin\frac{1}{2}x-\cos\frac{1}{2}x\right)^2$

   (d) $\operatorname{cosec} 2x+\cot 2x$

4. If $2A+B=45^\circ$, show that $\tan B=\frac{1-2\tan A-\tan^2 A}{1+2\tan A-\tan^2 A}$

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5. Eliminate θ in each of the following pairs of equations:
   (a) $x+1=\cos 2\theta,\ y=\sin\theta$
   (b) $x=\cos 2\theta,\ y=\cos\theta-1$
   (c) $x=\tan\theta,\ y=\tan 4\theta$
   (d) $x=4-3\sin 2\theta,\ y=7-9\cos 2\theta$

6. Solve each of the following equations for $0^\circ \le x \le 360^\circ$.
   (a) $2\sin x\cos x=\cos 2x$
   (b) $3\cot 2x+\cot x-1=0$
   (c) $2\cos 2x+13\sin x=5$
   (d) $4\sin x(7\cos x-3)=5\sin 2x$

7. In a triangle ABC, prove each of the following:
   (a) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
   (b) $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
   (c) If $\cos(A+B)=\cos A+\cos B$, then $\cos(A+B)=\cos A+\cos B+\cos C=0$.

8. Find the values of $\tan 2\beta$, $\sin 2\beta$, and $\sec 2\beta$, in each of the following cases when β is in the fourth quadrant:
   (a) $\sin\beta=-\frac{24}{25}$ (b) $\cot\beta=-\frac{120}{119}$

9. If $t=\tan\lambda$, show that $\tan 4\lambda=\frac{4t-4t^3}{t^4-6t^2+1}$. Hence, verify that, $t^4+4t^3-6t^2-4t+1=0$ when $\tan 4\lambda=1$.

10. Evaluate each of the following and leave your answers in surd form:
    (a) $\frac{2\tan 75^\circ}{1-\tan^2 75^\circ}$ (b) $\frac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ}$ (c) $2\sin 165^\circ\cos 165^\circ$ (d) $\sin 7.5^\circ$

11. Without using mathematical tables or a scientific calculator, evaluate each of the following:
    (a) $\frac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ}$ (b) $1-2\sin^2 67.5^\circ$ (c) $\sec 22.5^\circ\operatorname{cosec} 22.5^\circ$

12. Prove each of the following trigonometric identities:
    (a) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$
    (b) $\left(\cos\frac{B}{2}+\sin\frac{B}{2}\right)^2=1+\sin B$
    (c) $\frac{1+\cos\theta}{\sin\theta}=\cot\frac{\theta}{2}$
    (d) $\tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

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Trigonometric equations of the form $a\cos\theta + b\sin\theta = c$

A trigonometric equation of the form $a\cos\theta + b\sin\theta = c$, where a, b, and c are real numbers can be solved by using t-formulae or by expressing it in the form of $R\cos(\theta \pm \alpha) = c$ or $R\sin(\theta \pm \alpha) = c$.

Solutions of equations of the form $a\cos\theta + b\sin\theta = c$ by t-formulae

Recall the double angle formulae,

(i) $\sin 2\theta = 2\sin\theta\cos\theta$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

The equations for $\sin 2\theta$ and $\cos 2\theta$ can be expressed in terms of $\tan\theta$.

That is, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{2\sin\theta\cos\theta}{1}$$

But $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \sin 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$$

Divide both the numerator and denominator of the right-hand side of the equation by $\cos^2\theta$. That is,

$$\sin 2\theta = \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{2\tan\theta}{1 + \tan^2\theta}$$

Let $= \tan\theta$. It follows that,

$$\sin 2\theta = \frac{2t}{1 + t^2} \qquad (6.7)$$

Equation (6.7) is called the t-formula for $\sin 2\theta$, where $t = \tan\theta$.

Similarly, using the half angle formulae,

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1}$$

But $\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$. It implies that,

$$\sin\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}$$

Divide both the numerator and denominator of the right-hand side of the equation by $\cos^2\frac{\theta}{2}$.

That is,

$$\sin\theta = \frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \times \frac{1}{\cos^2\frac{\theta}{2}}}{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) \times \frac{1}{\cos^2\frac{\theta}{2}}}$$

Thus, $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$.

Let $t = \tan\frac{\theta}{2}$. It follows that,

$$\sin\theta = \frac{2t}{1 + t^2} \qquad (6.8)$$

Equation (6.8) is called the t-formula for $\sin\theta$, where $t = \tan\frac{\theta}{2}$.

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Activity 6.2: Deducing the *t*-formulae for $\cos\theta$ and $\tan\theta$

Individually or in a group, perform the following tasks:

1. Express $\cos 2\theta$ and $\tan 2\theta$ in terms of $\tan\theta$.
2. Use the concept applied in the derivation of *t*-formula for $\sin\theta$ to deduce the corresponding *t*-formulae for $\cos\theta$ and $\tan\theta$.
3. Give the general *t*-formulae for sine, consine, and tangent for any angle.

The relationship between the tangent, sine, and cosine of an angle in terms of tangent of half angle can be obtained using a right-angled triangle as shown in Figure 6.4.

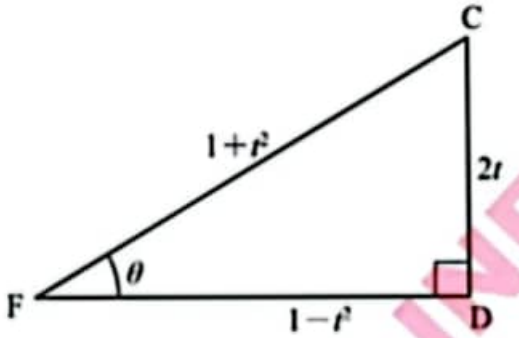


Figure 6.4: A *right-angled triangle DCF*

From Figure 6.4, the *t*-formulae of the trigonometric ratios are;

(a) $\sin\theta = \dfrac{2t}{1+t^2}$,

(b) $\cos\theta = \dfrac{1-t^2}{1+t^2}$,

(c) $\tan\theta = \dfrac{2t}{1-t^2}$.

Example 6.26

Solve the equation $5\cos\theta - 2\sin\theta = 2$ for $0° \le \theta \le 360°$ by using the substitution $t = \tan\frac{1}{2}\theta$.

Solution

Given $5\cos\theta - 2\sin\theta = 2$.

From the *t*-formulae, $\cos\theta = \dfrac{1-t^2}{1+t^2}$ and $\sin\theta = \dfrac{2t}{1+t^2}$, where $t = \tan\dfrac{\theta}{2}$.

$$\Rightarrow 5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$$

$$\Rightarrow 5(1-t^2) - 4t = 2(1+t^2)$$

$$\Rightarrow 5 - 5t^2 - 4t = 2 + 2t^2$$

$$\Rightarrow 7t^2 + 4t - 3 = 0$$

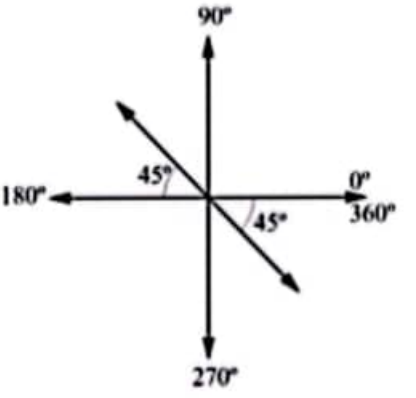
$$\Rightarrow t = -1 \text{ or } t = \frac{3}{7}$$

Hence, $\tan\frac{1}{2}\theta = -1$ or $\tan\frac{1}{2}\theta = \frac{3}{7}$

Now, $\tan\frac{1}{2}\theta = -1 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(-1)$

$$\Rightarrow \frac{1}{2}\theta = -45°$$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = -1$ in the second and fourth quadrants.



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From the sketch,

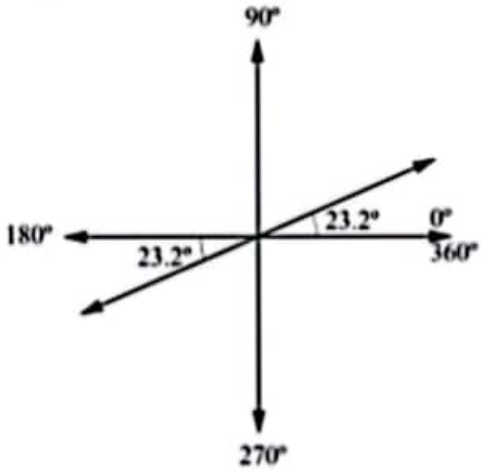
$\frac{\theta}{2} = -45°, 135°, \ldots$ It follows that,

$\theta = -90°, 270°, \ldots$

Now, $\tan\frac{1}{2}\theta = \frac{3}{7} \Rightarrow \frac{1}{2}\theta = \tan^{-1}\left(\frac{3}{7}\right)$

$\Rightarrow \frac{1}{2}\theta = 23.2°$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = \frac{3}{7}$ in the first and third quadrants.



From the sketch,

$\frac{\theta}{2} = -156.8°, 23.2°, \ldots$ It follows that,

$\theta = -313.6°, 46.4°, \ldots$

Therefore, the required solutions are 46.4° and 270°.

Example 6.27

Solve the equation $15\cos 2x + 2\sin 2x = 10$ for $-180° \le \theta \le 180°$.

Solution

Given $15\cos 2x + 2\sin 2x = 10$.

Substituting $\cos 2x = \frac{1-t^2}{1+t^2}$ and $\sin 2x = \frac{2t}{1+t^2}$, where $t = \tan x$ gives,

$$15\left(\frac{1-t^2}{1+t^2}\right) + 2\left(\frac{2t}{1+t^2}\right) = 10$$

Multiplying by $1+t^2$ on both sides gives,

$15(1-t^2) + 4t = 10(1+t^2)$

$\Rightarrow 15 - 15t^2 + 4t = 10 + 10t^2$

$\Rightarrow 25t^2 - 4t - 5 = 0$

Solve the quadratic equation to obtain,

$t = 0.5343$ or -0.3743.

Hence,

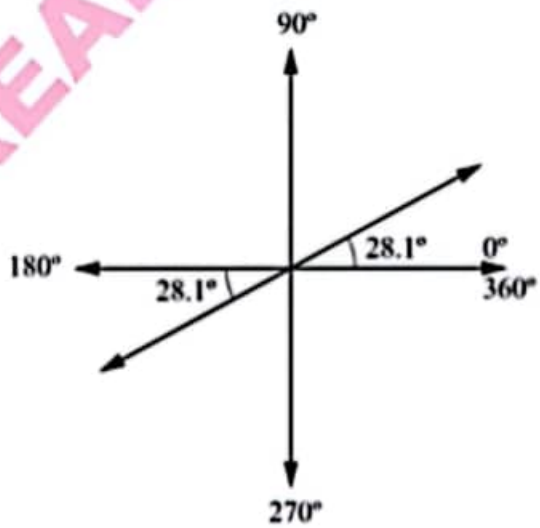
$\tan x = 0.5343$ or $\tan x = -0.3743$.

Now,

$\tan x = 0.5343 \Rightarrow x = \tan^{-1}(0.5343)$

$\Rightarrow x = 28.1°$

The following sketch illustrates the values of x satisfying $\tan x = 0.5343$ in the first and third quadrants.



From the sketch,

$x = -151.9°, 28.1°, \ldots$

Now,

$\tan x = -0.3743 \Rightarrow x = \tan^{-1}(-0.3743)$

$\Rightarrow x = -20.5°$

The following sketch illustrates the values of x satisfying $\tan x = -0.3743$ in the second and fourth quadrants.

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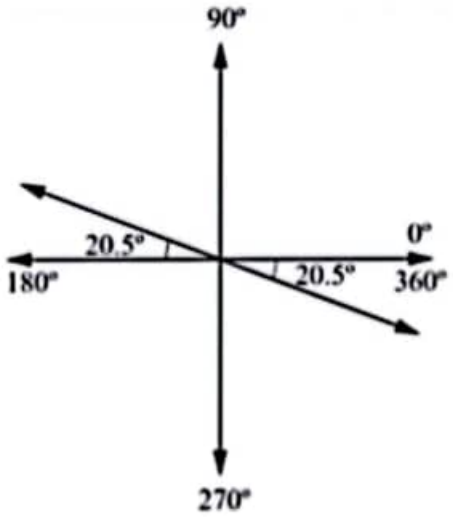

From the sketch, $x = -20.5°, 159.5°, ...$

Therefore, the required solution are $-151.90°$, $-20.5°$, $28.1°$, and $159.5°$.

Example 6.28

Solve the equation $3\cos\theta + 5\sin\theta = 2$ for $-360° \le \theta \le 360°$.

Solution

Substituting $t = \tan\frac{1}{2}\theta$, where $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$ gives,

$$3\left(\frac{1-t^2}{1+t^2}\right) + 5\left(\frac{2t}{1+t^2}\right) = 2$$

Multiply both sides by $1+t^2$ to get;

$3(1-t^2) + 10t = 2(1+t^2)$

$\Rightarrow 3 - 3t^2 + 10t = 2 + 2t^2$

$\Rightarrow 5t^2 - 10t - 1 = 0$

Solving the quadratic equation gives,

$t = -0.0954$ or 2.0954

Hence, $\tan\frac{1}{2}\theta = -0.0954$ or $\tan\frac{1}{2}\theta = 2.0954$.

Now,

$\tan\frac{1}{2}\theta = -0.0954 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(-0.0954)$

$\Rightarrow \frac{1}{2}\theta = -5.4°$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = -0.0954$ in the second and fourth quadrants.

From the sketch,

$\frac{1}{2}\theta = -5.4°, 174.6°, ...$

Thus, $\theta = -10.8°, 349.2°, ...$

Now,

$\tan\frac{1}{2}\theta = 2.0954 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(2.0954)$

$\Rightarrow \frac{1}{2}\theta = 64.5°$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = 2.0954$ in the first and third quadrants.

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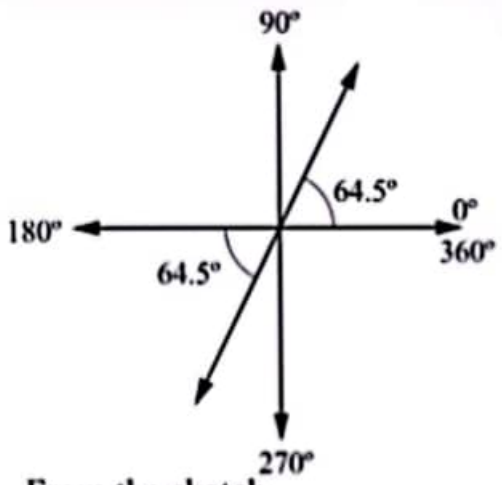

From the sketch,

$\frac{1}{2}\theta = -115.5°, 64.5°, ...$

Thus, $\theta = -231°, 129°, ...$

Therefore, the required solutions are $\theta = -231°, -10.8°, 129°$, and $349.2°$.

Expressing $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

The technique of expressing $a\cos\theta + b\sin\theta$ in the form of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ is useful in finding maximum and minimum values of trigonometric equations of the form $a\cos\theta + b\sin\theta = c$.

Expressing $a\cos\theta + b\sin\theta$ in the form of $R\cos(\theta - \alpha)$

Using the compound angle formula for $R\cos(\theta - \alpha)$ gives;

$R\cos(\theta - \alpha) \equiv a\cos\theta + b\sin\theta$

$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha.$

Comparing the coefficients of $\cos\theta$ gives;

$a = R\cos\alpha$ ..... (i)

Comparing the coefficients of $\sin\theta$ gives;

$b = R\sin\alpha$ ..... (ii)

Squaring and adding equations (i) and (ii) gives;

$$a^2 + b^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$$
$$= R^2(\cos^2\alpha + \sin^2\alpha)$$
$$= R^2$$

Now, $R = \sqrt{a^2 + b^2}$, where the value of R is taken as being positive.

The value of α is obtained by dividing equation (ii) by (i), that is,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$$

Thus, $\tan\alpha = \frac{b}{a} \Rightarrow \alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

Therefore,

$a\cos\theta + b\sin\theta = \sqrt{a^2 + b^2}\cos\left(\theta - \tan^{-1}\left(\frac{b}{a}\right)\right)$.

Note that, if $0° \le \alpha \le 90°$, both $\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$ give the same value of α. If they do not, then use the value of α which is not between $0°$ and $90°$.

Example 6.29

Express $3\cos\theta - 4\sin\theta$ in the form of $R\cos(\theta - \alpha)$.

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta - \alpha)$. It follows that

$3\cos\theta - 4\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Compare the coefficients of $\sin\theta$ and $\cos\theta$ to obtain the following pair of equations;

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$$\begin{cases} R\cos\alpha = 3 \\ R\sin\alpha = -4 \end{cases}$$

But $R = \sqrt{a^2+b^2} = \sqrt{3^2+(-4)^2} = 5$.

Thus, $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = -\frac{4}{5}$.

$\Rightarrow \alpha = 53.1°$ and $\alpha = -53.1°$.

Hence, $\alpha = -53.1°$ as it is not between $0°$ and $90°$.

Therefore,

$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1°)$.

Example 6.30

Show that $5\cos\theta - 12\sin\theta = 5$ can be expressed in the form $13\cos(\theta + 67.4°) = 5$.

Solution

Given $5\cos\theta - 12\sin\theta = 5$.

Consider the left-hand side.

It follows that

$5\cos\theta - 12\sin\theta = R\cos(\theta - \alpha)$

$\Rightarrow 5\cos\theta - 12\sin\theta =$

$R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Compare the coefficients of $\sin\theta$ and $\cos\theta$ to obtain;

$R\cos\alpha = 5$ ..... (i)

$R\sin\alpha = 12$ ..... (ii)

But

$R = \sqrt{a^2+b^2} = \sqrt{5^2+(-12)^2} = 13$

Dividing equation (ii) by equation (i), gives;

$\tan\alpha = \frac{12}{5} \Rightarrow \alpha = 67.4°$

Therefore,

$5\cos\theta - 12\sin\theta = 13\cos(\theta + 67.4°) = 5$.

Expressing $a\cos\theta + b\sin\theta$ in the form $R\sin(\theta+\alpha)$

Using the compound angle formula for $R\sin(\theta+\alpha)$ gives;

$a\cos\theta + b\sin\theta = R\cos\theta\sin\alpha + R\sin\theta\cos\alpha$

Comparing the coefficients of $\sin\theta$ gives;

$b = R\cos\alpha$ ..... (i)

Comparing the coefficients of $\cos\theta$ gives;

$a = R\sin\alpha$ ..... (ii)

Squaring and adding equations (i) and (ii) gives;

$a^2 + b^2 = R^2\sin^2\alpha + R^2\cos^2\alpha$

$\Rightarrow a^2 + b^2 = R^2(\sin^2\alpha + \cos^2\alpha)$

Thus, $a^2 + b^2 = R^2$.

$\Rightarrow R = \sqrt{a^2+b^2}$.

The value of α is obtained by dividing equation (ii) by equation (i) as follows.

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{a}{b}$

Thus, $\tan\alpha = \frac{a}{b} \Rightarrow \alpha = \tan^{-1}\left(\frac{a}{b}\right)$.

Therefore, $a\cos\theta + b\sin\theta$

$= \sqrt{a^2+b^2}\sin\left(\theta + \tan^{-1}\left(\frac{a}{b}\right)\right)$.

Example 6.31

Express $7\cos\theta + 24\sin\theta$ in the form of $R\sin(\theta+\alpha)$.

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Solution

Given $7\cos\theta + 24\sin\theta$, then

$R = \sqrt{a^2+b^2} = \sqrt{7^2+24^2} = 25$ and

$\tan\alpha = \frac{a}{b}$, where $a = 7$ and $b = 24$.

Thus, $\tan\alpha = \frac{7}{24} \Rightarrow \alpha = 16.3°$.

Therefore,

$7\cos\theta + 24\sin\theta = 25\sin(\theta + 16.3°)$.

Note that, in order to avoid the problem of obtaining two different values of α, select one among $R\sin(\theta+\alpha)$, $R\sin(\theta-\alpha)$, $R\cos(\theta+\alpha)$, and $R\cos(\theta+\alpha)$ that have the same sign as the expression $a\cos\theta + b\sin\theta$.

Example 6.32

Find the maximum and minimum values of $24\sin\theta - 7\cos\theta$, by stating the values of θ for which the maximum and minimum values are attained.

Solution

Given $24\sin\theta - 7\cos\theta$.

$\Rightarrow 24\sin\theta - 7\cos\theta$

$= R\sin(\theta - \alpha)$.

$\Rightarrow R = \sqrt{a^2+b^2} = \sqrt{24^2+(7)^2} = 25$

Thus, $\tan\alpha = \frac{b}{a} = \frac{7}{24}$

$\alpha = \tan^{-1}\left(\frac{7}{24}\right) = 16.3°$

Hence, $24\sin\theta - 7\cos\theta = 25\sin(\theta - 16.3°)$.

Note that, $\sin\theta$ has maximum value of 1 and minimum value of -1, that is, $-1 \le \sin\theta \le 1$.

Similarly, $-1 \le \sin(\theta - 16.3°) \le 1$

multiplying throughout by 25 gives,

$-25 \le 25\sin(\theta - 16.3°) \le +25$

At maximum value $\sin(\theta - 16.3°) = 1$, and at minimum value

$\sin(\theta - 16.3°) = -1$.

$\Rightarrow \sin(\theta - 16.3°) = 1$ and

$\sin(\theta - 16.3°) = -1$

$\Rightarrow \theta - 16.3° = \sin^{-1} 1$

$= 90°$

Thus, $\theta = 106.3°$.

Similarly, $\theta - 16.3° = \sin^{-1}(-1)$

$= 270°$

Thus, $\theta = 286.3°$.

Therefore, the maximum and minimum values of $24\sin\theta - 7\cos\theta$ are 25 and -25 which occur at $\theta = 106.3°$ and $286.3°$, respectively.

Example 6.33

Show that $13\cos\lambda + 7\sin\lambda$ can be expressed in the form $\sqrt{218}\cos(\lambda - \alpha)$, where $\tan\alpha = \frac{7}{13}$. Hence, find the maximum and minimum values of the function, giving the corresponding values of λ for which the maximum and minimum values occur.

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Solution

Given $13\cos\lambda + 7\sin\lambda$.

$\Rightarrow 13\cos\lambda + 7\sin\lambda = R\cos(\lambda - \alpha)$

$\Rightarrow 13\cos\lambda + 7\sin\lambda = R(\cos\lambda\cos\alpha + \sin\lambda\sin\alpha)$

Comparing the coefficients of $\sin\lambda$ gives;

$7 = R\sin\alpha$ ..... (i)

Comparing the coefficients of $\cos\lambda$ gives;

$13 = R\cos\alpha$ .....(ii)

Squaring and adding equations (i) and (ii) gives;

$R^2\cos^2\alpha + R^2\sin^2\alpha = 13^2 + 7^2$

$\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 169 + 49$

$\Rightarrow R^2 = 218 \Rightarrow R = \sqrt{218}$

Dividing equation (i) by equation (ii) gives;

$\tan\alpha = \frac{7}{13} \Rightarrow \alpha = \tan^{-1}\left(\frac{7}{13}\right) = 28.3°$.

Hence, $13\cos\lambda + 7\sin\lambda = \sqrt{218}\cos(\lambda - 28.3°)$.

At maximum and minimum values $\cos(\lambda - 28.3°) = 1$ and $\cos(\lambda - 28.3°) = -1$, respectively.

$\Rightarrow \sqrt{218}\cos(\lambda - 28.3°) = \sqrt{218}$ and $\sqrt{218}\cos(\lambda - 28.3°) = -\sqrt{218}$

$\Rightarrow \cos(\lambda - 28.3°) = 1$ and $\cos(\lambda - 28.3°) = -1$

$\Rightarrow \lambda - 28.3° = \cos^{-1}1$ and $\lambda - 28.3° = \cos^{-1}(-1)$

$\Rightarrow \lambda - 28.3° = 0$ and $\lambda - 28.3° = 180°$

$\Rightarrow \lambda = 28.3°$ and $\lambda = 208.3°$

Therefore, the maximum and minimum values of the function are $\sqrt{218}$ and $-\sqrt{218}$ attained at $\lambda = 28.3°$ and $208.3°$, respectively.

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Example 6.34

Solve the equation

$4\cos 2x + 3\sin 2x = \frac{5\sqrt{2}}{2}$,
for the values in the interval
$-180° \le x \le 180°$.

Solution

Given $4\cos 2x + 3\sin 2x = \frac{5\sqrt{2}}{2}$.

Let $4\cos 2x + 3\sin 2x = R\sin(2x+\alpha)$

$\Rightarrow 4\cos 2x + 3\sin 2x =$

$R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$.

Comparing coefficients of $\sin 2x$ gives;

$R\cos\alpha = 3$..... (i)

Comparing coefficients of $\cos 2x$ gives;

$R\sin\alpha = 4$..... (ii)

Dividing equation (ii) by equation (i) results to;

$\tan\alpha = \frac{4}{3} \Rightarrow \alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$

$\Rightarrow R = \sqrt{a^2+b^2} = \sqrt{4^2+3^2} = 5$

Thus, $5\sin(2x+53.1°) = \frac{5\sqrt{2}}{2}$.

$\Rightarrow \sin(2x+53.1°) = \frac{\sqrt{2}}{2}$

Applying inverse of sine on both sides gives,

$2x + 53.1° = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$\Rightarrow 2x + 53.1° = 45°$

$\Rightarrow 2x = 45° - 53.1°, 135° - 53.1°, 405° - 53.0°, -225° - 53.1°$

$\Rightarrow 2x = -8.1°, 81.9°, 351.9°, -278.1°$

$\Rightarrow x = -4.05°, 40.95°, 175.95°$, and $-139.05°$.

Therefore, the required solution are $x = -4.1°, -139.05°, 40.95°$ and $175.95°$.

Example 6.35

Solve the equation $3\cos x + \sin x = 2$ for $0° \le x \le 360°$.

Solution

Let $3\cos x + \sin x = R\cos(x-\alpha)$.

$\Rightarrow R = \sqrt{a^2+b^2} = \sqrt{3^2+1^2} = \sqrt{10}$

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Thus, $\tan\alpha = \frac{b}{a} = \frac{1}{3}$.

$\Rightarrow \alpha = \tan^{-1}\frac{1}{3}$

$\Rightarrow \alpha = 18.43°$

The equation $3\cos x + \sin x = 2$ can be written as,

$\sqrt{10}\cos(x - 18.43°) = 2$

$\Rightarrow \cos(x - 18.43°) = \frac{2}{\sqrt{10}}$

$\Rightarrow x - 18.43° = \cos^{-1}\frac{2}{\sqrt{10}}$

$\Rightarrow x - 18.43 = 50.77°$ or $309.23°$

$\Rightarrow x = 69.2°$ or $327.7°$

Therefore, the solutions are $x = 69.2°$ and $327.7°$.

Exercise 6.5

1. Use the t-formulae to solve each of the following equations for $0° \le x \le 360°$:
   (a) $2\cos x + 3\sin x - 2 = 0$
   (b) $7\cos x + \sin x - 5 = 0$
   (c) $3\cos x - 4\sin x + 1 = 0$
   (d) $3\cos x + 4\sin x = 2$
   (e) $2\tan x + 2 = \sec x$
   (f) $2\cos x + 7\sin x = -3$
   (g) $32\cos x + 36\sin x = 29$
   (h) $7\cot x - 2\operatorname{cosec} x = 6$
   (i) $12\cos x - 5\sin x = -3$
   (j) $1 + 7\tan x = 5\sec x$

2. Find the values of θ in each of the following equations for $-180° \le \theta \le 180°$:
   (a) $\sqrt{3}\cos\theta + \sin\theta = 1$
   (b) $3\tan\theta = 4 + 2\sec\theta$
   (c) $5\cos\theta - 12\sin\theta = 6$
   (d) $2\sin 2\theta + 15\cos 2\theta = 10$
   (e) $\cos\theta + \sin\theta = 0.5$
   (f) $\cos\theta(\cos\theta + \sin\theta) = 1$
   (g) $\cos\theta = 2 + 7\sin\theta$
   (h) $3\cos 4\theta - 2\sin 4\theta = 3$
   (i) $2\sin\theta + 7\cos\theta - 4 = 0$
   (j) $\operatorname{cosec}\theta - \sin\theta = 0.5$

3. Find the maximum and minimum values of the following expressions, stating the values of angle θ from $0°$ to $360°$ inclusive, for which the maximum and minimum values occur:
   (a) $\cos\theta + \sin\theta$
   (b) $\cos(\theta + 30°) - \cos\theta$
   (c) $4\sin\theta - 3\cos\theta$
   (d) $\sqrt{2}\cos(\theta + 60°) + 7\sin\theta$
   (e) $3\sin\theta + \cos\theta$
   (f) $3\sin\theta + 4\cos\theta$
   (g) $8\cos\theta - 15\sin\theta$
   (h) $3\sin\theta \quad 5\cos\theta$
   (i) $6\sin\theta - \cos\theta$
   (j) $\frac{1}{\cot\theta + \tan\theta}$

4. Show that $3\cos\theta + 2\sin\theta$ can be expressed in the form $\sqrt{13}\cos(\theta - a)$, where $\tan\alpha = \frac{2}{3}$. Hence, find the

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maximum and minimum values of $3\cos\theta + 2\sin\theta$ by calculating the corresponding values of θ in the range $-180° \leq \theta \leq 180°$.

5. Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where α is an acute angle. For what values of θ in the range $-180° \leq \theta \leq 180°$ does the expression is at maximum and minimum? State its maximum and minimum values.

General solutions

A general solution is a solution that contains all possible values that satisfy a given trigonometric equation.

Consider the graphs of the straight line $y = a$, where $-1 \leq a \leq 1$ and $y = f(\theta) = \cos\theta$ as shown in Figure 6.5

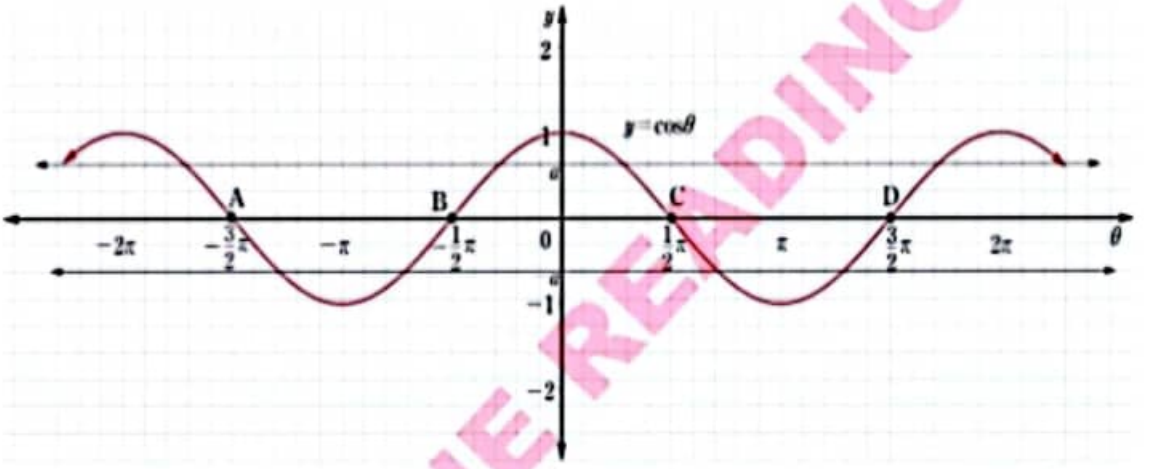

Figure 6.5: *Graphs of* $y = a$ *and* $f(\theta) = \cos\theta$

The principal value of the equation $a = \cos\theta$ is $\theta = \cos^{-1} a$ (which is the intercept of the line $y = a$ and the curve $y = \cos\theta$ as shown in Figure 6.5). Since the cosine function is periodic with period of 360°, then other solutions of the equation $a = \cos\theta$ corresponding to the principal value are obtained by adding or subtracting a multiple of 360° (the points of intersection in the θ-axis, that is, A, B, C, and D for $-2\pi \leq \theta \leq 2\pi$ as illustrated in Figure 6. 5). If a is the principal value of θ for which $\cos\theta = a$, then $-a$ is also a solution, and is not obtained by adding or subtracting a multiple of 360°. All the remaining solutions are obtained by adding or subtracting multiples of 360° to or from α. The general solution of the equation $\cos\theta = a$ where $-1 \leq a \leq 1$ is given as $\theta = 360°n \pm \alpha$, where $\alpha = \cos^{-1} a$ is a principal value and n is an integer.

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Activity 6.3: Finding the general solution of a trigonometric equation

Learning resources: Graph papers, ruler, pencil, and scientific calculator. Individually or in a group, perform the following tasks:

1. Draw the graphs of $y=\sin\theta$ and $y=\tan\theta$ for $-720^\circ \le \theta \le 720^\circ$ on different θy-planes.
2. Use the graphs of $y=\sin\theta$ and $y=\tan\theta$ to find the general solutions of the equations $\sin\theta=a$ for $-1\le a\le 1$ and $\tan\theta=a$ for $-\infty < a < \infty$. In each case, assume α to be the principal value and n to be an integer, then express the general solution in radian.
3. Give a suggestion between the formulas obtained in task 2.

Therefore, the general solutions for the equations of sine, cosine, and tangent can be summarized as follows:

In radians

(a) If $\sin\theta=\sin\alpha$, then $\theta=\pi n+(-1)^n\alpha$

(b) If $\cos\theta=\cos\alpha$, then $\theta=2\pi n\pm\alpha$

(c) If $\tan\theta=\tan\alpha$, then $\theta=\pi n+\alpha$ where $n=0,\pm1,\pm2,...$

In degrees

(a) If $\sin\theta=\sin\alpha$, then $\theta=180^\circ n+(-1)^n\alpha$

(b) If $\cos\theta=\cos\alpha$, then $\theta=360^\circ n\pm\alpha$

(c) If $\tan\theta=\tan\alpha$, then $\theta=180^\circ+\alpha$

Example 6.36

Find the general solution of the equation $\cos x+\sin x=1$ giving your answer in terms of π. Hence, determine the value of x in the interval $-2\pi\le x\le 2\pi$.

Solution

Let $\cos x+\sin x=R\cos(x-\alpha)$.

$\Rightarrow R=\sqrt{a^2+b^2}=\sqrt{1^2+1^2}=\sqrt{2}$

But $\alpha=\tan^{-1}\frac{b}{a}=\tan^{-1}1=\frac{\pi}{4}$.

Thus,

$\cos x+\sin x=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=1$.

It follows that,

$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=1$

$\Rightarrow \cos\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow x-\frac{\pi}{4}=\cos^{-1}\frac{1}{\sqrt{2}}$

$\Rightarrow x-\frac{\pi}{4}=\frac{\pi}{4}$

Using the general solution of cosine,

$\theta=2\pi n\pm\alpha$

But $\theta=x-\frac{\pi}{4}$ and $\alpha=\frac{\pi}{4}$.

$\Rightarrow x-\frac{\pi}{4}=2\pi n\pm\frac{\pi}{4}$

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$\Rightarrow x = 2\pi n \pm \frac{\pi}{4} + \frac{\pi}{4}$

Therefore, $x = 2\pi n + \frac{\pi}{2}$ and $x = 2\pi n$, where n is an integer.

For $x = 2\pi n + \frac{\pi}{2}$, choose appropriate values of n as follows.

$n = 0;\ x = \frac{\pi}{2}$

$n = -1;\ x = \frac{-3\pi}{2}$

For $x = 2\pi n$, choose appropriate values of n as follows.

$n = 0;\ x = 0$

$n = 1;\ x = 2\pi$

$n = -1;\ x = -2\pi$

Therefore,

$x = -2\pi, \frac{-3\pi}{2}, 0, \frac{\pi}{2}$ and 2π.

Example 6.37

Find the general solution of the equation $\cot x + \operatorname{cosec} x = \sqrt{3}$, giving the answer in terms of π.

Solution

Given $\cot x + \operatorname{cosec} x = \sqrt{3}$.

$\Rightarrow \frac{\cos x}{\sin x} + \frac{1}{\sin x} = \sqrt{3}$

Multiplying by $\sin x$ both sides gives;

$\cos x + 1 = \sqrt{3}\sin x$

Re-arrange the equation to obtain;

$\sqrt{3}\sin x - \cos x = 1$

Using $\sqrt{3}\sin x - \cos x = R\sin(x - \alpha)$

$\Rightarrow \sqrt{3}\sin x - \cos x =$

$R\cos\alpha \sin x - R\sin\alpha \cos x$

But

$R = \sqrt{a^2 + b^2} + \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2.$

Comparing coefficients of $\sin x$ gives;

$R\cos\alpha = \sqrt{3}$ ..... (i)

Comparing coefficients of $\cos x$ gives;

$R\sin\alpha = 1$ ..... (ii)

Dividing equation (ii) by equation (i) gives;

$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$

Thus,

$\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right) = 1.$

$\Rightarrow 2\sin\left(x - \frac{\pi}{6}\right) = 1$

$\Rightarrow \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$\Rightarrow x - \frac{\pi}{6} = \sin^{-1}\frac{1}{2}$

$\Rightarrow x - \frac{\pi}{6} = \frac{\pi}{6}$

Using the general solution for sine,

$\theta = \pi n + (-1)^n \alpha$

But $\theta = x - \frac{\pi}{6}$ and $\alpha = \frac{\pi}{6}$

Thus, $x - \frac{\pi}{6} = \pi n + (-1)^n \frac{\pi}{6}.$

$\Rightarrow x = \pi n + (-1)^n \frac{\pi}{6} + \frac{\pi}{6}$

$\Rightarrow x = \pi n + \left((-1)^n + 1\right)\frac{\pi}{6}$

Therefore, the general solution is $x = \pi n + \left((-1)^n + 1\right)\frac{\pi}{6}$, where n is an integer.

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Example 6.38

Use the t-formulae to find the general solution of the equation $2\cos x - \sin x = 1$, giving the answers in radians. Hence, find the values of x for $0 \le x \le 2\pi$.

Solution

Given $2\cos x - \sin x = 1$.

From the t-formulae;

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},$$

where $t = \tan\frac{1}{2}x$.

Thus, $2\left(\frac{1-t^2}{1+t^2}\right) - \frac{2t}{1+t^2} = 1.$

Multiply both sides by $1+t^2$ to obtain;

$2(1-t^2) - 2t = 1+t^2$

$\Rightarrow 3t^2 + 2t - 1 = 0$

$\Rightarrow t = -1$ or $t = \frac{1}{3}$

Thus, $\tan\frac{1}{2}x = -1$ or $\tan\frac{1}{2}x = \frac{1}{3}$.

Case 1

From $\tan\frac{1}{2}x = -1$,

$\Rightarrow \frac{1}{2}x = \tan^{-1}(-1) = -\frac{\pi}{4}.$

Using the general solution of tangent,

$\theta = n\pi + \alpha$

But $\theta = \frac{1}{2}x$ and $\alpha = -\frac{\pi}{4}$.

Thus, $\frac{1}{2}x = \pi n - \frac{\pi}{4}.$

$\Rightarrow x = 2\pi n - \frac{\pi}{2}$

Case 2

From $\tan\frac{1}{2}x = \frac{1}{3}$

$\Rightarrow \frac{1}{2}x = \tan^{-1}\frac{1}{3} = 0.32$

The general solution of tangent is given by; $\theta = \pi n + \alpha$

But $\theta = \frac{1}{2}x$ and $\alpha = 0.32$

Thus, $\frac{1}{2}x = \pi n + 0.32$

$\Rightarrow x = 2\pi n + 0.64$

Therefore, the general solutions of the equation $2\cos x - \sin x = 1$ is $x = 2\pi n - \frac{\pi}{2}$ and $x = 2\pi n + 0.64$, where is an integer.

For $x = 2\pi n - \frac{\pi}{2}$, choose appropriate values of n as follows.

$n = 1;\ x = \frac{3\pi}{2}$

For $x = 2\pi n + 0.64$, choose appropriate values of n as follows.

$n = 0;\ x = 0.64$

Therefore, $x = 0.64$ and $\frac{3\pi}{2}$.

Exercise 6.6

Find the general solution of each of the following equations (leave the answer in radians):

1. $2\sin 2\theta = 1$
2. $\tan 5\theta = 1$

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3. $\sec\theta = \sqrt{2}$
4. $3\cos\theta + 5\sin\theta = 2$
5. $2\sin 2\theta + 7\cos 2\theta = -7.23$
6. $3\sin^2 x + \cos x \sin x = 3\cos^2 x + 2$
7. $\cos 2\theta + \sin\theta = 0$
8. $8\sin\theta + 15\cos\theta = 6$
9. $3\cos 2\theta + 5\sin\theta\cos\theta = 2$
10. $\sec\theta - \tan\theta = \sqrt{3}$

Factor formulae

In algebra, factors are commonly used to solve equations and simplify expressions. Similarly, in trigonometry, factors are used to factorize the sum and difference of two terms. Moreover, factors are used to express a product as a sum or difference of two terms. The factor formulae are applied in solving equations, simplifying expressions, and proving trigonometric identities. Factor formulae are deduced from the compound angle formulae as follows:

Consider the compound angle formulae for sine.

$$\begin{cases} \sin(A+B) = \sin A\cos B + \cos A\sin B \\ \sin(A-B) = \sin A\cos B - \cos A\sin B \end{cases}$$

Adding the two equations gives;

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ ..... (i)

Subtracting the two equations gives;

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$ ..... (ii)

Again, from the compound angle formulae for cosine,

$$\begin{cases} \cos(A+B) = \cos A\cos B - \sin A\sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B \end{cases}$$

Adding the two equations gives;

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$ .....(iii)

Subtracting the two equations gives;

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$ .....(iv)

The left-hand sides of the identities (i) to (iv) are the factors.

Let $P = A + B$ and $Q = A - B$

Adding P and Q gives,

$P + Q = 2A$

Thus, $A = \frac{P+Q}{2}$.

Subtracting Q from P gives, $P - Q = 2B$

Thus, $B = \frac{P-Q}{2}$.

Substituting $A = \frac{P+Q}{2}$ and $B = \frac{P-Q}{2}$ in equations (i), (ii), (iii), and (iv); gives,

$$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \quad (6.9)$$

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \quad (6.10)$$

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \quad (6.11)$$

$$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \quad (6.12)$$

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Equations (6.9) to (6.12) are known as factor formulae. They express the sum or difference as a product.

Example 6.39

Prove that $\sin\theta+\sin 2\theta+\sin 3\theta=\sin 2\theta(2\cos\theta+1)$.

Solution

Consider the left-hand side.

$$\sin\theta+\sin 2\theta+\sin 3\theta=\sin 2\theta+(\sin 3\theta+\sin\theta)$$

$$=\sin 2\theta+2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)$$

$$=\sin 2\theta+2\sin 2\theta\cos\theta$$

$$=\sin 2\theta(1+2\cos\theta)$$

Therefore, $\sin\theta+\sin 2\theta+\sin 3\theta=\sin 2\theta(2\cos\theta+1)$.

Example 6.40

In any triangle ABC, prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Solution

Consider the left-hand side.

$$\sin A+\sin B+\sin C=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$$

But $A+B=180°-C\Rightarrow\frac{A+B}{2}=90°-\frac{C}{2}$.

$$\Rightarrow\sin A+\sin B+\sin C=2\sin\left(90°-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]$$

$$=2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90°-\frac{A+B}{2}\right)\right]$$

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$$= 2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2\cos\left(\frac{C}{2}\right)\left[2\cos\left(\frac{\frac{A+B}{2}+\frac{A-B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2}-\frac{A-B}{2}}{2}\right)\right]$$

$$= 2\cos\left(\frac{C}{2}\right)\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right]$$

$$= 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right).$$

Therefore, $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Example 6.41

If $\sin x+\sin y = a$ and $\cos x+\cos y = b$, show that $\cos^2\frac{1}{2}(x-y) = \frac{1}{4}(a^2+b^2)$.

Solution

Given $\sin x+\sin y = a$ and $\cos x+\cos y = b$. It follows that

$$\sin x+\sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$

$$\Rightarrow 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Similary, $\cos x+\cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$

$$\Rightarrow 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Squaring equations (i) and (ii) gives;

$$4\sin^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$4\cos^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

Adding equations (iii) and (iv) result to,

$$4\sin^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y)+4\cos^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = a^2+b^2$$

$$\Rightarrow 4\cos^2\frac{1}{2}(x-y)\left[\sin^2\frac{1}{2}(x+y)+\cos^2\frac{1}{2}(x+y)\right] = a^2+b^2$$

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But $\sin^2\frac{1}{2}(x+y)+\cos^2\frac{1}{2}(x+y)=1$.

Thus, $4\cos^2\frac{1}{2}(x-y)=a^2+b^2$

$\Rightarrow \cos^2\frac{1}{2}(x-y)=\frac{1}{4}(a^2+b^2)$.

Therefore, $\cos^2\frac{1}{2}(x-y)=\frac{1}{4}(a^2+b^2)$.

Example 6.42

Simplify $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta}{\cos\theta+\cos 3\theta+\cos 5\theta}$.

Solution

$$\frac{\sin\theta+\sin 3\theta+\sin 5\theta}{\cos\theta+\cos 3\theta+\cos 5\theta}=\frac{(\sin\theta+\sin 5\theta)+\sin 3\theta}{(\cos\theta+\cos 5\theta)+\cos 3\theta}$$

$$=\frac{2\sin\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)+\sin 3\theta}{2\cos\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)+\cos 3\theta}$$

$$=\frac{2\sin 3\theta\cos(-2\theta)+\sin 3\theta}{2\cos 3\theta\cos(-2\theta)+\cos 3\theta}$$

But $\cos(-2\theta)=\cos 2\theta$.

$$\frac{\sin\theta+\sin 3\theta+\sin 5\theta}{\cos\theta+\cos 3\theta+\cos 5\theta}=\frac{\sin 3\theta(2\cos 2\theta+1)}{\cos 3\theta(2\cos 2\theta+1)}=\frac{\sin 3\theta}{\cos 3\theta}$$

$$=\tan 3\theta$$

Therefore, $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta}{\cos\theta+\cos 3\theta+\cos 5\theta}=\tan 3\theta$.

Example 6.43

Simplify $\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+B)$.

Solution

$\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+B)$

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$$= \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B)} + \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}$$

$$= \frac{\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B) + \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$

$$= \frac{\frac{1}{2}(\sin A - \sin B) + \frac{1}{2}(\sin A + \sin B)}{\frac{1}{2}(\cos A + \cos B)}$$

$$= \frac{\sin A - \sin B + \sin A + \sin B}{\cos A + \cos B}$$

$$= \frac{2\sin A}{\cos A + \cos B}$$

Therefore, $\tan\frac{1}{2}(A-B) + \tan\frac{1}{2}(A+B) = \frac{2\sin A}{\cos A + \cos B}$.

Example 6.44

Solve the equation $\cos 5x + \cos x = \cos 3x$ for $0 \le x \le \pi$ giving the answer in terms of π.

Solution

Given $\cos 5x + \cos x = \cos 3x$. Using

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

That is,

$$2\cos\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) = \cos 3x$$

$$\Rightarrow 2\cos 3x\cos 2x = \cos 3x$$

$$\Rightarrow 2\cos 3x\cos 2x - \cos 3x = 0$$

$$\Rightarrow \cos 3x(2\cos 2x - 1) = 0$$

Thus, $\cos 3x = 0$ or $2\cos 2x = 1$.

Applying $\cos^{-1}$ on both sides of the equation $\cos 3x = 0$ gives,

$$3x = \cos^{-1} 0$$

$$\Rightarrow 3x = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$$

Thus, $x = \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi, \ldots$

Similarly, applying $\cos^{-1}$ on both sides of the equation $2\cos 2x = 1$ gives,

$$2x = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow 2x = \frac{1}{3}\pi, \frac{5}{3}\pi, \ldots$$

Thus, $x = \frac{1}{6}\pi, \frac{5}{6}\pi, \ldots$

Therefore, the required values of x are $\frac{1}{6}\pi, \frac{1}{2}\pi$, and $\frac{5}{6}\pi$.

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Exercise 6.7

1. Express each of the following as a product:
   (a) $\sin 50° + \sin 40°$
   (b) $\sin 70° - \sin 20°$
   (c) $\cos 55° + \cos 25°$
   (d) $\cos 35° - \cos 75°$

2. Express each of the following as sums or differences:
   (a) $\sin 55° \sin 40°$
   (b) $\cos 110° \sin 55°$
   (c) $\sin 40° \cos 30°$
   (d) $\cos 50° \cos 35°$

3. Find the general solution of each of the following equations, expressing the answer in multiples of π.
   (a) $\sin 4\theta + \sin 2\theta + \sin 6\theta = 0$
   (b) $\cos x + \cos 2x + \cos 3x = 0$
   (c) $\sin 7x + \sin x + \sin 4x = 0$
   (d) $\cos 7\theta - \cos\theta = 2\sin 3\theta$
   (e) $\sin 3x + \sin 5x = \sin 4x$
   (f) $\sin 7x - \sin 5x = \sqrt{2}\cos 6x$

4. Prove each of the following identities:
   (a) $\cos 130° + \cos 110° + \cos 10° = 0$
   (b) $\cos^3 x \sin^2 x = \frac{1}{16}(2\cos x - \cos 3x - \cos 5x)$
   (c) $\frac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = \cot\frac{1}{2}(\alpha - \beta)\cot\frac{1}{2}(\alpha + \beta)$
   (d) $\frac{\tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B)} = \frac{\sin A - \sin B}{\sin A + \sin B}$

5. Show that:
   (a) $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \frac{\sqrt{3}}{3}$
   (b) $2\sin 82\frac{1}{2}° \cos 37\frac{1}{2}° = \frac{\sqrt{3} + \sqrt{2}}{2}$
   (c) $\cos 465° + \cos 165° = -\frac{\sqrt{6}}{2}$
   (d) $2\sin 45° \cos 15° = \frac{\sqrt{3} + 1}{2}$

6. If $\tan(p + q) = a$ and $\tan(p - q) = b$, express $\frac{\sin 2p + \sin 2q}{\sin 2p - \sin 2q}$ in terms of a and b. Hence, show that $\tan 2q = \frac{a - b}{1 + ab}$ and use this result to obtain an expression for $\tan(p + 3q)$ in terms of a and b.

7. By expressing $2\sin 3\theta \sin\frac{1}{2}\theta$ and other similar expressions as the difference of two sines, prove the identity $(2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1)\sin\frac{1}{2}\theta = \sin\frac{7}{2}\theta$. Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$ and deduce the identity

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$(8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1)\sin\frac{1}{2}\theta = \sin\frac{7}{2}\theta$, hence show that $\cos\frac{2}{7}\pi, \cos\frac{4}{7}\pi$, and $\cos\frac{6}{7}\pi$ are roots of the equation $8x^3 + 4x^2 - 4x - 1 = 0$.

8. Solve each of the following equations for $0° \le \theta \le 180°$.
   (a) $\cos 4\theta + \cos 6\theta + \cos 2\theta = 0$
   (b) $\cos(2\theta + 40°) + \cos(2\theta - 60°) = 0$
   (c) $\cos 4\theta + \cos 2\theta = \cos\theta$
   (d) $\sin 5\theta + \sin 3\theta = \sin(-4\theta)$

9. If $p = \cos x + \cos 2x + \cos 3x$ and $q = \sin x + \sin 2x + \sin 3x$, prove that:
   (a) $p = q\cot 2x$
   (b) $p^2 + q^2 = 3 + 4\cos x + 2\cos 2x$

10. Solve the equation $\sin 2y + \cos 2y = \sin y + \cos y$ for $0 \le y \le 2\pi$.

11. If $f(\lambda) = \dfrac{\sin 4\lambda + \sin 3\lambda + \sin 2\lambda + \sin\lambda}{\cos 4\lambda + \cos 3\lambda + \cos 2\lambda + \cos\lambda}$, then:
   (a) Simplify $f(\lambda)$.
   (b) Solve the equation $f(\lambda) = 1$ for $-180° \le \lambda \le 180°$.

12. Simplify each of the following expressions:
   (a) $\dfrac{\sin 10A + \sin 9A + \sin 8A + \sin 7A}{\cos 10A + \cos 9A + \cos 8A + \cos 7A}$
   (b) $\dfrac{\sin\left(x + \frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3} - x\right)}{\cos\left(x + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4} - x\right)}$

Radians and small angles

Radian is a measure of an angle which is defined as the size of the central angle subtended by an arc equal in length l to the radius r of a circle.

Converting degrees into radians

Consider Figure 6.6

Figure 6.6: *Sketch describing an arc of a circle.*

From Figure 6.6, an arc of length equal to the circumference of a circle subtends a central angle of one complete revolution. Since the circumference, $C = \pi r$ subtends an angle of $360°$, while the arc of length l subtends a central angle θ, where θ is in degrees, then

$$\frac{\text{Length of arc AB}}{\text{Circumference of a circle}} = \frac{\text{Measure of the central angle}}{\text{Total measurement in a circle}}$$

$$\Rightarrow \frac{l}{2\pi r} = \frac{\theta}{360°}$$

Thus, $l = \dfrac{\theta}{360°} \times 2\pi r = \dfrac{\pi r\theta}{180°}$.

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Again, $\frac{l}{r}=\frac{\pi\theta}{180^\circ}$.

But $\frac{l}{r}=s$, where s is an angle in radian.

Therefore, $s=\frac{\pi\theta}{180^\circ}$.

Note that, the radian measure has no dimension. It is a dimensionless number. Radians can be converted into degrees by multiplying by the factor $\frac{180^\circ}{\pi}$. Also, degrees can be converted into radians by multiplying by the factor $\frac{\pi}{180^\circ}$.

Example 6.45

Convert each of the following into radians giving the answers in multiples of π:

(a) 60° (b) 2970° (c) 1°

Solution

(a) From $s=\frac{\pi\theta}{180^\circ}$.

But $\theta=60^\circ$, thus,

$s=\frac{\pi\times 60^\circ}{180^\circ}=\frac{\pi}{3}$.

Therefore, $60^\circ=\frac{\pi}{3}$ radians.

(b) From $s=\frac{\pi\theta}{180^\circ}$.

But $\theta=2970^\circ$, thus,

$s=\frac{\pi\times 2970^\circ}{180^\circ}=16.5\pi$.

Therefore, $2970^\circ=16.5\pi$ radians.

(c) From $s=\frac{\pi\theta}{180^\circ}$.

But $\theta=1^\circ$.

$\Rightarrow s=\frac{\pi\times 1^\circ}{180^\circ}$

$\Rightarrow s=\frac{1}{180}\pi$

Therefore, $1^\circ=\frac{1}{180}\pi$ radians.

Example 6.46

Convert each of the following into degrees:

(a) $\frac{2\pi}{3}$ radians (c) 1 radian

(b) $\frac{37\pi}{5}$ radians

Solution

(a) From $s=\frac{\pi\theta}{180^\circ}$, then $\theta=\frac{180^\circ}{\pi}s$.

Thus, $\theta=\frac{180^\circ}{\pi}\times\frac{2\pi}{3}=120^\circ$.

Therefore, $\frac{2\pi}{3}$ radians $=120^\circ$.

(b) From $s=\frac{\pi\theta}{180^\circ}$, then $\theta=\frac{180^\circ}{\pi}s$.

Thus, $\theta=\frac{180^\circ}{\pi}\times\frac{37\pi}{5}=1332^\circ$.

Therefore, $\frac{37\pi}{5}$ radians $=1332^\circ$.

(c) From $s=\frac{\pi\theta}{180^\circ}$, then $\theta=\frac{180^\circ}{\pi}s$.

Thus, $\theta=\frac{180^\circ\times 1}{\pi}\approx 57.296^\circ$

Therefore, 1 radian $\approx 57.296^\circ$.

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Example 6.47

Find in radians an interior angle of a regular nonagon.

Solution

A regular nonagon has 9 equal sides. The sum of the 9 exterior angles = 360°.

$\Rightarrow$ Each exterior angle $= \frac{360°}{9} = 40°$.

$\Rightarrow$ Each interior angle $= 180° - 40° = 140°$.

From, $s = \frac{\pi\theta}{180°} = \frac{\pi \times 140°}{180°} = \frac{7\pi}{9}$ radians.

Therefore, an interior angle of a regular nonagon is $\frac{7\pi}{9}$ radians.

Example 6.48

Find the angle between the minute- hand and the hour-hand of a clock, when time is 5:45 pm. Give your answers;

(a) in degrees (b) in radians

Solution

(a) In the following figure, the hour-hand and the minute-hand are along the arcs RP and RQ, respectively.

Angle POQ = (angle subtended by the minute-hand in 45 minutes) – (angle subtended by the hour- hand in 5 hours and 45 minutes i.e. $\frac{23}{4}$)

$= \left(\frac{360}{60} \times 45\right)^{\circ} - \left(\frac{360}{12} \times \frac{23}{4}\right)^{\circ}$

$= 270° - 172.5°$

$= 97.5°$

Therefore, the required angle is 97.5°.

Altenatively,

$$\text{Angle POQ} = \left(\frac{\text{Position of a minute-hand} - \text{Position of an hour-hand}}{12}\right) \times 360°$$

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$$= \left(\frac{9-5.75}{12}\right) \times 360°$$

$$= 97.5°$$

Therefore, the required angle is 97.5°.

(b) Let $s = \frac{\pi\theta}{180°}$.

$$\Rightarrow s = \frac{\pi \times 97.5°}{180°} = \frac{13\pi}{24} \text{ radians}$$

Therefore, the required angle is $\frac{13\pi}{24}$ radians.

Exercise 6.8

1. Convert each of the following angles into radians, giving the answer in terms of π:
   (a) 30° (d) −135° (g) 315°
   (b) 240° (e) 40° (h) 405°
   (c) 90° (f) 270°

2. Convert each of the following angles into degrees:
   (a) $\frac{\pi}{3}$ (d) $\frac{-2\pi}{3}$ (g) $\frac{17\pi}{8}$
   (b) $\frac{5\pi}{3}$ (e) 3π (h) 2π
   (c) $\frac{7\pi}{6}$ (f) $\frac{7\pi}{5}$

3. Solve each of the following for $0 \le x \le 2\pi$, giving the answer in multiple of π.
   (a) $\cos x = \frac{\sqrt{3}}{2}$ (d) $\sin x = -\frac{1}{2}$
   (b) $\tan x = 1$ (e) $\cos x = -\frac{1}{\sqrt{2}}$
   (c) $\sin x = \frac{1}{\sqrt{2}}$ (f) $\tan x = \sqrt{3}$

4. Convert each of the following agles into degrees, giving the answer correct to 2 decimal places:
   (a) 0.49 radians (d) −0.85radians
   (b) 1.72 radians (e) 4.39 radians
   (c) −2.36 radians (f) −5.08 radians

5. Convert each of the following angles into radians, giving the answer in 3 significant figures:
   (a) 65.4° (c) 84°32′25″
   (b) 32°45′

6. The difference between two angles is 60° and their sum is $\frac{5\pi}{6}$ radians. Find the value of each angle, giving the answer in radians.

7. Find the angle in radians between the minute- hand and the hour-hand of a clock at:
   (a) 3:45 p.m (c) 7:20 a.m
   (b) 11:50 p.m

8. Find the interior angles in radian of a regular:
   (a) heptagon (c) octagon
   (b) hexagon

9. The difference between two angles is $\frac{5\pi}{4}$ and their sum is $\frac{11\pi}{4}$. Find the value of each angle in degrees.

10. Find the degree measures corresponding to each of the following radians:
    (a) $-\frac{5\pi}{14}$ (c) −2
    (b) $\frac{21\pi}{14}$ (d) $7\frac{2}{3}$

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Approximating small angles

If the measured angle in radians is small, the concept of small-angle approximations can be used to approximate the values of the principal trigonometric functions. The geometry of small angles has some unique properties.

Activity 6.4: Recognizing geometrical properties of small angles

Learning resources: Graph papers, scientific calculator, ruler, and pencil.
Individually or in a group, perform the following tasks:

1. Construct a table of values for $y = \sin\theta$, $y = \theta$, and $y = \cos\theta$ for convenient values of θ in radians.
2. Draw on the same xy-plane the graphs of the functions in task 1.
3. State the possible conclusions suggested by the graphs drawn in task 2.
4. What have you observed in task 1 about the values of θ (radians) for $\sin\theta$ and $\cos\theta$? Give comments.
5. Deduce the relationship between $\cos\theta$ and θ versus $\sin\theta$ and θ.
6. Share your findings with your fellow students for more inputs.

In Figure 6.7, the chord $\overline{AB}$ subtends an angle θ in radians at the centre O of a circle with radius r and the tangent at A meets $\overline{OB}$ at C.

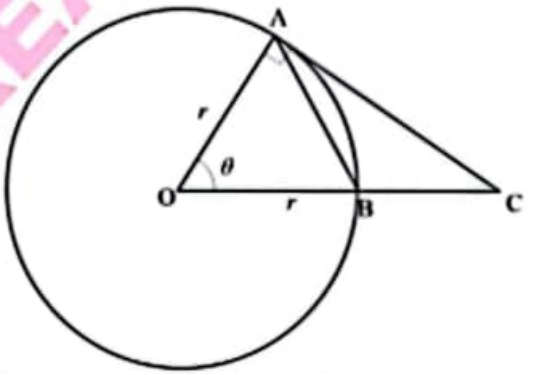

Figure 6.7: *Sketch for small angle formula derivation*

Figure 6.7, shows a sector of a circle, AOB centred at O with radius r. The line segment $\overline{OB}$ is extended to C and $\overline{CA}$ is a tangent to the circle at A. From Figure 6.7, it can be deduced that:

Area of triangle AOB < Area of sector AOB < Area of right-angled triangle AOC

But Area of triangle $AOB = \frac{1}{2}\overline{OA} \times \overline{OB}\sin\theta = \frac{1}{2}r^2\sin\theta$.

Area of sector $AOB = \frac{1}{2}r^2\theta$ and Area of right-angled triangle $AOC = \frac{1}{2}\overline{AC} \times r$.

But $\overline{AC} = r\tan\theta$.

Thus, Area of right-angled triangle $AOC = \frac{1}{2}r^2\tan\theta$.

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$\Rightarrow \frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$.

Dividing each term by $\frac{1}{2}r^2$ gives;

$\sin\theta < \theta < \tan\theta$, where $0 < \theta < \frac{\pi}{2}$.

Dividing each term by $\sin\theta$ gives;

$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$.

$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$.

For small angles, as $\theta \to 0$, $\cos\theta \to 1$ and $\frac{1}{\cos\theta} \to 1$, thus,

$1 < \frac{\theta}{\sin\theta} < 1$

Thus, $\frac{\theta}{\sin\theta} \approx 1$ as $\theta \to 0$.

Therefore, for small values of θ, $\sin\theta \approx \theta$.

An approximation for $\cos\theta$ is obtained from the identity

$\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$, where for small angles $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$. Thus,

$\cos\theta \approx 1 - 2\left(\frac{1}{2}\theta\right)^2$

$\Rightarrow \cos\theta \approx 1 - \frac{1}{2}\theta^2$.

Therefore, for small angles

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

The approximation of $\tan\theta$ in terms of θ is derived from the fact that

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ as $\theta \to 0$, $\cos\theta \to 1$,

and $\sin\theta \to \theta$

Therefore, for small values of θ, $\tan\theta \approx \theta$.

Example 6.49

Approximate the value in each of the following expressions when θ is small:

(a) $\frac{\sin 4\theta - \tan 2\theta}{3\theta}$ (b) $\frac{1-\cos 2\theta}{\tan 2\theta \sin\theta}$

(c) $\frac{3\tan\theta - \theta}{\sin 2\theta}$.

Solution

(a) Given $\frac{\sin 4\theta - \tan 2\theta}{3\theta}$.

As $\theta \to 0$, $\sin 4\theta \approx 4\theta$, $\tan 2\theta \approx 2\theta$.

Hence, $\frac{\sin 4\theta - \tan 2\theta}{3\theta} \approx \frac{4\theta - 2\theta}{3\theta} \approx \frac{2}{3}$.

Therefore, for small values of θ, $\frac{\sin 4\theta - \tan 2\theta}{3\theta} \approx \frac{2}{3}$.

(b) Given $\frac{1-\cos 2\theta}{\tan 2\theta \sin\theta}$.

As

$\theta \to 0$, $\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2 \approx 1 - 2\theta^2$,

$\sin\theta \approx \theta$, and $\tan 2\theta \approx 2\theta$.

Hence, $\frac{1-\cos 2\theta}{\tan 2\theta \sin\theta} \approx \frac{1-(1-2\theta^2)}{2\theta \times \theta}$

$\approx \frac{2\theta^2}{2\theta^2}$

≈ 1

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Therefore, for small values of θ, $\dfrac{1-\cos 2\theta}{\tan 2\theta \sin\theta} \approx 1$.

(c) Given $\dfrac{3\tan\theta-\theta}{\sin 2\theta}$.

As $\theta \to 0$, $\sin 2\theta \approx 2\theta$, and $\tan\theta \approx \theta$

Hence, $\dfrac{3\tan\theta-\theta}{\sin 2\theta} \approx \dfrac{3\theta-\theta}{2\theta} \approx 1$.

Therefore, for small values of θ, $\dfrac{3\tan\theta-\theta}{\sin 2\theta} \approx 1$.

Example 6.50

Find an approximation of $\dfrac{\cos(\alpha+\theta)-\cos\alpha}{\theta}$, when θ is small.

Solution

Given $\dfrac{\cos(\alpha+\theta)-\cos\alpha}{\theta}$.

By using the factor formula,

$$\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta} = \frac{-2\sin\left(\dfrac{\alpha+\theta+\alpha}{2}\right)\sin\left(\dfrac{\alpha+\theta-\alpha}{2}\right)}{\theta}$$

$$= \frac{-2\sin\left(\alpha+\dfrac{\theta}{2}\right)\sin\left(\dfrac{\theta}{2}\right)}{\theta}$$

As $\theta \to 0$, $\sin\left(\dfrac{\theta}{2}\right) \approx \dfrac{\theta}{2}$, and $\sin\left(\alpha+\dfrac{\theta}{2}\right) \approx \sin\alpha$.

$$\Rightarrow \frac{\cos(\alpha+\theta)-\cos\alpha}{\theta} \approx \frac{-2\sin\alpha}{\theta} \times \frac{\theta}{2}$$

$$\approx -\sin\alpha$$

Therefore, $\dfrac{\cos(\alpha+\theta)-\cos\alpha}{\theta} \approx -\sin\alpha$, when θ is small.

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Example 6.51

If θ is small, show that

$$\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2.$$

Solution

Required to show that

$$\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2.$$

Apply the compound angle formula for sine as follows.

$$\sin\left(\theta+\frac{\pi}{6}\right)=\sin\frac{\pi}{6}\cos\theta+\sin\theta\cos\frac{\pi}{6}$$

$$=\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta$$

Thus, as $\theta\to 0$, $\sin\theta\approx\theta$

and $\cos\theta\approx 1-\frac{1}{2}\theta^2$. It follows that,

$$\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}\left(1-\frac{1}{2}\theta^2\right)+\frac{\sqrt{3}}{2}\theta$$

$$\approx\frac{1}{2}-\frac{1}{4}\theta^2+\frac{\sqrt{3}}{2}\theta.$$

Therefore,

$$\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2.$$

Exercise 6.9

1. Approximate each of the following expressions for small values of θ:
   (a) $\dfrac{21+7\tan\theta-20\cos\theta}{1+\sin 2\theta}$
   (b) $\dfrac{\sin 3\theta+\tan 5\theta}{2\theta}$
   (c) $\dfrac{1+\sin\theta}{5+3\tan\theta-4\cos\theta}$
   (d) $\dfrac{\sin 3\theta+\tan\theta}{\cos 2\theta}$
   (e) $\sin(\theta+45°)$
   (f) $\dfrac{\cos\theta-1}{\sin\theta}$
2. Suppose that x is small, use the small angle approximation formulae to show that $4\cos 4x+\cos^2 2x\approx 5-36x^2+4x^4$ hence, find the approximation of $4\cos 4x+\cos^2 2x$ when $x=3$ radians.
3. If θ is small, show that $\tan 3\theta\cos 2\theta$ can be approximated by $3\theta-6\theta^3$, hence approximate the value of $\tan 0.3\cos 0.2$.
4. If θ is small, prove that $\dfrac{\tan(\alpha+\theta)-\tan\alpha}{\theta}\approx 1+\tan^2\alpha.$
5. If θ is small, show that $\cos\left(\dfrac{2\pi}{3}-\theta\right)\approx\dfrac{1}{4}(\theta^2+2\sqrt{3}\theta-2).$
6. Find an approximate value of $\dfrac{1-\cos 4\theta+\sin\theta-\sin\theta\cos 4\theta}{1+\sin\theta}$ when θ is small.
7. Simplify the expression, $\dfrac{\sin(3\alpha+2\theta)\sin 2\theta}{3\theta}$ when θ is small.
8. Use the identity $\sin^2 x+\cos^2 x=1$, the binomial theorem, and the

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approximation $\sin x \approx x$ to verify that $\cos x \approx 1-\frac{1}{2}x^2$, where x is small. Hence, deduce that $\cos 4° \approx 0.9976$.

9. If θ is a small angle in radians, find the limit of $\frac{\sin 3\theta + \tan 6\theta}{\cot 2\theta + 1 + 3\theta}$ as θ tends to zero.

10. Evaluate each of the following when θ is small:

(a) $\frac{1-\sin\theta-7\sin^2\theta+3\sin^3\theta}{1-3\sin\theta}$

(b) $\tan(\alpha+\theta)$

Trigonometric functions

Trigonometric functions are the periodic functions which denote the relationship between an angle and the sides of a right-angled triangle. They are recognized as circular functions that can be interpreted as the functions of an angle of triangle.

Trigonometric functions are important because they are periodic or repeating, which can be used to model many naturally occurring periodic processes.

If *f(x)* is any function of x, then the function obtained by replacing x by any of the trigonometric ratios is an example of a trigonometric function.

In general, any function of the trigonometric ratios is a trigonometric function.

Domain and range of trigonometric functions

The domain of a given function is the set of all possible inputs (x-values) to the functions, while the range is the set of all possible outputs (y-values) of the function.

The graph of sine and cosine functions behave like waves which oscillate between the amplitudes -1 and 1, inclusive with a periodic unit of 2π.

The domain of $f(x)=\sin x$ is the set of all real numbers, that is $\{x : x \in \mathbb{R}\}$, while the range is the set $\{f(x) : -1 \le f(x) \le 1\}$.

Similarly, the domain of $f(x)=\cos x$ is the set of all real numbers, that is $\{x : x \in \mathbb{R}\}$ while the range is the set $\{f(x) : -1 \le f(x) \le 1\}$.

Moreover, for the function $f(x)=\tan x=\frac{\sin x}{\cos x}$, then the domain of $f(x)=\tan x$ is the set of all real values except the values where $\cos x$ is zero. Therefore, the domain of $f(x)=\tan x$ is $\left\{x : x \in \mathbb{R}, x \neq \frac{\pi}{2}+\pi n\right\}$, for all integer numbers n and the range is $\{f(x) : f(x) \in \mathbb{R}\}$.

Example 6.52

Give the domain and range of $f(x)=2\sin x$.

Solution

Given $f(x)=2\sin x$.

Domain of $f(x)=2\sin x$ is the set of all real numbers, since the sine function is defined over the entire set of real numbers.

Thus, domain $=\{x : x \in \mathbb{R}\}$.

The range of $f(x)=2\sin x$ is determined by using the following procedures:

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Since, $-1 \le \sin x \le 1$, multiplying by 2 throughout gives $-2 \le 2\sin x \le 2$. Thus, the range of $f(x) = 2\sin x$ is $\{f(x) : -2 \le f(x) \le 2\}$.

Therefore, the domain $= \{x : x \in \mathbb{R}\}$ and range $= \{f(x) : -2 \le f(x) \le 2\}$.

Example 6.53

Determine the domain and range of $g(x) = \frac{2}{3}\cos\frac{x}{2}$, for $-2\pi \le x \le 2\pi$.

Solution

Given

$g(x) = \frac{2}{3}\cos\frac{x}{2}$, for $-2\pi \le x \le 2\pi$.

The domain is the set $\{x : -2\pi \le x \le 2\pi\}$.

Range: Since $-1 \le \cos\frac{x}{2} \le 1$.

Multiplying $\frac{2}{3}$ throughout the inequalities gives $-\frac{2}{3} \le \frac{2}{3}\cos\frac{x}{2} \le \frac{2}{3}$.

Thus,

$\text{Range} = \left\{g(x) : -\frac{2}{3} \le g(x) \le \frac{2}{3}\right\}$.

Therefore,

the domain $= \{x : -2\pi \le x \le 2\pi\}$ and

the range $= \left\{g(x) : -\frac{2}{3} \le g(x) \le \frac{2}{3}\right\}$.

The graphs of sine and cosine functions

The sine and cosine functions are periodic functions with a period of 2π and are defined for all real values of θ. For $\theta \ge 0$, the first period of the function $y = \sin\theta$ is between 0 and 2π radians, and the second period is between 2π and 4π radians and so on. In general, the n^{th} period is between $2(n-1)\pi$ and $2\pi n$ radians.

Similarly, for $y = \cos\theta$ the n^{th} period is between $\frac{\pi}{2} + 2(n-1)\pi$ and $\frac{\pi}{2} + 2\pi n$.

Activity 6.5: Drawing graphs of $f(\theta) = \sin\theta$

Learning resources: Graph paper, pencil, scientific calculator, and ruler.

Individually or in a group, perform the following tasks:

1. Construct a table of values of $f(\theta) = \sin\theta$ for $-2\pi \le \theta \le 2\pi$.
2. Draw the graph of $f(\theta) = \sin\theta$ on the graph paper.
3. Identify the properties of the graph drawn in task 2.
4. Share your results with your fellow students.

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Example 6.54

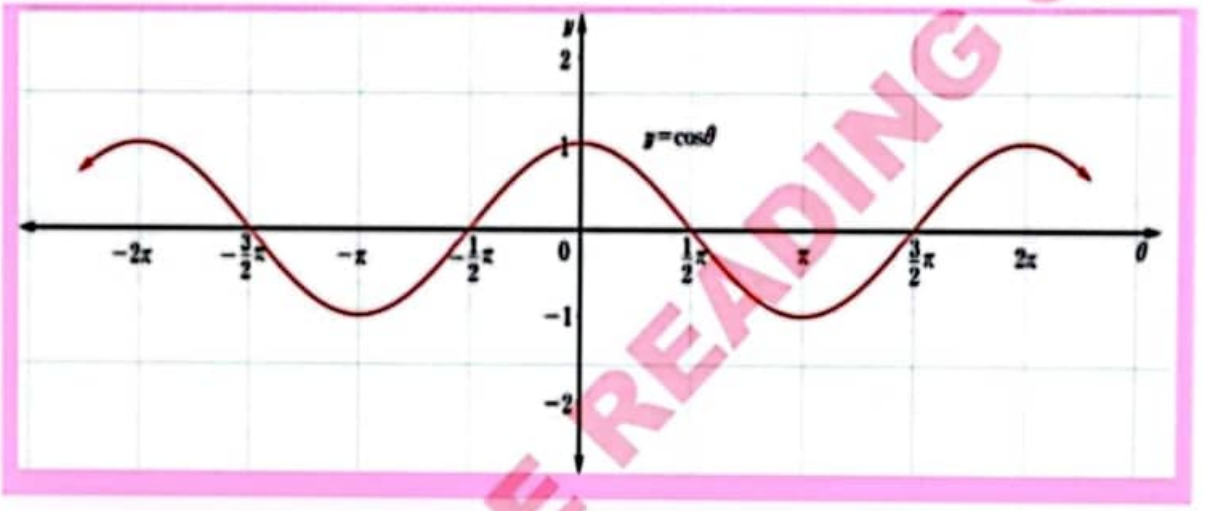
Draw the graph of $f(\theta) = \cos\theta$ for $-2\pi \le \theta \le 2\pi$.

Solution

In order to draw the graph of $f(\theta) = \cos\theta$ prepare a table of values as follows.

θ	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$f(\theta) = \cos\theta$	1	0	−1	0	1	0	−1	0	1

Use the values in the table to draw the following graph.



From the graph in Example 6.54, the properties of the cosine function are summarized as:

1. The minimum and maximum values are -1 and 1, respectively.
2. The cosine function has a period of 2π radians.
3. The cosine is an even function, that is $f(-x) = \cos(-x) = f(x)$.
4. The amplitudes of the function are -1 and 1.

Graph of the tangent function

The graph of $f(\theta) = \tan\theta$ is symmetrical about the origin (0, 0), since it is an odd function. The function $f(\theta) = \tan\theta$ is a periodic function with a period of π and it is defined at all points except where $\theta = \frac{\pi}{2} + n\pi$, n is any integer. The lines $\theta = \frac{\pi}{2} + n\pi$ are vertical asymptotes since $f(\theta) = \tan\theta$ tends to $\pm\infty$ as $\theta \to \frac{\pi}{2} + n\pi$.

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Example 6. 55

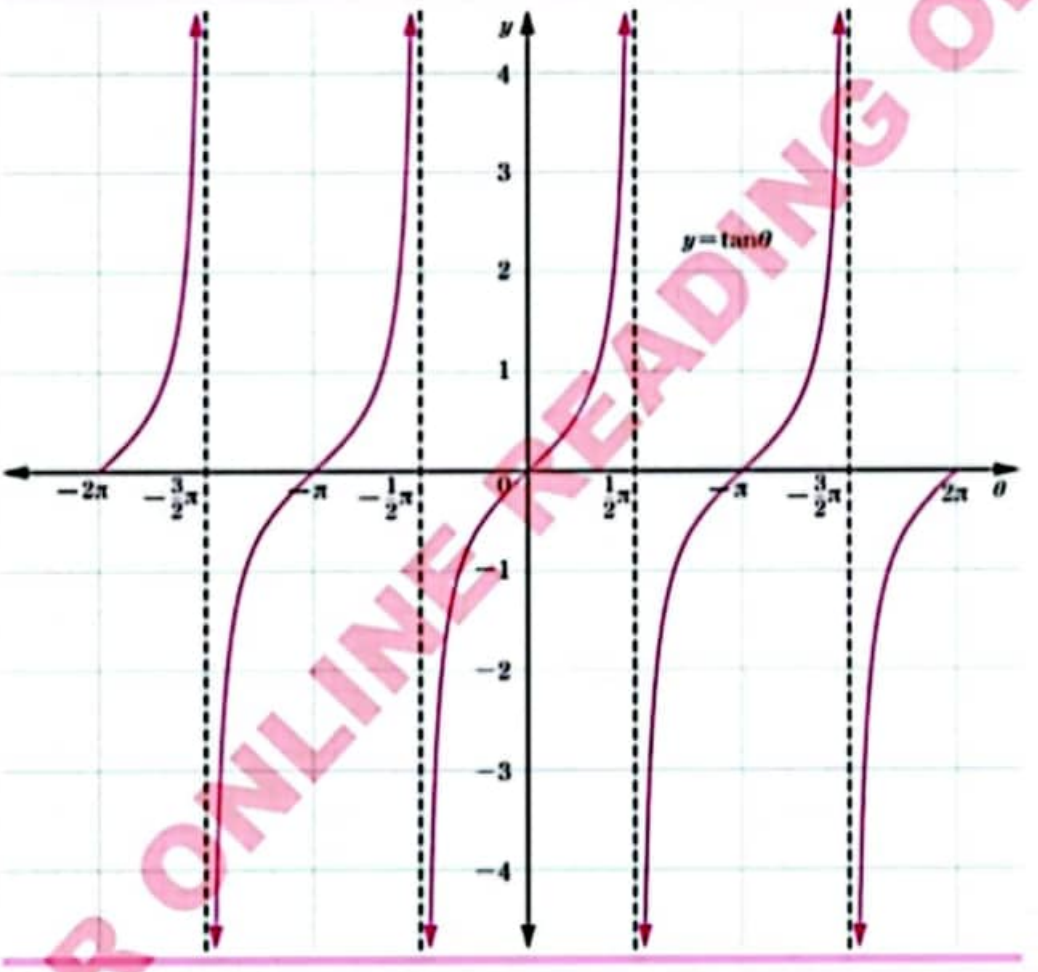
Draw the graph of $f(\theta) = \tan\theta$.

Solution

In order to draw the graph of $f(\theta) = \tan\theta$ prepare a table of values as follows.

θ	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$f(\theta) = \tan\theta$	0	Undefined	0	Undefined	0	Undefined	0	Undefined	0

Use the values in the table to draw the following graph.



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Exercise 6.10

Draw the graph in each of the following, hence identify the domain and range:

1. $f(x) = \sec x$
2. $f(x) = \tan 2x$
3. $g(x) = \cot x$
4. $h(x) = \operatorname{cosec} x$
5. $g(x) = \sec\frac{x}{2}$
6. $f(x) = -2\cos x$, for $-\pi \le x \le \pi$

7. $g(x) = -4\tan 2x$

8. $h(x) = \frac{1}{2}\sec x$

9. $f(x) = 6\sin\frac{x}{4}$, for $0 \le x \le \pi$

10. $f(x) = \tan x$, *for* $-2\pi \le x \le 2\pi$

Inverse trigonometric functions

The inverse trigonometric functions are inverses of sine, cosine, tangent, cotangent, secant, and cosecant. In trigonometry, inverse functions are used to find the angle of any trigonometric ratios. The inverse of sine, cosine, and tangent are defined as follows;

(i) $\theta = \sin^{-1} x$ or $\theta = \arcsin x$ if and only if $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(ii) $\theta = \cos^{-1} x$ or $\theta = \arccos x$ if and only if $x = \cos\theta$ and $0 \le \theta \le \pi$.

(iii) $\theta = \tan^{-1} x$ or $\theta = \arctan x$ if and only if $x = \tan\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The inverses of reciprocal of trigonometric functions can be defined using the inverse of sine, cosine and tangent. For instance, if $y = \sec\theta$, then $\theta = \sec^{-1} y$ can be expressed in terms $\cos^{-1} y$ as follows,

Given $y = \sec\theta$.

$\Rightarrow \theta = \sec^{-1} y$ ..... (i)

Also, $y = \frac{1}{\cos\theta}$

$\Rightarrow \cos\theta = \frac{1}{y}$. That is,

$\theta = \cos^{-1}\left(\frac{1}{y}\right)$ ..... (ii)

Equate equations (i) and (ii) to obtain;

$\theta = \sec^{-1} y = \cos^{-1}\frac{1}{y}$, for $y \ge 1$ or $y \le 1$

Therefore, $\sec^{-1} y = \cos^{-1}\frac{1}{y}$.

Using a similar approach the following results can be derived:

(i) If $y = \text{cosec}\theta$, then $\theta = \text{cosec}^{-1} y = \sin^{-1}\frac{1}{y}$, for $y \ge 1$ or $y \le 1$.

(ii) If $y = \cot\theta$, then $\theta = \cot^{-1} y = \tan^{-1}\frac{1}{y}$, for $0 < y < \pi$.

Example 6.56

Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$.

Solution

Consider the left-hand side.

Let $x = \sin\theta$. It follows that $\theta = \sin^{-1} x$.

Thus,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$
$$= \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$
$$= \sin^{-1}(2\sin\theta\cos\theta)$$
$$= \sin^{-1}(\sin 2\theta) = 2\theta$$
$$= 2\sin^{-1} x$$

Therefore, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$.

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Example 6.57

If $\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)=\frac{\pi}{4}$, then find the positive value of x correct to four significant figures.

Solution

Given $\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)=\frac{\pi}{4}$.

Applying tangent on both sides gives, $\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)=\tan\frac{\pi}{4}$

Let $A=\tan^{-1}\left(\frac{1-4x}{1+6x}\right)$ and $B=\tan^{-1}\left(\frac{1+2x}{1-3x}\right)$.

From the identity, $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$.

$$\Rightarrow\frac{\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)\right)-\tan\left(\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)}{1+\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)\right)\tan\left(\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)}=1$$

From the relation $\tan\left(\tan^{-1}B\right)=B$.

$$\Rightarrow\frac{\left(\frac{1-4x}{1+6x}\right)-\left(\frac{1+2x}{1-3x}\right)}{1+\left(\frac{1-4x}{1+6x}\right)\left(\frac{1+2x}{1-3x}\right)}=1$$

$$\Rightarrow\frac{1-7x+12x^2-1-8x-12x^2}{1+3x-18x^2+1-2x-8x^2}=1$$

$$\Rightarrow 26x^2-16x-2=0$$

Either $x=0.7219$ or $x=-0.1066$.

Therefore, the positive value of x is 0.7219.

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Example 6.58

Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ in its simplest form.

Solution

To simplify $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$.

From the identities,

$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$, $\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 1$,

and $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$, the given expression can be written as,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)$$

Divide by $\cos\frac{x}{2}$ to both the numerator and denominator to obtain;

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

Therefore, $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \frac{\pi}{4} + \frac{x}{2}$.

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Example 6.59

If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$, prove that, $x^2+y^2+z^2+2xyz=1$.

Solution

Given $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$.

Let $A=\sin^{-1}x \Rightarrow x=\sin A$,

$B=\sin^{-1}y \Rightarrow y=\sin B$,

$C=\sin^{-1}z \Rightarrow z=\sin C$,

But $A+B+C=\frac{\pi}{2} \Rightarrow A+B=\frac{\pi}{2}-C$.

Applying cosine on both sides of the equation, $A+B=\frac{\pi}{2}-C$ gives;

$$\cos(A+B)=\cos\left(\frac{\pi}{2}-C\right)$$

Using compound angle formula gives,

$\cos A\cos B-\sin A\sin B=\sin C$

But

$\cos A=\sqrt{1-x^2}$ and $\cos B=\sqrt{1-y^2}$.

Thus, $\sqrt{1-x^2}\times\sqrt{1-y^2}-xy=z$

$\Rightarrow\sqrt{1-x^2}\times\sqrt{1-y^2}=z+xy$

Square both sides to get,

$$\left(\sqrt{1-x^2}\times\sqrt{1-y^2}\right)^2=(z+xy)^2$$

$$(1-x^2)(1-y^2)=(z+xy)(z+xy)$$

$$\Rightarrow 1-x^2-y^2+x^2y^2=z^2+2xyz+x^2y^2$$

Simplify the equation to obtain,

$x^2+y^2+z^2+2xyz=1$

Therefore, $x^2+y^2+z^2+2xyz=1$.

Example 6.60

Solve the equation

$\cos^{-1}x+\cos^{-1}\left(x\sqrt{3}\right)=\frac{\pi}{2}$.

Solution

Given $\cos^{-1}x+\cos^{-1}\left(x\sqrt{3}\right)=\frac{\pi}{2}$.

Let $A=\cos^{-1}x \Rightarrow \cos A=x$ and

$B=\cos^{-1}\left(x\sqrt{3}\right) \Rightarrow \cos B=\left(x\sqrt{3}\right)$

But $\sin A=\sqrt{1-\cos^2 A}$.

$\Rightarrow \sin A=\sqrt{1-x^2}$

Also, $\sin B=\sqrt{1-\cos^2 B}$.

$\Rightarrow \sin B=\sqrt{1-3x^2}$

Now, $A+B=\frac{\pi}{2}$

Apply cosine on both sides of the equation $A+B=\frac{\pi}{2}$ to obtain:

$\cos(A+B)=\cos\frac{\pi}{2}$

Thus, $\cos A\cos B-\sin A\sin B=0$.

Substituting the values gives;

$$\left(x^2\sqrt{3}\right)-\left(\sqrt{1-x^2}\right)\left(\sqrt{1-3x^2}\right)=0$$

$$\Rightarrow\sqrt{1-4x^2+3x^4}=x^2\sqrt{3}$$

Squaring both sides gives;

$1-4x^2+3x^4=3x^4$

Simplify the equation to obtain;

$4x^2-1=0$

Hence, $x=\pm\frac{1}{2}$.

But $x=-\frac{1}{2}$ does not satisfy the given equation.

Therefore, the value of x is $\frac{1}{2}$.

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Exercise 6.11

1. Evaluate each of the following expressions without using a non-programmable scientific calculator:

   (a) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

   (b) $\cos\left(\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{\sqrt{2}}{2}\right)$

   (c) $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$

   (d) $\cos\left(\cos^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{4}{5}\right)\right)$

2. Without using a non-programmable scientific calculator, find the value in each of the following:

   (a) $\tan x$, if $x=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{7}{24}$

   (b) $\sin x$, if $x=\sin^{-1}\frac{4}{5}-\cos^{-1}\frac{8}{17}$

   (c) $\cos x$, if $x=\cos^{-1}\frac{24}{25}+\sin^{-1}\frac{15}{17}$

   (d) $\sin x$, if $x=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}$

3. Prove each of the following trigonometric equations:

   (a) $\frac{1}{4}\pi-\tan^{-1}\frac{2}{3}=\tan^{-1}\frac{1}{5}$

   (b) $\tan^{-1}p+\tan^{-1}q=\tan^{-1}\left(\frac{p+q}{1-pq}\right)$

   (c) $\cos\left(2\tan^{-1}x\right)=\frac{1-x^2}{1+x^2}$

   (d) $\cot^{-1}\left(\frac{\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}}\right)=\frac{\theta}{2}$ where $\theta\in\left(0,\frac{\pi}{4}\right)$

   (e) $\cot^{-1}\left(\frac{\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}}\right)=\frac{\theta}{2},\ \theta\in\left(0,\frac{\pi}{4}\right)$

   (f) $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{2}\le x\le 1.$

   (g) $3\tan^{-1}2-\tan^{-1}\frac{2}{11}=\pi$

   (h) $2\tan^{-1}2+\tan^{-1}3=\pi+\tan^{-1}\frac{1}{3}$

   (i) $\frac{\pi}{4}+\tan^{-1}x=\tan^{-1}\left(\frac{1+x}{1-x}\right)$

   (j) $\tan^{-1}\left(\frac{\cos\alpha}{1+\sin\alpha}\right)=\frac{\pi}{4}-\frac{\alpha}{2}$

4. Simplify each of the following expressions:

   (a) $\cos\left(\sin^{-1}x\right)$

   (b) $\sin\left(\tan^{-1}x\right)$

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5. Solve each of the following equations:

(a) $\arcsin x + \arctan x = \frac{\pi}{2}$

(b) $\sin\left(2\cos^{-1} x\right) = \sqrt{1-x^2}$

(c) $\arctan 2x + \arctan x = \frac{\pi}{4}$

(d) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(x\sqrt{2}\right) = \frac{\pi}{2}$

(e) $\cos^{-1} x + \cos^{-1}\left(x\sqrt{8}\right) = \frac{\pi}{2}$

(f) $2\sin^{-1}\left(x\sqrt{6}\right) + \sin^{-1}(4x) = \frac{\pi}{2}$

(g) $\tan^{-1}\left(\frac{z-2}{z-4}\right) + \tan^{-1}\left(\frac{z+2}{z+4}\right) = \frac{\pi}{4}$

(h) $\arccos x + \arctan x = \frac{\pi}{2}$

(i) $\tan^{-1}\left(\frac{2y}{1-y^2}\right) = \pi - \cot^{-1}\left(\frac{1-y^2}{2y}\right)$

(j) $\cos\left(\tan^{-1} x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

6. If $\sin^{-1} y = 2\cos^{-1} x$, show that $y^2 = 4x^2\left(1-x^2\right)$.

7. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

8. For each of the following expressions, show that:

(a) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

(b) $2\sin^{-1}\left(\sqrt{x-a+\frac{1}{2}}\right) = \cos^{-1}(2a-2x)$

(c) $\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2} = \cot^{-1} y + \tan^{-1} y$

9. If $\tan^{-1}\left(\frac{1}{U}\right) + \tan^{-1}\left(\frac{1}{V}\right) = \tan^{-1}\left(\frac{1}{W}\right)$, verify that,
$(U-W)(V-W) = 1+W^2$,

10. Evaluate $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{13}{9}\right)$ without using a non-programmable scientific calculator.

Graphs of inverse trigonometric functions

The graphs of inverse trigonometric functions differ from those of trigonometric functions, the roles of y and θ are interchanged. For example, the graph of $y = \sin^{-1}\theta$ is a sine curve drawn on the y-axis instead of the x-axis. The domains of the inverse of trigonometric functions are restricted.

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Activity 6.6: Drawing the graph of $y = \sin^{-1} x$

Learning resources: Graph papers, pencil, scientific calculator, and ruler.
Individually or in a group, perform the following tasks:

1. Construct a table of values of $y = \sin^{-1} x$ for $-1 \le x \le 1$.
2. Draw the graph of $y = \sin^{-1} x$ on the xy-plane.
3. Identify the domain and range of $y = \sin^{-1} x$ from the graph drawn in task 2.
4. On a different xy-plane draw the graph of $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
5. Give the suggestions on the graphs drawn in tasks 2 and 4.
6. What have you observed from the tasks? Give comments.

Example 6.61

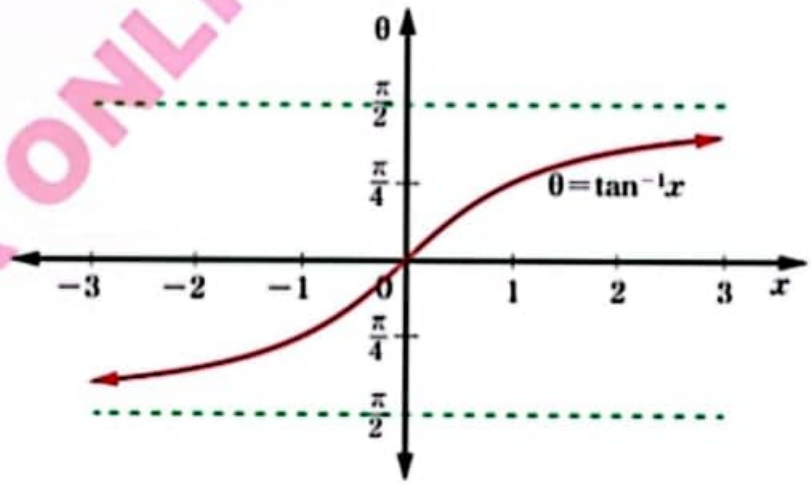
Draw the graph of $\theta = \tan^{-1} x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and identify its domain and range.

Solution

In order to draw the graph of $\theta = \tan^{-1} x$ prepare a table of values as follows;

x	Undefined	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	Undefined
$\theta = \tan^{-1} x$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Use the values in the table to draw the following graph.



From the graph, it implies that Domain $= \{x : x \in \mathbb{R}\}$ and Range $= \left\{\theta : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$.

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Example 6.62

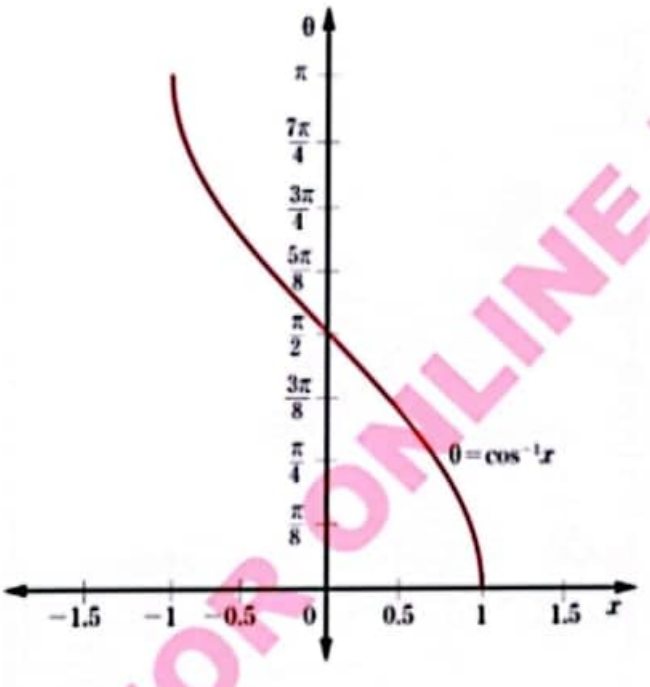
Draw the graph of $\theta = \cos^{-1} x$ for $-1 < x < 1$ and identify its domain and range.

Solution

In order to draw the graph of $\theta = \cos^{-1} x$ prepare a table of values as follows;

x	-1	0	1
$\theta = \cos^{-1} x$	π	$\frac{\pi}{2}$	0

Use the values in the table to draw the following graph.



From the graph, it implies that

Domain $= \{x \in \mathbb{R} : -1 \leq x \leq 1\}$

Range $= \{\theta \in \mathbb{R} : 0 \leq \theta \leq \pi\}$

Exercise 6.12

1. Draw the graph in each of the following functions:
   (a) $h(x) = \sin\left(\cos^{-1} x\right)$
   (b) $f(x) = \cos^{-1}(\tan x)$
   (c) $g(x) = \cot^{-1} x$
   (d) $g(x) = \sec^{-1} x$
   (e) $g(x) = \text{cosec}^{-1} x$
   (f) $h(x) = \cos\left(\sin^{-1}\frac{x}{2}\right)$
2. State the domain and range of each of the following functions:
   (a) $y = \sin^{-1} x$ (c) $y = \tan^{-1} 2x$
   (b) $y = \cot^{-1} x$
3. (a) To define an arccotangent function, first restrict cotangent to an interval on which it is one to one and takes on all real numbers. What is the interval for such restrictions?
   (b) Using the interval found in part (a) define $\cot x$ and $\cot^{-1} x$ functions, hence draw the graphs of both functions on the same axes.
4. Simplify
   $$x = \frac{\sin(y+\pi)+\sin(y-\pi)}{\sin\frac{\pi}{6}}$$ and draw the graph of the simplified equation.
5. Given $x = 2\cos^2 y - 1$, express y in terms of x and draw the graph of the resulting function.

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Chapter summary

1. The trigonometric ratios are $\sin\theta$, $\cos\theta$, and $\tan\theta$. Their corresponding reciprocals are $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$, respectively.

2. The trigonometric identities (Pythagorean identities) are:
   (a) $\cos^2\theta + \sin^2\theta = 1$
   (b) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
   (c) $1 + \tan^2\theta = \sec^2\theta$

3. The compound angle formulae are:
   (a) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
   (b) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
   (c) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

4. The double angle formulae are:
   (a) $\sin 2A = 2\sin A\cos A$
   (b) $\cos 2A = \cos^2 A - \sin^2 A$
   (c) $\cos 2A = 2\cos^2 A - 1$
   (d) $\cos 2A = 1 - 2\sin^2 A$
   (e) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

5. The t-formulae are:
   (a) $\sin\theta = \dfrac{2t}{1+t^2}$ (b) $\cos\theta = \dfrac{1-t^2}{1+t^2}$ (c) $\tan\theta = \dfrac{2t}{1-t^2}$ where $t = \tan\dfrac{1}{2}\theta$.

6. The factor formulae are given by:
   (a) $\sin P + \sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$
   (b) $\sin P - \sin Q = 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$
   (c) $\cos P + \cos Q = 2\cos\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$
   (d) $\cos P - \cos Q = -2\sin\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$.

7. For a small angle θ, the approximation of sine, cosine, and tangent are:
   (a) $\sin\theta \approx \theta$ (b) $\cos\theta \approx 1 - \dfrac{1}{2}\theta^2$ (c) $\tan\theta \approx \theta$

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Revision exercise 6

1. If $\cos A = \frac{21}{24}$ and $\sin B = -\frac{8}{17}$, where A and B are angles in the fourth and third quadrants, respectively, evaluate each of the following:
   (a) $\tan(A - B)$ (b) $\cos(A - B)$ (c) $\sin(A + B)$

2. Eliminate θ in each of the following pairs of equations:
   (a) $x = 9 + 4\cos\theta,\ y = 7 + 15\sin\theta$ (c) $x = 1 + \cos 2\theta,\ y = \sin\theta$
   (b) $x = 5 + 3\tan\theta,\ y = 4 + \tan 2\theta$ (d) $x = 4\sec\theta,\ y = \cos 3\theta$

3. Simplify each of the following expressions:
   (a) $\tan(45° + \theta) - \tan(45° - \theta)$ (c) $\frac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x}$
   (b) $\sin^3\theta + \sin^3(120° + \theta) + \sin^3(240° + \theta)$ (d) $\frac{\sin 5y - \sin y}{\sin 10y - \sin 6y}$

4. Verify each of the following identities:
   (a) $\frac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x} = \tan x$ (b) $\frac{\sin x}{\sin^2 x}\ \frac{\sin 3x}{\cos^2 x}\ 2\sin x$

5. Convert each of the following radian angles into degrees:
   (a) $\frac{23\pi}{9}$ (b) $\frac{51\pi}{29}$ (c) 7.5 (d) $-\frac{5\pi}{6}$

6. Find in radians the angle between the minute-hand and the hour-hand of a clock at:
   (a) 4:55 pm (b) 5:48 pm

7. Convert each of the following into radians:
   (a) $-540°$ (b) $-1050°$ (c) $120°30'45''$

8. In each of the following expressions, find its approximation when θ is a small angle:
   (a) $\frac{\sin^2(3\theta) + 2\theta}{\tan\theta}$ (b) $\frac{1 - \cos 4\theta}{2\theta\sin 3\theta}$ (c) $\frac{\cos 3\theta - 1}{\theta\sin 4\theta}$ (d) $\tan 3\theta\sin 2\theta$

9. Prove each of the following identities:
   (a) $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{1}{\sec\theta - \tan\theta}$
   (b) $\frac{1 - \sec x}{\sec x} = \frac{\sin^2 x}{1 - \cos x}$
   (c) $\frac{\tan\theta}{1 - \cot\theta} + \frac{\cot\theta}{1 - \tan\theta} = 1 + \sec\theta\operatorname{cosec}\theta$

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(d) $\dfrac{\tan\theta+\tan 7\theta}{\tan 3\theta+\tan 5\theta}=\dfrac{\cos 3\theta\cos 5\theta}{\cos\theta\cos 7\theta}$

(e) $\left(\dfrac{\cos\theta-\cos 3\theta}{\sin 5\theta-\sin\theta}\right)\left(\dfrac{\sin 8\theta+\sin 2\theta}{\cos 4\theta-\cos 6\theta}\right)=1$

(f) $\cos\left(\dfrac{1}{4}\pi+\theta\right)\cos\left(\dfrac{1}{4}\pi+\theta\right)=\dfrac{1}{2}\cos 2\theta$

(g) $\sqrt{\dfrac{1+\sin x}{1-\sin x}}=\sec x+\tan x$

(j) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta$

(h) $(\operatorname{cosec} x-\sin x)(\sec x-\cos x)=\dfrac{1}{\tan x+\cot x}$

(i) $\dfrac{\sin x-2\sin^3 x}{2\cos^3 x-\cos x}=\tan x$

(k) $\dfrac{\sin A\sin B}{\cos A+\cos B}=\dfrac{2\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan^2\frac{A}{2}\tan^2\frac{B}{2}}$

10. Use the t-formula to solve the following equations for $-180°\le\theta\le 180°$:

(a) $5\sin 2\theta-12\cos 2\theta=1$ (b) $2\sin\theta+7\cos\theta=-4$

11. Find the general solution in each of the following equations:

(a) $\sin x-\sin 2x=\sin 4x-\sin 3x$

(b) $\cos\theta+\sin\theta=1$

(c) $\cos\theta-\sin 4\theta=0$

(d) $3\cos\theta+\sin\theta=2$

(e) $\cos ax+\cos bx=0$, where a and b are constants

(f) $6\cos\theta-4\sin\theta=7$

(g) $\sin\theta+\sin 3\theta+\sin 5\theta=0$

(h) $\tan 3\theta=\cot\theta+\cot 2\theta$

12. Find the solution in each of the following equations for the angle $0°$ and $360°$, inclusive.

(a) $\begin{cases}\tan x+\tan y=4\\ \tan 2x+\tan 2y=0\end{cases}$

(b) $\begin{cases}\cos x+\cos y=\cos\dfrac{7\pi}{12}\\ \sin x+\sin y=\sin\dfrac{7\pi}{12}\end{cases}$

13. If $\sin\theta=\dfrac{4}{5}$, find the value of $\dfrac{5\sec\theta+4\operatorname{cosec}\theta+3\tan\theta}{4\cot\theta+3\sec\theta+5\sin\theta}$.

14. Prove the identity $\cos 4\theta+4\cos 2\theta=8\cos^4\theta-3$, hence solve the equation $\cos 4\theta+4\cos 2\theta=2$ for θ between $0°$ and $360°$.

15. Show that $\operatorname{cosec}2\theta+\cot 2\theta=\cot\theta$, hence solve the equation $\operatorname{cosec}2\theta+\cot 2\theta=2$, for $0°\le\theta\le 360°$.

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16. Solve each of the following equations by using t-formulae for $0° \le x \le 360°$:

(a) $3\sin x + 4\cos x = 2$

(b) $2\cos x - \sin x = 1$

(c) $2\cos x + 3\sin x = 2$

(d) $4\cos x \sin x + 15\cos 2x = 10$

(e) $3\tan x - 2\sec x = 4$

(f) $\sqrt{3}\cos x + \sin x = 1$

17. Find the values of R and α in each of the following equation:

(a) $5\cos\theta + 12\sin\theta = R\cos(\theta - \alpha)$

(b) $3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha)$

(c) $6\sin 3\theta + 8\cos 3\theta = \mathrm{R}\sin(3\theta + \alpha)$

(d) $\cos 2\theta + \sin 2\theta = R\cos(2\theta - \alpha)$

18. Find the maximum and minimum values in each of the following expressions, and their corresponding values of θ for $0° \le \theta \le 360°$.

(a) $\cos\theta + \sin\theta$

(b) $3\sqrt{2}\cos(\theta + 45°) + 7\sin\theta$

(c) $8\cos\theta - 15\sin\theta$

(d) $\sin\theta - 6\cos\theta$

19. Express $3\cos x + 4\sin x$ in the form $R\sin(x + \alpha)$, where α is an acute angle. Hence, find the maximum and minimum values of the expression and their corresponding values of x for $-180° \le x \le 180°$ at which the maximum and minimum occur.

20. If θ is small, simplify each of the following:

(a) $\dfrac{\cos(\theta + \alpha) - \cos(\theta - \alpha)}{4\theta}$ (b) $\dfrac{(1 - \cos 2\theta)(1 + \tan\theta)}{3\tan\theta\sin\theta}$ (c) $\tan\left(\dfrac{\pi}{3} + \theta\right)$

21. Express $5\sin^2 x - 3\sin x\cos x + \cos^2 x$ in the form $a + b\cos(2x - \alpha)$, where a, b, α are independent of x. Hence or otherwise, find the maximum and minimum value of $5\sin^2 x - 3\sin x\cos x + \cos^2 x$ as x varies.

22. Express $\tan(45° + x) - \tan x = 2$ in the form of $\tan^2 x + 2\tan x - 1 = 0$. Hence, solve the equation $\tan(45° + x) - \tan x = 2$, giving all solutions in the interval $0° < x < 180°$.

23. (a) If $a\cos^2 x + b\sin^2 x = c$, show that $\tan^2 x = \dfrac{c - a}{b - c}$. Hence, deduce that $\tan^2 x = \dfrac{1}{3}$ given that $6\cos^2 x + 2\sin^2 x = 5$.

(b) If $\cos\theta + \cot\theta - x = 0$ and $\cos\theta - \cot\theta + y = 0$. Prove that $\left(x^2 - y^2\right)^2 = 16xy$.

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24. If $t = \tan\frac{1}{2}\theta$, find the values of t which satisfy the equation $(p+2)\sin\theta + (2p-1)\cos\theta = 2p+1$, where p is a non-zero constant.
Hence, find the angles which satisfy the equation when $p = \sqrt{3}$, for $-180° < \theta < 180°$.

25. Express each of the following expressions in factor form:
(a) $\sin 16\theta - \sin 7\theta$
(b) $\cos\left(-\frac{27}{5}\theta\right) - \cos\left(\frac{16}{5}\theta\right)$
(c) $\sin 13\theta - \sin(-9\theta)$
(d) $\cos\left(\frac{5\theta}{2}\right) + \cos\left(\frac{9\theta}{2}\right)$

26. If $\sin x + \sin 2x = a$ and $\cos x + \cos 2x = b$, prove that $(a^2+b^2)(a^2+b^2-3) = 2b$.

27. Prove each of the following:
(a) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
(b) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$
(c) $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1} x\right)\right)\right) = \left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$
(d) $\tan^{-1} x + \tan^{-1} y = \frac{1}{2}\sin^{-1}\left[\frac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right]$
(e) $\cot^{-1}\frac{1}{3} = \cot^{-1} 3 + \cot^{-1}\frac{3}{4}$
(f) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$

28. Find the values of x which satisfy the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-6x}{1+6x}\right) = \frac{5\pi}{4}$.

29. If the equation $m\cos 2\theta + n\sin 2\theta = p$ has θ_1 and θ_2 as its root, prove that $\tan\theta_1 + \tan\theta_2 = \frac{2n}{p+m}$.

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30. If $x = p\cos\theta + q\sin\theta$ and $y = p\sin\theta - q\cos\theta$, show that $x^2 + y^2 = p^2 + q^2$.

31. Show that $\dfrac{1+\sin x-\cos x}{1+\sin x+\cos x} + \dfrac{1+\sin x+\cos x}{1+\sin x-\cos x} = \dfrac{2}{\sin x}$.

32. Prove that $\dfrac{\sin(x+3y)+\sin(3x+y)}{\sin 2x+\sin 2y} = 2\cos(x+y)$.

33. Show that $\tan\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right) = \sqrt{y}$ given that $\sin\theta = \dfrac{1-y}{1+y}$.

34. Show that the general solution of $\tan\left(3x - \dfrac{\pi}{4}\right) = \tan x$ is $x = \dfrac{(4n+1)\pi}{8}$, where n is an integer.

35. Prove that $\cos x - \sin x = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right) = -\sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)$. Hence, evaluate $\cos x - \sin x = 1$, for $-2\pi \le x \le 2\pi$.

36. Express $3\sin 6\theta + 5\cos 6\theta$ in the form $R\cos(6\theta - \alpha)$. Hence, or otherwise solve the equation $3\sin 6\theta + 5\cos 6\theta = 5$, for $0° \le \theta \le 360°$.

37. Given that $t = \tan x$, write an expression for $\tan 2x$ in terms of t. Hence, or otherwise, find the general solution in radians, of the equation $\tan x + \tan 2x = 0$.

38. Prove each of the following identities:

(a) $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$ (b) $\dfrac{\cos\theta+\cos 2\theta+\cos 3\theta+\cos 4\theta}{\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta} = \cot\dfrac{5}{2}\theta$

39. If $\sin x = \dfrac{1}{4}$ and x is in the second quadrant, find each of the following equations (leaving the answers in surd form):

(a) $\sin\dfrac{x}{2}$ (b) $\cos\dfrac{x}{2}$ (c) $\tan\dfrac{x}{2}$

40. Use the compound angle formula to show that $\cot(\lambda - \beta) = \dfrac{1+\cot\lambda\cot\beta}{\cot\beta - \cot\lambda}$.

Hence, deduce that, if $\cot\lambda = 0.5$, $\cot\beta = 2$, and $\cot\gamma = 3$, then

$\cot(\lambda+\beta+\gamma) = 3$.

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Chapter Seven

Linear programming

Introduction

Most of the production companies, industries and businesses focus on optimization of profit in such a way that operational costs are minimized in order to maximize profit. Linear programming is a mathematical method used to determine optimal solutions to problems that can be expressed as linear equations and inequalities. In this chapter, you will learn about formulation and graphical solutions of linear transportation problems. The competencies developed will help you to decide, allocate, select, schedule, and evaluate resources in the best possible way for the purpose of optimizing the available resources especially in the fields of agriculture, economics, finance, business, engineering, energy, manufacturing, and transportation, among many other applications.

Think

Decision-making in transportation without knowledge of linear programming.

Formulation of linear programming problems

Formulation of a linear programming problem involves translating a real life problem into a mathematical algebraic equations or inequalities, aiming at optimizing the scarce resources available. Formulation of a linear programming problem requires decision variables, an objective function, and a number of constraints.

Decision variables

The decision variables are unknown quantities that determine the output. These are a set of quantities that needs to be determined in order to solve the problem. They represent the ultimate solution. To solve any linear programming problem, the decision variables need to be identified. These variables are typically denoted by letters such as $x_1, x_2, \cdots, x_n$. For instance, x_1

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may represent the number of products of type A manufactured per month, and x_2 can represent the number of products of type B manufactured per month.

Objective function

An objective function is a linear function whose value is to be either minimized or maximized subject to a number of constraints imposed on the set of decision variables. For instance, in a maximization problem, the objective function is written as: Maximize $z = ax + by$, while for a minimization problem, it is written as: Minimize $z = ax + by$.

Constraints

Constraints are inequalities or equations which connect the decision variables under certain restrictions or limitations. Usually, constraints limit the values of the decision variables due to availability of resources. The constants for a maximization problem have the form $ax + by \leq c$, while for a minimization problem they have the form $ax + by \geq c$. The constants a and b are proportional contributions of each decision variable to both the objective function and the constraints.

Non-negativity constraints

The decision variables should always take non-negative values for all linear programming problems. That is, the values for decision variables should be greater than or equal to zero. That is, $x_1 \geq 0,\ x_2 \geq 0,\ \cdots,\ x_n \geq 0$.

Graphical solution

The constraints of the problem can be represented graphically. The representation is obtained by treating the inequalities as linear equations, but the set of inequalities is used to determine the region satisfying all the constraints. The following terminologies are used when graphing linear programming problems.

Optimization problem

An optimization problem is a problem where a particular objective function is maximized or minimized subject to constraints over the set of feasible solutions. It is the problem of finding the most desirable solution from the feasible solutions. An optimal problem may involve maximization of profit of production, or minimization of cost using the available resources.

Feasible region

A feasible region is the region containing the set of all possible solutions that satisfy all the constraints. The feasible region includes also the boundary lines. If the feasible region is enclosed by a polygon, it is said to be bounded, otherwise it is unbounded.

Feasible solution

A feasible solution is a solution that satisfies all constraints. Points within and on the boundary of the feasible region represent feasible solutions of the problem.

Optimal solution

An optimal solution is the solution in the feasible region that gives the optimal

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value. It is the largest or smallest value of the objective function.

Optimal value

The optimal value is the value of the optimal solution that maximizes or minimizes the objective function of the linear programming problem.

Optimal point

The optimal point is a point in the feasible region where the objective function attains its optimal value.

Steps for solving a linear programming problem graphically

The following are the steps of graphing linear programming problems:

1. Formulate the linear programming problem.
2. Replace an inequality symbol with an equals sign to form an equation of a boundary line of the graph.
3. Draw the straight line, this line represents the boundary line. Use dotted lines if an inequality sign $<$ or $>$ is included. Use a solid line if an inequality sign $\leq$ or $\geq$ is included.
4. Identify the feasible region by testing any convenient point (coordinates) which does not lie on the boundary. Substitute the coordinates of the point in the inequality. If the inequality is satisfied, then the point lies in the feasible region, otherwise the point does not lie in the feasible region.
5. Identify the coordinates of an optimum or corner point.
6. Evaluate the objective function at the optimum points to obtain the optimal value which may either be maximum or minimum value depending on the linear programming problem.

Transportation problems

In most cases, products or goods produced are transported from a storage location (source) to the consumer (destination) at a minimum cost in order to attain the desired profit. In order to minimize the total transportation costs, it is necessary to make the best routing decisions in the distribution processes. The transportation problem is a type of linear programming problem designed to minimize the cost of distributing a product from sources to respective destinations.

Note that, a transportation problem is considered to be balanced, if the total supply from the sources to each destination is equal to the total demands, otherwise the transportation problem is unbalanced.

There are several kinds of transportation problems, but all serve the purpose of minimization of cost. Figure 7.1(a) illustrates a transportation problem which involves two sources, S_1 and S_2 with two destinations, D_1 and D_2 while Figure 7.1(b) illustrates a transportation

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problem which involves two sources S_1 and S_2 with three destinations D_1, D_2, and D_3.

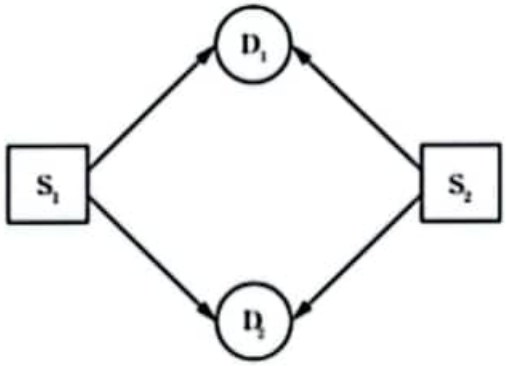

Figure 7.1 (a): *Two sources and two destinations*

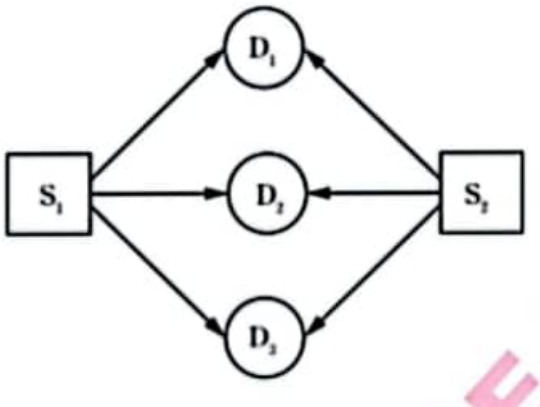

Figure 7.1 (b): *Two sources and three destinations*

Formulation of transportation problems

Formulation of a transportation problem depends on the number of destinations from the given sources. Consider the transportation problem which involves two sources having 'k' products in source P and 'h' products in source Q and two destinations which require 'a' products in destination A and 'b' products in destination B as shown in Figure 7.2.

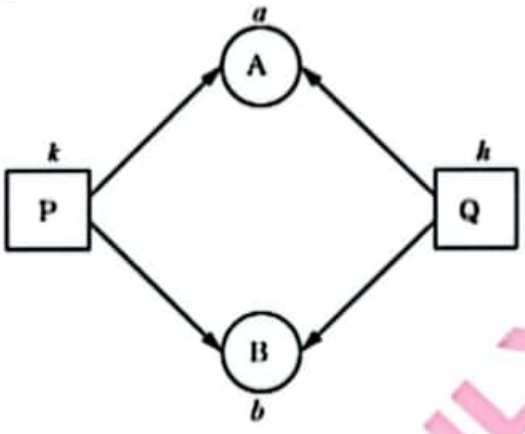

Figure 7.2: *Two sources and two destinations*

From Figure 7.2, the decision variables will be defined as follows:

Let: x be the number of products to be transported from source P to destination A,

y be the number of products to be transported from source P to destination B.

Since, there are already products transported from source P to destinations A and B, then $(a-x)$ will be the number of products to be transported from source Q to destination A, and $(b-y)$ will be the number of products to be transported from source Q to destination B.

Similarly, consider the transportation problem involves two sources which has 'k' products in source P and 'h' products in source Q and three destinations which require 'a' products in destination A, 'b' products in destination B, and 'c' products in destination C as shown in Figure 7.3.

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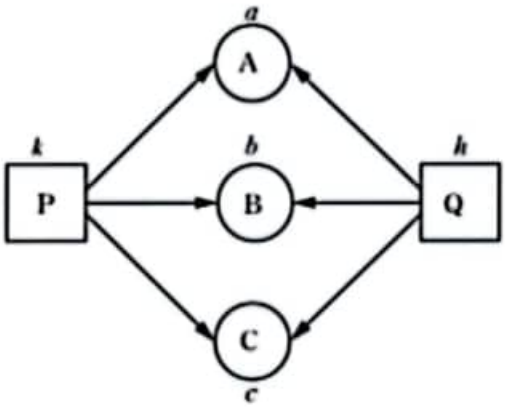


Figure 7.3: *Two sources and three destinations*

From Figure 7.3, the decision variables will be as follows:

Let: x be the number of products to be transported from source P to destination A,

y be the number of products to be transported from source P to destination B,

z be the number of products to be transported from source P to destination C.

Since, there are already products transported from source P to destinations A, B, and C, then $(a-x)$ will be the number of products to be transported from source Q to destination A, $(b-y)$ will be the number of products to be transported from source Q to destination B, and $(c-z)$ will be the number of products to be transported from source Q to destination C.

After deciding on the variables, other procedures of writing objective functions and constraints on the Cartesian coordinate system are the same for all linear programming problems.

Example 7.1

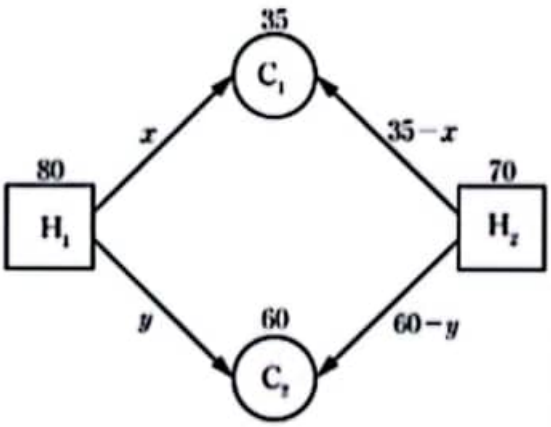
A manufacturer of certain products has two warehouses, H_1 and H_2 for storing the products. She stores 80 units of the products in H_1 and 70 units in H_2. Two customers, C_1 and C_2 placed orders for 35 and 60 units from H_1 and H_2, respectively. The transportation cost in Tanzanian shillings for each unit is as shown in the following table.

From \ To	Transport Costs	
Warehouses	C_1	C_2
H_1	80	120
H_2	100	130

Formulate the linear programming problem for minimizing the total transportation cost.

Solution

This transportation problem involves two sources and two destinations as illustrated in the following figure.



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The decision variables are as follows:
Let: x be the number of products to be transported from warehouse H_1 to customer C_1, y be the number of products to be transported from warehouse H_1 to customer C_2, then $(35-x)$ will be the number of products to be transported from warehouse H_2 to customer C_1, and $(60-y)$ will be the number of products to be transported from warehouse H_2 to customer C_2. Thus, the objective function is formulated as;
Minimize

$$z = 80x+120y+100(35-x)+130(60-y)$$
$$= -20x-10y+11{,}300$$

The constraints are formulated as follows:

$x \le 35$
$y \le 60$
$x+y \le 80$
$35-x+60-y \le 70 \Rightarrow x+y \ge 25$
$35-x \le 35 \Rightarrow x \ge 0$
$60-y \le 60 \Rightarrow y \ge 0$

Therefore, the formulated linear programming problem is to;

Minimize $z = -20x - 10y + 11300$
Subject to:

$x \le 35$
$y \le 60$
$x+y \le 80$
$x+y \ge 25$
$x \ge 0,\ y \ge 0$

Example 7.2

A medical company has factories at two locations, A and B. From these locations, supply is assigned to each of the three agencies P, Q, and R. The monthly requirements of the agencies are 5, 5, and 4 packets of medicine, respectively. While the production capacities of the factories A and B are 8 and 6 units per month, respectively. The transportation cost per packet from the factories is shown in the following table:

To \ From	A	B
P	15	10
Q	10	12
R	15	10

Formulate a linear programming problem for minimizing the transportation cost.

Solution

This transportation problem involves two sources and three destinations as illustrated in the following figure:

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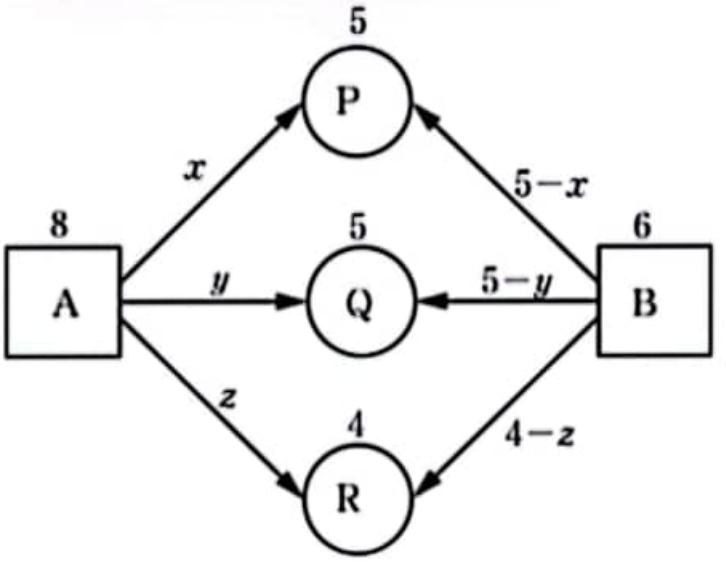

The decision variables are defined as follows;

Let x be the number of packets to be transported from factory A to agency P,

y be the number of packets to be transported from factory A to agency Q,

z be the number of packets to be transported from factory A to agency R,

$(5-x)$ be the number of packets to be transported from factory B to agency P,

$(5-y)$ be the number of packets to be transported from factory B to agency Q, and

$(4-z)$ be the number of packets to be transported from factory B to agency R.

Thus, the objective function is given by,

Minimize $f = 15x+10y+15z+10(5-x)+12(5-y)+10(4-z)$

$= 5x-2y+5z+150$

The constraints are formulated as follows:

$x \le 5$

$y \le 5$

$z \ge 0$ and $4-z \ge 0$.

But $x+y+z=8$. This implies that

$z = 8-(x+y)$.

$8-(x+y) \ge 0 \Rightarrow 8 \ge x+y$

Thus, $x+y \le 8$.

Also, $4-z \ge 0$.

Thus, $4-(8-(x+y)) \ge 0$.

Simplification gives,

$-4+x+y \ge 0$

Thus, $x+y \ge 4$.

Also, $5-x \le 5 \Rightarrow x \ge 0$ and

$5-y \ge 5 \Rightarrow y \ge 0$.

Therefore, the formulated linear programming problem is:

Minimize

$f = 5x-2y+5[8-(x+y)]+150$

$= 190-7y$

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Subject to: $x \leq 5$

$y \leq 5$

$x + y \geq 4$

$x + y \leq 8$

$x \geq 0, y \geq 0$

Exercise 7.1

1. Agnes, the civil engineer has two warehouses W_1 and W_2 containing 1,500 and 1,200 tonnes of bricks, respectively. She is constructing houses at three sites S_1, S_2, and S_3. She needs 1,100 tonnes of bricks at S_1, 1,200 tonnes at S_2, and 1,000 tonnes at S_3. The transport costs per tonne of bricks from each warehouse to each of the three sites S_1, S_2, and S_3 are as follows:

From \ To	W_1	W_2
S_1	12,000	8,000
S_2	10,000	7,000
S_3	9,000	11,000

Formulate a linear programming problem for minimization of the cost of transportation.

2. Joseph has two factories F_1 and F_2 for supplying products at three market centres P_1, P_2, and P_3. The weekly requirements of the centres are 60, 60, and 48 packets of products, respectively. While the production capacities of the factories F_1 and F_2 are 160 and 120 units per week, respectively. The transportation cost in Tanzania shillings per packet from the factories is shown in the following table.

From \ To	F_1	F_2
P_1	10,400	10,200
P_2	10,300	10,500
P_3	11,500	12,500

Formulate a linear programming problem for minimization of the cost of transportation.

3. Two suppliers of sweets, F and G are capable of supplying not more than 40 kg and 35 kg of sweets, respectively. The supplier is required to supply sweets to three day care schools, X, Y, and Z. The daily school requirements are 30 kg, 25 kg, and 20 kg for schools X, Y, and Z, respectively. The distance in kilometres from the suppliers to the respective schools are tabulated in the following table.

To \ From	F	G
X	6,000	5,000
Y	8,500	7,000
Z	9,000	5,500

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If the transportation costs are proportional to the distance travelled per kilometre, formulate a linear programming problem to minimize the transportation cost for the sweets supplier.

4. Two strawberry producers, A and B, with 50 tonnes and 45 tonnes of strawberries, respectively are requested to supply strawberries to three food processing companies, P, Q, and R demanding 30 tonnes, 25 tonnes, and 40 tonnes of strawberries, respectively. The transportation costs in Tanzanian shillings from the sources to the demanding sites are tabulated as follows:

To \ From	A	B
P	30,000	25,000
Q	45,000	30,000
R	20,000	40,000

Formulate a linear programming problem representing the given transportation problem.

5. Nana and Nina are poultry farmers who can supply a maximum of 80 and 65 trays of eggs every month, respectively. Three hotels, STAR, MOON, and CLOUD, each places a monthly order of 40, 55, and 50 trays of eggs, respectively. The transportation costs of eggs in Tanzanian shillings from the poultry farms to the hotels are tabulated in the following table.

To \ From	Nana	Nina
STAR	7,500	7,500
MOON	9,000	9,500
CLOUD	12,000	12,500

Formulate a linear programming problem representing the given information.

6. A certain electric bulb manufacturing company has two factories located at cities, F_1 and F_2 and three retail centres located at C_1, C_2, and C_3. The monthly demand at the retail centres C_1, C_2, and C_3 are 800, 500, and 200, electric bulbs, respectively, while the monthly supply at the factories are 600 and 900, electric bulbs, respectively. The cost of transportation in Tanzanian shillings of one electrical bulb between each factory and each retail centre is shown in the following table:

From \ To	C_1	C_2	C_3
F_1	50	50	30
F_2	60	40	10

Formulate a linear programming problem from the given information.

7. A sand dealer has two quarries, Q_1 and Q_2 which produce 3,000 tonnes and 1,500 tonnes of sand

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per day, respectively. Three road constructors, A, B, and C require each day 2,000 tonnes, 1,500 tonnes, and 1,000 tonnes of sand, respectively. The distances in kilometres from the quarries to each constructor's site are shown in the following table.

From \ To	A	B	C
Q_1	7	4	2
Q_2	3	2	2

If the transportation costs are proportional to the distance travelled per kilometre formulate a linear programming problem for minimization of the cost.

8. Two sugar production factories located in Morogoro and Kagera have weekly production capacities of 8 and 6 tonnes, respectively. The products from the factories are delivered to depots situated in Mwanza, Dodoma, and Arusha. The weekly requirements of the depots for Mwanza, Dodoma, and Arusha are 5, 5, and 4 tonnes, respectively. Formulate a linear programming problem to minimize costs if the transportation cost in Tanzanian shillings per tonne is as shown in the following table.

From \ To	Mwanza	Dodoma	Arusha
Morogoro	16,000	10,000	15,000
Kagera	10,000	12,000	10,000

9. Mr. Bakari has two warehouses, W_1 and W_2 that contain 900 tonnes and 600 tonnes of wheat flour, respectively. He is planning to supply flour to three customers, T_1, T_2, and T_3 who are in need of 500 tonnes, 600 tonnes, and 400 tonnes, respectively. Formulate a linear programming problem given that the transportation costs in Tanzanian shillings per tonne of wheat flour from each warehouse to each customer are as shown in the following table.

From \ To	T_1	T_2	T_3
W_1	600	300	400
W_2	400	200	600

Graphical solutions of transportation problems

Transportation problems can be solved graphically in a similar way as it is done when solving any other linear programming problem. After a transportation problem is formulated, a graph is drawn to show the feasible region and its corner points. The corner points are then used to obtain the optimal solution of the given transportation problem.

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Example 7.3

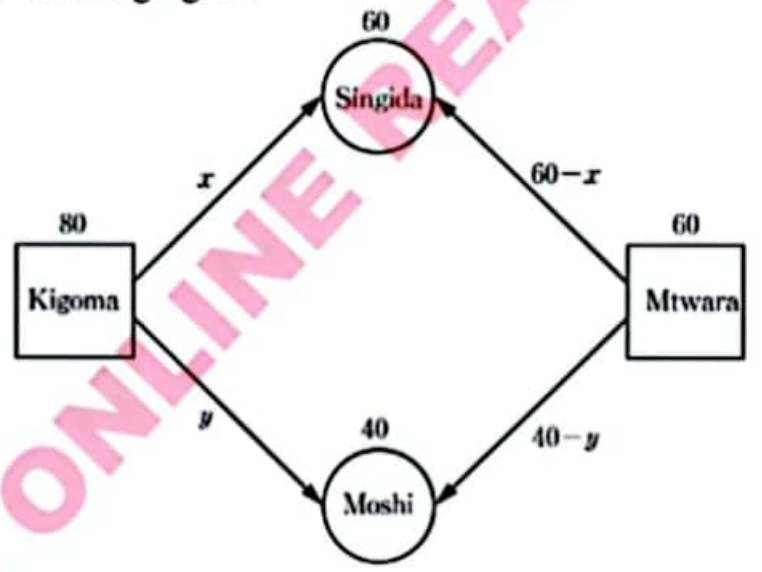
Mariam has two warehouses, one in Kigoma and other in Mtwara. She is required to transport 60 tonnes and 40 tonnes of goods to Singida and Moshi, respectively. The warehouses at Kigoma and Mtwara contain 80 and 60 tonnes of goods, respectively. The transportation costs in Tanzanian shillings per tonne of goods are shown in the following table.

From / To	Kigoma	Mtwara
Singida	180,000	240,000
Moshi	120,000	100,000

How should the transportation of tonnes of goods be done at a minimum cost?

Solution

The transportation problem involves two sources and two destinations as illustrated pictorially in the following figure.



The decision variables are defined as follows;

Let x be the number of tonnes of goods to be transported from Kigoma to Singida,
y be the number of tonnes of goods to be transported from Kigoma to Moshi,
$(60-x)$ be the number of tonnes of goods to be transported from Mtwara to Singida, and $(40-y)$ be the number of tonnes of goods to be transported from Mtwara to Moshi.

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Thus, the objective function is formulated as:

Minimize $z = 180000 + 120000 + 240000(60 - x) + 100000(40 - y)$

$$z = 18400000 - 60000x + 20000y$$

The constraints are formulated as follows:

$x \leq 60$

$y \leq 40$

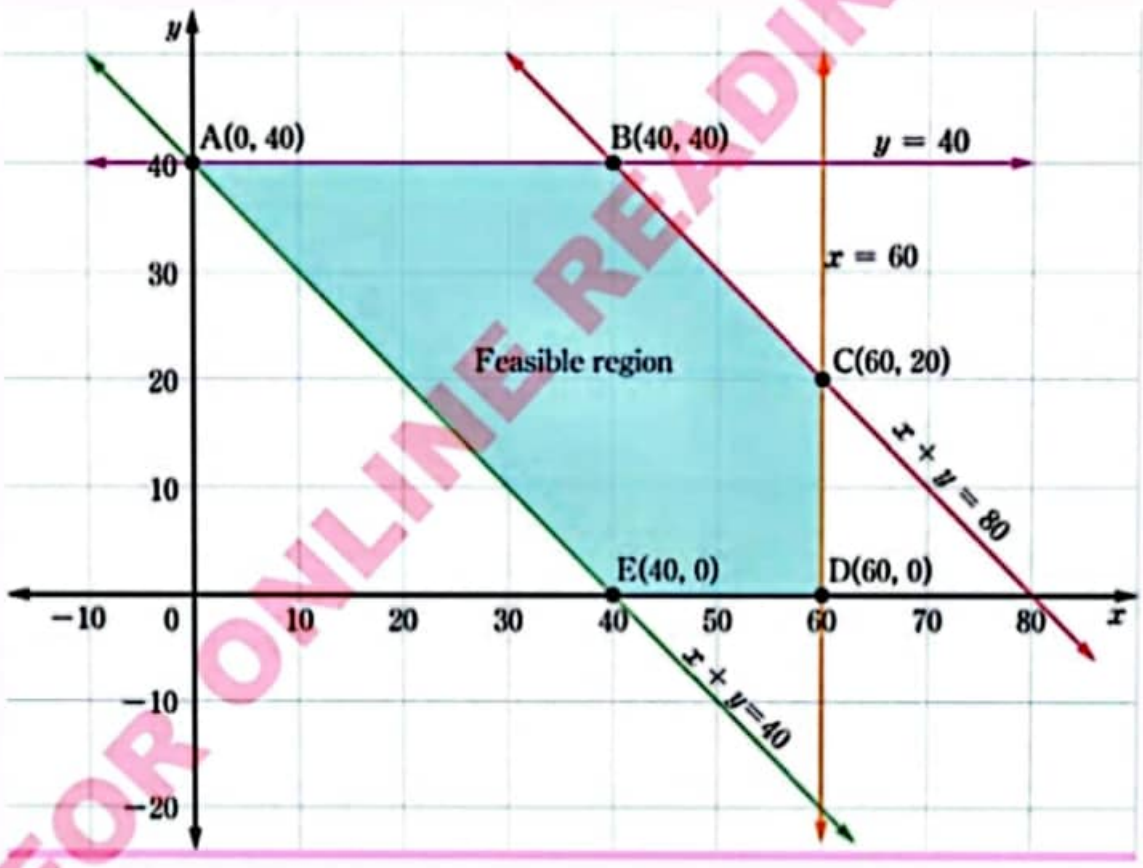
$x + y \leq 80$

$60 - x + 40 - y \leq 60 \Rightarrow x + y \geq 40$

$60 - x \leq 60 \Rightarrow x \geq 0$

$40 - y \leq 40 \Rightarrow y \geq 0$

The graph of the constraints is as shown in the following figure.



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The corner points and values of the objective function are shown in the following table.

Corner points	Values of the objective function $z = 18400000 - 60000x + 20000y$
A(0, 40)	$z = 18400000 - 60000(0) + 20000(40) = 19200000$
B(40, 40)	$z = 18400000 - 60000(40) + 20000(40) = 16800000$
C(60, 20)	$z = 18400000 - 60000(60) + 20000(20) = 15200000$
D(60, 0)	$z = 18400000 - 60000(60) + 20000(0) = 14800000$
E(40, 0)	$z = 18400000 - 60000(40) + 20000(0) = 16000000$

The minimum cost is at the point D(60, 0) which is 14,800,000 Tanzanian shillings.

Therefore, in order for Mariam to minimize the cost, the transportation should be as follows:
60 tonnes of goods to be transported from Kigoma to Singida,
0 tonnes of goods to be transported from Kigoma to Moshi,
0 tonnes of goods to be transported from Mtwara to Singida, and
40 tonnes of goods to be transported from Mtwara to Moshi.

Example 7.4

A supplier of rice has stores at Mbagala and Ubungo. Three customers located at Mbezi, Manzese, and Kinyerezi order sacks of rice weekly. The customers weekly requirements are 32, 24, and 20 sacks of rice, respectively. The stores' output capacities at Mbagala and Ubungo are 40 and 36 sacks of rice, respectively. The cost of transportation per sack in Tanzanian shillings are shown in the following table:

From / To	Mbagala	Ubungo
Mbezi	160	100
Manzese	100	120
Kinyerezi	150	100

(a) How many sacks should be transported from each store to each customer at the minimum cost? Find the minimum transportation cost.

(b) Is the linear programming problem balanced or not and why?

Solution

The transportation problem involves two sources and three destinations as illustrated in the following figure.

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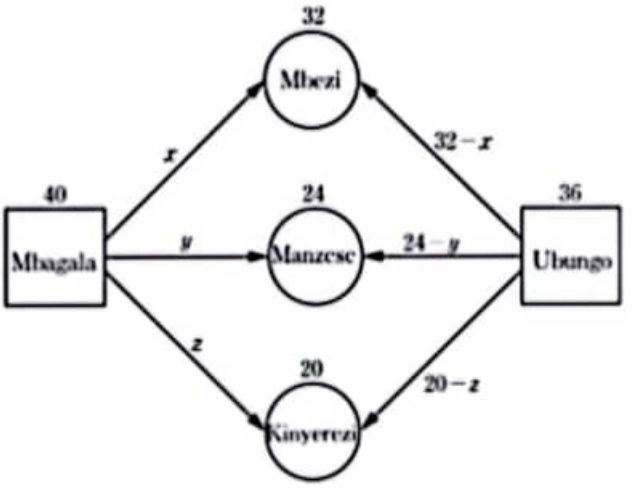

The decision variables are defined as follows;
Let: x be the number of sacks of rice to be transported from Mbagala to Mbezi,
y be the number of sacks of rice to be transported from Mbagala to Manzese,
z be the number of sacks of rice to be transported from Mbagala to Kinyerezi,
$(32-x)$ be the number of sacks of rice to be transported from Ubungo to Mbezi,
$(24-y)$ be the number of sacks of rice to be transported from Ubungo to Manzese,
$(20-z)$ be the number of sacks of rice to be transported from Ubungo to Kinyerezi.

Thus, the objective function is given by;

$$w = 160x + 100y + 150z + 100(32-x) + 120(24-y) + 100(20-z)$$
$$= 60x - 20y + 50z + 8080$$

Subjects: $x \le 32$

$y \le 24$

$z \le 20$

$32 - x \le 32 \Rightarrow x \ge 0$

$24 - y \le 24 \Rightarrow y \ge 0$

$20 - z \le 20 \Rightarrow z \ge 0$

But $x + y + z = 40 \Rightarrow z = 40 - (x + y)$.

From $z \le 20 \Rightarrow z = 40 - (x + y) \le 20$.

$\Rightarrow x + y \ge 20$

Also, $z \ge 20 \Rightarrow z = 40 - (x + y) \ge 0$.

$\Rightarrow x + y \le 40$

Thus, the linear programming problem is:

Minimize $w = 60x - 20y + 50\big(40 - (x + y)\big) + 8080$ or

Minimize $w = 10x - 70y + 10080$

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Subject to : $x \leq 32$

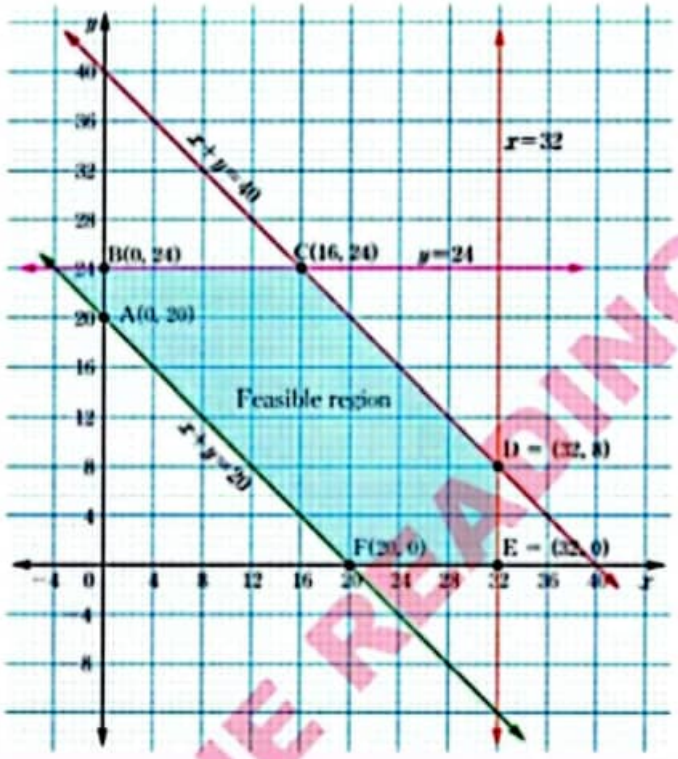
$y \leq 24$

$x + y \geq 20$

$x + y \leq 40$

$x \geq 0,\ y \geq 0$

The graph of the formulated linear programming problem is as shown in the following figure.



The corner points and the values of the objective functions are shown in the following table.

Corner points of the feasible region	Value of the objective function $w = 10x - 70y + 10080$
A (0, 20)	$w = 10(0) - 70(20) + 10080 = 8,680$
B (0, 24)	$w = 10(0) - 70(24) + 10080 = 8,400$
C (16, 24)	$w = 10(16) - 70(24) + 10080 = 8,560$
D (32, 8)	$w = 10(32) - 70(8) + 10080 = 9,840$
E (32, 0)	$w = 10(32) - 70(0) + 10080 = 10,400$
F (20, 0)	$w = 10(20) - 70(0) + 10080 = 10,280$

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The minimum value is at the point B (0, 24) which is 8,400 Tanzanian shillings.

Therefore, in order for the supplier to minimize cost, the transportation should be done as follows:

0 sacks of rice to be transported from Mbagala to Mbezi

24 sacks of rice to be transported from Mbagala to Manzese

16 sacks of rice to be transported from Mbagala to Kinyerezi

32 sacks of rice to be transported from Ubungo to Mbezi

0 sacks of rice to be transported from Ubungo to Manzese

4 sacks of rice to be transported from Ubungo to Kinyerezi

(b) The transportation problem is balanced since the total supply is equal to the demand. That is,

Total supply = 40 + 37 = 76 sacks.

Total demand = 32 + 24 = 76 sacks.

Exercise 7.2

1. A cement dealer has two depots D_1 and D_2 which hold 120 and 40 tonnes, respectively. He has two customers, C_1 and C_2 who have ordered 80 and 50 tonnes of cement, respectively. Their distances in kilometres are shown in the following table.

To \ From	D_1	D_2
C_1	20	40
C_2	15	30

If the transportation costs are proportional to the distance travelled per kilometre, how should the cement dealer supply the cement to the two customers in order to minimize the total transportation cost?

2. A manufacturer has a warehouse at Misungwi which contains 25 units of his product. She also has another warehouse at Ngudu which contains 30 units. She has to supply to shops in Mabuki and Misasi with 20 and 15 units, respectively. The transportation costs in Tanzanian shillings per unit are tabulated as follows.

To \ From	Misungwi	Ngudu
Mabuki	150	220
Misasi	130	180

How should the manufacturer supply her product in order to minimize the transportation costs?

3. Almasi has made 900 and 600 bricks at his houses H_1 and H_2, respectively. He plans to build new houses at three sites P, Q, and R. He expects to use 500 bricks

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at P, 600 bricks at Q, and 400 bricks at R. The transportation costs in Tanzanian shilling per brick from each of his houses to each of the three sites are shown in the following table.

From / To	H_1	H_2
P	60	40
Q	30	20
R	40	60

(a) Determine the number of bricks that he will transport to each site at a minimum cost.
(b) What will be the overall minimum cost?

4. Two factories are located at two places M_1 and M_2. From these locations, a certain product is to be delivered to each of the three destinations, A, B, and C. The weekly requirements of the product at the destinations are respectively 5, 5, and 4 units, while the production capacities of the factories at M_1 and M_2 are respectively 8 and 6 units. The costs of transportation in Tanzanian shillings per unit are given in the following table.

From / To	M_1	M_2
A	160	100
B	100	120
C	150	100

(a) How many units of the product should be transported from each factory to each destination in order to minimize transportation cost?
(b) What will be the minimum transportation cost?

5. Mrs. Mlunda has two depots, A and B with a stock of 30,000 and 20,000 bricks, respectively. She receives orders from three customers. Mr. Malaga requires 15,000 bricks, Mr. Falulu requires 20,000 bricks and Ms. Tina requires 15,000 bricks. The costs of transportation per brick in Tanzanian shillings are tabulated in the following table.

From / To	A	B
Mr. Malaga	40	20
Mr. Falulu	20	60
Ms. Tina	30	40

How should Mrs. Mlunda deliver the orders so as to minimize the cost of transportation?

6. A soft drink dealer has two sources, C_1 and C_2 with capacities of 50,000 and 30,000 crates, respectively. The dealer wishes to supply drinks to three shops, S_1, S_2, and S_3 whose requirements are 35,000, 20,000, and 25,000 crates, respectively. The costs

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of transportation per crate in Tanzanian shillings from the sources to the shops are as summarized in the following table.

To \ From	C_1	C_2
S_1	70	30
S_2	60	40
S_3	30	20

How should the delivery be scheduled at a minimum cost of transportation? Find the minimum cost.

7. A business person has maize storages in two wards, W_1 and W_2 with capacities of 100 tonnes and 50 tonnes, respectively. The storages supply maize to three centres, C_1, C_2, and C_3 whose requirements are 60, 50, and 40 tonnes, respectively. The transportation costs in Tanzanian shillings per tonne from each ward to each centre are shown in the following table.

From \ To	C_1	C_2	C_3
W_1	60,000	30,000	25,000
W_2	40,000	20,000	30,000

(a) How should the supply be made to minimize the cost of transportation?

(b) Determine the minimum cost of each supply. What is the overall minimum cost?

8. Three jogging clubs, C_1, C_2, and C_3 need 20, 15, and 20 sets of equipments, respectively. The clubs are sponsored by two benefactors, B_1 and B_2 who have only 25 and 30 sets of equipments, respectively. The transportation costs per set of equipments in Tanzania shillings are shown in the following table.

Benefactors	Jogging clubs		
	C_1	C_2	C_3
B_1	300	200	100
B_2	250	100	150

(a) Determine how the benefactors should supply the equipments to each of the jogging clubs at a minimum cost.

(b) Find the minimum cost.

9. A packaging company has two market locations, L_1 and L_2 to sell coconuts. The company can supply 80,000 bags from L_1 and 70,000 bags from L_2. The retail traders, Msigwa and Asha can place orders for 35,000 and 60,000 bags, respectively. The transportation costs in Tanzania shillings per bag from each market location to each of the

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retail traders are as shown in the following table.

From \ To	Msigwa	Asha
L_1	8,000	12,000
L_2	10,000	13,000

How many bags of coconuts should the company deliver to each retail trader from each market location in order to minimize the total cost of transportation?

Chapter summary

1. Linear programming is a mathematical method used to determine optimal solutions to problems that can be expressed as linear equations and inequalities.
2. The transportation problem is a linear programming problem designed to minimize the cost of distributing products from different sources to different destinations.
3. Decision variables are the unknown quantities that decide the output of the linear programming problem.
4. An objective function is a linear function whose value is to be either minimized or maximized subject to some constraints defined over the set of feasible solutions.
5. Constraints are inequalities or equations which connect the decision variables under certain restrictions or limitations on the decision variables.
6. A feasible region is the set of all possible solutions of the linear programming problem.
7. An optimal solution is the value at a point where the objective function reaches its optimal value.
8. A feasible solution is any solution that satisfies all constraints of the linear programming problem.
9. An optimal value is the quantity from the optimal solution that maximizes or minimizes the objective function of the linear programming problem.

Revision exercise 7

1. A certain manufacturing company has two factories at towns E and F. From these factories, a certain commodity is delivered to each of the three depots situated at D_1, D_2, and D_3. The daily requirements of the depots are

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respectively, 100, 100, and 80 units of the commodity while the production capacity of the factories at E and F are 160 and 120 units, respectively. The cost of transportation in Tanzanian shillings per unit is given in the following table.

From \ To	D_1	D_2	D_3
E	14,700	11,200	13,800
F	11,400	12,900	11,500

(a) Formulate the objective function to be minimized by the factory in supplying the units of the commodity to each depot.

(b) Formulate the constraints associated with the transportation problem.

2. A cement company has two depots, C_1 and C_2 with stocks of 20,000 and 25,000 bags of cement, respectively. It receives orders from three building sites B_1, B_2, and B_3 for 12,000; 15,000 and 18,000 bags of cement, respectively. The cost of transporting bags of cement to the sites from the depots are shown in the following table.

From \ To	B_1	B_2	B_3
C_1	38,000	28,500	27,900
C_2	28,600	42,300	39,500

Formulate a linear programming problem for the given information.

3. Two godowns, MEDIUM and LARGE have a beans storage capacity of 4,200 bags and 3,800 bags, respectively. The beans are supplied to three shops at Umoja, Uhuru, and Ukombozi whose requirements are 3,000; 2,400; and 2,600 bags, respectively. The costs of transportation per bag from the godowns to the shops are given in the following table.

To \ From	MEDIUM	LARGE
Umoja	600	500
Uhuru	400	280
Ukombozi	300	350

Formulate a linear programming problem representing the given transportation problem.

4. An oil company has two depots, H and J with capacities of 40,000 litres and 30,000 litres, respectively. The company is to supply oil to three petrol pumps,

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M, N, and P whose requirements are 25,000; 20,000; and 25,000 litres, respectively. The distances in kilometres between the depots and the petrol pumps are given in the following table.

From / To	H	J
M	65	45
N	60	50
P	40	35

Assuming that the transportation costs are proportional to the distance travelled per kilometer, formulate a linear programming problem for minimization of the cost.

5. Dr. Ally has two traditional stores T_1 and T_2 for storing groundnuts. He stored 860 bags in T_1 and 800 bags in T_2. Two customers M_1 and M_2 placed orders for 700 and 600 bags, respectively. The transportation costs per bag from each store are summarized in the following table.

To / From	M_1	M_2
T_1	40	45
T_2	35	55

(a) How many bags of groundnuts should Dr. Ally deliver to each customer at minimum transportation cost?

(b) Determine the minimum cost of transportation.

6. Ms. Hamida has two warehouses, one is in Dar es Salaam and the other in Morogoro. She wants to transpot tonnes of goods to Kibaha and Chalinze. She needs to transport 180 tonnes and 140 tonnes of goods to Kibaha and Chalinze, respectively. The warehouses at Dar es Salaam and Morogoro contain 200 and 170 tonnes of goods, respectively. The transportation cost in Tanzanian shillings per tonne of goods is given in the following table.

From / To	Dar es-Salaam	Morogoro
Kibaha	370,000	345,000
Chalinze	298,500	282,500

(a) Formulate the linear programming problem for minimizing the cost of transportation.

(b) How should the transportation of tonnes of goods be done at a minimum cost?

7. An oil company has two depots, N and Q with capacities 7,000 litres and 4,000 litres of oil, respectively. The company is to supply oil to three petrol pumps,

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D, E, and F whose requirements are 4,500 litres, 3,000 litres, and 3,500 litres, respectively. The distances between the depots and the petrol pumps in kilometres are given in the following table.

From / To	N	Q
D	7	3
E	6	4
F	3	2

Assuming the transportation cost is 1 Tanzanian shilling per kilometre, how should the delivery be scheduled in order to minimize the cost? Determine the minimum cost of the delivery.

5. A certain factory has two warehouses, W_1 and W_2 each containing 900 crates of soda to be supplied to three regions, R_1, R_2, and R_3. There are 600 crates of soda to be transported to R_1, 500 crates of soda to be transported to R_2 and 700 crates to be transported to R_3. The transport costs per crate to each region are shown in the following table.

To / From	R_1	R_2	R_3
W_1	60/=	30/=	40/=
W_2	40/=	20/=	60/=

How should the factory supply the crates of soda to the three regions at a minimum cost?

6. Bangulo company has factories at towns, A and B which supply goods to warehouses, P_1 and P_2. The weekly factory capacities are 160 and 140 units, respectively and the warehouses requirements are 70 and 120 units, respectively. The cost of transportation of 1 unit from A to P_1 is Tshs 160 and from A to P_2 is Tshs 240. Similarly, the cost of transportation from B to P_1 is Tshs 200 and from B to P_2 is Tshs 260. How should the company make supplies to the warehouses at a minimum cost?

7. Mr. Nyahuye have two storage warehouses. He stores 200 tonnes of rice at warehouse 1 and 300 tonnes of rice at warehouse 2. The rice has to be sent to three marketing centres A, B, and C. The demands at A, B, and C are 150, 150, and 200 tonnes of rice, respectively. The transportation costs in Tanzanian shillings per tonne from the warehouses to each market centre are shown in the following table.

Warehouses	Market centres		
	A	B	C
Warehouse 1	50	100	70
Warehouse 2	80	150	40

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How many tonnes of rice should be sent from the warehouses to each marketing centre so that the transportation cost is minimum?

8. A cement company has factories at P and Q which supply tonnes of cement to regions, A and B. The weekly factory capacities are 160 and 140 tonnes of cement, respectively and the regions require 70 and 120 tonnes of cement, respectively. The cost of transportation of one tonne of cement from factory P to region A is Tshs 160 and the cost from factory P to region B is Tshs 240. Similarly, the transportation cost from factory Q to region A is Tshs 200 and from factory Q to region B is Tshs 260.

   (a) Find the objective function to be minimized by the company so as to supply tonnes of cement to each region.

   (b) Find the constraints associated with the transportation problem.

   (c) Does the problem balance or not and why?

12. A bricks manufacturing company has two sites, Mlalo and Lukozi with capacity of producing 6000 and 3000 bricks, respectively. The company is to supply bricks to three construction areas, Gare, Mtae and Bondeni whose demands are 4000, 3000 and 2000 bricks, respectively. The distance in kilometres between the sites and the construction areas is given in the following table.

From / To	Mlalo	Lukozi
Gare	14	8
Mtae	10	6
Bondeni	5	4

   (a) If the transportation costs of 50 bricks is 4,000 Tanzanian shillings per kilometer, formulate a linear programming problem for minimizing the total transport cost.

   (b) Is the problem balanced or not? Give reason.

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Chapter Eight

Introduction

The concept of differentiation refers to the method of finding the derivative of a continuous function. It is the process of finding the slope or gradient of a continuous function. For a straight line, the slope remains the same along the line. The slope of a curve changes continuously along the curve. In this case, the gradient of the curve is considered at the point of tangent to the curve. In this chapter, you will learn about derivatives, differentiation of functions, applications of differentiation in real life problems, Taylor's and Maclaurin's series, and introduction to partial derivatives of functions. The competencies developed are applicable in various real life situations such as in business, science, engineering, economics, building and constructions, population modeling, dynamical systems, among many other applications.

Think

The World without the concept of differentiation.

Derivatives

The derivative of a function is defined as the rate of change of a function with respect to a given variable. Given the graph of the function, the derivative can be interpreted as the slope of the graph of the function, or more precisely as the slope of the tangent line at a given point. In the coordinate plane, slope of a straight line is given by the ratio of the change in the y coordinate to the change in the x coordinate. In Figure 8.1, if (x_0, y_0) and (x_1, y_1) are two points on the line, then the slope is given by the ratio: $\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$.

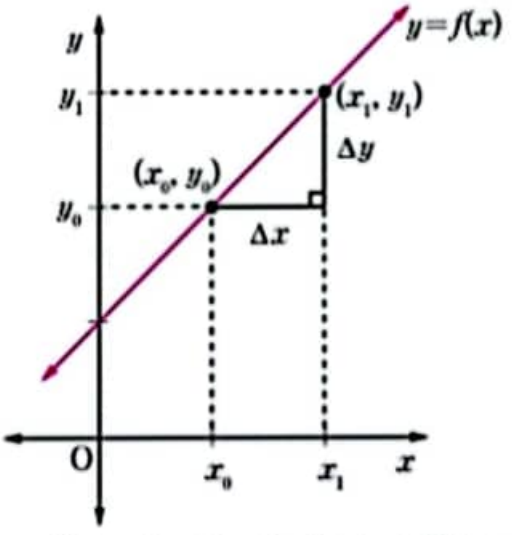

Figure 8.1: *Graph of a straight line*

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Let h be a small increase in x_0, and $x_1 = x_0 + h$, then $h = x_1 - x_0$.

If $y = f(x)$, then $y_0 = f(x_0)$ and $y_1 = f(x_1) = f(x_0 + h)$.

Thus, in terms of function value, slope is given by $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$. This change in notation is useful for advancing from the idea of the slope of a line to the more general concept of the derivative of a function.

Given a point (x, y) on the graph of $f(x)$, the process of finding a function $f'(x)$ (read as "f prime") whose expression is $f'(x)$ gives the slope of the graph at the point (x, y). The expression $f'(x)$ is called the derivative of the function, $f(x)$ at a point (x, y). The process of finding the derivative of a function is called differentiation.

Notations of the derivative

The derivative of a function $y = f(x)$ with respect to the variable x is denoted either by y' or $f'(x)$ or $\frac{dy}{dx}$ or $\frac{df}{dx}$, and it is known as the first derivative. The notation $\frac{dy}{dx}$ is read as "Dy Dx".

Generally, $\frac{dy}{dx}$ means the rate at which y changes when x changes. It defines the gradient of the curve representing $y = f(x)$.

Note that, derivative is the rate of change of one quantity with respect to another quantity. For example, $\frac{dy}{dz}$ is the rate of change of y with respect to z. The second derivative of a function of $y = f(x)$ is obtained by differentiating the first derivative, and it is denoted either by y'' or $f''(x)$ or $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$.

That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } f''(x) = \frac{d}{dx}(f'(x))$$

$$\text{or } \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

The third derivative is denoted either by y''' or $f'''(x)$ or $\frac{d^3y}{dx^3}$. That is,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \text{ or } f'''(x) = \frac{d}{dx}(f''(x))$$

$$\text{or } \frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right).$$

Note that, the derivative of a constant function is zero. That is, $\frac{d}{dx}(c) = 0$, where c is any constant. This is because, in a constant function there is no change in the values of the function.

Differentiation of a function from first principles

The slope of a secant line is used to derive the formula for differentiation of functions from first principles. Figure 8.2 shows a curve of the function $y = f(x)$ with a secant line passing through the points Q and P, where h is a small increase in the value of the x coordinate at point Q.

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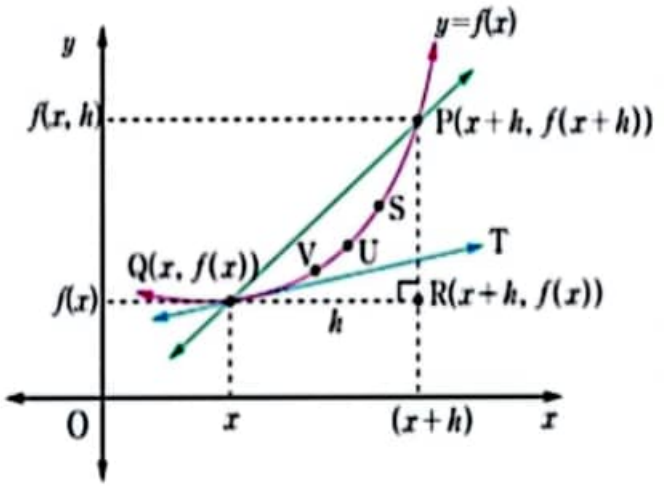

Figure 8.2: *Gradient of a curve* $y = f(x)$ *between the points* $P(x+h, f(x+h))$ *and* $Q(x, f(x))$

The slope of the secant line

$$\text{Slope of } \overline{PQ} = \frac{\overline{PR}}{\overline{RQ}} = \frac{f(x+h)-f(x)}{(x+h)-x}$$

$$\text{Slope of } \overline{PQ} = \frac{f(x+h)-f(x)}{h}$$

The concept of limit as applied in derivatives

The limit of a function $f(x)$ at a point is the value $f(x)$ approaches as x approaches a certain value. For example, $\lim_{x \to a} f(x) = L$ means that, as x approaches a, $f(x)$ approaches the value L. The concept of limit is used to describe the behaviour of graphs as the variable approaches a certain point. The slope of a tangent line to a function can be defined using the concept of limits. The following activity illustrates the concept of limits.

Activity 8.1: Application of limits in derivatives of functions

Individually or in a group, perform the following tasks:

1. Sketch the graph of the curve $y = f(x)$ of your choice on the xy-plane.
2. Locate a point $Q(x_0, f(x_0))$ on the curve of your choice.
3. Draw a tangent line to the curve at point Q.
4. Pick a point $P(x_0 + h, f(x_0 + h))$, near $Q(x_0, f(x_0))$, where h is a small change in x. Next, draw a secant line to join the points Q and P.
5. Find the gradient of the secant line $\overline{PQ}$ in task 4.
6. Discuss how you can use the gradient of a secant line to approximate the slope of the tangent line at point Q.
7. What did you observe from the tasks?

In Figure 8.2, if h is small, the gradient of the secant line is an approximation of the gradient of the tangent line. Therefore, it can be concluded that the gradient of the tangent line is the limit of the gradient of the secant line as h becomes smaller and smaller (means as h approaches zero).

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If point P is moved closer to Q, say to points S, U, V,..., then the chords $\overline{QS}, \overline{QU}, \overline{QV}, \supset$ will be getting closer and closer to the tangent line drawn at point Q. The action of moving point P towards point Q decreases the value of h. Thus, as h approaches zero, (that is, $h \to 0$) the line $\overline{PQ}$ approaches the tangent line at point Q. Hence, the gradient of the chord $\overline{PQ}$ approaches the gradient of the tangent at point Q. The gradient of the tangent line at point Q can then be said to be the limiting value of the gradient of the chord $\overline{PQ}$ as $h \to$.

This can be written as, gradient of the curve at $Q = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Thus, the gradient of the curve at any point $Q(x, f(x))$ on the curve is the gradient of the tangent line at the same point Q. Therefore, the gradient of the curve is $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that this limit exists.

The gradient of the curve is called the gradient function or derived function because it is derived from the original function. Thus, the formula for differentiation from first principles of $f(x)$ is given by;

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists. This means that the function is differentiated with respect to x.

Activity 8.2: Illustrating the concept of derivative of a function from first principles

Individually or in a group, write down any two polynomial functions $f(x)$ and $g(x)$ of degrees 2 and 3, respectively. Then, perform the following tasks:

1. Write the general formula for differentiating a function from first principles.
2. Use the formula in task 1 to find the derivative of each of the polynomial functions.
3. Use the results in task 2 to deduce the general formula for differentiating the polynomial $f(x) = x^n$, where n is a positive integer
4. What have you observed from task 3?
5. Share your findings with other students through discussion.

Example 8.1

Differentiate from first principles the function $f(x) = 7x + 6$.

Solution

Given $f(x) = 7x + 6$. From first principles it implies that,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But $f(x+h) = 7(x+h) + 6$.

Thus,

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$$f'(x)=\lim_{h\to 0}\frac{7(x+h)+6-(7x+6)}{h}$$
$$=\lim_{h\to 0}\frac{7x+7h+6-7x-6}{h}$$
$$=\lim_{h\to 0}\frac{7h}{h}$$
$$=\lim_{h\to 0}7=7$$

Therefore, $f'(x)=7$.

Example 8.2

Find $f'(x)$ from first principles if, $f(x)=x^2-6x+1$.

Solution

Given $f(x)=x^2-6x+1$. From first principles, it implies that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $f(x+h)=(x+h)^2-6(x+h)+1$.

Thus, $f'(x)=\lim_{h\to 0}\frac{(x+h)^2-6(x+h)+1-(x^2-6x+1)}{h}$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{x^2+2xh+h^2-6x-6h+1-x^2+6x-1}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{2xh+h^2-6h}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}(2x+h-6)=2x-6$$

Therefore, $f'(x)=2x-6$.

Example 8.3

Find the gradient function of the function $f(x)=x^3+7$ using first principles, hence evaluate the gradient at the point where $x=2$.

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Solution

Given $f(x)=x^3+7$. From first principles it implies that,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $f(x+h)=(x+h)^3+7$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{(x+h)^3+7-(x^3+7)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+7-x^3-7}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}(3x^2+3xh+h^2)$$

$$\Rightarrow f'(x)=3x^2$$

Thus, at $x=2$, $f'(2)=3(2)^2=12$.

Therefore, $f'(x)=3x^2$ and $f'(2)=12$.

Example 8.4

Differentiate from first principles the function $f(x)=\frac{1}{\sqrt{x}}$.

Solution

Given $f(x)=\frac{1}{\sqrt{x}}$, then from first principles;

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $f(x+h)=\frac{1}{\sqrt{x+h}}$

Thus, $f'(x)=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$

$$=\lim_{h\to 0}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

Rationalizing the numerator gives,

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$$f'(x)=\lim_{h\to 0}\left(\frac{\sqrt{x}-\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{\left(\sqrt{x}-\sqrt{x+h}\right)\times\left(\sqrt{x}+\sqrt{x+h}\right)}{\sqrt{x}\left(\sqrt{x+h}\right)\times\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to}\left(\frac{x-(x+h)}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{}{h}$$

$$=\lim_{h\to 0}\left(\frac{-h}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{-1}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

Now, as $h\to 0$ gives,

$$f'(x)=\left(\frac{-1}{\sqrt{x}\left(\sqrt{x}\right)\left(\sqrt{x}+\sqrt{x}\right)}\right)$$

$$\Rightarrow f'(x)=\frac{-1}{2x\sqrt{x}}$$

Therefore, $f'(x)=\frac{-1}{2x\sqrt{x}}$.

Exercise 8.1

1. From first principles, find the derivative in each of the following functions:
   (a) $f(x)=5x+6$ (b) $f(t)=t^3-t^2$ (c) $f(x)=-3x+5$
   (d) $f(x)=2x^2-3x+5$ (e) $f(v)=2v-500$ (f) $f(x)=\frac{1}{3-x}$
   (g) $f(t)=kt^5$ (h) $f(x)=\sqrt{2x}$
2. Using first principles of differentiation, find the gradient function in each of the following functions and hence compute the gradient of each curve $y=f(x)$ at the given value of x.

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(a) $y = 5x; x = 1$ (b) $y = x^3 - x^2 + 4; x = 2$ (c) $y = x - \frac{1}{4}; x = 2$

(d) $y = x^2 + 6; x = 3$ (e) $y = \frac{1}{x^2}; x = 4$

3. If $f(x) = 5x - 2x^2$, find $f'(x)$ using first principles of differentiation and evaluate $f'(3)$ and $f'(-1)$.

4. For each of the following curves, find the y-coordinate for the given value of x and use first principles to find the gradient.

(a) $y = 5x^3; x = 6$ (b) $y = x(x^2 - 3); x = 2$ (c) $y = x(x-2)^2; x = 1$

5. From first principles of differentiation, show that:

(a) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$ (b) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = -\frac{3}{x^4}$

6. Use first principles to show that $\frac{d}{dx}\left(x^2\sqrt{1-x}\right) = \frac{\sqrt{1-x}}{2}\left(\frac{4x-5x^2}{1-x}\right)$.

7. From first principles, show that $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{\left(1+x^2\right)^2}$. Hence, verify that $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{16}{25}$ at $x = \frac{1}{2}$.

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Differentiation of a function

The process of finding the derivative of a function or rate of change of one variable with respect to another variable is called differentiation. It is important to note that, not all functions are differentiable. A function $f(x)$ is said to be differentiable at a point (x, y) if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

Derivatives of polynomial functions

The derivative of the polynomial function $f(x) = x^n$, where n is a positive integer can be determined from first principles.

Given $f(x) = x^n$, then from first principles,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But $f(x+h)=(x+h)^n$

Thus, $f'(x)=\lim_{h\to 0}\dfrac{(x+h)^n-x^n}{h}$.

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\left(x\left(1+\frac{h}{x}\right)\right)^n-x^n}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\left(x^n\left(1+\frac{h}{x}\right)^n\right)-x^n}{h} \quad \text{.....(i)}$$

On expanding $\left(1+\frac{h}{x}\right)^n$ by using binomial theorem gives,

$$\left(1+\frac{h}{x}\right)^n=1+n\frac{h}{x}+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\text{.....(ii)}$$

Substituting equation (ii) into equation (i) gives;

$$f'(x)=\lim_{h\to 0}\frac{\left[x^n+nhx^{n-1}+\frac{n(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots\right]-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\left[x^n+nhx^{n-1}+\frac{n(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots-x^n\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[nhx^{n-1}+n\frac{(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots\right]}{h}$$

$$=\lim_{h\to 0}\left[nx^{n-1}+n\frac{(n-1)}{2!}hx^{n-2}+\frac{n(n-1)(n-2)}{3!}h^2x^{n-3}+\cdots\right]$$

$$=\left[nx^{n-1}+n\frac{(n-1)}{2!}(0)x^{n-2}+\frac{n(n-1)(n-2)}{3!}(0)^2x^{n-3}+\cdots\right] \text{ as } h\to 0$$

Thus, $f'(x)=nx^{n-1}$.

Therefore, $\dfrac{d}{dx}(x^n)=f'(x)=nx^{n-1}$. This is called the power rule of differentiation.

Note that, this formula is true even for negative values of n and fractional values such as $n=\dfrac{p}{q}$, where p and q are integers but $q\neq 0$.

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Example 8.5

Differentiate $y = x^5$ with respect to x.

Solution

From $\frac{d}{dx}(x^n) = nx^{n-1}$, it follows that,

$$\frac{d}{dx}(x^5) = 5x^{5-1}$$

$$= 5x^4$$

Therefore, $\frac{d}{dx}(x^5) = 5x^4$.

Example 8.6

Differentiate the function $f(x) = (x-2)(x+3)$ with respect to x at the point $x = 1$.

Solution

Given, $f(x) = (x-2)(x+3)$.

$\Rightarrow f(x) = x^2 + x - 6$

Using the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, it follows that,

$f'(x) = 2x^{2-1} + x^{1-1} - 0$

$\Rightarrow f'(x) = 2x + 1$

At the point $x = 1$, $f'(1) = 2(1) + 1 = 3$.

Therefore, at the point $x = 1$, $f'(x) = 3$.

Properties of derivatives

The following are properties of derivatives for polynomial functions:

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
2. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

$$= f'(x) \pm g'(x)$$

3. If k is any scalar, then $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)] = k f'(x)$.

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Example 8.7

Find the derivative of the function $f(x)=\dfrac{4x^3+2x^2}{\sqrt[3]{x}}$.

Solution

Since the derivative of sum of two functions is the sum of their derivatives, then

$$f(x)=\frac{4x^3+2x^2}{\sqrt[3]{x}}=\frac{4x^3}{\sqrt[3]{x}}+\frac{2x^2}{\sqrt[3]{x}}$$

$$=\frac{4x^3}{x^{\frac{1}{3}}}+\frac{2x^2}{x^{\frac{1}{3}}}=4x^{\frac{8}{3}}+2x^{\frac{5}{3}}$$

Using $\dfrac{d}{dx}\left(x^n\right)=nx^{n-1}$,

$$\Rightarrow f'(x)=4\left(\frac{8}{3}\right)x^{\frac{8}{3}-1}+2\left(\frac{5}{3}\right)x^{\frac{5}{3}-1}$$

$$=\frac{32}{3}x^{\frac{5}{3}}+\frac{10}{3}x^{\frac{2}{3}}$$

Therefore, $f'(x)=\dfrac{32}{3}x^{\frac{5}{3}}+\dfrac{10}{3}x^{\frac{2}{3}}$.

Exercise 8.2

1. Find the derivative of each of the following functions with respect to x at the given values of x.
   (a) $f(x)=2x\left(x^2-4\right);\ x=1$
   (b) $f(x)=3x^4-2x^3+x^2+9;\ x=0$
   (c) $f(x)=\dfrac{6x^5+2x^3}{x^2};\ x=-1$
   (d) $f(x)=x^4-\dfrac{1}{3}x^3+\dfrac{1}{4}x^2+9;\ x=-2$
2. Differentiate each of the following functions with respect to x:
   (a) $f(x)=\dfrac{1}{4}ax^2-3bx+c$
   (b) $f(x)=\dfrac{1}{5}\left(\dfrac{x^2+4}{\sqrt[5]{x}}\right)$
   (c) $f(x)=\dfrac{\sqrt{x}+x^2-4x}{\sqrt{x}}$
3. Find the derivative of each of the following functions with respect to x:
   (a) $f(x)=2(x-1)(x-5)$
   (b) $g(x)=\dfrac{3(x-1)(x-2)}{4}$
   (c) $h(x)=\dfrac{x^3-2x^2}{3x}$
4. If the gradient of the curve $y=f(x)=(x-3)(x^2+a)$ at $x=1$ is -1. Determine the value of a.
5. Find the coordinates of the point on the curve $y=\dfrac{2}{x^2}$ at which its gradient is $\dfrac{1}{2}$.
6. If $g(x)=2x^3-8x+5$, find the values of x for which $g'(x)=0$.
7. Find the values of t for which the gradient of the curve $x=2t^3+3t^2$ is zero. Hence, determine the gradient at $t=2$.

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8. Determine the gradient of the function $h(t)=9t^4-7t^3+8t^2-\frac{8}{t}+\frac{10}{t^3}$ at $=1$.

9. Differentiate $2x^2+5+4x^{-2}$ with respect to x and then find the value of the derivative when $x=4$.

10. If $y(x+2)^{-2}=-100$, find $\frac{dy}{dx}$ and the gradient of the curve at $x=-1$.

Derivative of product of polynomials

Let y be the product of two functions u and v of an independent variable x. That is, $y=uv$. The derivative of the product of functions can be obtained as follows.

Let $y=u(x)v(x)$. If δy, δu and δv denote small increments in y, u, and v, respectively, it implies that

$y+\delta y=(u+\delta u)(v+\delta v)$

$\Rightarrow y+\delta y=uv+v\delta u+u\delta v+\delta u\delta v$

But $y=uv$

$\Rightarrow \delta y=v\delta u+u\delta v+\delta u\delta v$

Dividing by δx on both sides gives,

$\frac{\delta y}{\delta x}=v\frac{\delta u}{\delta x}+u\frac{\delta v}{\delta x}+\frac{\delta u}{\delta x}\delta v$

Now, as $\delta x\to 0$, $\frac{\delta y}{\delta x}=\frac{dy}{dx}$, $\frac{\delta v}{\delta x}=\frac{dv}{dx}$, $\frac{\delta u}{\delta x}=\frac{du}{dx}$, and $\delta v\to 0$.

Thus, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}+\frac{du}{dx}(0)$

$=v\frac{dv}{dx}+u\frac{dv}{dx}$.

The result, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$ is called the product rule of differentiation.

Therefore, the product rule for differentiation is given by

$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$.

Example 8.8

Use the product rule to differentiate the function $y=(x^3-4x^2+6x)(2x^4-19x+5)$ with respect to x.

Solution

Let $u=x^3-4x^2+6x$ and $v=2x^4-19x+5$.

$\Rightarrow \frac{du}{dx}=3x^2-8x+6$ and $\frac{dv}{dx}=8x^3-19$.

From $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$.

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$$\Rightarrow \frac{dy}{dx} = (2x^4 - 19x + 5)(3x^2 - 8x + 6) + (x^3 - 4x^2 + 6x)(8x^3 - 19)$$

$$= 14x^6 - 48x^5 + 60x^4 - 76x^3 + 243x^2 - 268x + 30$$

Therefore, $\frac{dy}{dx} = 14x^6 - 48x^5 + 60x^4 - 76x^3 + 243x^2 - 268x + 30$.

Example 8.9

Differentiate with respect to x the function $f(x) = (x^3 - 2)(x+4)^2$, then find $f'(x)$ at $x = 1$.

Solution

The function $f(x) = (x^3 - 2)(x+4)^2$ is a product of two functions.

Using $f'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$.

Let $v = (x+4)^2$ and $u = x^3 - 2$.

$\Rightarrow \frac{dv}{dx} = 2(x+4)$ and $\frac{du}{dx} = 3x^2$.

$\Rightarrow f'(x) = (x+4)^2(3x^2) + (x^3 - 2)(2x+8)$

$\Rightarrow f'(x) = 5x^4 + 32x^3 + 48x^2 - 4x - 16$

At the point $x = 1$,

$f'(1) = 5(1)^4 + 32(1)^3 + 48(1)^2 - 4(1) - 16$

$= 65$

Therefore, $f'(x) = 5x^4 + 32x^3 + 48x^2 - 4x - 16$ and $f'(x)$ at $x = 1$ is 65.

Exercise 8.3

1. Differentiate each of the following functions with respect to the given variables:

   (a) $f(x) = (1+x)^2(x^2 - 1)$

   (b) $f(s) = (6s^2 - 4s)(s^2 + 4s + 8)$

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(c) $f(y)=\left(6-3y^2+\frac{1}{y^2}\right)\left(\frac{1}{y}-4y+8y^2\right)$

(d) $f(t)=\left(7t^2-4t+9\right)\left(2t^3-2t+1\right)$

2. Find the x-coordinate of the point where the gradient of the curve $y=\left(x^2-2\right)\sqrt{3+x}$ is zero.
3. Evaluate the derivative of the curve $y=x^3(4-x)^{\frac{1}{2}}$ at $x=1$.
4. Find the derivative in each of the following functions.

(a) $f(x)=\left(x^2+4x+1\right)\sqrt{x-3}$

(b) $g(x)=2x^6\left(2+x\right)^5$

(c) $h(x)=x^{-3}\left(1+x\right)^{\frac{1}{2}}$

(d) $k(x)=\left(1-x^3\right)\left(2-x^5+x^6\right)$

5. Differentiate $g(x)=\left(x^2+1\right)(x+4)^{-2}$ with respect to x.
6. If $f(x)=2x^{\frac{3}{2}}\left(2+\sqrt{x}\right)\left(-1+\sqrt{x}\right)$, show that $f'(x)=4x+5x\sqrt{x}-6\sqrt{x}$.
7. The equation of a curve is $y=(x-3)(x+4)$. Find the gradient of the curve;
   (a) at the point where the curve crosses the y-axis.
   (b) at each of the points where the curve crosses the x-axis.
8. Use the product rule to find $f'(r)$, where $f(r)=\left(1+r^2\right)^2\left(1-r^2\right)$.
9. Differentiate $z=\left(y-2\right)^2\left(2y+3\right)$ with respect to y.
10. Given that $f(u)=\left(u+1\right)^2\left(u-1\right)^4$, find $f'(u)$ and $f'(-5)$.

The derivative of quotient of two functions

If u and v are two differentiable functions of x and $y=\frac{u}{v}$, then the derivative of y can be determined as follows:

Given $y=\frac{u}{v}$, it implies that

$y+\delta y=\frac{u+\delta u}{v+\delta v}$, where δy, δu, and δv denote small increments in y, u, and v, respectively.

Thus, $\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}$.

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$$\Rightarrow \delta y = \frac{(vu + v\delta u) - (uv + u\delta v)}{v(v + \delta v)}$$

$$\Rightarrow \delta y = \frac{v\delta u - u\delta v}{v(v + \delta v)}$$

Divide by δx both sides to get

$$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}$$

Taking the limit as $\delta x \to 0$ and $\delta v \to 0$ gives

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta x \to 0}\left(v\dfrac{\delta u}{\delta x} - u\dfrac{\delta y}{\delta x}\right)}{\lim\limits_{\delta x \to 0}\left(v^2 + v\delta v\right)}$$

$$= \frac{\lim\limits_{\delta x \to 0} v\dfrac{\delta u}{\delta x} - \lim\limits_{\delta x \to 0} u\dfrac{\delta v}{\delta x}}{\lim\limits_{\delta x \to 0}\left(v^2 + v\delta v\right)}$$

$$= \frac{v\lim\limits_{\delta x \to 0}\dfrac{\delta u}{\delta x} - u\lim\limits_{\delta x \to 0}\dfrac{\delta v}{\delta x}}{\lim\limits_{\delta x \to 0} v^2 + v\lim\limits_{\delta x \to 0}\delta v}$$

But $\lim\limits_{\delta x \to 0}\dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$, $\lim\limits_{\delta x \to 0}\dfrac{\delta u}{\delta x} = \dfrac{du}{dx}$,

$\lim\limits_{\delta x \to 0}\dfrac{\delta v}{\delta x} = \dfrac{dv}{dx}$, $\lim\limits_{\delta v \to 0} v^2 = v^2$, and $\lim\limits_{\delta v \to 0}\delta v = 0$

Thus, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ is called the quotient rule of differentiation.

Therefore, the quotient rule of differentiation is given by;

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Example 8.10

Find the derivative of $y = \dfrac{x^2 + 4}{x^2 - 4}$.

Solution

Given $y = \dfrac{x^2 + 4}{x^2 - 4}$.

Let $u = x^2 + 4$ and $v = x^2 - 4$.

$\Rightarrow \dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = 2x$.

Using quotient rule, it implies that

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Thus,

$$\frac{dy}{dx} = \frac{(x^2 - 4)(2x) - (x^2 + 4)(2x)}{(x^2 - 4)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-16x}{(x^2 - 4)^2}$$

Therefore, $\dfrac{dy}{dx} = \dfrac{-16x}{(x^2 - 4)^2}$.

Example 8.11

If $y = \dfrac{5x^3 + 2x^2}{x + 1}$, find $\dfrac{dy}{dx}$.

Solution

Given $y = \dfrac{5x^3 + 2x^2}{x + 1}$.

Let $u = 5x^3 + 2x^2$ and $v = x + 1$.

$\Rightarrow \dfrac{du}{dx} = 15x^2 + 4x$ and $\dfrac{dv}{dx} = 1$.

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Using the quotient rule,

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Thus,

$$\frac{dy}{dx} = \frac{(x+1)(15x^2+4x)-(5x^3+2x^2)(1)}{(x+1)^2}$$

$$= \frac{15x^3+4x^2+15x^2+4x-5x^3-2x^2}{(x+1)^2}$$

$$= \frac{10x^3+17x^2+4x}{(x+1)^2}$$

Therefore, $\frac{dy}{dx} = \frac{10x^3+17x^2+4x}{(x+1)^2}$.

Exercise 8.4

1. Find $\frac{dy}{dx}$ in each of the following:

   (a) $y = \frac{2x+4}{3x-2}$

   (b) $y = \left(\frac{x+1}{x-1}\right)^2$

   (c) $y = \frac{2x+1}{x^2-2}$

2. Differentiate the following functions with respect to x. For each case, find $f'(2)$ correct to 3 decimal places.

   (a) $f(x) = \sqrt{\left(\frac{1+x}{3+x}\right)}$

   (b) $f(x) = \frac{\sqrt{x}}{\sqrt{x+3}}$

   (c) $f(x) = \frac{2x^3-x^2}{\sqrt{x+3}}$

3. Find the derivative in each of the following:

   (a) $s = \frac{t^{-1}+2t^2}{3t^2}$

   (b) $p = \frac{q^3-2q^2+3q}{q^2}$

   (c) $u = \frac{v+2v^2}{3v}$

4. Given that $t = \frac{3z^2+z-6}{2z}$, find $\frac{dt}{dz}$ at $z = 2$.

5. If $g(x) = x-1+\frac{1}{x+1}$, where $x \neq -1$, find the values of x for which $g'(x) = 0$.

6. Show that the gradient of the curve $y = \frac{2x}{x^2-5}$ at $x = 2$ is -18.

7. If $y = \frac{x}{x+5}$, prove that $x\frac{dy}{dx} = y(1-y)$.

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8. Verify that, the derivative of $g(y)=\frac{1+y^2}{1-y^2}$ is $g'(y)=\frac{4y}{(1-y^2)^2}$.

9. Determine $\frac{d}{ds}\left(\frac{(s-1)(s-2)}{(s-3)(s-4)}\right)$, then find the gradient when $s=4.5$.

10. If $z^2(1+w^2)=1-w^2$, show that $\left(\frac{dz}{dw}\right)^2=\frac{1-z^4}{1-w^4}$.

The chain rule

The chain rule is used to find the derivative of a composite function.

Consider the following functions:

(i) $f(x)=(4x-1)^3$ is a cubic of a linear function $g(x)=(4x-1)$.

(ii) $n(x)=\sqrt{(x^4-3)}$ is a square root of the quartic function $m(x)=(x^4-3)$.

(iii) $h(x)=(x^2-2)^5$ is the quintic of the quadratic function $g(x)=(x^2-2)$.

If y is a function of u, say $y=f(u)$, where u itself is a function of x, say $u=g(x)$, then y is called a composite function of x.

Given $y=f(u)$, then

$y+\delta y=f(u+\delta u)$, where δy and δu denote small increments in y and u, respectively.

$\delta y=f(u+\delta u)-f(u)$

Dividing by δx both sides gives,

$$\frac{\delta y}{\delta x}=\frac{f(u+\delta u)-f(u)}{\delta x}$$

$$\Rightarrow \frac{\delta y}{\delta x}=\frac{f(u+\delta u)-f(u)}{\delta u}\times\frac{\delta u}{\delta x}$$

Introducing limits on both sides gives,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta u\to 0}\frac{f(u+\delta u)-f(u)}{\delta u}\times\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{\delta u}[f(u)]\times\frac{du}{dx}$$

Thus, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$, which is the chain rule of differentiation.

Therefore, the chain rule of differentiation is given by;

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Example 8.12

Given that $y=(2x^2+4)^6$, find $\frac{dy}{dx}$.

Solution

Given $y=(2x^2+4)^6$.

Let $u=2x^2+4$, then $y=u^6$.

$$\Rightarrow \frac{du}{dx}=4x \text{ and } \frac{dy}{du}=6u^5.$$

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Using the chain rule, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$.

$\Rightarrow \frac{dy}{dx}=6u^5\times 4x$

$\Rightarrow \frac{dy}{dx}=24xu^5$. But $u=2x^2+4$.

Therefore, $\frac{dy}{dx}=24x\left(2x^2+4\right)^5$.

Example 8.13

If $y=6(x^3-3x+2)^2$ and $u=(x^3-3x+2)$, express y in terms of u. Hence, find $\frac{dy}{dx}$ at $x=2$.

Solution

Given $y=6(x^3-3x+2)^2$ and

$u=(x^3-3x+2)$

$\Rightarrow y=6u^2$

$\Rightarrow \frac{dy}{du}=12u$ and $\frac{du}{dx}=3x^2-3$

Using the chain rule,

$\Rightarrow \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Thus, $\frac{dy}{dx}=12u\times\left(3x^2-3\right)$

$\Rightarrow \frac{dy}{dx}=12\left(x^3-3x+2\right)\left(3x^2-3\right)$

At $x=2$

$\Rightarrow\left[\frac{dy}{dx}\right]_{x=2}=12\left((2)^3-3(2)+2\right)\left(3(2)^2-3\right)=432$.

Therefore, $y=6u^2$ and $\frac{dy}{dx}=432$, at $x=2$.

Example 8.14

Find the derivative of $\left(x^4-3x^2+4\right)^{\frac{2}{3}}$ at $x=1$.

Solution

Given $y=\left(x^4-3x^2+4\right)^{\frac{2}{3}}$.

Let $u=(x^4-3x^2+4)\Rightarrow y=u^{\frac{2}{3}}$.

$\Rightarrow \frac{dy}{dx}=\frac{2}{3}u^{-\frac{1}{3}}$ and $\frac{du}{dx}=4x^3-6x$.

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=\frac{2}{3}u^{-\frac{1}{3}}\times\left(4x^3-6x\right)$

$\Rightarrow \frac{dy}{dx}=\frac{2}{3}\left(x^4-3x^2+4\right)^{-\frac{1}{3}}\times\left(4x^3-6x\right)$

At $x=1$, it implies that

$\left[\frac{dy}{dx}\right]_{x=1}=\frac{2}{3}\left((1)^4-3(1)^2+4\right)^{-\frac{1}{3}}\times\left(4(1)^3-6(1)\right)$

$=-1.058$

Therefore, $\left[\frac{dy}{dx}\right]_{x=1}=-1.058$.

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Exercise 8.5

1. Using the chain rule, find the derivative of each of the following functions with respect to x:

   (a) $f(x)=\sqrt{x^4-2}$ (b) $f(x)=\dfrac{1}{x^3+1}$

2. Differentiate each of the following functions using the chain rule:

   (a) $y=\left(x^2+1\right)^{16}$ (c) $y=\left(3x^2-4\right)^3$

   (b) $y=\left(z^3+4z^2-3z-3\right)^{-6}$ (d) $y=\sqrt{x^2+5}$

3. Differentiate each of the following functions using the chain rule:

   (a) $f(t)=\sqrt{t^2-6t+7}$ (c) $g(z)=\dfrac{1}{\sqrt{3-z^3}}$ (e) $y=\left(t^3-\sqrt{t}\right)^{29}$

   (b) $f(t)=(3-t)^{21}$ (d) $z=\left(x+\dfrac{1}{x}\right)^{\frac{2}{5}}$

4. Use the chain rule of differentiation to show that,

   $$\frac{d}{dx}\left[\frac{1}{\sqrt[3]{6x^5-7x^3+9}}\right]=-x^2\left(10x^2-7\right)\left(6x^2-7x^3+9\right)^{-\frac{4}{3}}.$$

5. Given that $f(t)=\dfrac{1}{\sqrt[3]{2t^4+3t^3-5t+6}}$, find the following:

   (a) $f'(t)$

   (b) $f'(-1)$

6. Find $\dfrac{d}{dr}\left(\sqrt{r^2-\dfrac{1}{r^2}}\right)$ using the chain rule.

7. Find $\dfrac{d}{d\theta}(4-2\theta)^5$.

8. Find the derivative of the function $x=\left(1-\dfrac{2}{\sqrt{u}}\right)^{\frac{1}{3}}$ using the chain rule.

9. Evaluate $\dfrac{d}{dy}\left(y+1-\dfrac{1}{y}\right)^3$ at $y=-2$.

10. Verify that $\dfrac{d}{dt}\left[\dfrac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+t}\right]=\dfrac{-2}{\sqrt{1+t^2}\left(\sqrt{1+t^2}+t\right)^2}$.

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Differentiation of implicit functions

An implicit function is a function written in terms of both dependent and independent variables. It is a function of the form $f(x, y, \ldots)$ whose variables are connected by an equation which does not express one variable in terms of the other variables. For example, $x^2y^3 = 8$ is an implicit function because each variable in the equation $x^2y^3 = 8$ is not expressed as a function of the other. Also, the equation $x = y^2$ does not define y as a function of x, since each positive value of x has two values of y.

In this case, $x = y^2$ can be differentiated with respect to y as follows.

$$\frac{d(x)}{dy} = \frac{d}{dy}(y^2)$$

$$\Rightarrow \frac{dx}{dy} = 2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Generally, $\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$, where δx and δy are small increments in x and y, respectively.

Now, as $\delta x \to 0$, $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ and $\frac{\delta x}{\delta y} = \frac{dx}{dy}$.

Therefore, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Thus, implicit differentiation is a method for finding the slope of a curve when the equation of the curve is not written in explicit form. The chain rule and the product rule are useful in differentiating implicit functions.

To differentiate a function of y with respect to x requires to differentiate the function with respect to y, and then multiply by $\frac{dy}{dx}$ to complete the chain rule. That is,

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}.$$

Suppose z is in terms of y and y is in terms of x. It can be written as; $z = z(y)$ and $y = y(x)$. Using the chain rule, it gives $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$.

If an expression has a term of the form $x^m y^n$, then

$$\frac{d}{dx}(x^m y^n) = x^m \frac{d}{dx}(y^n) + y^n \frac{d}{dx}(x^m).$$

$$\Rightarrow \frac{d}{dx}(x^m y^n) = nx^m y^{n-1} \frac{dy}{dx} + mx^{m-1} y^n.$$

Example 8.15

Differentiate $z = y^2$ with respect to x.

Solution

Given $z = y^2 \Rightarrow \frac{dz}{dy} = 2y$.

Using the chain rule,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} = 2y \times \frac{dy}{dx}$$

Therefore, $\frac{dz}{dx} = 2y\frac{dy}{dx}$.

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Example 8.16

Find $\frac{dy}{dx}$ for the curve $3x^2 - 4y^2 = 12$ at the point (4, 3).

Solution

Given the curve $3x^2 - 4y^2 = 12$.

Differentiate with respect to x:

$$\frac{d}{dx}(3x^2 - 4y^2) = \frac{d}{dx}(12)$$

$$\Rightarrow \frac{d}{dx}(3x^2) - \frac{d}{dx}(4y^2) = 0$$

$$\Rightarrow 6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{4y}$$

At the point (4, 3),

$$\frac{dy}{dx} = \frac{3\times 4}{4\times 3}$$

$$= 1$$

Therefore, $\frac{dy}{dx}$ at (4, 3) is 1.

Example 8.17

If $y^2 = \frac{x-6}{2x+3}$, show that $\left(\frac{dy}{dx}\right)^2 = \frac{225}{4(x-6)(2x+3)^3}$.

Solution

Given $y^2 = \frac{x-6}{2x+3}$.

Differentiate with respect to x:

$$2y\frac{dy}{dx} = \frac{(2x+3)(1)-(x-6)(2)}{(2x+3)^2}$$

Thus, $2y\frac{dy}{dx} = \frac{15}{(2x+3)^2}$.

Squaring both sides gives,

$$4y^2\left(\frac{dy}{dx}\right)^2 = \frac{225}{(2x+3)^4}.$$

But $y = \frac{x-}{2x+3}$

$$\left(\frac{dy}{dx}\right)^2 = \frac{225}{4\left(\frac{x-6}{2x+3}\right)(2x+3)^4}$$

$$= \frac{225}{4(x-6)(2x+3)^3}$$

Therefore, $\left(\frac{dy}{dx}\right)^2 = \frac{225}{4(x-6)(2x+3)^3}$.

Exercise 8.6

1. Differentiate each of the following functions with respect to x:
   (a) $\sqrt{y^6}$ (b) $x + y$

2. By implicit differentiation of $x^2 - y^2 = 1$, show that $\frac{dy}{dx} = \frac{x}{y}$.

3. Given $y^2 = \frac{x-2}{x+3}$, find $\frac{dy}{dx}$.

4. Differentiate $(3x+6)^4 = 5y^3$ with respect to x.

5. If $x^2 + y^2 = 4x$, show that $\frac{dy}{dx} = \frac{2-x}{y}$.

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6. If $\frac{1}{x}+\frac{1}{y}=y$, show that $-\frac{1}{x^2}=\left(1+\frac{1}{y^2}\right)\frac{dy}{dx}$.

7. Find the gradient of the curve $x^2+y^2=9$ at the point $\left(1,-2\sqrt{2}\right)$.

8. Find $\frac{dy}{dx}$ for the curve $x^2-3y^2=9$ at the point (6, 3).

9. If $x^2-3y^2=9$ find y'.

10. Given that $x^2+z^2=25$, determine the values of $\frac{dz}{dx}$ when $x=-4$.

Further implicit differentiation of functions

Implicit differentiation can also be done without rearranging the terms in the expression of the function. In this case, each term is differentiated as a function of x by applying the chain rule and the product rule if needed.

Example 8.18

Using implicit differentiation, find $\frac{dy}{dx}$ for $y^2+x^3-y^3+5=4y$.

Solution

Differentiate each term with respect to x as follows;

$\frac{d}{dx}(y^2)+\frac{d}{dx}(x^3)-\frac{d}{dx}(y^3)+\frac{d}{dx}(5)=\frac{d}{dx}(4y)$. It implies that,

$$\frac{d}{dy}(y^2)\times\frac{dy}{dx}+\frac{d}{dx}(x^3)-\frac{d}{dy}(y^3)\times\frac{dy}{dx}+\frac{d}{dx}(5)=\frac{d}{dy}(4y)\times\frac{dy}{dx}$$

Thus, $2y\frac{dy}{dx}+3x^2-3y^2\frac{dy}{dx}+0=4\frac{dy}{dx}$.

Collect together all the terms containing $\frac{dy}{dx}$ yields:

$$\frac{dy}{dx}=\frac{3x^2}{3y^2-2y+4}$$

Therefore, $\frac{dy}{dx}=\frac{3x^2}{3y^2-2y+4}$.

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Example 8.19

If $x^5 + 4xy^3 - 3y^5 = 2$, find $\frac{dy}{dx}$.

Solution

Differentiate each term with respect to x as follows;

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(4xy^3) - \frac{d}{dx}(3y^5) = \frac{d}{dx}(2)$$

For the term $4xy^3$, the product rule is used as follows:

$$5x^4 + 4\left(x\frac{d}{dy}(y^3) \times \frac{dy}{dx} + y^3\frac{dx}{dx}\right) - 3\frac{d}{dy}(y^5)\frac{dy}{dx} = 0$$

$$\Rightarrow 5x^4 + 4\left(3xy^2\frac{dy}{dx} + y^3\right) - 15y^4\frac{dy}{dx} = 0$$

Collect all terms containing $\frac{dy}{dx}$:

$$\Rightarrow (12xy^2 - 15y^4)\frac{dy}{dx} = -(5x^4 + 4y^3)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{5x^4 + 4y^3}{12xy^2 - 15y^4}$$

Therefore, $\frac{dy}{dx} = -\frac{5x^4 + 4y^3}{12xy^2 - 15y^4}$.

Example 8.20

Find the gradient of the curve $x^2y - 2xy^2 + y^2 = 1$ at the point (2, 1).

Solution

Differentiate each term with respect to x as follows:

$$\frac{d}{dx}(x^2y) - \frac{d}{dx}(2xy^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} - \left(2y^2 + 4xy\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

Collect together all terms containing $\frac{dy}{dx}$:

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$$\Rightarrow (x^2-4xy+2y)\frac{dy}{dx}=2y^2-2xy$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-2xy}{x^2-4xy+2y}$$

At the point $(2,1)$, it follows that

$$\left[\frac{dy}{dx}\right]_{(2,1)}=\frac{2(1)^2-2(2)(1)}{(2)^2-4(2)(1)+2(1)}=1.$$

Therefore, the gradient of the curve at the point $(2,1)$ is 1.

Exercise 8.7

1. Find $\frac{dy}{dx}$ in each of the following:
   (a) $y^2(1+x^2)=3-x^2$
   (b) $2xy+y^2=x+y$
   (c) $4x^2y^3-6xy^2+4xy-\frac{1}{yx}=40$
   (d) $x^2+xy^2+y^3=2$
2. If $x^2y^3=x-6$, find $\frac{dy}{dx}$.
3. Determine the gradient of the graph of $3(x^2+y^2)^2=100xy$ at the point $(3, 1)$.
4. Given that $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$.
5. Find y', given that $x^2y-xy^2+x^2+y^2=0$.
6. Find $\frac{dz}{dx}$, if $x^2+z^2-6xz+3x-2z+15=0$.
7. Given the implicit function $x^2+xy=2$, show that $\frac{dy}{dx}=-\frac{2x+y}{x}$.
8. Show that the gradient to the curve $x^3+3xy+y^3=5$ at the point $(1, 1)$ is -1.
9. Given that $x^2y-xy^2+x^2+y^2=0$, find $\frac{dy}{dx}$.
10. If $x^3+y^3=3axy$, where a is an arbitrary constant, find $\frac{dy}{dx}$.
11. Given $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=6$, show that $\frac{dy}{dx}=\frac{3y-\sqrt{xy}}{\sqrt{xy}-3x}$.
12. If $\sqrt{1-p^2}+\sqrt{1-q^2}=k(p-q)$, show that $\frac{dq}{dp}=\frac{\sqrt{1-q^2}}{1-p^2}$.
13. Differentiate each of the following with respect to x:
    (a) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=0$
    (b) $y+\sqrt{xy}=x^2$
    (c) $(x^2+y^2)^2=xy$

Derivatives of trigonometric functions

Trigonometric functions involve sine, cosine, tangent, secant, cosecant, and cotangent with their units in radians..

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Derivatives of trigonometric functions can be derived using first principles of differentiation.

Example 8.21

Use first principles to find the derivative of each of the following:

(a) $f(x)=\sin x$ (b) $y=\cos x$

Solution

(a) From first principles of differentiation,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

where $f(x)=\sin x$ and $f(x+h)=\sin(x+h)$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Using the compound angle formula, it implies that

$$f'(x)=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$

$$=\lim_{h\to 0}\sin x\frac{(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h}$$

$$=\sin x\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$

But

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \text{ and } \lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)=0$$

$$\Rightarrow f'(x)=0\times\sin x+1\times\cos x$$

$$=\cos x$$

Therefore, $f'(x)=\cos x$.

(b) From first principles of differentiation:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{y(x+h)-y(x)}{h},$$ where

$y(x+h)=\cos(x+h)$ and $y(x)=\cos x$.

$$\Rightarrow\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

Using the compound angle formula, it implies that

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$=\cos x\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$

But $\lim_{h\to 0}\frac{(\cos h-1)}{h}=0$ and $\lim_{h\to 0}\frac{\sin h}{h}=1$

Thus, $\frac{dy}{dx}=0\times\cos x-1\times\sin x$

$$=-\sin x$$

Therefore, $\frac{dy}{dx}=-\sin x$.

Example 8.22

Find the derivative of $\sin^3\theta$ with respect to θ.

Solution

Let $y=\sin^3\theta$.

Also, let $u=\sin\theta\Rightarrow y=u^3$.

$$\Rightarrow\frac{du}{d\theta}=\cos\theta \text{ and } \frac{dy}{du}=3u^2.$$

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Using the chain rule,

$$\frac{dy}{d\theta}=\frac{du}{d\theta}\times\frac{dy}{du}$$

$$\Rightarrow \frac{dy}{d\theta}=\cos\theta\times 3u^2$$

$$=\cos\theta\times 3\sin^2\theta$$

Therefore, $\frac{dy}{d\theta}=3\cos\theta\sin^2\theta.$

Example 8.23

Find the gradient of the curve $y=5\sin x-x^2$ when $x=-\pi$.

Solution

Differentiate with respect to x as follows:

$$\frac{dy}{dx}=\frac{d}{dx}(5\sin x-x^2)$$

$$=5\cos x-2x$$

At $x=-\pi$, it follows that

$$\left[\frac{dy}{dx}\right]_{x=-\pi}=5\cos(-\pi)-2(-\pi)$$

$$=5(-1)-2(-\pi)$$

$$=2\pi-5.$$

Therefore, the gradient of the curve at $x=-\pi$ is $2\pi-5$.

Example 8.24

Using first principles, show that $f'(\theta)=\sec\theta\tan\theta$, if $f(\theta)=\sec\theta$.

Solution

Let $f(\theta)=\sec\theta=\frac{1}{\cos\theta}$.

From first principles,

$$f'(\theta)=\lim_{h\to 0}\frac{f(\theta+h)-f(\theta)}{h}$$

$$\Rightarrow f'(\theta)=\lim_{h\to 0}\frac{\frac{1}{\cos(\theta+h)}-\frac{1}{\cos(\theta)}}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos\theta-\cos(\theta+h)}{h\cos\theta\cos(\theta+h)}\right]$$

Using the factor formula, it implies that

$$f'(\theta)=\lim_{h\to 0}\frac{-2\sin\left(\theta+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\cos\theta\cos(\theta+h)}$$

$$=\lim_{h\to 0}\frac{-2\sin\left(\theta+\frac{h}{2}\right)}{\cos\theta\cos(\theta+h)}\times\lim_{h\to 0}\frac{\sin\left(-\frac{h}{2}\right)}{h}$$

$$=\frac{-2\sin\theta}{\cos\theta\cos\theta}\times\left(-\frac{1}{2}\right)$$

$$\Rightarrow f'(\theta)=\sec\theta\tan\theta$$

Therefore, $f'(\theta)=\sec\theta\tan\theta.$

Example 8.25

Differentiate $y=\cos\left(\frac{1}{x}\right)$ with respect to x.

Solution

Given $y=\cos\left(\frac{1}{x}\right)$.

Let $u=\frac{1}{x}=x^{-1}\Rightarrow y=\cos u$.

Thus, $\frac{dy}{du}=-\sin u$ and $\frac{du}{dx}=-x^{-2}$.

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Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx}=-\sin u\times\left(-x^{-2}\right)$$

Thus, $\frac{dy}{dx}=x^{-2}\sin\left(\frac{1}{x}\right)=\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$.

Therefore, $\frac{dy}{dx}=\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$.

Example 8.26

If $f(x)=\frac{\cos x}{x}$, show by using first principles that $f'(x)=-\frac{x\sin x+\cos x}{x^2}$.

Solution

Given $f(x)=\frac{\cos x}{x}\Rightarrow f(x+h)=\frac{\cos(x+h)}{x+h}$

From first principles,

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\frac{\cos(x+h)}{x+h}-\frac{\cos x}{x}}{h}$$

$$=\lim_{h\to 0}\frac{x\cos(x+h)-(x+h)\cos x}{h(x+h)x}$$

$$=\lim_{h\to 0}\frac{x[\cos(x+h)-\cos x]-h\cos x}{hx(x+h)}$$

$$=\lim_{h\to 0}\frac{x\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-h\cos x}{hx(x+h)}$$

$$=-\lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)}{(x+h)}\times\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}-\lim_{h\to 0}\frac{\cos x}{x(x+h)}$$

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$$= -\frac{\sin x}{x} - \frac{\cos x}{x \times x}$$

$$= -\frac{\sin x}{x} - \frac{\cos x}{x^2}$$

$$\Rightarrow f'(x) = -\frac{x\sin x + \cos x}{x^2}$$

Therefore, $f'(x) = -\dfrac{x\sin x + \cos x}{x^2}$.

Exercise 8.8

1. Differentiate each of the following functions with respect to x:
   (a) $\sin 2x$
   (b) $x^2 \sin x$
   (c) $\sin\left(\dfrac{2}{x}\right)$
   (d) $6\cos 2x$
   (e) $\cos(1-x^2)$
   (f) $\dfrac{\cos x}{x^3}$
   (g) $\sin x(1+\cos x)$
   (h) $\tan\sqrt{x+2}$
   (i) $\tan 4x$
   (j) $\sin^2(3x^2-1)$
   (k) $\dfrac{\sin x}{1+\cos x}$
   (l) $\cos\left(\dfrac{1-\sin x}{1+\cos x}\right)$
2. Find the gradient of each of the following curves at the given value of x:
   (a) $y = 2\sin x - x^2$; $x = -\pi$
   (b) $y = -4\cos x$; $x = \dfrac{1}{2}\pi$
3. If $y = \sqrt{\dfrac{1+\sin x}{1-\sin x}}$, show that $\dfrac{dy}{dx} = \dfrac{1}{1-\sin x}$.
4. Show that $\dfrac{d}{dx}\left(\dfrac{\tan x}{1-\tan^2 x}\right)^{\frac{1}{2}} = \dfrac{\sec^4 x}{2\sqrt{\tan x}\left(1-\tan^2 x\right)^{\frac{3}{2}}}$.
5. Differentiate $f(x) = \dfrac{\sin x}{x}$ with respect to x.
6. If $g(x) = 3\cot x + 5\operatorname{cosec} x$, show that $g'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$.
7. Find $\dfrac{d}{d\theta}\left(\sin\sqrt{\theta^2-1}\right)$.
8. Prove that $\dfrac{d}{dx}\left(\tan\sqrt{6x^3+2}\right) = \dfrac{9x^2}{6x^3+2}\sec^2\sqrt{6x^3+2}$.
9. Given that $x = \dfrac{1+\sin\theta+\cos\theta}{1-\sin\theta+\cos\theta}$, show that $x' = (1-\sin\theta)^{-1}$.
10. Find $\dfrac{dy}{dx}$ for each of the following:
    (a) $xy^2 + \cos 2y = 4$
    (b) $y^4 + x^3 + \cos(x+y^2) = 0$
    (c) $y = \dfrac{1-\cos 2x}{\sin 2x}$
    (d) $y = \dfrac{1+\tan x}{1-\tan x}$
11. Using first principles, show that:
    (a) $\dfrac{d}{d\theta}(1+2\cos 4\theta) = -8\sin 4\theta$
    (b) $\dfrac{d}{dx}(\tan x) = \sec^2 x$
    (c) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

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(d) $\frac{d}{dx}(\cos^2 x) = -\sin 2x$

(e) $\frac{d}{dx}\left(\frac{\operatorname{cosec} 2\theta}{2\theta}\right) = -\frac{\operatorname{cosec} 2\theta(2\theta \cot 2\theta + 1)}{2\theta^2}$

Derivatives of inverse trigonometric functions

The inverse trigonometric functions include $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$. This section discusses how to determine the derivatives of inverse trigonometric functions. Activity 8.3 illustrates the concept of derivatives of inverse trigonometric functions.

Activity 8.3: Recognizing derivatives of inverse trigonometric functions

Individually or in a group, perform this activity using the following tasks:

1. Think and write down any inverse trigonometric function.
2. Write the formula for differentiation from first principles.
3. By showing all steps clearly, differentiate the function in task 1 from first principles.
4. What have you observed from task 3?
5. Think and use any other alternative method of performing task 3.
6. Give an opinion between the methods in tasks 3 and 5.

Derivatives of some inverses of trigonometric functions are obtained as follows:

(a) Consider the function $y = \sin^{-1} x$.

Then, $x = \sin y$.

Thus, $\frac{dx}{dy} = \cos y$

From $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, then

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Since $\sin^2 y + \cos^2 y = 1$, then

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

But $\sin y = x$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Therefore, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$.

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Example 8.27

Differentiate $y=\sin^{-1}(2x-3)$ with respect to x.

Solution

Given $y=\sin^{-1}(2x-3)$.

Let $u=2x-3 \Rightarrow y=\sin^{-1}u$.

$\Rightarrow \frac{du}{dx}=2$ and $\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}$.

Using the chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Substituting the expressions for $\frac{dy}{du}$ and $\frac{du}{dx}$ gives,

$\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\times 2$. But $u=2x-3$.

Thus, $\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x-3)^2}}\times 2$.

Therefore, $\frac{dy}{dx}=\frac{1}{\sqrt{3x-x^2-2}}$.

Example 8.28

Find the derivative of each of the following functions with respect to x:

(a) $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(b) $y=\cos^{-1}x$

Solution

(a) Given $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

Differentiate with respect to x:

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$=\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{-4x}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-4x}{(1+x^2)^2\cos y}$$

But $\sin^2 y+\cos^2 y=1$.

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}$$

$$\Rightarrow \cos y=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$\Rightarrow \cos y=\sqrt{\frac{4x^2}{(1+x^2)^2}}=\frac{2x}{1+x^2}$$

Thus, $\frac{dy}{dx}=\frac{-4x}{(1+x^2)^2\times\frac{2x}{1+x^2}}$.

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{(1+x^2)}$$

Therefore,

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=\frac{-2}{(1+x^2)}.$$

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(b) Given $y = \cos^{-1} x$.

$\Rightarrow x = \cos y$, which implies that

$\frac{dx}{dy} = -\sin y$

Using the relation $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$,

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$

But $\sin^2 y + \cos^2 y = 1$

$\Rightarrow \sin y = \sqrt{1-\cos^2 y}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$

Therefore,

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$.

Example 8.29

If $y = \cos^{-1}(3x-1)$, find $\frac{dy}{dx}$.

Solution

Given $y = \cos^{-1}(3x-1)$.

$\Rightarrow 3x-1 = \cos y$

Thus, $\frac{d}{dx}(3x-1) = \frac{d}{dx}(\cos y)$

$\Rightarrow 3 = -\sin y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{-\sin y}$

But $\sin^2 y + \cos^2 y = 1$

$\Rightarrow \sin y = \sqrt{1-\cos^2 y}$

Thus, $\frac{dy}{dx} = -\frac{3}{\sqrt{1-\cos^2 y}}$

$= -\frac{3}{\sqrt{1-(3x-1)^2}}$

$= -\frac{3}{\sqrt{1-(9x^2-6x+1)}}$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{\sqrt{-9x^2+6x}}$

Therefore, $\frac{dy}{dx} = -\frac{3}{\sqrt{3x(2-3x)}}$.

Example 8.30

(a) Given $y = \cos^{-1}(3x-5x^3)$, show that

$\frac{dy}{dx} = \frac{3(5x^2-1)}{\sqrt{1-9x^2+30x^4-25x^6}}$.

(b) If $y = \tan^{-1} x$, show that

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Solution

(a) Given $y = \cos^{-1}(3x-5x^3)$.

$\Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}(3x-5x^3)$

$\Rightarrow -\sin y \frac{dy}{dx} = 3-15x^2$

$\Rightarrow \frac{dy}{dx} = -\frac{3-15x^2}{\sin y}$

But $\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \sqrt{1-\cos^2 y}$

$\Rightarrow \frac{dy}{dx} = -\frac{3-15x^2}{\sqrt{1-\cos^2 y}}$

$\Rightarrow \frac{dy}{dx} = -\frac{(3-15x^2)}{\sqrt{1-(3x-5x^3)^2}}$

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Therefore,

$$\frac{dy}{dx}=\frac{3\left(5x^2-1\right)}{\sqrt{1-9x^2+30x^4-25x^6}}.$$

(b) Given $y=\tan^{-1}x$, then,

$x=\tan y$

$\Rightarrow \frac{dx}{dy}=\sec^2 y$

Using the relation $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec^2 y}$

But $\sec^2 y=1+\tan^2 y$.

Thus, $\frac{dy}{dx}=\frac{1}{1+\tan^2 y}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{1+x^2}$

Therefore, $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$.

Example 8.31

Differentiate $y=\tan^{-1}\left(\frac{1}{x^2}\right)$ with respect to x.

Solution

Given $y=\tan^{-1}\left(\frac{1}{x^2}\right)$.

$\Rightarrow \frac{1}{x^2}=\tan y$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}\right)=\frac{d}{dx}(\tan y)$

$\Rightarrow -2x^{-3}=\sec^2 y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{-2}{x^3\sec^2 y}$

But $\sec^2 y=1+\tan^2 y$.

Thus, $\frac{dy}{dx}=\frac{-2}{x^3\left(1+\tan^2 y\right)}$.

$\Rightarrow \frac{dy}{dx}=\frac{-2x}{x^4+1}$

Therefore, $\frac{dy}{dx}=-\frac{2x}{x^4+1}$.

Example 8.32

If $y=\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, show that $\frac{dy}{dx}=\frac{1}{2}$.

Solution

Given $y=\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$.

$\Rightarrow \tan y=\frac{\cos x}{1-\sin x}$

$\Rightarrow \frac{d}{dx}(\tan y)=\frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right)$

$\Rightarrow \sec^2 y\frac{dy}{dx}=\frac{-\sin x(1-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2}$

$\Rightarrow \frac{dy}{dx}=\frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2\sec^2 y}$

$=\frac{1-\sin x}{(1-\sin x)^2\sec^2 y}$

But $\sec^2 y=1+\tan^2 y$

$\Rightarrow \sec^2 y=1+\left(\frac{\cos x}{1-\sin x}\right)^2$

$\Rightarrow \frac{dy}{dx}=\frac{1-\sin x}{(1-\sin x)^2\left(1+\left(\frac{\cos x}{1-\sin x}\right)^2\right)}$

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Upon rearranging, it gives

$$\frac{dy}{dx}=\frac{1-\sin x}{2-2\sin x}=\frac{1-\sin x}{2(1-\sin x)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Therefore, $\frac{dy}{dx}=\frac{1}{2}$.

Exercise 8.9

1. If $\sqrt{y}=\tan^{-1}x$, show that
$$\frac{d}{dx}\left[\left(1+x^2\right)\frac{dy}{dx}\right]\left(1+x^2\right)=2.$$
2. Differentiate $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ with respect to x.
3. If $y=\sin^{-1}\left(3\theta-4\theta^3\right)$, show that
$$\sqrt{1-\theta^2}\,\frac{dy}{d\theta}=3.$$
4. Differentiate
$$y=\frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\left(\frac{x}{a}\right)$$ with respect to x.
5. Find the first derivative of
$y^3\sin x+y+6=\tan^{-1}x$.
6. Given that $y=\tan^{-1}\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$.
7. Given that, x and y satisfy the equation
$\tan^{-1}x+\tan^{-1}y+\tan^{-1}xy=\frac{11}{12}\pi$.
Show that if $x=1$, then
$\frac{dy}{dx}=-1-\frac{1}{2}\sqrt{3}$.
8. Show that
$$\frac{d}{dx}\left(\sin(\tan^{-1}x)\right)=\left(1+x^2\right)^{-\frac{3}{2}}.$$
9. If $y=x^2\sin^{-1}x$, find $\frac{dy}{dx}$.
10. Differentiate each of the following with respect to x:
    (a) $y=\sin^{-1}5x$
    (b) $y=\cos^{-1}x\sqrt{2}$

Computer packages in differentiating polynomial and trigonometric functions

Computer packages can be used to differentiate polynomial and trigonometric functions. In this section, applications of Maple and MATLAB packages in differentiation are discussed.

(a) Maple 18

Maple is a computing software which uses the function "*diff*" that allows differentiation of a function. The "*diff*" function works similar to that of the function D in Mathematica.

The function "*diff*" in Maple 18 software has two arguments;

1. The function to be differentiated.
2. The independent variable to be considered in differentiation.

Example 8.33

Using Maple 18 software, differentiate $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$ with respect to x.

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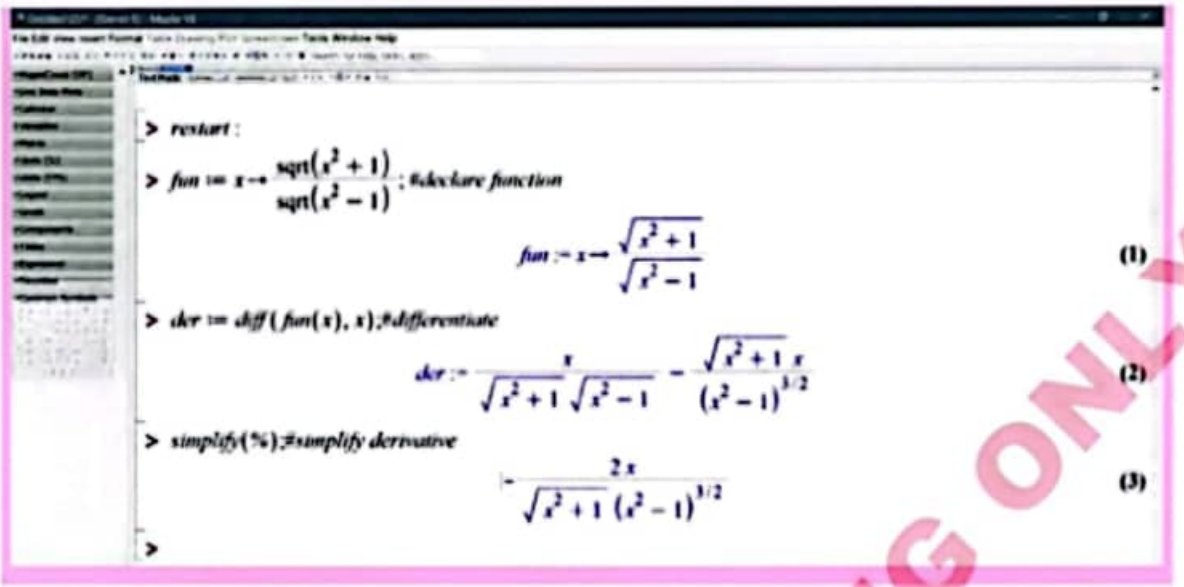
Solution



```
> restart:
> fun := x→sqrt(x^2+1)/sqrt(x^2-1); #declare function
```

$$fun := x \rightarrow \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} \quad (1)$$

```
> der := diff(fun(x), x); #differentiate
```

$$der := \frac{x}{\sqrt{x^2+1}\sqrt{x^2-1}} - \frac{\sqrt{x^2+1}\,x}{(x^2-1)^{3/2}} \quad (2)$$

```
> simplify(%); #simplify derivative
```

$$-\frac{2x}{\sqrt{x^2+1}\,(x^2-1)^{3/2}} \quad (3)$$

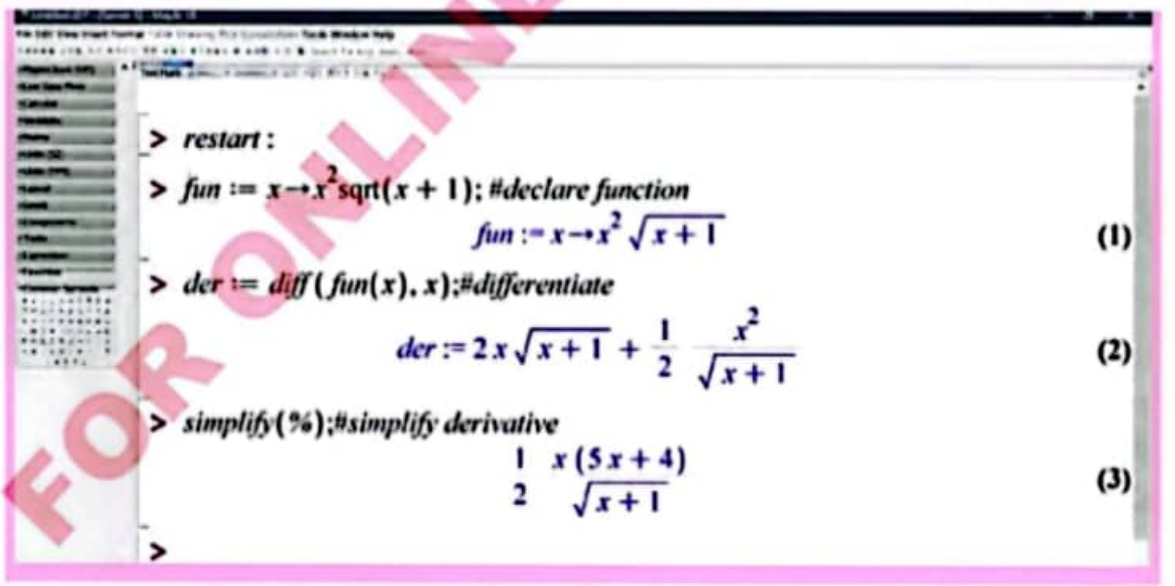
```
>
```

Therefore, $\frac{d}{dx}\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}\right) = -\frac{2x}{\sqrt{x^2+1}\left(x^2-1\right)^{\frac{3}{2}}}$.

Example 8.34

Using Maple 18 software, differentiate $x^2\sqrt{x+1}$ with respect to x.

Solution



```
> restart:
> fun := x→x^2 sqrt(x + 1); #declare function
```

$$fun := x \rightarrow x^2\sqrt{x+1} \quad (1)$$

```
> der := diff(fun(x), x); #differentiate
```

$$der := 2x\sqrt{x+1} + \frac{1}{2}\frac{x^2}{\sqrt{x+1}} \quad (2)$$

```
> simplify(%); #simplify derivative
```

$$\frac{1}{2}\frac{x(5x+4)}{\sqrt{x+1}} \quad (3)$$

```
>
```

Therefore, $\frac{d}{dx}\left(x^2\sqrt{x+1}\right) = \frac{1}{2}\frac{x(5x+4)}{\sqrt{x+1}}$.

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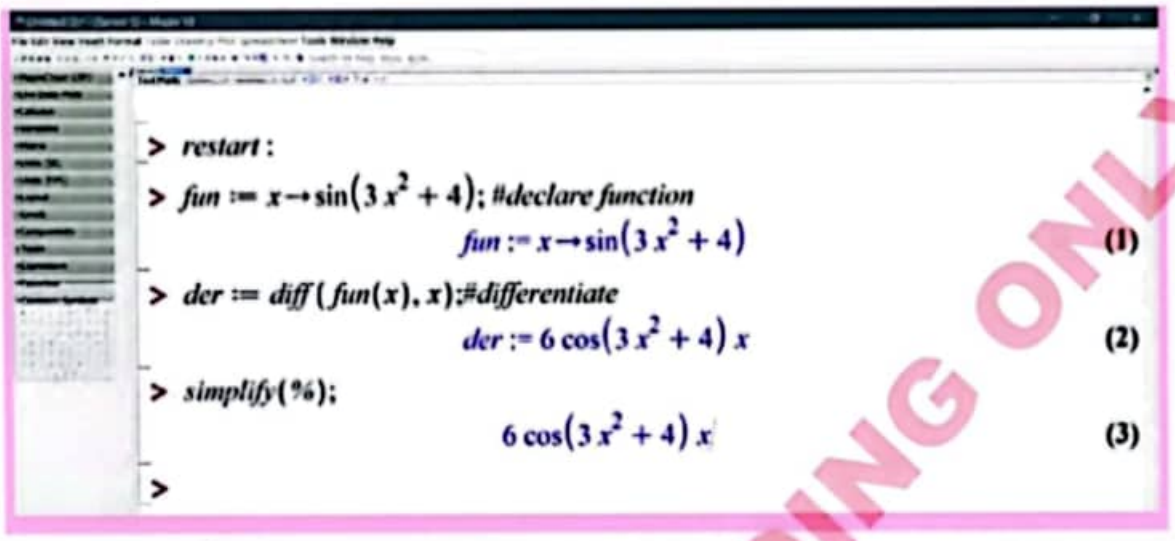
Example 8.35

Using Maple 18 software, find the derivative of $\sin(3x^2+4)$.

Solution



Therefore, $\frac{d}{dx}\left(\sin(3x^2+4)\right)=6x\cos(3x^2+4)$.

(b) MATLAB

MATLAB is another computer software that can be used to find the derivative of a function. It involves declaring variables, defining the function whose derivative is needed, and finally the "diff" function is applied to find the derivative.

Differentiation using MATLAB R2014a software involves finding the rate of change of a quantity with respect to the other, such as to find the rate at which y changes with respect to x.

Example 8.36

Using MATLAB R2014a, differentiate $y=4x^4-6x^2-10$ with respect to x.

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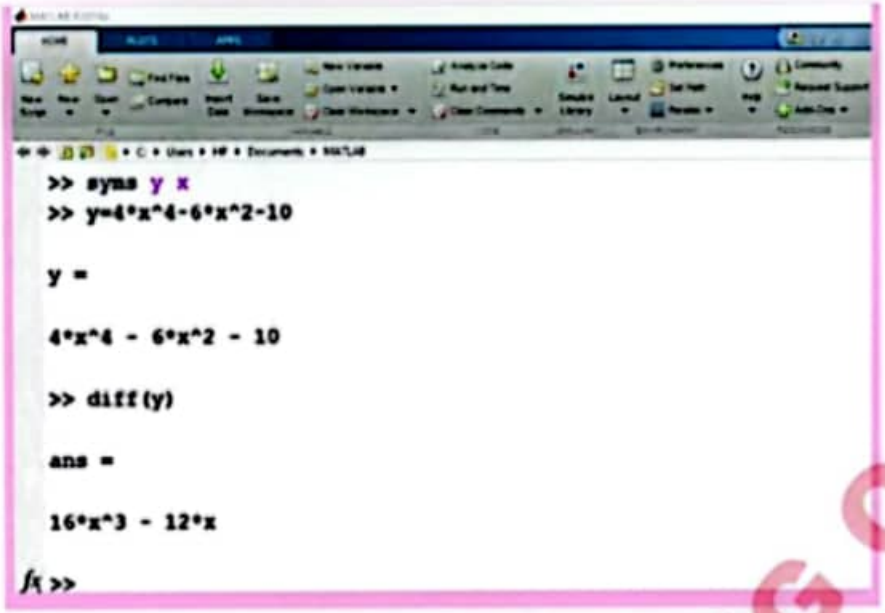
Solution

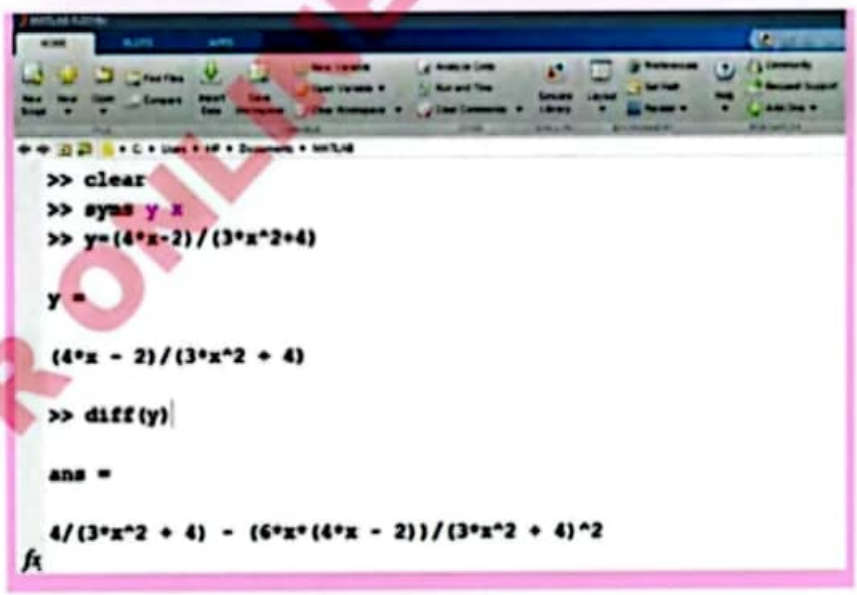


Therefore, $\frac{d}{dx}\left(4x^4-6x^2-10\right)=16x^3-12x.$

Example 8.37

Use MATLAB R2014a software to differentiate the function $y=\frac{4x-2}{3x^2+4}$ with respect to x.

Solution



Therefore, $\frac{dy}{dx}=\frac{4}{3x^2+4}-\frac{6x(4x-2)}{\left(3x^2+4\right)^2}.$

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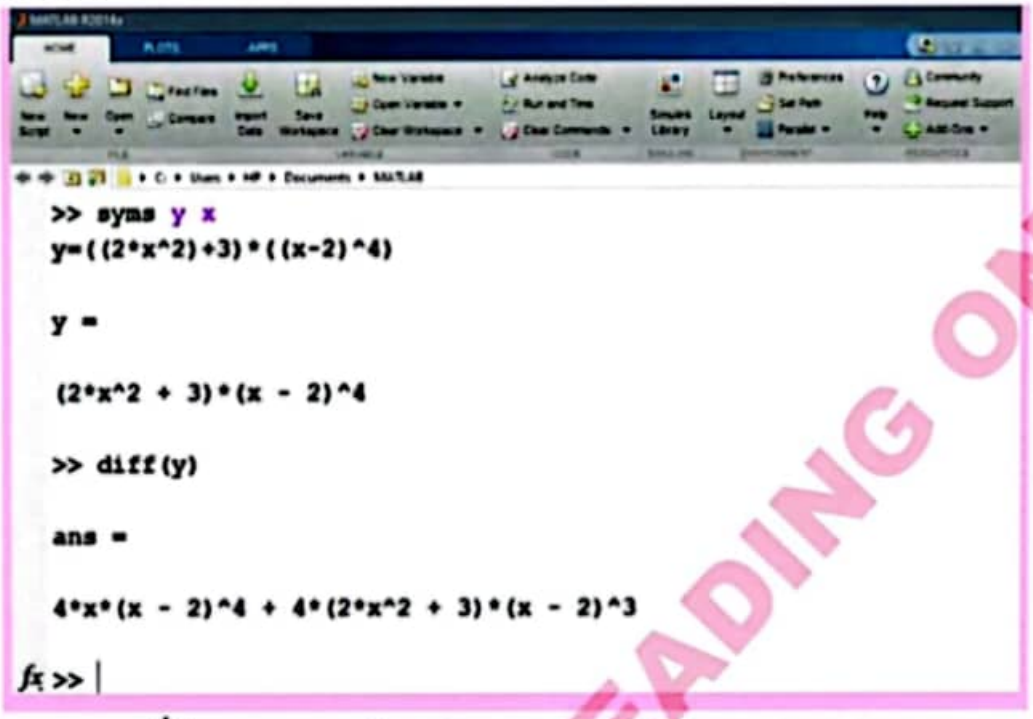
Example 8.38

Use MATLAB R2014a software to differentiate the function $y=(2x^2+3)(x-2)^4$ with respect to x.

Solution



Therefore, $\frac{dy}{dx}=4x(x-2)^4+4(2x^2+3)(x-2)^3$.

Note that, other versions of Maple and MATLAB packages can be used to differentiate polynomial and trigonometric functions.

Exercise 8.10

1. Differentiate each of the following functions with respect to x using the Maple 18 software.

   (a) $y=4x^2(x-8)^3$ (c) $y=\frac{3+x}{x^2-4}$ (e) $y=\sin(7x+4)$

   (b) $y=\tan 2x+\tan^2 5x$ (d) $y=\cos(x^3+2x+1)$

2. Differentiate each of the following functions with respect to x using MATLAB R2014a:

   (a) $y=\sin 2x\cos 4x$ (c) $y=\sec(3x+6)$ (e) $y=\cot(\operatorname{cosec}2x)$

   (b) $y=\sin^{-1}2x+\cos^{-1}(x^2-1)$ (d) $y=\frac{\sqrt[5]{2x+1}}{2(x-8)^{\frac{2}{3}}}$

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Differentiation of logarithmic functions

Logarithmic functions are functions of the form $y = \log_b x$, where $x > 0$, $b > 0$, $b \neq 1$. Logarithmic functions are inverses of the exponential functions. Thus, any logarithmic function can be written in exponential form. The derivatives of logarithmic functions to base e (natural logarithm) can easily be found by re-writing the logarithmic function in its exponential form. If the base is different from e, it should first be converted to base e, then the appropriate technique for differentiation is applied. Activity 8.4 illustrates the concept of derivatives of logarithmic functions.

Activity 8.4: Recognizing derivatives of logarithmic functions

Individually or in a group, perform the following tasks:

1. Write down a natural logarithmic function of any independent variable.
2. Determine the derivative of the function in task 1 from first principles.
3. Determine the derivative of the function in task 1 by using any alternative method.
4. Give your opinion regarding the answers obtained in tasks 2 and 3.
5. What have you observed from this activity? Give reasons.

Example 8. 39

Differentiate $y = \log_e x$ with respect to x.

Solution

Given $y = \log_e x$, then

$$e^y = x.$$

$$\Rightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\Rightarrow e^y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Therefore, $\frac{dy}{dx} = \frac{1}{x}$.

Generally, $\frac{d}{dx}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)}$.

Example 8.40

If $y = \log_e(x^3 - 2)$, show that

$$\frac{dy}{dx} = \frac{3x^2}{\left(x^3 - 2\right)}.$$

Solution

Given $y = \log_e\left(x^3 - 2\right)$.

Let $u = x^3 - 2 \Rightarrow y = \log_e u$.

Thus, $\frac{du}{dx} = 3x^2$ and $\frac{dy}{du} = \frac{1}{u}$.

Using the chain rule of differentiation,

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 \times \frac{1}{u}$$

But $u = x^3 - 2$.

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$$\Rightarrow \frac{dy}{dx} = 3x^2 \times \frac{1}{x^3 - 2}$$

Therefore, $\frac{dy}{dx} = \frac{3x^2}{x^3 - 2}$.

Example 8.41

Show that

$$\frac{d}{dx}\left(x^{\sin x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

Solution

Let $y = x^{\sin x}$.

Introducing natural logarithms both sides gives,

$$\ln y = \ln x^{\sin x}$$

$$\Rightarrow \ln y = \sin x \ln x$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \ln x).$$

Implicit differentiation gives,

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{1}{x}\sin x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\cos x \ln x + \frac{1}{x}\sin x\right)$$

But $y = x^{\sin x}$.

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{1}{x}\sin x\right)$$

Therefore,

$$\frac{d}{dx}\left(x^{\sin x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

Example 8.42

Given $y = \log_e\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$, evaluate y' at $x = \frac{1}{2}$.

Solution

Given $y = \log_e\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$.

$$\Rightarrow y = \log_e\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_e\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{1}{2}\left[\log_e(1-x^2) - \log_e(1+x^2)\right]$$

$$\Rightarrow y' = \frac{1}{2}\left[\frac{-2x}{(1-x^2)} - \frac{2x}{(1+x^2)}\right]$$

$$\Rightarrow y' = \frac{-2x}{(1-x^2)(1+x^2)}$$

At $x = \frac{1}{2}$, it follows that

$$\left[\frac{dy}{dx}\right]_{x=\frac{1}{2}} = \frac{-2\left(\frac{1}{2}\right)}{\left(1-\left(\frac{1}{2}\right)^2\right)\left(1+\left(\frac{1}{2}\right)^2\right)}$$

Therefore, y' at $x = \frac{1}{2}$ is $\frac{-16}{15}$.

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Example 8.43

Find $\frac{dy}{dx}$ if $y = e^{x^2}\sqrt{\sin x}$.

Solution

Given $y = e^{x^2}\sqrt{\sin x}$.

Introduce natural logarithms both sides:

$$\Rightarrow \ln y = \ln e^{x^2}\sqrt{\sin x}$$

$$= \ln e^{x^2} + \ln(\sin x)^{\frac{1}{2}}$$

$$= x^2 \ln e + \frac{1}{2}\ln \sin x$$

$$\Rightarrow \ln y = x^2 + \frac{1}{2}\ln \sin x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x^2 + \frac{1}{2}\ln \sin x\right)$$

Implicit differentiation gives,

$$\frac{1}{y}\frac{dy}{dx} = 2x + \frac{1}{2}\frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(2x + \frac{1}{2}\cot x\right)$$

But $y = e^{x^2}\sqrt{\sin x}$.

$$\Rightarrow \frac{dy}{dx} = e^{x^2}\sqrt{\sin x}\left(2x + \frac{1}{2}\cot x\right)$$

Therefore, $\frac{dy}{dx} = \left(2x + \frac{1}{2}\cot x\right)e^{x^2}\sqrt{\sin x}$.

Example 8.44

Prove that the derivative of

$y = \ln(x^4 \sin^2 x)$ is $y' = \frac{4}{x} + 2\cot x$.

Solution

Given $y = \ln\left(x^4 \sin^2 x\right)$.

$$\Rightarrow y = \ln x^4 + \ln \sin^2 x$$

$$\Rightarrow y = 4\ln x + 2\ln \sin x$$

Taking derivative on both sides:

$$y' = 4\times\frac{1}{x} + 2\times\frac{1}{\sin x}\times\cos x$$

$$\Rightarrow y' = \frac{4}{x} + 2\cot x$$

Therefore, $y' = \frac{4}{x} + 2\cot x$.

Example 8.45

Find y' if $y = \log_a x^2$.

Solution

Given $y = \log_a x^2$.

Convert the base of the logarithmic function to base e.

$$\Rightarrow y = \frac{\ln x^2}{\ln a}$$

$$\Rightarrow y = \frac{2\ln x}{\ln a}$$

Now, differentiation with respect to x gives,

$$y' = \frac{2}{\ln a}\frac{d}{dx}(\ln x)$$

$$\Rightarrow y' = \frac{2}{x\ln a}$$

Therefore, $y' = \frac{2}{x\ln a}$.

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Exercise 8.11

1. Differentiate each of the following functions with respect to x:
   (a) $y = \ln(5x-4)$
   (b) $y = x\ln x - x$
   (c) $y = x\left(\sin(\ln x) - \cos(\ln x)\right)$

2. Find the derivative of $y = \ln\left(\frac{x+3}{x+5}\right)$ with respect to x.

3. Given that $y = \ln\left[(x^2+3)(x^3+2)\right]$, find the expression for $\frac{dy}{dx}$.

4. Differentiate each of the following functions with respect to x.
   (a) $f(x) = \ln\left(x+\sqrt{1+x^2}\right)$
   (b) $f(x) = \ln\left(\frac{x^4}{(3x-5)^2}\right)$
   (c) $f(x) = (\sin x)^x$
   (d) $f(x) = \ln\left[(x^2+2)^3(1-x^3)^4\right]$

5. Find the derivative of each of the following functions with respect to x:
   (a) $y = (\sin x)^{\tan x}$
   (b) $y = \ln\left[\frac{(x^2+3)\sqrt{\cot x}}{\tan^3 x}\right]$
   (c) $y = \ln\left(x\sqrt{x^2-4}\right)$
   (d) $y = x^x$

6. Given that $f(x) = \ln(\ln\tan x)$, find $f'(x)$.

7. Differentiate $x = \ln\left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)$ with respect to θ.

8. Show that the derivative of the function $f(x) = \frac{1}{5}x^5\left(\ln x - \frac{1}{5}\right)$ is $f'(x) = x^4\ln x$.

9. Given $f(x) = \ln\left[\frac{x^2\sqrt{x^2-2}}{x+1}\right]$, show that $f'(x) = \frac{2}{x} + \frac{x}{x^2-2} - \frac{1}{x+1}$.

10. If $x^y = y^x$, show that $\frac{dy}{dx} = \frac{y(x\ln y - y)}{x(y\ln x - x)}$.

11. Find the derivative of the function $y = \log_2(3x+1)$ at $x = 1$.

12. If $y = \ln\sqrt{\frac{x-1}{x+1}}$, show that $\frac{dy}{dx} = \frac{1}{x^2-1}$. Hence, find $\frac{dy}{dx}$ when $x = \sqrt{3}$.

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Derivatives of exponential functions

The derivative of an exponential function $y = e^{u(x)}$, where u is any differentiable function, can be determined using the rules of differentiation.

Activity 8.5 illustrates the derivative of an exponential function.

Activity 8.5: Recognizing derivative of an exponential function

Individually or in a group, perform the following tasks:

1. Determine the derivative of $f(x) = e^x$ from first principles.
2. Determine the derivative in task 1 using any alternative method.
3. Give your opinion in answers obtained in tasks 1 and 2.
4. What have you observed from this activity? Give reasons.
5. Share your results with other students for more inputs.

Consider the following series expansion of the exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The derivative of the right-hand side is determined by the rule for derivative of sums of functions. That is, the derivative of a sum of equals the sum of the derivatives of each of the terms.

That is,

$$\frac{d}{dx}(e^x) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \frac{d}{dx}\left(\frac{x^4}{4!}\right) + \cdots$$

$$\Rightarrow \frac{d}{dx}(e^x) = 0 + 1 + \frac{2}{2\times 1}x + \frac{3}{3\times 2\times 1}x^2 + \frac{4}{4\times 3\times 2\times 1}x^3 + \cdots$$

$$\Rightarrow \frac{d}{dx}(e^x) = 1 + x + \frac{1}{2\times 1}x^2 + \frac{1}{3\times 2\times 1}x^3 + \cdots$$

$$\Rightarrow \frac{d}{dx}(e^x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The expression for the derivative is the same as the original function.

Therefore, $\frac{d}{dx}(e^x) = e^x$.

Alternatively;

Let $y = e^x$

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Introducing natural logarithm on both sides gives,

$\ln y = \ln e^x$

$\Rightarrow \ln y = x \ln e$

$\Rightarrow \ln y = x$, since $\ln e = \log_e e = 1$.

$\Rightarrow \frac{d}{dx} \ln y = \frac{d(x)}{dx}$

Implicit differentiation gives,

$\frac{1}{y}\frac{dy}{dx} = 1$.

$\Rightarrow \frac{dy}{dx} = y$

But, $y = e^x$

Therefore, $\frac{dy}{dx} = e^x$

Generally, $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}\frac{d}{dx}f(x)$.

Note that, e^x is its own derivative. In fact e^x, is the only function which is its own derivative.

Example 8.46

If $f(x) = e^{ax}$, where a is a constant, find $f'(x)$.

Solution

Let $u = ax$ and $y = e^u$.

$\Rightarrow \frac{du}{dx} = a$ and $\frac{dy}{du} = e^u$.

Using the chain rule of differentiation,

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow f'(x) = \frac{d}{dx}\left(e^u\right) \times \frac{d}{dx}(ax)$

$= e^u \times a$

$= ae^{ax}$

Therefore, $f'(x) = ae^{ax}$.

Example 8.47

Find $\frac{d}{dx}(e^{x^3+5x})$.

Solution

Given $\frac{d}{dx}(e^{x^3+5x})$.

Let $y = e^{x^3+5x}$ and $u = x^3 + 5x$.

$\Rightarrow y = e^u$

$\Rightarrow \frac{dy}{du} = e^u$ and $\frac{du}{dx} = 3x^2 + 5$.

Using the chain rule of differentiation;

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u(3x^2 + 5)$

$\Rightarrow \frac{dy}{dx} = \left(3x^2 + 5\right)e^u$

$= \left(3x^2 + 5\right)e^{x^3+5x}$

Therefore, $\frac{d}{dx}(e^{x^3+5x}) = (3x^2 + 5)e^{x^3+5x}$.

Example 8.48

Find the derivative of $y = \frac{5^x}{1 + e^{-x}}$ with respect to x.

Solution

Given $y = \frac{5^x}{1 + e^{-x}}$.

Let $u = 5^x$ and $v = 1 + e^{-x}$.

$\Rightarrow \frac{dv}{dx} = -e^{-x}$

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For $u=5^x$

Introducing natural logarithm on both sides gives,

$\ln u = \ln 5^x$

$\Rightarrow \ln u = x \ln 5^x$

$\Rightarrow \frac{d(\ln u)}{dx} = \frac{d(x \ln 5)}{dx}$

Implicit differentiation gives,

$\frac{1}{u}\frac{du}{dx} = \ln 5$

$\Rightarrow \frac{du}{dx} u \ln 5$, but $u = 5^x$

Thus, $\frac{du}{dx}$ $5^x \ln 5$

Using quotient rule of differentiation.

$\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

$\frac{dy}{dx} = \frac{(1+e^{-x})(5^x \ln 5) - (5^x)(-e^{-x})}{(1+e^{-x})^2}$

$\frac{dy}{dx} = \frac{5^x \ln 5 + 5^x e^{-x}(1+\ln 5)}{(1+e^{-x})^2}$

Therefore,

$\frac{dy}{dx} = \frac{5^x \ln 5 + 5^x e^{-x}(1+\ln 5)}{(1+e^{-x})^2}$

Example 8.49

Find y' given that $y = e^{-x} \ln x$.

Solution

Given $y = e^{-x} \ln x$.

Let $u = e^{-x}$ and $v = \ln x$.

$\Rightarrow \frac{du}{dx} = -e^{-x}$ and $\frac{dv}{dx} = \frac{1}{x}$.

Using the product rule for differentiation,

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\Rightarrow y' = \ln x(-e^{-x}) + e^{-x}\left(\frac{1}{x}\right)$

$\Rightarrow y' = -e^{-x} \ln x + \frac{e^{-x}}{x}$.

Therefore, $y' = -e^{-x} \ln x + \frac{e^{-x}}{x}$.

Exercise 8.12

1. Find the derivative with respect to x of each of the following functions:
   (a) $f(x) = e^{x^2}$
   (b) $g(x) = e^x(1-6x^2)$
   (c) $h(x) = e^x \ln x$
   (d) $f(x) = x^3 9^x$
2. Differentiate $y = e^{\cos x} \sin x$ with respect to x.
3. If $y = a^{x^2}$, show that $y' = 2xa^{x^2} \ln a$.
4. If $y = e^{-2x} \sin 3x$, show that $y' = e^{-2x}(3\cos 3x - 2\sin 3x)$.
5. Given that $f(x) = x^2 e^x$, find $f'(x)$.
6. Find the derivative of each of the following functions:
   (a) $y = e^{\sin^2 4x}$
   (b) $y = 3^{2x}$

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(c) $y = x(3^x)$

(d) $x = e^{\tan\theta}$

(e) $x = t^3 - e^{-3t^2} + \ln e^{\sec t}$

(f) $v = e^{u^2+2u-8}$

7. Show that
$$\frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right) = \frac{-8}{\left(e^{2x}-e^{-2x}\right)^2}.$$

8. If $\frac{x}{x-y} = \ln\left(\frac{a}{x-y}\right)$, show that $\frac{dy}{dx} = 2 - \frac{x}{y}$.

9. If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{x-y}{x(1+\ln x)}$.

10. Differentiate $y = \frac{x^3 \ln 2x}{e^x \sin x}$ with respect to x.

The second derivative of a function

In general, the derivative of a function $f(x)$ is also a function of x. Suppose $g(x) = f'(x)$. If $g(x)$ is differentiable, then $g'(x)$ is called the second derivative of $f(x)$, and it is denoted by $\frac{d^2f}{dx^2}$.

Thus, $g'(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}$.

The second derivative of a variable y with respect to x can be written as $\frac{d^2y}{dx^2}$.

Example 8.50

If $y = x^2\cos 8x$, find $\frac{d^2y}{dx^2}$ at $x = 0$.

Solution

Given $y = x^2\cos 8x$.

Let $u = x^2$ and $v = \cos 8x$.

$\Rightarrow \frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -8\sin 8x$.

Using the product rule of differentiation,

$\Rightarrow \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Thus, $\frac{dy}{dx} = 2x\cos 8x - 8x^2\sin 8x$.

Again, differentiate $\frac{dy}{dx}$ to obtain $\frac{d\,y}{dx}$, that is,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\cos 8x - 8x^2\sin 8x\right)$$

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$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos 8x - 16x\sin 8x - 16x\sin 8x - 64x^2\cos 8x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos 8x - 32x\sin 8x - 64x^2\cos 8x$$

At $x = 0$,

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = 2\cos(8\times 0) - 32\times 0\times \sin(8\times 0) - 64\times (0)^2 \times \cos(8\times 0)$$
$$= 2$$

Therefore, $\left[\frac{d^2y}{dx^2}\right]_{x=0} = 2.$

Example 8.51

If $y = \frac{\sin x}{x}$, prove that

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0.$$

Solution

Let $u = \sin x$ and $v = x$.

$$\Rightarrow \frac{du}{dx} = \cos x \text{ and } \frac{dv}{dx} = 1$$

Using the quotient rule of differentiation,

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

Multiplying by x^2 both sides gives;

$$x^2\frac{dy}{dx} = x\cos x - \sin x$$

Differentiate again with respect to x gives,

$$2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = -x\sin x$$

Divide by x^2 both sides to obtain,

$$\frac{2}{x}\frac{dy}{dx} + \frac{d^2y}{dx^2} = -\frac{\sin x}{x}.$$

But $y = \frac{\sin x}{x}$.

$$\Rightarrow \frac{2}{x}\frac{dy}{dx} + \frac{d^2y}{dx^2} = -y$$

Therefore, $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0.$

Derivatives of parametric functions

Parametric functions are functions which have a pair of equations in which two variables, x and y are related by two equations of the form $x = f(t)$ and $y = g(t)$, where x and y are functions of the parameter t. Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$, but $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

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Thus, $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$.

Therefore, a useful formula for finding the first derivative of a parametric function is given by;

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$$

The second derivative of a parametric function is obtained by differentiating $\frac{dy}{dx}$ with respect to t followed by dividing the result by the derivative of x with respect to t. That is, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$. Since $\frac{dy}{dx}$ is a function of t, then $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$

This formula is useful in finding second derivatives of parametric functions.

Example 8.52

If $x=\sin t$ and $y=\cos 2t$, show that $\frac{d^2y}{dx^2}=-4$.

Solution

Given $x=\sin t$ and $y=\cos 2t$.

$\Rightarrow \frac{dx}{dt}=\cos t$ and

$\frac{dy}{dt}=-2\sin 2t=-4\sin t\cos t$

But $\frac{du}{dx}=\frac{dy}{dx}\times\frac{1}{\frac{dx}{dt}}$.

Thus, $\frac{dy}{dx}=(-4\sin t\cos t)\times\frac{1}{\cos t}$.

$\Rightarrow \frac{dy}{dx}=-4\sin t$

Again, $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dt}(-4\sin t)\times\frac{1}{\cos t}$

$\Rightarrow \frac{d^2y}{dx^2}=-4\cos t\times\frac{1}{\cos t}$

$\Rightarrow \frac{d^2y}{dx^2}=-4$

Therefore, $\frac{d^2y}{dx^2}=-4$.

Example 8.53

Given that $2x=\theta+\frac{1}{\theta}$ and $2y=\theta-\frac{1}{\theta}$, show that:

(a) $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$ (b) $\frac{d^2y}{dx^2}=-\frac{8\theta^3}{(\theta^2-1)^3}$

Solution

(a) Given $2x=\theta+\frac{1}{\theta}$ and $2y=\theta-\frac{1}{\theta}$.

$\Rightarrow 2\frac{dx}{d\theta}=1-\frac{1}{\theta^2}$ and $2\frac{dy}{d\theta}=1+\frac{1}{\theta^2}$

Thus, $\frac{du}{dx}=\frac{dy}{d\theta}\times\frac{1}{\frac{dx}{d\theta}}$.

$\Rightarrow \frac{dy}{dx}=\frac{1+\theta^{-2}}{2}\times\frac{2}{1-\theta^{-2}}$

$\frac{dy}{dx}=\frac{1+\theta^{-2}}{1-\theta^{-2}}$

$\Rightarrow \frac{dy}{dx}=\frac{1+\theta^{-2}}{1-\theta^{-2}}=\frac{\theta^2+1}{\theta^2-1}$

Therefore, $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$.

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(b) From $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$, it implies that

$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\times\frac{d\theta}{dx}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{\theta^2+1}{\theta^2-1}\right)\times\frac{2}{1-\theta^{-2}}$$

$$=\frac{2\theta(\theta^2-1)-2\theta(\theta^2+1)}{(\theta^2-1)^2}\times\frac{2\theta^2}{\theta^2-1}$$

$$=\frac{2\theta^3-2\theta^3-2\theta}{(\theta^2-1)^3}\times 2\theta^2$$

$$=-\frac{8\theta^3}{(\theta^2-1)^3}$$

Therefore, $\frac{d^2y}{dx^2}=-\frac{8\theta^3}{(\theta^2-1)^3}$.

Example 8.54

If $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Give your answer in terms of half angle of θ.

Solution

Given $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

$\Rightarrow\frac{dx}{d\theta}=a(1+\cos\theta)$ and $\frac{dy}{d\theta}=a\sin\theta$.

$\Rightarrow\frac{du}{dx}=\frac{dy}{d\theta}\times\frac{1}{\frac{dx}{d\theta}}$. But $\frac{d\theta}{dx}=\frac{1}{a(1+\cos\theta)}$

$\Rightarrow\frac{dy}{dx}=a\sin\theta\times\frac{1}{a(1+\cos\theta)}$

$\Rightarrow\frac{dy}{dx}=\frac{\sin\theta}{1+\cos\theta}$

Using half angles formulae for sine and cosine gives,

$$\frac{dy}{dx}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$\Rightarrow\frac{dy}{dx}=\tan\frac{1}{2}\theta$

Also, $\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\times\frac{d\theta}{dx}$

$$=\frac{d}{d\theta}(\tan\tfrac{1}{2}\theta)\times\frac{1}{a(1+\cos\theta)}$$

$$=\frac{1}{2}\sec^2\tfrac{1}{2}\theta\times\frac{1}{a(2\cos^2\frac{1}{2}\theta)}$$

$$=\frac{\sec^2\frac{1}{2}\theta}{4a\cos^2\frac{1}{2}\theta}$$

$$=\frac{1}{4a}\sec^4\tfrac{1}{2}\theta.$$

Therefore, $\frac{d^2y}{dx^2}=\frac{1}{4a}\sec^4\frac{1}{2}\theta$.

Exercise 8.13

1. If $x=\sin 3\theta$ and $y=\cos3\theta\sin\theta$, show that

$$\frac{d^2x}{d\theta^2}-\frac{d^2y}{d\theta^2}-10y=3x(2\cos\ -3).$$

2. Given that $y=\frac{\sin x}{x^2}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence, verify that

$$x^4\frac{d^2y}{dx^2}+4x^3\frac{dy}{dx}+(x^2+2)\sin x=0$$

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3. If $x=\frac{t+2}{2t+1}$ and $y=\frac{2t+3}{t}$, find:
   (a) $\frac{dy}{dx}$ (c) $\frac{d^2y}{dx^2}$, when $x=0$
   (b) $\frac{d^2y}{dx^2}$

4. If $x=e^t(\cos t+\sin t)$ and $y=e^t(\cos t-\sin t)$, find:
   (a) $\frac{dy}{dx}$ (c) $\frac{d^2y}{dx^2}$, when $t=0$
   (b) $\frac{d^2y}{dx^2}$

5. Show that $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0$, given that $y=e^x\cos 2x$.

6. A curve is define parametrically by the equations
   $x=\frac{t^2}{1+t^3}$ and $y=\frac{t^3}{1+t^3}$. Find:
   (a) $\frac{dy}{dx}$, when $t=2$
   (b) $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

7. A curve is defined parametrically by $x=2t^2+5t+1$ and $y=t+6$. Find its gradient at the point $(1, 6)$.

8. Verify that $\frac{dy}{dx}=1$, if
   $x=\tan^{-1}\left(\frac{2t}{1-t^2}\right)$ and
   $y=\sin^{-1}\left(\frac{2t}{1+t^2}\right)$.

9. Find $\frac{d^2y}{dx^2}$ if $y=\frac{\ln x}{x}$. Hence, evaluate $\frac{d^2y}{dx^2}$ when $x=2$ (leave your answer in logarithmic form).

10. Given that $x=\frac{1}{2}\left(\sin^{-1}\theta\right)^2$, find the value of
    $\left(1-\theta^2\right)\frac{d^2x}{d\theta^2}-\theta\frac{dx}{d\theta}$.

11. Given $y=ae^{nx}+be^{-nx}$, show that $\frac{d^2y}{dx^2}=n^2y$.

Applications of differentiation

Differentiation is useful in solving real life problems, particularly in physical sciences, computer science, engineering, economics, and in many other fields. For instance, in ecology, derivatives are used in predictions of the population of species in a habitat after a certain period of time. In seismology, it is used to determine earthquakes. In business, the concept of derivative is used in calculating profit and loss.

Problems involving small changes

Small changes of quantities are related to the concept of differentiation when δx is small, then $\frac{\delta y}{\delta x}\approx\frac{dy}{dx}$, which implies, $\delta y\approx\frac{dy}{dx}\delta x$. This notation can also be written as $\Delta y\approx\frac{dy}{dx}\Delta x$.

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If y is a function of x, then the change in y towercase corresponding to a given small change in x can be determined.

Example 8.55

If $y = 2x^2 - 3x$, find an approximate change in y when x increases from 7 to 7.02.

Solution

Given $y = 2x^2 - 3x$.

$\Rightarrow \frac{dy}{dx} = 4x - 3$

But $\Delta y \approx \frac{dy}{dx}\Delta x$

$\Rightarrow \Delta y \approx (4x-3)\Delta x$, where

$\Delta x = 7.02 - 7 = 0.02$ and $x = 7$.

$\Rightarrow \Delta y \approx (4\times 7 - 3)(0.02)$

$\Rightarrow \Delta y \approx 0.5$

Therefore, the approximate change in y is 0.5.

Example 8.56

The length of one side of a square is 10 cm. Find the increase in the area of the square if its side expands by 0.01cm.

Solution

Let A denote the area of the square and l denote the length of its side.

$\Rightarrow A = l^2$ and $\frac{dA}{dl} = 2l$.

Using small changes;

$\frac{\Delta A}{\Delta l} \approx \frac{dA}{dl}$

$\Rightarrow \Delta A \approx \frac{dA}{dl}\Delta l$

$\Rightarrow \Delta A \approx 2l\Delta l$

But $l = 10$ cm and $\Delta l = 0.01$ cm.

Thus,

$\Delta A \approx 2\times 10 \text{ cm} \times 0.01 \text{ cm} \approx 0.2 \text{ cm}^2$

Therefore, the increase in area is 0.2 cm^2.

Example 8.57

Use the technique of small changes to find an approximate value of $\sqrt{25.08}$.

Solution

Given $\sqrt{25.08}$.

Let $y = \sqrt{x}$ and $(x, y) = (25, 5)$

$\Rightarrow (x+\delta x,\ y+\delta y) = (25+0.08,\ 5+\delta y)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

But $\delta y \approx \frac{dy}{dx}\delta x$.

$\Rightarrow \delta y \approx \frac{1}{2\sqrt{25}}(0.08)$

$\Rightarrow \delta y \approx 0.008$

Thus, $y + \delta y = 5 + 0.008 \approx 5.008$

Therefore, $\sqrt{25.08} \approx 5.008$.

Example 8.58

If $R = kr^n$, where k is a constant and an error of $y\%$ is made in measuring the radius r. Prove that the resulting error in R is $ny\%$.

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Solution

Given $R = kr^n$.

$$\Rightarrow \frac{dR}{dr} = knr^{n-1} = \frac{knr^n}{r}$$

$$\Rightarrow \frac{dR}{dr} = kr^n \frac{n}{r} = \frac{Rn}{r}$$

But $\Delta R = \frac{dR}{dr}\Delta r$.

$$\Rightarrow \Delta R = \frac{Rn}{r}\Delta r$$

$$\Rightarrow \frac{\Delta R}{R} = n\left(\frac{\Delta r}{r}\right)$$

$$\Rightarrow \frac{\Delta R}{R} = n\left(\frac{\Delta r}{r}\right)$$

But $\frac{\Delta R}{r} = y\%$.

$$\Rightarrow \frac{\Delta R}{R} = ny\%$$

Therefore, the resulting error in R is $ny\%$.

Exercise 8.14

1. Find the approximate increase in y when x increases from 8 to 8.01, given $y = x^2 + 2x$.
2. Find the approximate value of each of the following:
   (a) $\sqrt[3]{27.27}$ (b) $(5.03)^3$ (c) $\sqrt[6]{64.96}$
3. Find the approximate change in the volume of a cube whose side has length x cm caused by increasing the length of its sides by 1%.
4. Approximate the decrease in volume and surface area if the radius of an ice sphere shrinks from 10 m to radius 9.8 m.
5. Find the approximate increase in the volume of a sphere if its radius increases from 10 cm to 10.1 cm.
6. If $y = x + \frac{1}{x}$, find an approximate increase in y when x changes from 2 to 2.02.
7. A solid circular cylinder has a base of radius 5 cm and height 12.5 cm. If the radius of the base increases to 5.04 cm, find the approximate increase in surface area of the cylinder (Leave π in your answer).
8. An error of $2\frac{1}{2}\%$ is made in calculating of the area of a circle. Find the corresponding percentage increase in the circumference.
9. As x increases, prove that the area of a circle of radius x and the area of a square of side length x increase by the same percentage, provided that the increase in x is small.
10. Let y centimetres be the length of a pendulum and t seconds be the time of one complete swing. It is known that $y = kt^2$, where k is a constant. If the length of the pendulum is increased by $y\%$, where y is small, find the corresponding percentage increase in time of swing.

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Problems on rates of change

The rate of change is the change of one quantity with respect to another. Understanding the nature of change and the rate at which it takes place enables experts to make important predictions and decisions about the atmospheric pressure, humidity, wind patterns, temperature, and many others. Also, rates of change of quantities such as volumes, areas, motion, and patterns can be determined. The chain rule can is used when dealing with problems involving rates of changes.

Distance, velocity, and acceleration

The concept of rate of change is also used in motion of objects to find velocity and acceleration when the displacement function is known. The derivative of a function representing the position of a particle along a curve at time t is the velocity at that time, and the derivative of the velocity, which is the second derivative of the position function, represents the acceleration of the particle at time t.

Let $s(t)$ be the position of a particle at time t. The derivative, $s'(t)$ represents the instantaneous velocity v and the second derivative, $a(t) = v'(t) = s''(t)$ represents the instantaneous acceleration of the particle at time t.

Acceleration, $\frac{d^2s}{dt^2}$ may be written in another form by using the fact that $\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v$.

Therefore, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$.

Example 8.59

The position of a particle on a curve is given by $s(t) = t^3 - 2t^2 + 6t + 5$, where t is measured in seconds and s is in metres. Find the velocity and acceleration of the particle at the end of 2 seconds.

Solution

Given $s(t) = t^3 - 2t^2 + 6t + 5$.

$\Rightarrow v = s'(t) = 3t^2 - 4t + 6$

At $t = 2$ sec,

$v = s'(2) = 3(2)^2 - 4(2) + 6$

$\Rightarrow v = 10$ m/s

Also, $a = v'(t) = 6t - 4$.

Also, at $t = 2$ sec,

$\Rightarrow a = 6(2) - 4 = 8$ m/s^2.

Therefore, at the end of 2 seconds, the velocity is 10 m/s and acceleration is 8 m/s^2.

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Example 8.60

A particle moves along a straight line $\overline{OB}$ so that it is s metres from O in t seconds, where $s = t(2t-3)(t-4)$. Deduce the expressions for its velocity and acceleration in terms of t, hence describe the following motion at $t = 2$ seconds:

(a) The position of the particle.
(b) The direction of the particle with reference to point B.
(c) Its speed.
(d) State whether the speed is increasing or decreasing.
(e) The rate of change of the speed.

Solution

Given

$s = t(2t-3)(t-4) = 2t^3 - 11t^2 + 12t.$

$\Rightarrow s'(t) = 6t^2 - 22t + 12$

Also, $a = v'(t) = 12t - 22.$

At time $t = 2$ seconds,

(a) $s = 2(2(2)-3)(2-4) = -4.$
Therefore, the particle is 4 m from O on $\overline{BO}$.

(b) The particle is moving away from B in the opposite direction.

(c) Its speed:

$s'(t) = v = 6t^2 - 22t + 12$

$\Rightarrow v = 6(2)^2 - 22(2) + 12$

$= -8$ m/s

Since speed is the magnitude of velocity, then its speed is 8 m/s.

(d) Its speed is decreasing.

(e) Rate of change of speed,

$a = v'(t) = 12t - 22$

$\Rightarrow \left[\frac{dv}{dt}\right]_{t=2} = 12(2) - 22$

$= 2$ m/s^2.

Therefore, the rate of change of speed is 2 m/s^2.

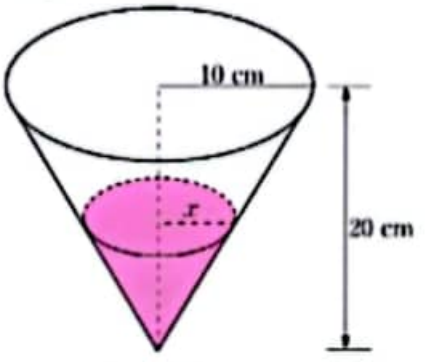
Example 8.61

A liquid is running out of a conical funnel at the rate of 5 cm^3/s. The radius of the funnel is 10 cm and its height is 20 cm. How fast is the liquid level dropping when the liquid is 10 cm deep? (Leave your answer in π).

Solution

Let h be the depth, r the radius, and V the volume of the liquid at time t.

Thus, $\frac{dV}{dt} = -5$ cm^3/s (liquid is decreasing). Consider the following diagram:



Given $\frac{r}{h} = \frac{10}{20}$, then $r = \frac{h}{2}$.

Volume of the cone is $V = \frac{1}{3}\pi r^2 h.$

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Upon substitution, it gives

$$V = \frac{1}{12}\pi h^3$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{4}h^2$$

Apply chain rule of differentiation:

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\Rightarrow -5 = \frac{\pi}{4}h^2 \times \frac{dh}{dt}$$

But $h = 10$.

$$\Rightarrow \frac{dh}{dt} = -\frac{20}{100\pi}$$

$$= -\frac{1}{5\pi}$$

Therefore, the liquid level is dropping at the rate of $\frac{1}{5\pi}$ cm^3/s.

Example 8.62

The volume of a spherical balloon increases at a constant rate of 1.5 cm^3/s when the balloon is blown up. If the volume of the balloon is 62 cm^3, find the rate of increase of its radius.

Solution

Let r be the radius of the balloon, V the volume, and t the time in seconds.

Since $V = \frac{4}{3}\pi r^3$, then $\frac{dV}{dr} = 4\pi r^2$.

Using the chain rule of differentiation:

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}.$$

Thus, $1.5 = 4\pi r^2 \times \frac{dr}{dt}$.

$$\Rightarrow \frac{dr}{dt} = \frac{1.5}{4\pi r^2}$$

But $V = \frac{4}{3}\pi r^3$.

$$\Rightarrow 62 = \frac{4}{3}\pi r^3$$

$$\Rightarrow r = \left(\frac{62 \times 3}{4\pi}\right)^{\frac{1}{3}}$$

$$= 2.4553 \text{ cm}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1.5}{4\pi(2.4553)^2}$$

$$= 0.02 \text{ cm/s}$$

Therefore, the rate of increase of the radius is 0.02 cm/s.

Exercise 8.15

1. The surface area of a sphere is $4\pi r^2$, where r is its radius. If the radius increases at the rate of 0.1 cm /s, find the rate of change of the area in cm^2/s when the radius of the sphere is 40 cm.
2. The equation $s(t) = -4.9t^2 + 49t + 20$ represents the height in metres of an object after it is thrown vertically upward from a point 15 m above the ground at a velocity of 49 m/s. How high above the ground will the object reach?

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3. A hollow circular cone is held downward beneath a water tap leaking at the rate of $2\ cm^3/s$. Find the rate of rise of water level when the depth of the cone is 6 cm, given that its height is 18 cm and its radius is 12 cm.
4. The radius of a spherical ballon increases at the rate of 6 cm/min. Find the rate of increase of its volume when the radius of the ballon is 4 cm.
5. At what rate is the surface area of a spherical baloon increasing when the radius is 4 cm, given chat the radius of the balloon increases at the rate of 6 cm/sec?
6. The area of a circle is increasing at the rate of $3\ cm^2/s$. Find the rate of change of the circumference of the circle when its radius is 2 cm.
7. A container in the form of right circular cone of height 10 cm and base radius 1 cm is receiving drops of liquid from a leaking tap at the rate of $0.1\ cm^3/s$. Find the rate at which the surface area of the liquid is increasing when the liquid is half way up the cone.
8. An inverted right circular cone of vertical angle $120°$ is collecting water from a tap at the rate of $18\pi\ cm^3/min$. Find the water level after 12 minutes and the rate of increase of water level in the cone.
9. If the volume of a sphere increases at a rate of $6\ cm^3/s$, find the rate of increase of its surface area when the radius is 4 cm.
10. The position of a particle along the x-axis moving is given by $s(t) = 5 + 4\sin 2t + 3\cos 2t$. Show that its velocity is zero when $s = 10$ and its acceleration is $20 - 4s$.
11. If $s(t) = 3t + t^3$ is a displacement function of a particle in metres, calculate the velocity and acceleration of the particle after 2 seconds.
12. A particle moves back and forth along a horizontal line defined by the distance function $s(t) = t^3 - 12t^2 + 36t - 30$, for $t \geq 0$.
    (a) Find the velocity and acceleration functions.
    (b) When is the velocity zero?
    (c) When is the acceleration zero?
    (d) At what time is the velocity negative? Explain its physical meaning.

Turning points and points of inflexion of a curve

Turning points are points on a curve where its graph changes direction. Activity 8.6 illustrates the concepts of turning points and points of inflexion.

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Activity 8.6: Identification of turning points and points of inflexion

Individually or in a group, perform the following tasks:

1. Draw graphs of curves of your choice. The curves should be of degrees two and three.
2. Identify the turning points and points of inflexion of the graphs in task 1.
3. Identify the values of the turning points and points of inflexion found in task 2.
4. Use the results in task 3 to determine the following:
   (a) Maximum points
   (b) Minimum points
   (c) Points of inflexion
5. Share your results with other students for more inputs.

A turning point is a point at which values of the function change from decreasing to increasing, or vice versa. It is a point at which the gradient of the curve is zero. If a curve is defined by $y = f(x)$, then a turning point is determined when $f'(x) = 0$. In other words, a turning point is obtained when the derivative of the curve is zero.

Turning points and points of inflexion are also known as stationery points or critical points.

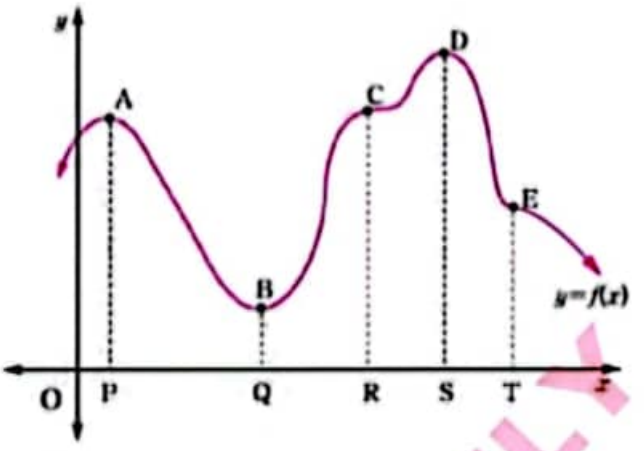

Figure 8.3: *Turning points and points of inflexion*

Figure 8.3 illustrates the concepts of turning points and points of inflexion. Points A and D are maximum points of the curve, point B is a minimum point and points C and E are points of inflexion. If O is the origin, the function has maximum values PA and SD when x = OP and x = OS respectively. Also, the function has a minimum value QB when x = OQ. The values at the of inflexion are RC and TE where x = OR and x = OT, respectively.

The following are steps for determining turning points:

1. Differentiate the given function to obtain $f'(x)$.
2. Equate the expression for $f'(x)$ to zero.
3. Solve the equation $f'(x) = 0$, for the x-coordinate(s) of the turning point(s).
4. Find the x values $(x_1, x_2, x_3, \ldots)$ which satisfy the equation $f'(x) = 0$.
5. Calculate the corresponding values of y using $y = f(x)$.

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Example 8.63

Given the function defined by the equation $y = x^2 - 4x + 5$, find the coordinates of the turning points.

Solution

Given $y = x^2 - 4x + 5$.

$\Rightarrow \frac{dy}{dx} = 2x - 4$

Solve the equation $\frac{dy}{dx} = 0$:

$\Rightarrow 2x - 4 = 0$

$\Rightarrow x = 2$

Calculate y-coordinate from $y = f(x)$:

$\Rightarrow y = (2)^2 - 4(2) + 5$

$\Rightarrow y = 1$

Therefore, the turning point is $(2,1)$.

Example 8.64

Determine the turning points of the curve $y = 4x^3 + 3x^2 - 60x - 12$.

Solution

Given $y = 4x^3 + 3x^2 - 60x - 12$.

$\Rightarrow \frac{dy}{dx} = 12x^2 + 6x - 60$

At the turning points, $\frac{dy}{dx} = 0$.

$\Rightarrow 12x^2 + 6x - 60 = 0$

Upon solving the equation, $x = 2$ and $x = -2.5$.

Calculate the y-coordinates from $y = f(x)$:

When $x = 2$

$\Rightarrow y = 4(2)^3 + 3(2)^2 - 60(2) - 12 = -88$

When $x = -2.5$

$\Rightarrow y = 4(-2.5)^3 + 3(-2.5)^2 - 60(-2.5) - 12$

$= 94.25$

Therefore, the turning points are $(2, -88)$ and $(-2.5, 94.25)$.

Example 8.65

Determine the turning points of the curve $f(x) = 7 + 24x - 9x^2 - 2x^3$.

Solution

Given $f(x) = 7 + 24x - 9x^2 - 2x^3$.

$\Rightarrow f'(x) = 24 - 18x - 6x^2$.

At the turning points, $f'(x) = 0$.

Solving $f'(x) = 0$ for x

$\Rightarrow 24 - 18x - 6x^2 = 0$.

Upon solving, $x = -4$ and $x = 1$.

Determine the corresponding values of y.

When $x = -4$:

$f(-4) = 7 + 24(-4) - 9(-4)^2 - 2(-4)^3$

$= -150$

When $x = 1$:

$f(1) = 7 + 24(1) - 9(1)^2 - 2(1)^3$

$= 20$

Therefore, the turning points are $(-4, -105)$ and $(1, 20)$.

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Exercise 8.16

1. Determine the turning points of the curve $y = 4x^3 + 3x^2 - 6x - 2$.
2. Find the turning points of the curve $y = x^2(x-6)$.
3. Determine the turning points of the function $y = 2x^3 - 12x^2 - 30x - 10$.
4. In each of the following curves, find the coordinates of the turning points:
   (a) $y = x^2 - 2x - 8$
   (b) $y = 3x - x^2$
   (c) $y = x + \frac{1}{x}$
5. Find the coordinates of the turning points in each of the following curves:
   (a) $y = x^3 + x^2 - x + 3$
   (b) $y = 8 + 4x - 2x^2 - x^3$
   (c) $y = 2x^2 - 8x$
   (d) $y = (1-x)(x-5)$
6. The curve $f(x) = x^3 - kx^2 + x - 3$ has two turning points. If one point occurs at $x = 1$, find the value of k and the coordinates of the turning points.
7. For the curve $y = x^3 - 9$, find its turning points.
8. Show that the turning points of the curve $f(t) = t^3 - \frac{1}{2}t^2 - 2t + 4$ are $(1, 2.5)$ and $\left(-\frac{2}{3}, 4\frac{22}{27}\right)$.
9. Verify that the turning point of the curve $y = 2x - e^x$ is $(0.6931, -0.6136)$.
10. Determine the equation of the cubic curve whose turning points are $(1, 3)$ and $(-1, 7)$.

Real life problems on maximum and minimum values

The turning points and points of inflexion of functions (curves) have numerous applications in real life situations. For instance, stationary points (maximum, minimum and points of inflexion) are useful in sketching curves. In economics, stationary points are used to determine the equilibrium prices, break-even points, marginal costs and marginal revenues. The following are steps for classifying stationary points.

1. Given $y = f(x)$, determine $\frac{dy}{dx}$ (that is, $y' = f'(x)$).
2. Let $\frac{dy}{dx} = 0$ and solve for the values of x.
3. Substitute the values of x into the given equation, $y = f(x)$ in order to obtain the corresponding y-coordinate values. This establishes the coordinates of the stationary points.

To determine the nature of stationary points: Either,

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4. Find $\frac{d^2y}{dx^2}$ and substitute into it the values of x in order to obtain the nature of the stationary point as follows:
   (a) If the value of $\frac{d^2y}{dx^2}$ at the point is positive, then the point is a minimum.
   (b) If the value of $\frac{d^2y}{dx^2}$ at the point is negative, then the point is a maximum
   (c) If the value of $\frac{d^2y}{dx^2}$ at the point is zero, then the point is either a maximum, a minimum or a point of inflexion.

   Or

5. Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign of the gradient of the curve changes from positive to negative, then the point is a maximum. If the sign of the gradient changes from negative to positive, the point is a minimum. When the gradient sign does not change, then the point is a point of inflexion.

Example 8.66

Identify the nature of the stationary points of the function $f(x)=3x^4-8x^3+6x^2-5$. Hence, sketch its graph.

Solution

Given $f(x)=3x^4-8x^3+6x^2-5$.

$\Rightarrow f'(x)=12x^3-24x^2+12x.$

At stationary points, $f'(x)=0$.

$\Rightarrow 12x^3-24x^2+12x=0$

$\Rightarrow 12x(x-1)(x-1)=0$

$\Rightarrow x=0$ and $x=1$.

When, $x=0 \Rightarrow f(x)=-5$.

When, $x=1 \Rightarrow f(x)=-4$.

The stationary points are $(0,-5)$, and $(1,-4)$.

Classify the stationary points by finding the second derivative. That is,

$$f''(x)=\frac{d}{dx}\left(12x^3-24x^2+12x\right)$$

$\Rightarrow f''(x)=36x^2-48x+12.$

When $x=0 \Rightarrow f''(0)=12>0$. Thus, the point $(0,-5)$ is a minimum point.

When $x=1 \Rightarrow f''(0)=0$.

The nature of the point is determined by variation of signs of the gradient of the curve. That is,

$f'(x)=12x^3-24x^2+12x.$

If x is slightly less than 1, say $x=0.9$, then

$$f'(0.9)=12(0.9)^3-24(0.9)^2+12(0.9)$$

$$=0.108>0$$

Also, if If x is slightly greater than 1, say $x=1.1$, then

$$f'(1.1)=12(1.1)^3-24(1.1)^2+12(1.1)$$

$$=0.132>0$$

Since the gradient sign does not change, then the point $(1,-4)$ is an inflexion point.

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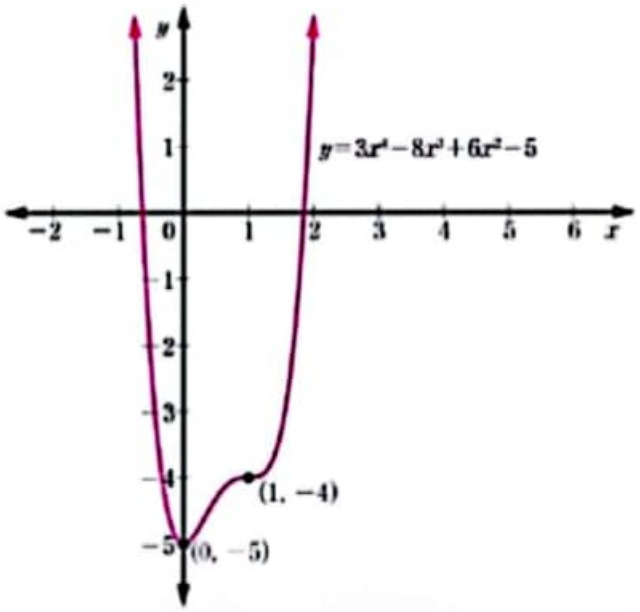
Therefore, $(0,-5)$ is a minimum point and $(1,-4)$ is a point of inflexion.

The following is the graph of the curve $f(x)=3x^4-8x^3+6x^2-5$.



Example 8.67

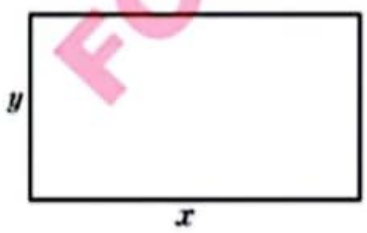
A gardener has 400 m of fencing material and wishes to build a rectangular field completely enclosed by the fence.

(a) Find the dimensions of the field that the gardener should make so as to maximize the area.

(b) Find the maximum area to be enclosed.

Solution

(a) Let x be the length of the field and y be the width as shown in the following figure.



Area = length × width

$\Rightarrow$ Area, $A=xy$

But the fencing does not exceed 400 m. So the field must have a maximum perimeter of 400 m.

$\Rightarrow x+x+y+y=400$

$\Rightarrow 2x+2y=400$

$\Rightarrow y=200-x.$

Substituting $y=200-x$ into the equation $A=xy$ gives,

$A=x(200-x)=200x-x^2$

$\Rightarrow \frac{dA}{dx}=200-2x$

At the turning points, $\frac{dA}{dx}=0$.

Thus, $200-2x=0 \Rightarrow x=\frac{200}{2}=100.$

$\Rightarrow y=200-100=100.$

Therefore, the dimensions of the field should be 100 m by 100 m.

(b) When $x=100$, it implies that

$\left[\frac{d^2A}{dx^2}\right]_{x=100}=-2<0 \Rightarrow (100,100)$

is a maximum point.

The area is maximum when

$x=100\,\text{m}$ and $y=100\,\text{m}$

Thus, $A=xy=100\,\text{m}\times 100\,\text{m}$

$\Rightarrow A=10{,}000\ \text{m}^2$

Therefore, the maximum area is $10{,}000\ \text{m}^2$.

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Example 8.68

Find the coordinates and the nature of turning points of the curve $y = x^3 + 3x^2 - 9x + 6$. Hence, sketch the curve.

Solution

Given $y = x^3 + 3x^2 - 9x + 6$.

The gradient of the curve is given by

$y' = 3x^2 + 6x - 9$

At turning points, $y' = 0$.

$\Rightarrow 0 = 3x^2 + 6x - 9$

$\Rightarrow (x-1)(x+3) = 0$

$\Rightarrow x = 1$ and $x = -3$.

The corresponding y coordinates are obtained as follows:

When $x = 1 \Rightarrow y = (1)^3 + 3(1)^2 - 9(1) + 6$.

$\Rightarrow y = 1$

When

$x = -3 \Rightarrow y = (-3)^3 + 3(-3)^2 - 9(-3) + 6$.

$\Rightarrow y = 33$

Thus, the turning points are $(1,1)$ and $(-3,33)$.

To test the nature of the turning points, use the second derivative test that is,

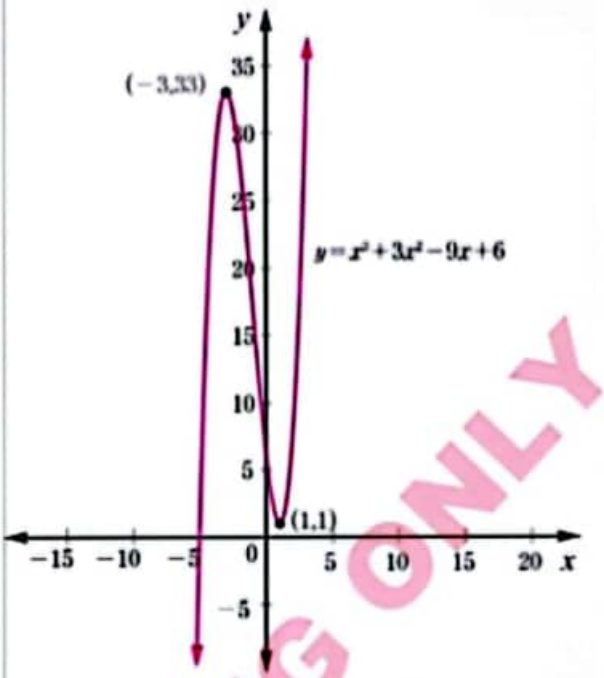
$y'' = 6x + 6$ when

$x = 1 \Rightarrow y''(1) = 6(1) + 6 = 12 > 0 \Rightarrow (1,1)$ is a minimum point.

$y'' = 6x + 6$ when

$x = -3 \Rightarrow y''(-3) = 6(-3) + 6 = -12 < 0 \Rightarrow (-3, 33)$ is a maximum point.

Therefore, $(1,1)$ is a minimum point and $(-3, 33)$ is a maximum point.



Example 8.69

Find the maximum and minimum values of the curve $y = 3x^2 + 3 - x^3$ by examining the variation of signs for its gradient.

Solution

Given

$y = 3x^2 + 3 - x^3 \Rightarrow \frac{dy}{dx} = 6x - 3x^2$.

For turning points, $\frac{dy}{dx} = 0$.

Thus,

$6x - 3x^2 = 0 \Rightarrow x = 0$ and $x = 2$.

When

$x = 0 \Rightarrow y = 3(0)^2 + 3 - (0)^3 = 3$,

thus $(x, y) = (0,3)$.

When

$x = 2 \Rightarrow y = 3(2)^2 + 3 - (2)^3 = 7$,

thus $(x, y) = (2,7)$.

Hence, $(0,3)$ and $(2,7)$ are the coordinates of the turning points.

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Next, examine the nature of the turning points using variation of signs for the gradient.

Considering the point $(0,3)$, proceed as follows.

If x is slightly less than 0, say $x=-0.9$, then

$$\left[\frac{dy}{dx}\right]_{x=-0.9} = 6(-0.9)-3(-0.9)^2 = -7.8 < 0.$$

If x is slightly greater than 0, say $x=0.1$, then

$$\left[\frac{dy}{dx}\right]_{x=0.1} = 6(0.1)-3(0.1)^2 = 0.57 > 0.$$

Since the gradient changes from negative to positive, the point $(0,3)$ is a minimum point and the minimum value is 3.

Considering the point (2, 7), proceed as follows.

If x is slightly less than 2, let say $x=1.9$, then

$$\left[\frac{dy}{dx}\right]_{x=1.9} = 6(1.9)-3(1.9)^2 = 0.57 > 0.$$

If x is slightly greater than 2, let say $x=2.1$, then

$$\left[\frac{dy}{dx}\right]_{x=2.1} = 6(2.1)-3(2.1)^2 = -0.63 < 0.$$

Since the gradient changes from positive to negative, then the point (2, 7) is a maximum point and the maximum value is 7.

Therefore, the maximum value is 7 and the minimum value is 3.

Example 8.70

Find and classify the nature of the turning points for the function $y=4x^5-5x^4$. Hence, sketch its graph.

Solution

Given $y=4x^5-5x^4$. It implies that, $\frac{dy}{dx}=20x^4-20x^3$.

At the turning point, $\frac{dy}{dx}=0$.

That is, $20x^4-20x^3=0$.

Thus,

$20x^3(x-1)=0 \Rightarrow x=0$ or $x=1$.

When $x=0$, $y=4(0)^5-5(0)^4=0$.

$\Rightarrow (x,y)=(0,0)$

When $x=1$, $y=4(1)^5-5(1)^4=-1$.

$\Rightarrow (x,y)=(1,-1)$

Thus, the coordinates of the turning points are $(0,0)$ and $(1,-1)$.

Classify the stationary points by finding the second derivative.

That is, $\frac{d^2y}{dx^2}=80x^3-60x^2$.

When

$x=0$, $\frac{d^2y}{dx^2}=80(0)^3-60(0)^2=0$.

Since $\frac{d^2y}{dx^2}=0$, then the point

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$(0,0)$ is either a maximum, minimum, or a point of inflexion. To verify this, use the variation signs of the first derivative as follows.

If x is slightly less than 0, say -0.5, then

$$\frac{dy}{dx} = 20(-0.5)^4 - 20(-0.5)^3 = 3.75,$$

which is positive.

If is slightly more than 0, say 0.5, then

$$\frac{dy}{dx} = 20(0.5)^4 - 20(0.5)^3 = -1.25,$$

which is negative.

Since the gradient changes from positive to negative, then the point $(0,0)$ is a maximum point.

When $x = 1$,

$$\frac{d^2y}{dx^2} = 80(1)^3 - 60(1)^2 = 20,$$

which is positive.

Hence, $(1,-1)$ is a minimum point.

Thus, $(1,-1)$ is a minimum point and $(0,0)$ is a maximum point.

For x-intercept, $y = 0$. That is,

$0 = 4x^5 - 5x^4 \Rightarrow x^4(4x-5) = 0.$

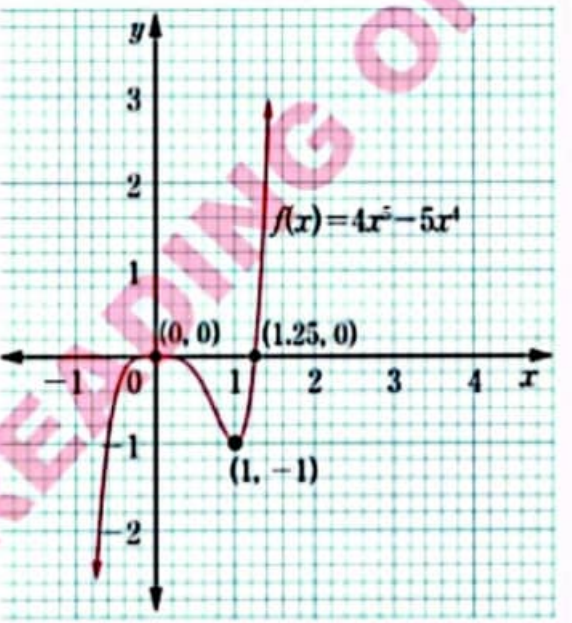
Hence, $x = 0$ or $x = \frac{5}{4}$.

Thus, the x-intercepts are $(0,0)$ and $\left(\frac{5}{4},0\right)$.

For y-intercept; $x = 0$. That is, $y = 4(0)^5 - 5(0)^4 = 0$.

Thus, the y-intercept is at the point $(0,0)$.

Therefore, $(1,-1)$ is a minimum point, $(0,0)$ is a maximum point, and the graph of $y = 4x^5 - 5x^4$ is as shown in the following figure.



Exercise 8.17

1. An open rectangular prism is to be constructed from a square base and four vertical rectangular shapes in order to have the capacity of 32 m³. Find the smallest area of the cuboid to be used in the construction of the rectangular prism.

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2. A rectangular sheet of cardboard has length 12 cm and width 7.5 cm. Equal squares of length x cm are cut from each corner. The flaps are then folded to make an open box in the form of a cuboid. If the volume of the box is V cm^3:
   (a) Show that $V = 4x^3 - 39x^2 + 90x$.
   (b) Find the value of x which gives the maximum value of V.
   (c) What is the maximum value of V?

3. The product of two positive numbers is 100. Find their smallest possible sum.

4. Find the coordinates of the turning points of the function $\frac{15+10x}{\sqrt{x^2+1}}$ and classify them.

5. Find the values of x for which the curve $y = (x-2)(x-3)^2$ has turning points and determine the maximum and minimum turning points of the curve.

6. For each of the following curves, find the coordinates of the turning points, classify the turning points, and determine where the curves cross the axes.
   (a) $y = (x-2)^2(x-1)$
   (b) $y = x^4 - 4x^3$
   (c) $y = 36x - 3x^2 - 2x^3$
   (d) $y = 5x^4 - 12x^5$
   (e) $y = x^2 - 4$
   (f) $y = 4x^3 - 2x^2 - x - 1$

7. A rectangle of length x centimetres has a constant area of 12 cm^2. Find the perimeter of the rectangle in terms of x and the smallest possible perimeter of the rectangle.

8. A lidless box with square ends is to be made from a sheet of metal. Determine the smallest area of the metal for which its volume is 3.5 m^3.

9. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for maximum volume.

10. Determine the area of the largest fence of a rectangular ground that can be enclosed by 100 m of fencing, if part of an existing straight wall is used as one side.

Taylor and Maclaurin series

Taylor's series is an expression of a function f(x) near $x = a$ which consists of an infinite sum of terms involving derivatives of the function at $x = a$, where $a \neq 0$. If $a = 0$, the infinite series is known as Maclaurin's series.

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(a) Taylor's theorem

Consider the polynomial in power of $(x-a)$ given by;

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+\cdots+c_n(x-a)^n.$$

The successive derivatives of $f(x)$ will be as follows;

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\cdots+nc_n(x-a)^{n-1}$$

$$f''(x)=2\times c_2+3\times 2c_3(x-a)+4\times 3c_4(x-a)^2+\cdots+n(n-1)c_n(x-a)^{n-2}$$

$$f'''(x)=3\times 2c_3+4\times 3\times 2c_4(x-a)+\cdots+n(n-1)(n-2)c_n(x-a)^{n-3}$$, and so on.

The constants $c_0, c_1, c_2,\ldots, c_n$ are calculated by substituting $x=a$ in the function $f(x)$ and to the respective derivatives.

Putting $x=a$ gives,

$$c_0=f(a),\ c_1=f'(a),\ c_2=\frac{f''(a)}{2!},\ c_3=\frac{f'''(a)}{3!},\ c_4=\frac{f^{iv}(a)}{4!},\ldots,\ c_n=\frac{f^n(a)}{n!}$$

Substituting these coefficients in the polynomial gives Taylor's theorem.

Thus, Taylor's theorem states that,

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^n(a)}{n!}(x-a)^n$$

If $x=a+h\Rightarrow h=x-a$, then Taylor's theorem becomes,

$$f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\cdots+\frac{f^n(a)}{n!}h^n$$

Taylor's theorem is useful in expanding the function $f(x)$ about $x=a$.

Example 8.71

Use Taylor's theorem to expand $\sin\left(\frac{\pi}{3}+h\right)$ in ascending powers of h as far as the term in h^4.

Solution

Let $f(x)=\sin x=\sin\left(\frac{\pi}{3}+h\right)$ and $f\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

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$$\Rightarrow f'(x)=\cos x \text{ and } f'\left(\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

$$\Rightarrow f''(x)=-\sin x \text{ and } f''\left(\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

$$\Rightarrow f'''(x)=-\cos x \text{ and } f'''\left(\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$$

$$\Rightarrow f^{(iv)}(x)=\sin x \text{ and } f^{(iv)}\left(\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

By Taylor's theorem, it implies that

$$f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\frac{f^{iv}(a)}{4!}h^4+\cdots+\frac{f^n(a)}{n!}$$

$$\text{Thus, } \sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h+\frac{\left(-\frac{\sqrt{3}}{2}\right)}{2!}h^2+\frac{\left(-\frac{1}{2}\right)}{2!}h^3+\frac{\left(\frac{\sqrt{3}}{2}\right)}{4!}h^4+\cdots$$

$$\Rightarrow \sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2-\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$$

$$\text{Therefore, } \sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2-\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$$

Example 8.72

Use Taylor's theorem to expand $\cos\left(\frac{\pi}{6}+h\right)$ in ascending powers of h as far as the term in h^4. Hence, find the value of $\cos 31°$ correct to four significant figures, taking $1°$ as 0.01745 radians.

Solution

Let $f(x)=\cos x$, $f(a+h)=\cos\left(\frac{\pi}{6}+h\right)$ and $f\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

$$\Rightarrow f'(x)=-\sin x \text{ and } f'\left(\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

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$$\Rightarrow f''(x) = -\cos x \text{ and } f''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow f'''(x) = \sin x \text{ and } f'''\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow f^{(iv)}(x) = \cos x \text{ and } f^{(iv)}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

By Taylor's theorem, it follows that

$$f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f^{iv}(a)}{4!}h^4 + \cdots$$

Thus, $\cos\left(\frac{\pi}{6}+h\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}h + \frac{\left(-\frac{\sqrt{3}}{2}\right)}{2!}h^2 + \frac{\left(\frac{1}{2}\right)}{3!}h^3 + \frac{\left(\frac{\sqrt{3}}{2}\right)}{4!}h^4 + \cdots$

$$\Rightarrow \cos\left(\frac{\pi}{6}+h\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}h - \frac{\sqrt{3}}{4}h^2 + \frac{1}{12}h^3 + \frac{\sqrt{3}}{48}h^4 + \cdots$$

For the value of $\cos 31° = \cos(30° + 1°)$, it implies that, $h = 1° = 0.01745$

Thus,

$$\cos 31° = \frac{\sqrt{3}}{2} - \frac{1}{2}(0.01745) - \frac{\sqrt{3}}{4}(0.01745)^2 + \frac{1}{12}(0.01745)^3 + \frac{\sqrt{3}}{48}(0.01745)^4 + \cdots$$

$$= 0.8572.$$

Therefore, $\cos\left(\frac{\pi}{6}+h\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}h - \frac{\sqrt{3}}{4}h^2 + \frac{1}{12}h^3 + \frac{\sqrt{3}}{48}h^4 + \cdots$ and

$\cos 31° = 0.8572.$

Example 8.73

Use Taylor's theorem to expand $\tan\left(\frac{\pi}{4}+h\right)$ in ascending powers of h as far as the term in h^3.

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Solution

Let $f(x) = \tan x = \tan\left(\frac{\pi}{4} + h\right)$ and $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1.$

$\Rightarrow f'(x) = \sec^2 x$ and $f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$

$\Rightarrow f''(x) = 2\sec^2 x \tan x$ and $f''\left(\frac{\pi}{4}\right) = 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = 4$

$\Rightarrow f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ and

$f'''\left(\frac{\pi}{4}\right) = 4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) + 2\sec^4\left(\frac{\pi}{4}\right) = 16$

By Taylor's theorem,

$$\Rightarrow f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f^{iv}(a)}{4!}h^4 + \cdots$$

Thus, $\tan\left(\frac{\pi}{4} + h\right) = 1 + 2h + \frac{4}{2!}h^2 + \frac{16}{3!}h^3 + \cdots$

$$= 1 + 2h + 2h^2 + \frac{8}{3}h^3 + \cdots$$

Therefore, $\tan\left(\frac{\pi}{4} + h\right) = 1 + 2h + 2h^2 + \frac{8}{3}h^3 + \cdots$

Maclaurin's series

Considering a relationship $x = a + h$, where a is a constant, and x and h are variables, there is a special case given by $a = 0$ when $x = h$. In this case, Taylor's theorem can be reduced to:

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots$ which is called Maclaurin's series. Maclaurin's series is a special case of Taylor's theorem for $a = 0$.

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Example 8.74

Use Maclaurin's series to expand $\ln(1+x)$ in ascending powers of x as far as the term in x^4.

Solution

Given $f(x)=\ln(1+x)$.

$\Rightarrow f(0)=\ln(1)=0$

$\Rightarrow f'(x)=(1+x)^{-1}$ and $f'(0)=1$

$\Rightarrow f''(x)=-(1+x)^{-2}$ and $f''(0)=-1$

$\Rightarrow f'''(x)=2(1+x)^{-3}$ and $f'''(0)=2$

$\Rightarrow f^{iv}(x)=-6(1+x)^{-4}$ and $f^{(iv)}(0)=-6$

By Maclaurin's series,

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$$

Thus, $\ln(1+x)=0+1\times x+\frac{(-1)}{2!}x^2+\frac{2}{3!}x^3+\frac{(-6)}{4!}x^3+\cdots$

$$\Rightarrow \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Therefore, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

Example 8.75

Find Maclaurin's series for e^{5x} as far as the term in x^3.

Solution

Given $f(x)=e^{5x}$.

$\Rightarrow f(0)=1$

$\Rightarrow f'(x)=5e^{5x}$ and $f'(0)=5e^{5(0)}=5$

$\Rightarrow f''(x)=5^2e^{5x}$ and $f''(0)=5^2e^{5(0)}=5^2$

$\Rightarrow f'''(x)=5^3e^{5x}$ and $f'''(0)=5^3e^{5(0)}=5^3$.

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By Maclaurin' series,

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$$

Thus, $e^{5x}=1+5x+\frac{25}{2}x^2+\frac{125}{6}x^3+\cdots$

Therefore, $e^{5x}=1+5x+\frac{25}{2}x^2+\frac{125}{6}x^3+\cdots$

Exercise 8.18

1. Show that the first three terms of Maclaurin's expansion of $\log_e(1+e^x)$ are $\log_e 2+\frac{1}{2}x+\frac{1}{8}x^2$. Hence, show that there is no term in x^3.
2. Using Taylor's theorem, obtain a series expansion of $\sin(\theta+x)$, where θ is an acute angle measured in radians such that $\sin\theta=\frac{4}{5}$ (Give the first four non-zero terms).
3. Apply Taylor's theorem to expand the following:
   (a) $\ln(1+4x)$ in ascending powers of x as far as the term in x^4 and deduce its corresponding Maclaurin's series.
   (b) $\operatorname{cosec} x$ in ascending powers of $\left(x-\frac{\pi}{2}\right)$ as far as the term in $\left(x-\frac{\pi}{2}\right)^4$.
   (c) $\sec x$ in ascending powers of $\left(x+\frac{\pi}{4}\right)$ as far as the term in $\left(x+\frac{\pi}{4}\right)^4$.
   (d) $\cos\left(\frac{\pi}{3}+x\right)$ in ascending powers of x as far as the term in x^4.
4. Prove that $\log_e\left(\frac{1-2x}{(1+2x)^2}\right)=6x+2x^2-8x^3+4x^4+\cdots$ and state the range of values of x for which the series is valid.
5. Find the Maclaurin's series for the following functions up to the term containing x^4.
   (a) $\cos x$ (b) $\sin x$
   (c) a^x (d) $\sqrt{1+4x}$
6. Construct the power series for $\cos^2 2\theta$ as far as the term in θ^6.
7. Using Maclaurin's series, develop a series for $e^{\frac{3}{2}x}$ as far as the fourth term.

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8. Show that the first three non-zero terms of the Maclaurin series for $\sin(\sin x)$ are $x-\frac{x^3}{3}+\frac{x^5}{10}+\cdots$

9. Use Taylor's theorem to expand $\sin\left(\frac{\pi}{2}+h\right)$ in ascending powers of h as far as the term in h^4.

10. Use Taylor's theorem to show that the expansion of $x^2 \ln x$ in powers of $(x-1)$ as far as the term in $(x-1)^3$ is $(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3$.

Introduction to partial derivatives

The function $z=f(x,y)$ depends on two variables, x and y, where x and y are independent to each other. Thus, f depends on x and y. The derivative of f with respect to x or y is called a partial derivative.

Identifying functions of two variables

The definition of a function of two variables is very similar to the definition of a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, the ordered pairs of variables are mapped to another variable.

(a) $f(x,y)$ means f is a function of x and y

(b) $h(q,r)$ means h is a function of q and r.

Let $z=f(x,y)$ be a function of two independent variables, x and y. The partial derivatives of z are obtained as follows:

If x varies while y remains constant, then z is a function of x. The partial derivative of z with respect to x is given by,

$$\frac{\partial z}{\partial x}=f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,\,y)-f(x,y)}{h}.$$

If y varies while x remains constant, then z is a function of y. The partial derivative of z with respect to y is given by;

$$\frac{\partial z}{\partial y}=f_y(x,y)=\lim_{h\to 0}\frac{f(x,\,y+h)-f(x,y)}{h}.$$

Other notations used for partial derivatives include,

$$\frac{\partial^2 z}{\partial x^2}=z_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\ \frac{\partial^2 z}{\partial y^2}=z_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),$$

$$\frac{\partial^2 z}{\partial y\partial x}=z_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),\text{ and }\frac{\partial^2 z}{\partial x\partial y}=z_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

where z_{xx} is the second partial derivative of z with respect to x, z_{yy} is the second partial derivative of z with respect to y, and z_{xy} and z_{yx} is a mixed second partial derivative of z with respect to x and y.

Example 8.76

Find the first partial derivatives of $z=x^2+3xy+y^2$.

Solution

The first partial derivatives are given by:

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(x^2+3xy+y^2\right)$$

$$=\frac{\partial}{2x}\left(x^2\right)+3y\frac{\partial}{\partial x}(x)+y^2\frac{\partial}{\partial x}(1)$$

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$$= 2x + 3y(1) + y^2(0)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x + 3y$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y^2\right)$$

$$= x^2\frac{\partial}{\partial y}(1) + 3x\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}\left(y^2\right)$$

$$= x^2(0) + 3x(1) + 2y$$

$$\Rightarrow \frac{\partial z}{\partial y} = 3x + 2y$$

Therefore, $\frac{\partial z}{\partial x} = 2x + 3y$ and $\frac{\partial z}{\partial y} = 3x + 2y$.

Example 8.77

Find the first partial derivatives of $z = \sin 3x \cos 4y$.

Solution

Given $z = \sin 3x \cos 4y$.

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\sin 3x \times \cos 4y)$$

$$= \cos 4y\frac{\partial}{\partial x}(\sin 3x)$$

$$= 3\cos 3x \cos 4y$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\sin 3x \cos 4y)$$

$$= \sin 3x\frac{\partial}{\partial y}(\cos 4y)$$

$$= -4\sin 3x \sin 4y$$

Therefore $\frac{\partial z}{\partial x} = 3\cos 3x \cos 4y$ and $\frac{\partial z}{\partial y} = -4\sin 3x \sin 4y$.

Example 8.78

Find $\frac{\partial^2 z}{\partial x^2}$ if $z = e^{x^3+y^2}$.

Solution

Given $z = e^{x^3+y^2}$.

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{x^3+y^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2 e^{x^3+y^2}$$

Thus, $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 e^{x^3+y^2}\right)$.

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \left(9x^4 + 6x\right)e^{x^3+y^2}$$

Therefore, $\frac{\partial^2 z}{\partial x^2} = \left(9x^4 + 6x\right)e^{x^3+y^2}$.

Example 8.79

If $f(x, y) = x^3 \sin 4y - y^2 \tan^{-1} x$, find;

(a) $\frac{\partial^2 f}{\partial y \partial x}$ (b) $\frac{\partial^2 f}{\partial x^2}$

Solution

(a) $$\frac{\partial f}{\partial x} = \sin 4y\frac{\partial}{\partial x}\left(x^3\right) - y^2\frac{\partial}{\partial x}\left(\tan^{-1} x\right)$$

$$= 3x^2 \sin 4y - \frac{1}{1+x^2}y^2$$

From

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 \sin 4y - \frac{1}{1+x^2}y^2\right)$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 3x^2\frac{\partial}{\partial y}(\sin 4y) - \frac{1}{1+x^2}\frac{\partial}{\partial y}\left(y^2\right)$$

$$= 12x^2 \cos 4y - \frac{2}{1+x^2}y$$

Therefore, $\frac{\partial^2 f}{\partial y \partial x} = 12x^2 \cos 4y - \frac{2y}{1+x^2}$.

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(b) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 \sin 4y - \frac{y^2}{1+x^2}\right)$

$= \sin 4y \frac{\partial}{\partial x}(3x^2) - y^2 \frac{\partial}{\partial x}\left(\frac{1}{1+x^2}\right)$

$= 6x \sin 4y - y^2 \frac{-2x}{(1+x^2)^2}$

$= 6x \sin 4y + \frac{2xy^2}{(1+x^2)^2}$

Therefore, $\frac{\partial^2 f}{\partial x^2} = 6x \sin 4y + \frac{2xy^2}{(1+x^2)^2}$.

Exercise 8.19

1. If $p = \frac{kT}{V}$, find the first partial derivatives of p, where k is a constant.

2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^2 y^3$.

3. Find the first partial derivatives of the function $z = \frac{x}{y^2} - \frac{y}{x^2}$ with respect to x and y.

4. Find $\frac{\partial z}{\partial x}$ in each of the following functions:
   (a) $z = (x^2 + 4y^2)\sin(x^3 y)$
   (b) $z = \frac{x^2 + y^2}{x + y}$
   (c) $z = (3x + y)^2$

5. In each of the following functions, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
   (a) $f(x, y) = x^3 y + x^2 y^2$
   (b) $f(x, y) = x^2 + 3xy$
   (c) $f(x, y) = x^4 y^3 + 8x^2 y + y^4 + 5x$
   (d) $f(x, y) = \frac{x^2}{y} + \frac{y^2}{x}$

6. Given $z = x^2 \sin(x - 2y)$, find:
   (a) $\frac{\partial^2 z}{\partial x^2}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$
   (b) $\frac{\partial^2 z}{\partial y^2}$ (d) $\frac{\partial^2 z}{\partial y \partial x}$

7. Find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ if $z = \cos^{-1}\left(\frac{x}{y}\right)$.

8. If $z = \sqrt{\frac{3x}{y}}$, show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, and hence evaluate $\frac{\partial^2 z}{\partial x^2}$ at $\left(\frac{1}{2}, 3\right)$.

9. Verify that if $z = \frac{x}{y} \ln y$, then $\frac{\partial z}{\partial y} = x \frac{\partial^2 z}{\partial y \partial x}$. Find the value of $\frac{\partial^2 z}{\partial y^2}$ when $x = -3$ and $y = 1$.

10. The time of oscillation T of a pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length and g is the acceleration due to gravity. Determine:
    (a) $\frac{\partial T}{\partial l}$ (b) $\frac{\partial T}{\partial g}$

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Chapter summary

1. Differentiation from first principles is done by using the formula, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided this limit exists.

2. The derivative of a polynomial term is given by $\frac{d}{dx}(x^n)=nx^{n-1}$.

3. The derivative of the sum or difference of two or more polynomial functions is given by:
$$\frac{d}{dx}[f(x)\pm g(x)\pm h(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]\pm\frac{d}{dx}[h(x)]$$

4. If $y = u(x)\ v\ (x)$, then the product rule for differentiation is given by $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$ and the quotient rule for differentiation of $y=\frac{u(x)}{v(x)}$ is given by, $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$. where u and v are two differentiable function of x.

5. To find the nature of turning point, using the second derivative:
   (a) If $f''(x)>0$, then the turning point is minimum
   (b) If $f''(x)<0$, then the turning point is maximum
   (c) If $f''(x)=0$, then the point is an inflexion, minimum or maximum point.

6. Taylor's theorem is given by;
$$f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\frac{f^{iv}(a)}{4!}h^4+\cdots+\frac{f^n(a)h^n}{n!}$$

7. Maclaurin's theorem is a special case of Taylor's theorem and it is given by; $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$

8. The first partial derivatives of $z=f(x,y)$ are given by;
$$z_x=f_x(x,y)=\frac{\partial z}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\text{ and}$$
$$z_y=f_y(x,y)=\frac{\partial z}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

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Revision exercise 8

1. Differentiate each of the following functions from first principles:
   (a) $\frac{\sin x}{x}$ (b) $\cos^2 x$ (c) $x\sin x$ (d) $\cos(5x+2)$

2. Given that $y=\tan\left(x(2x+3)^{-1}\right)$, show that
   $y'=3(2x+3)^{-2}\sec^2\left(x(2x+3)^{-1}\right)$.

3. Differentiate each of the following:
   (a) $y=5\sin x-4\cos x+8$ (c) $y=\tan^{-1}(\sin x+\cos x)$
   (b) $4x^4\ln\sin 2y-e^{2y}(1+x^2)=7$ (d) $f(x)=\sqrt{1+8x-3x^2}$

4. Given that $x=1-\sin 2\theta$ and $y=1+\cos 2\theta$, find:
   (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$ (c) $\frac{dy}{dx}$ when $\theta=\frac{\pi}{16}$

5. Find $\frac{dy}{dx}$ in each of the following:
   (a) $2xy^3+3x^2y=7$ (c) $y^2\ln x-3=1-x^2\ln x$
   (b) $\tan^{-1}\left(\frac{y}{x}+3\right)=\ln\left(x^2+y^2+2\right)$ (d) $x^3\sin y-y^2\tan 2x=20x^2y^2$

6. Differentiate each of the following functions:
   (a) $\left(1+e^x\right)^4$ (b) $\frac{x}{e^x}$ (c) $e^x\left(x^2-3\right)$ (d) $\frac{\ln x}{x^3}$ (e) $5e^{1-x}$

7. Find the derivative of each of the following functions:
   (a) x^2e^{4x} (b) $\sqrt{e^x-x}$ (c) x^6-4x^4+2x-6 (d) $\frac{1+4x^2}{1+x^2}$

8. A woman wishes to fence in a rectangular enclosure of area 400 m^2. One side of the enclosure is formed by part of a brick wall already in position. Find the least possible length of fencing required for the other three sides.

9. Expand each of the following:
   (a) $(2+x)^2e^{-x}$ in ascending powers of x as far as the term in x^3.
   (b) $\log_e y$ at $y=1$ in ascending powers of $(y-1)$ as far as the term in $(y-1)^3$.
   (c) $\frac{\cos x}{\sqrt{1-x}}$ in ascending powers of x as far as the term in x^4.

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10. If $y = e^x \tan x$, show that $\frac{d^2y}{dx^2} - 2(1+\tan x)\frac{dy}{dx} + (1+2\tan x)y = 0$.

11. Given $y = e^{4x}\cos 3x$, find the value of $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y$.

12. If $x > 1$, show that $\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots$, hence find $\log_e 2$.

13. Use Taylor's theorem to find the expansion of each of the following:
    (a) $\sin(x+h)$
    (b) $\sin\left(\frac{\pi}{6}+h\right)$ in ascending powers of h as far as the term in h^4
    (c) $\tan\left(\frac{\pi}{3}+h\right)$ in ascending powers of h as far as the term in h^4
    (d) $\tan\left(\frac{\pi}{6}+x\right)$ in ascending powers of x as far as the term in x^4

14. From first principles of differentiation, find the derivative in each of the following functions.
    (a) $f(x) = \frac{\cos 2x}{2x}$
    (c) $h(x) = \frac{1}{1-6x}$
    (e) $h(\theta) = \sin 4\theta - \cos 2\theta$
    (b) $g(x) = 8x^2 + 4x - 5$
    (d) $g(x) = \sqrt{5-9x}$

15. Use Maple 18 software to differentiate each of the following:
    (a) $f(x) = e^{4x}\cos\left(4x^2 - 2x + 1\right)$
    (c) $h(x) = \tan^{-1}\left(\frac{1-x}{e^x}\right)$
    (b) $g(x) = \sqrt{\frac{x^2-4}{x^2+8}}$
    (d) $x^2y^2 - 4xy + \sin x = 10$

16. Given that $f(x) = \ln(3 + x)$;
    (a) Find $f(0)$ and $f'(0)$, hence show that $f''(0) = -\frac{1}{9}$.
    (b) Write down the first three terms of the Maclaurin's series for $f(x)$, where $-3 < x \leq 3$.

17. Find the coordinates of the turning points of the curve $y = x - e^x$ and sketch its graph.

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18. A cycloid has the parametric equations $x = 5(20 + \sin 20)$ and $y = 5(2 - \cos 20)$. Evaluate each of the following at $0 = 0.75$ radians, correct to four significant figures.

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

19. Oil is dropping onto a surface at the rate of $\frac{1}{10}\pi$ cm^3 / s and forms a circular film which may be considered to have a uniform depth of 0.1 cm. Find the rate at which the radius of the circular film is increasing when the radius is 7 cm.

20. The radius of a circle is to be measured and its area calculated. If the radius is 0.001 m and the area must be accurate to 0.1 m^2, find the maximum radius.

21. Suppose the volume of a sphere is increasing at a constant rate of 7 cm^3/s.

(a) Find in terms of π the rate at which the radius of the sphere is increasing when the radius is 5 cm.

(b) Show that at any instant, the rate of increase of the surface area of the sphere is inversely proportional to the radius of the sphere.

22. Show that $\sin\left(\frac{1}{6}\pi + x\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{4}x^2 + \cdots$ where x is measured in radians.

23. The radius of a sphere decreases from 12.5 cm to 10.25 cm. Determine correct to four decimal places the approximate change of

(a) the surface area (b) the volume

24. Differentiate each of the following functions:

(a) $\ln\left(\frac{2+\cos x}{3-\sin x}\right)$ (c) $\ln(\sec 2x + \tan 2x)$

(b) $\arctan\left(\frac{1-x}{1+x}\right)$ (d) $\frac{1}{x^2}\sqrt{1+x^3}$

25. If $g(x) = \tan x$,

(a) Show that $g''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$.

(b) Find the Maclaurin series expansion for $g(x)$ up to the term in x^3.

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26. Give the first three non-zero terms of Maclaurin's series in each of the following functions:

(a) $\cos 3x$ (b) $\sin 2x$ (c) $(3+2t)^4$

27. Use MATLAB 2014a software to differentiate each of the following functions:

(a) $y=\dfrac{3x+7}{x^2-9}$ (c) $y=\cos^{-1}(1+\tan x)$

(b) $y=(1+e^{3x})(x-4)^6$ (d) $4x\ln\sin y-e^x\sqrt{1+y}=4\tan x$

28. The distance x metres moved by a car in a time t seconds is given by $x=t^3-2t^2-5t+6$. Determine the velocity and acceleration when

(a) $t=0$ seconds (b) $t=1.5$ seconds

29. Find the first order partial derivatives of z with respect to the independent variables x and y in each of the following:

(a) $x^2+y^2+z^2=25$ (d) $z=\dfrac{x^3}{y^2}-\dfrac{y^3}{x}$

(b) $z=e^{x^2+xy}$ (e) $z=2x^2-6xy+y^2$

(c) $z=x^3+2xy+y^4$ (f) $z=\sin(2x+3y)$

30. A waste paper basket consists of an open circular top. If the volume of the basket is to be 20 cm^3, find the radius of its base if the material used is minimum.

31. A wire of length 100 cm is to be cut into two parts. One portion is bent into the shape of a circle and the other into a square and then fixed on a tray to occupy a minimum area. Find the length of the two parts.

32. A solid is formed by a cylinder of radius r and height h, together with two hemispheres of radius r attached at each end. If the volume v of the solid is constant, but r is increasing at the rate of $\dfrac{1}{2\pi}$ metres per minute, how fast must h be changing when r and h are both 10 metres.

33. The position function of a stone thrown from a cliff is given by $s(t)=10t-16t^2$ metres (below the cliff) after t seconds. Find the average velocity of the stone between $t=1$ second and $t=5$ seconds.

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34. A particle moving in a straight line has its displacement governed by the equation $s(t) = -2t^2 + 10t - 1,\ t \geq 0,$ where s is in metres and t in seconds. Find the velocity and acceleration of the particle at any time t.

35. Given $z = x^3y^2 - yx^{-2} + y^{-1}$, find each of the following:

(a) $\dfrac{\partial z}{\partial x}$, when $x = 1$ and $y = 2$

(b) $\dfrac{\partial^2 z}{\partial x \partial y}$ at the point $(-1, -1)$

36. If $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{2}{\left(x^2 + y^2 + z^2\right)^2}$$

37. A chemical factory wishes to make a cylindrical container of thin metal to hold 126 cm^3, using the least possible area of a metal. If the outside surface area is A cm^2 and the radius is r cm, then

(a) Show that $A = 2r^{-1}\left(\pi r^3 + 126\right)$

(b) Find the required radius and height of the container (leave π your answer).

38. Investigate the stationary points of the curve $y = f(x) = 3x^5 - 5x^3$ and then sketch the graph.

39. The time, T of a swing of a pendulum is given by $T = k\sqrt{l}$, where k is a constant. Determine the percentage change in time of the swing if the length l of the pendulum decreases from 32.1 cm to 32.0 cm.

40. Determine the value of $\dfrac{d^2y}{dx^2}$, correct to 4 significant figures, at $\theta = \dfrac{\pi}{6}$ radians for the cardioid defined by $x = 5(2\theta - \cos 2\theta)$ and $y = 5(2\sin\theta - \sin 2\theta)$.

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Chapter Nine

Introduction

Integration is one of the main operations in calculus and it is an important concept in Mathematics. Integration is the inverse process of differentiation. In this chapter, you will learn about the inverse process of differentiation, integration of functions, and applications of integration. The competencies developed have many real life applications such as determining the total cost and total revenue of goods produced, finding displacement, velocity and acceleration of moving bodies, moment of inertia of vehicles, and the rate of a chemical reaction. It is also used in building constructions, in graphical representations where three-dimensional models are demonstrated in the analysis of spread of infectious diseases, among many other applications.

Think

The World without the concept of integration.

Inverse process of differentiation

Suppose $f(x)$ is a known function, then its derivative $f'(x)$ with respect to x can be obtained. The inverse process gives the class of all functions with the same derivative as $f(x)$. This inverse process of differentiation is called integration. For instance, the derivative of $f(x) = x^3$ with respect to x gives $f'(x) = 3x^2$. Therefore, the integral of $3x^2$ is $x^3 + c$.

Anti-derivative and integral notation

The opposite of the derivative of a function is called the anti-derivative. The process of finding an anti-derivative of a function is called integration and is denoted by the integral symbol $\int$. When this symbol includes a function $f(x)$, it is written as $\int f(x)\,dx$ and read, "the integral of $f(x)$ with respect to the variable x". In this case, the function $f(x)$ is called the integrand.

Note that, the derivative of any constant is zero. This implies that, differentiation makes the function lose the constant. For instance, the derivatives of $y = x^3 + 1$ and

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$y = x^3 + 2$ are both equal to $3x^2$. Thus, the integration of $3x^2$ gives $x^3 + c$ where c is an arbitrary constant. This means that, integration of $3x^2$ cannot give back the functions, $y = x^3 + 1$ or $y = x^3 + 2$. This is because during differentiation the constant becomes 0 as it does not change. Therefore, to show the presence of a constant, an arbitrary constant is added after completion of the integration process. That is, $\int 3x^2 dx = x^3 + c$, where c is an arbitrary constant. Therefore, the function $x^3 + c$ defines the class of all functions whose derivatives is $3x^2$.

Example 9.1

Find the anti-derivative in each of the following:

(a) $\frac{d}{dx}(x^4) = 4x^3$

(b) $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$

(c) $\frac{d}{dx}(\cos x) = -\sin x$

(d) $\frac{d}{dx}(\sin x) = \cos x$

(e) $\frac{d}{dx}(e^x) = e^x$

(f) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Solution

(a) If $\frac{d}{dx}(x^4) = 4x^3$, it means that,

$d(x^4) = 4(x^3)dx$

$\Rightarrow 4(x^3)dx = d(x^4)$

Introduce the integral sign on both sides of the equation to obtain,

$\int 4x^3 dx = \int d(x^4)$

$= x^4 + c$

Therefore, $\int 4x^3 dx = x^4 + c$.

(b) If $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$, it means that,

$d(x^{n+1}) = (n+1)x^n dx$

$\Rightarrow x^n dx = \frac{d(x^{n+1})}{n+1}$

Introduce the integral sign on both sides of the equation to obtain,

$\int x^n dx = \int \frac{d(x^{n+1})}{n+1}$

$= \frac{x^{n+1}}{n+1} + c$

Therefore, $\int x^n dx = \frac{x^{n+1}}{n+1} + c$.

(c) If $\frac{d}{dx}(\cos x) = -\sin x$, it means that,

$d(\cos x) = -\sin x\, dx$

$\Rightarrow \sin x\, dx = -d(\cos x)$

Introduce the integral sign on both sides of the equation to obtain,

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$$\int \sin x\,dx = -\int d(\cos x)$$
$$= -\cos x + c$$

Therefore, $\int \sin x\,dx = -\cos x + c$.

(d) If $\frac{d}{dx}(\sin x) = \cos x$, it means that,

$$d(\sin x) = \cos x\,dx$$
$$\Rightarrow \cos x\,dx = d(\sin x)$$

Introduce the integral sign on both sides of the equation to obtain,

$$\int \cos x\,dx = \int d(\sin x)$$
$$= \sin x + c$$

Therefore, $\int \cos x\,dx = \sin x + c$.

(e) If $\frac{d}{dx}(e^x) = e^x$, it means that,

$$d(e^x) = e^x\,dx$$
$$\Rightarrow e^x\,dx = d(e^x)$$

Introduce the integral sign on both sides of the equation to obtain,

$$\int e^x\,dx = \int d(e^x)$$
$$= e^x + c$$

Therefore, $\int e^x\,dx = e^x + c$.

(f) If $\frac{d}{dx}(\ln x) = \frac{1}{x}$, it means that,

$$d(\ln x) = \frac{1}{x}dx$$
$$\Rightarrow \frac{1}{x}dx = d(\ln x)$$

Introduce the integral sign on both sides of the equation to obtain,

$$\int \frac{1}{x}dx = \int d(\ln x)$$
$$= \ln x + c$$

Therefore, $\int \frac{1}{x}dx = \ln x + c$.

There are several standard integrals that are known from the rules of differentiation. Table 9.1 shows some useful standard integrals. **Table 9.1:** *Some useful standard integrals.*

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$; where $n \neq -1$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + c$
$\int \frac{1}{x}dx = \ln\lvert x\rvert + c$	$\int \sec^2 x\,dx = \tan x + c$
$\int e^x dx = e^x + c$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\int \sin x\,dx = -\cos x + c$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + c$

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$\int \cos x\,dx = \sin x + c$	$\int \operatorname{cosec} x\,dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert + c$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + c$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{dx}{\sqrt{a^2-x^2}} = -\cos^{-1}\left(\frac{x}{a}\right) + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

Types of integrals

There are two types of integrals, namely; indefinite and definite integrals.

Indefinite integrals

An indefinite integral of a function $f(x)$ is a differentiable function $F(x)$ such that its derivative is the given function $f(x)$. An indefinite integral of a function $f(x)$ is expressed as $\int f(x)\,dx = F(x) + c$, where c is an arbitrary constant of integration.

Example 9.2

Find $\int 2x^2 dx$.

Solution

Given $\int 2x^2 dx$.

$$\Rightarrow \int 2x^2 dx = 2\int x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right) + c$$

Therefore, $\int 2x^2\,dx = \frac{2x^3}{3} + c$.

Example 9.3

Find $\int 12\cos\theta\,d\theta$.

Solution

Given $\int 12\cos\theta\,d\theta$.

$$\Rightarrow \int 12\cos\theta\,d\theta = 12\int \cos\theta\,d\theta$$

$$= 12\sin\theta + c$$

Therefore, $\int 12\cos\theta\,d\theta = 12\sin\theta + c$.

Exercise 9.1

1. Find the anti-derivative in each of the following:

(a) $\frac{d}{dx}\left(x^6\right) = 6x^5$

(b) $\frac{d}{dx}\left(2x^{\frac{1}{2}}\right) = x^{\frac{-1}{2}}$

(c) $\frac{d}{dx}\left(3x^{-7}\right) = -21x^{-8}$

(d) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(e) $\frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{2}{3}x^{\frac{-1}{3}}$

(f) $\frac{d}{dx}\left(\frac{3}{2}x^{-2}\right) = -3x^{-3}$

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2. Determine each of the following:

(a) $\int x^2 dx$ (d) $\int 2\cos\theta d\theta$

(b) $\int 5e^x dx$ (e) $\int \frac{1}{2}\sin\theta\ \theta$

(c) $\int \frac{1}{4x} dx$ (f) $\int 4\tan\theta d\theta$

3. Find the integral in each of the following functions:

(a) $f(x) = \frac{3}{x^2}$ (d) $h(x) = -2e^x$

(b) $g(x) = \frac{1}{\sqrt{x}}$ (e) $f(\theta) = 10\sin\theta$

(c) $f(t) = 3\sqrt[4]{t}$ (f) $f(x) = 2\sqrt[4]{x^3}$

Integration of functions

Functions or expressions can be simple or complex. Thus, integration of functions depends on their nature. Simple expressions can be easily integrated while complex expressions are integrated by applying appropriate techniques.

Integration of simple expressions

Expressions which involve mathematical operations on simple polynomials, fractions, and exponential functions are referred to as simple expressions. Simple expressions can be integrated by treating each term separately using some integration rules. The rules governing mathematical operations such as scalar multiplication, additional, and subtraction of integral functions are also used, that is;

(i) $\int af(x)dx = a\int f(x)dx$

(ii) $\int (af(x) \pm bg(x))dx$
$= a\int f(x)dx \pm b\int g(x)dx$, where a and b are constants.

Note that,

(i) $\int 0dx = c$

(ii) $\int adx = ax + b$, where a and b are constants.

Example 9.4

Integrate $x^3 + 2x - 1$ with respect to x.

Solution

Required to find $\int (x^3 + 2x - 1)dx$.

$$\Rightarrow \int (x^3 + 2x - 1)dx = \int x^3 dx + 2\int x\,dx - 1\int x^0 dx$$

$$= \frac{x^{3+1}}{3+1} + \frac{2x^{1+1}}{1+1} - 1\frac{x^{0+1}}{0+1} + c$$

$$= \frac{x^4}{4} + \frac{2x^2}{2} - x + c$$

Therefore, $\int (x^3 + 2x - 1)dx = \frac{x^4}{4} + x^2 - x + c$.

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Example 9.5

Integrate each of the following with respect to x.

(a) $\sqrt{x^3}+\frac{2}{x^2}$ (b) $\frac{1}{3}e^x-2e^{-x}$

Solution

(a) Given $\int\left(\sqrt{x^3}+\frac{2}{x^2}\right)dx.$

$$\Rightarrow\int\left(\sqrt{x^3}+\frac{2}{x^2}\right)dx=\int\sqrt{x^3}\,dx+\int\frac{2}{x^2}dx$$

$$=\int x^{\frac{3}{2}}dx+\int 2x^{-2}dx$$

$$=\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{2x^{-2+1}}{-2+1}+c$$

$$=\frac{2x^{\frac{5}{2}}}{5}-2x^{-1}+c$$

$$=\frac{2x^{\frac{5}{2}}}{5}-\frac{2}{x}+c$$

Therefore, $\int\left(\sqrt{x^3}+\frac{2}{x^2}\right)dx=\frac{2\sqrt{x^5}}{5}-\frac{2}{x}+c.$

(b) Given $\int\left(\frac{1}{3}e^x-2e^{-x}\right)dx.$

$$\Rightarrow\int\left(\frac{1}{3}e^x-2e^{-x}\right)dx=\begin{cases}=\int\frac{1}{3}e^x dx-\int 2e^{-x}dx\\ =\frac{1}{3}e^x-2e^{-x}+c\end{cases}$$

Therefore $\int\left(\frac{1}{3}e^x-2e^{-x}\right)dx=\frac{1}{3}e^x+2e^{-x}+c.$

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Example 9.6

Integrate $3+\frac{5x^3-2x}{3x^4}$ with respect to x.

Solution

Given $\int\left(3+\frac{5x^3-2x}{3x^4}\right)dx.$

$$\Rightarrow\int\left(3+\frac{5x^3-2x}{3x^4}\right)dx=\int 3dx+\int\frac{5x^3-2x}{3x^4}dx$$

$$=3\int x^0dx+\int\left(\frac{5x^3}{3x^4}-\frac{2x}{3x^4}\right)dx$$

$$=3x+\frac{5}{3}\int\frac{1}{x}dx-\frac{2}{3}\int x^{-3}dx$$

$$=3x+\frac{5}{3}\ln|x|+\frac{1}{3}x^{-2}+c;\ \text{since}\frac{d}{dx}(\ln x)=\frac{1}{x}.$$

Therefore, $\int\left(3+\frac{5x^3-2x}{3x^4}\right)dx=3x+\frac{5}{3}\ln|x|+\frac{1}{3}x^{-2}+c.$

Example 9.7

Find $\int\left(\frac{3}{2}\cos\theta+\frac{5}{3}\sin\theta\right)d\theta.$

Solution

Given $\int\left(\frac{3}{2}\cos\theta+\frac{5}{3}\sin\theta\right)d\theta.$

$$\Rightarrow\int\left(\frac{3}{2}\cos\theta+\frac{5}{3}\sin\theta\right)d\theta=\int\frac{3}{2}\cos\theta d\theta+\int\frac{5}{3}\sin\theta d\theta$$

$$=\frac{3}{2}\int\cos\theta d\theta+\frac{5}{3}\int\sin\theta d\theta$$

$$=\frac{3}{2}\sin\theta-\frac{5}{3}\cos\theta+c$$

Therefore, $\int\left(\frac{3}{2}\cos\theta+\frac{5}{3}\sin\theta\right)d\theta=\frac{3}{2}\sin\theta-\frac{5}{3}\cos\theta+c.$

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Exercise 9.2

Determine each of the following:

1. $\int(5x^3+8x^2-3x+5)dx$
2. $\int(-6x^3+9x^2+4x-3)dx$
3. $\int\left(x^{\frac{3}{2}}+2x+3\right)dx$
4. $\int\left(12t^{\frac{3}{4}}-9t^{\frac{5}{2}}\right)dt$
5. $\int\left(\frac{1}{3}x^3+5x^2-7x+2\right)dx$
6. $\int\left(t^{-3}-2t^4+\frac{7}{t}+1\right)dt$
7. $\int\frac{(1-x)^3}{\sqrt{x}}dx$
8. $\int\left(\frac{3}{x^2}+\frac{\cos x}{2}\right)dx$
9. $\int\left(\sqrt{x}+\frac{1}{3\sqrt{x}}\right)dx$
10. $\int\left(\frac{8}{x}-\frac{5}{x^2}+\frac{6}{x^3}\right)dx$
11. $\int\frac{t^2+5t^6-4}{t^3}dt$
12. $\int\left(\frac{1}{x}+\frac{\sin x}{5}-\frac{\cos x}{3}\right)dx$
13. $\int(6-2x)dx$
14. $\int(x^2+4x+1)dx$
15. $\int(x^4-2x-3\sqrt{x})dx$
16. $\int(4t^3-3t^2-2t+1)dt$
17. $\int(7e^x+4e^x)dx$
18. $\int\frac{y^3+2}{y^2}dy$
19. $\int(2\cos\theta+2\sin\theta)d\theta$
20. $\int\frac{4\sin x}{3\tan x}dx$
21. $\int\left(4x^3+7x^2-\frac{3}{2}\cos x\right)dx$
22. $\int\frac{x^4-x^3+\sqrt[3]{x}-1}{x^2}dx$
23. $\int\frac{t^2+5t^6-4}{t^3}dt$
24. $\int\left(\frac{1}{r^2}-\frac{1}{r^4}+3r\right)dr$

Techniques of integration

There are several techniques of integration depending on the given function. The most useful techniques of integration include integration by inspection, by substitution, by parts, and by partial fractions.

Integration by substitution method

This is a method which is used in reversing the chain rule. It is sometimes called chain rule backwards. The following are steps to be followed when integrating a given function by substitution method:

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Step 1: Choose a new variable for which the given function is to be reduced.

Step 2: Solve for dx of the given integral, where the function is integrated with respect to x.

Step 3: Rewrite the integral in terms of the new variable.

Step 4: Integrate the resulting function with respect to the new variable.

Step 5: Substitute back the value of the new variable to get the function in terms of x.

Consider the following cases where integration by the substitution method is applicable:

(a) ***Integrals of the form*** $\int (ax \pm b)^n dx$, ***where*** a ***and*** b ***are constants***

Let $u = ax \pm b$, then $du = adx$

$\Rightarrow dx = \frac{du}{a}$.

$$\text{Thus, } \int (ax \pm b)^n dx = \int u^n \times \frac{du}{a}$$

$$= \frac{1}{a}\int u^n du$$

$$= \frac{1}{a}\left(\frac{u^{n+1}}{n+1}\right) + c$$

But $u = ax \pm b$.

Therefore,

$$\int (ax \pm b)^n dx = \frac{1}{a}\left(\frac{(ax \pm b)^{n+1}}{n+1}\right) + c.$$

(b) ***Integrals of the form*** $\int \sqrt[m]{(ax \pm b)^n}\, dx$, ***where*** a ***and*** b ***are constants***

Let $u = ax \pm b$, then $du = adx$.

$\Rightarrow dx = \frac{du}{a}$.

$$\text{Thus, } \int \sqrt[m]{(ax \pm b)^n}\, dx = \int (u)^{\frac{n}{m}} \times \frac{du}{a}$$

$$= \frac{1}{a}\int u^{\frac{n}{m}} du$$

$$= \frac{1}{a}\left(\frac{u^{\frac{n}{m}+1}}{\frac{n}{m}+1}\right) + c$$

But $u = ax \pm b$.

Therefore,

$$\int \sqrt[m]{(ax \pm b)^n}\, dx = \frac{1}{a}\left(\frac{(ax \pm b)^{\frac{n}{m}+1}}{\frac{n}{m}+1}\right) + c.$$

Example 9.8

Find each of the following:

(a) $\int (2x+3)^3 dx$

(b) $\int \sqrt[3]{(1-2x)^7}\, dx$

Solution

(a) Given $\int (2x+3)^3 dx$.

Let $u = 2x+3$, then $du = 2dx$.

$\Rightarrow dx = \frac{du}{2}$.

Thus,

$$\int (2x+3)^3 dx = \int u^3 \times \frac{du}{2}$$

$$= \frac{1}{2}\int u^3 du$$

$$= \frac{1}{2}\left(\frac{u^{3+1}}{3+1}\right) + c = \frac{1}{2}\left(\frac{u^4}{4}\right) + c$$

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But $u = 2x+3$.

Therefore,

$$\int (2x+3)^3 dx = \frac{1}{8}(2x+3)^4 + c.$$

(b) Given $\int \sqrt[3]{(1-2x)^7}\, dx$.

Let $u = 1-2x$, then $du = -2dx$.

$$\Rightarrow dx = -\frac{du}{2}.$$

$$\int \sqrt[3]{(1-2x)^7}\, dx = \int (u)^{\frac{7}{3}} \times -\frac{du}{2}$$

$$= -\frac{1}{2}\int u^{\frac{7}{3}} du$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{10}{3}}}{\frac{10}{3}}\right) = -\frac{3}{20}u^{\frac{10}{3}} + c$$

But $u = 1-2x$.

Therefore,

$$\int \sqrt[3]{(1-2x)^7}\, dx = -\frac{3}{20}(1-2x)^{\frac{10}{3}} + c.$$

(c) ***Integrals of the form*** $\int \frac{dx}{ax \pm b}$, ***where a and b are constants***

Let $u = ax \pm b$, then $du = adx$.

$$\Rightarrow dx = \frac{du}{a}.$$

$$\text{Thus} \int \frac{dx}{ax \pm b} = \int \frac{1}{u} \times \frac{du}{a}$$

$$= \frac{1}{a}\int \frac{du}{u}$$

$$= \frac{1}{a}\ln|u| + c$$

But $u = ax \pm b$.

$$\text{Therefore,} \int \frac{dx}{ax \pm b} = \frac{1}{a}\ln|ax \pm b| + c.$$

Example 9.9

Find each of the following:

(a) $\int \frac{4dx}{2x+3}$ (b) $\int \frac{dx}{5-7x}$

Solution

(a) Given $\int \frac{4dx}{2x+3}$.

Let $u = 2x+3$, then $du = 2dx$.

$$\Rightarrow dx = \frac{du}{2}.$$

$$\text{Thus,} \int \frac{4dx}{2x+3} = 4\int \frac{1}{u} \times \frac{du}{2}$$

$$= \frac{4}{2}\int \frac{du}{u}$$

$$= 2\ln|u| + c.$$

But $u = 2x+3$.

$$\text{Therefore,} \int \frac{4dx}{2x+3} = 2\ln|2x+3| + c.$$

(b) Given $\int \frac{dx}{5-7x}$.

Let $u = 5-7x$, then $du = -7dx$.

$$\Rightarrow dx = -\frac{du}{7}.$$

$$\text{Thus} \int \frac{dx}{5-7x} = \int \frac{1}{u} \times \left(-\frac{du}{7}\right)$$

$$= -\frac{1}{7}\int \frac{du}{u}$$

$$= -\frac{1}{7}\ln|u| + c$$

But $u = 5-7x$.

Therefore,

$$\int \frac{dx}{5-7x} = -\frac{1}{7}\ln|5-7x| + c.$$

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(d) *Integrals of the form $\int \sin(ax \pm b)\,dx$ where a and b are constants*

Let $u = ax \pm b$, then $du = a\,dx$,

$\Rightarrow dx = \frac{du}{a}$.

$$\text{Thus, } \int \sin(ax \pm b)\,dx = \int \sin u \times \frac{du}{a}$$

$$= \frac{1}{a}\int \sin u\,du$$

$$= -\frac{1}{a}\cos u + c$$

But $u = ax \pm b$.

Therefore,

$$\int \sin(ax \pm b)\,dx = -\frac{1}{a}\cos(ax \pm b) + c.$$

Example 9.10

Find $\int 5\sin(8x+2)\,dx$.

Solution

Given $\int 5\sin(8x+2)dx$.

Let $u = 8x + 2$, then $du = 8dx$.

$\Rightarrow dx = \frac{du}{8}$.

$$\Rightarrow \int 5\sin(8x+2)\,dx = \int 5\sin u \times \frac{du}{8}$$

$$= \frac{5}{8}\int \sin u\,du$$

$$= -\frac{5}{8}\cos u + c$$

But $u = 8x + 2$.

Therefore,

$$\int 5\sin(8x+2)\,dx = -\frac{5}{8}\cos(8x+2) + c.$$

(e) *Integrals of the form $\int e^{ax \pm b}dx$, where a and b are constants*

Let $u = ax \pm b$, then $du = a\,dx$.

Thus, $dx = \frac{du}{a}$.

$$\Rightarrow \int e^{ax \pm b}dx = \int e^u \times \frac{du}{a}$$

$$= \frac{1}{a}\int e^u du$$

$$= \frac{1}{a}e^u + c.$$

But $u = ax \pm b$.

Therefore, $\int e^{ax \pm b}dx = \frac{1}{a}e^{ax \pm b} + c$.

Example 9.11

Determine $\int 5e^{5x+2}dx$.

Solution

Given $\int 5e^{5x+2}dx$.

Let $u = 5x + 2$, then $du = 5dx$.

$\Rightarrow dx = \frac{du}{5}$.

$$\Rightarrow \int 5e^{5x+2}dx = \int 5e^u \times \frac{du}{5}$$

$$= \frac{5}{5}\int e^u du$$

$$= e^u + c$$

But $u = 5x + 2$.

Therefore, $\int 5e^{5x+2}dx = e^{5x+2} + c$.

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Exercise 9.3

Find each of the following:

1. $\int (6x-9)^8 dx$
2. $\int (1-2x)^{\frac{3}{2}} dx$
3. $\int (3t-1)^{-3} dt$
4. $\int 3(1-x)^{-1} dx$
5. $\int 4e^{\frac{x}{3}} dx$
6. $\int \frac{1}{4-3t} dt$
7. $\int 6\cos(1-3x) dx$
8. $\int \frac{dx}{5-7x}$
9. $\int \tan(2\theta+1) d\theta$
10. $\int \frac{dx}{2x-1}$
11. $\int \frac{2}{7} e^{3-4x} dx$
12. $\int 5\sin\left(\frac{1}{2}x-1\right) dx$
13. $\int e^{2x+2} dx$
14. $\int \cos(ax \pm b) dx$
15. $\int \tan(ax \pm b) dx$

Integration by the inspection method

This method is applicable when integrating functions such as polynomials, trigonometric, logarithmic, and exponential functions. Consider the following cases where integration by inspection method is applicable.

(a) *Integrals of the form* $\int f(x)g(x)dx$

Let $u = g(x)$ such that

$$\frac{du}{dx} = f(x) \Rightarrow du = f(x)dx$$

Thus, $dx = \frac{du}{f(x)}$.

$$\Rightarrow \int f(x)g(x)dx = \int f(x)u \times \frac{du}{f(x)}$$
$$= \int u\, du$$
$$= \frac{1}{2}u^2 + c$$

But $u = g(x)$.

Therefore, $\int f(x)g(x)dx = \frac{1}{2}[g(x)]^2 + c$.

(b) *Integrals of the form* $\int \frac{f(x)}{g(x)} dx$

Let $u = g(x)$ such that

$$\frac{du}{dx} = f(x) \Rightarrow du = f(x)dx.$$

Thus, $dx = \frac{du}{f(x)}$.

$$\Rightarrow \int \frac{f(x)}{g(x)} dx = \int \frac{f(x)}{u} \times \frac{du}{f(x)}$$
$$= \int \frac{du}{u}$$
$$= \ln|u| + c$$

But $u = g(x)$.

Therefore, $\int \frac{f(x)}{g(x)} dx = \ln|g(x)| + c$.

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Example 9.12

Find each of the following:

(a) $\int(2x-1)(x^2-x+1)\,dx$ (b) $\int \frac{2x}{1+x^2}dx$

Solution

(a) Given $\int(2x-1)(x^2-x+1)\,dx$.

By inspection, it can be seen that the derivative of x^2-x+1 is equal to $2x-1$.

Now, let $u=x^2-x+1$, then $\frac{du}{dx}=2x-1 \Rightarrow du=(2x-1)dx$.

Substitute $x^2-x+1=u$ and $dx=\frac{du}{(2x-1)}$ into the given integral as follows,

$$\int(2x-1)(x^2-x+1)\,dx=\int (2x-1)u\times\frac{du}{(2x-1)}$$

$$=\int u\,du$$

$$=\frac{u^2}{2}+c$$

But $u=x^2-x+1$.

Therefore, $\int(2x-1)(x^2-x+1)\,dx=\frac{1}{2}(x^2-x+1)^2+c$.

(b) Given $\int\frac{2x}{1+x^2}dx$.

Let $u=1+x^2$, then $\frac{du}{dx}=2x \Rightarrow du=2xdx$.

Substitute $1+x^2=u$ and $dx=\frac{du}{2x}$ into the given integral as follows,

$$\int \frac{2x}{1+x^2}dx=\int \frac{2x}{u}\times\frac{du}{2x}$$

$$=\int \frac{du}{u}$$

$$=\ln|u|+c$$

But $u=1+x^2$.

Therefore, $\int\frac{2x}{1+x^2}dx=\ln|1+x^2|+c$.

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Example 9.13

Find $\int xe^{x^2}dx$.

Solution

Given $\int xe^{x^2}dx$.

Let $u = x^2$, then $\frac{du}{dx} = 2x$

$\Rightarrow du = 2xdx$.

Substitute $x^2 = u$ and $dx = \frac{du}{2x}$ into the given integral as follows,

$$\int xe^{x^2}dx = \int xe^u \frac{du}{2x}$$

$$= \frac{1}{2}e^u du$$

$$= \frac{1}{2}e^u + c$$

But $u = x^2$.

Thus, $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$.

Therefore, $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$.

Exercise 9.4

Find each of the following:

1. $\int (x^3 + x^2 - 3)(3x^2 + 2x)dx$
2. $\int e^{-t}(1 + e^{-t})dt$
3. $\int 8t^2(3t^3 - 1)dt$
4. $\int \frac{e^x}{1 + e^x}dx$
5. $\int \frac{1 + \ln x}{x \ln x}dx$
6. $-\int \sin\theta \cos\theta \, d\theta$
7. $\int \frac{1 + \tan x}{1 - \tan x}dx$
8. $\int x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}} + 3\right)dx$
9. $\int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}d\theta$
10. $\int \frac{1 + \ln x}{x}dx$
11. $\int \frac{4 + 18x - 18x^2}{3 + 4x + 9x^2 - 6x^3}dx$
12. $\int \frac{\ln(x+2)}{2x+4}dx$
13. $\int \frac{x^{\frac{1}{2}}}{1 + x^{\frac{3}{2}}}dx$
14. $\int \frac{\sqrt{x}}{1 + \sqrt{x}}dx$
15. $\int x\sqrt{x^2 - 1}dx$
16. $\int (x+2)e^{x^2+4x}dx$
17. $\int xe^{x^2}\cos(3e^{x^2})dx$
18. $\int \frac{1 - e^t}{1 + e^t}dt$
19. $\int \frac{1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right)dx$
20. $\int \frac{1}{x(\ln x)^2}dx$
21. $\int \frac{e^{\tan^{-1}x}}{1 + x^2}dx$
22. $\int \frac{\sec^2(x+1)}{\tan(x+1)}dx$

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23. Show that $\int \frac{2\sin 2x}{1+\cos x}dx = 4\ln(1+\cos x) - 4\cos x - 4 + c$, where c is an arbitrary constant.

24. Show that $\int \frac{e^{3x}}{1+e^x}dx = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) - \frac{3}{2} + c$, where c is an arbitrary constant.

Integration by parts

If $u(x)$ and $v(x)$ are any two differentiable functions, then the derivative of the product of the two functions is given by;

$$\frac{d}{dx}[u(x)v(x)] = v(x)\frac{d}{dx}u(x) + u(x)\frac{d}{dx}v(x).$$

Introduce the integral sign on both sides to obtain,

$$\int \frac{d}{dx}[u(x)v(x)]dx = \int v(x)\frac{d}{dx}u(x)dx + \int u(x)\frac{d}{dx}v(x)dx.$$

Applying the definition of indefinite integral gives,

$$u(x)v(x) = \int v(x)\frac{d}{dx}u(x)dx + \int u(x)\frac{d}{dx}v(x)dx.$$

Thus, $\int u(x)\frac{d}{dx}v(x)dx = u(x)v(x) - \int v(x)\frac{d}{dx}u(x)dx$.

Since du and dv are differentials of a function of one variable, then the formula for the integration by parts is given by,

$$\int u(x)\frac{d}{dx}v(x)dx = u(x)v(x) - \int v(x)\frac{d}{dx}u(x)dx.$$

If $\frac{d}{dx}v(x) = v'(x)$ and $\frac{d}{dx}u(x) = u'(x)$, then the formula for integration by parts can be written as

$$\int uv'dx = uv - \int vu'dx \text{ or simply } \int u\,dv = uv - \int v\,du.$$

Generally, if $u(x)$ and $v(x)$ are any two differentiable functions of x, then $\int u\,dv = uv - \int v\,du$, which is the formula for integration by parts.

In this case, dv must be a function that can be integrated and u must be a function that can be differentiated.

Note that, the integrand is in the form of a product of two functions, where the left hand part of the integrand is considered as a first function and its right hand part is considered as a second function. The first function is chosen in such a way that

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its derivative can easily be integrated. Generally, the order of preference can be chosen in such a way that the first function comes first in the word ILATE, where I stands for inverse trigonometric functions, L stands for logarithmic functions, A stands for algebraic functions, T stands for trigonometric functions, and E stands for exponential functions.

Consider the following cases where the method of integration by parts is applicable.

(a) *Integrals of the form*

$\int x^n \sin(ax)dx$ ***or*** $\int x^n \cos(ax)dx$,

where a is a constant

In this case, let the polynomial part be the function u and the trigonometric part be dv.

Example 9.14

Find $\int x \sin x\, dx$.

Solution

Given $\int x \sin x\, dx$.

Let $u = x$, then $\frac{du}{dx} = 1$.

Also, let $\frac{dv}{dx} = \sin x \Rightarrow dv = \sin x\, dx$.

Introduce the integral sign on both sides and integrating gives,

$\int dv = \int \sin x\, dx \Rightarrow v = -\cos x$

Apply the formula for integrating by parts:

$\int u\, dv = uv - \int v\, du$.

$$\Rightarrow \int x \sin x\, dx = -x\cos x - \int (-\cos x) \times 1 dx$$
$$= -x\cos x + \int \cos x\, dx$$
$$= -x\cos x + \sin x + c$$

Therefore,

$\int x \sin x\, dx = -x\cos x + \sin x + c$.

Example 9.15

Find $\int x^2 \cos(4x-1)dx$.

Solution

Given $\int x^2 \cos(4x-1)dx$.

The integral can be obtained by integration by parts as follows:

Let $u = x^2$, then $du = 2xdx$.

Also, let $\frac{dv}{dx} = \cos(4x-1)$

$\Rightarrow dv = \cos(4x-1)dx$

Introduce the integral sign on both sides to obtain,

$\int dv = \int \cos(4x-1)dx$

$\Rightarrow v = \frac{1}{4}\sin(4x-1)$

From the formula $\int udv = uv - \int vdu$, it implies that

$\int x^2 \cos(4x-1)dx$

$= \frac{x^2}{4}\sin(4x-1) - \int \frac{2x}{4}\sin(4x-1)dx$

$= \frac{x^2}{4}\sin(4x-1) - \frac{1}{2}\int x \sin(4x-1)dx$

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But $\int x\sin(4x-1)dx$ requires further integration and it is integrated by parts as follows,

Let $u = x$, then $\frac{du}{dx} = 1$.

Also, let $\frac{dv}{dx} = \sin(4x-1) \Rightarrow dv = \sin(4x-1)dx$

Introducing the integral sign on both sides gives,

$$\int dv = \int \sin(4x-1)dx$$

$$\Rightarrow v = -\frac{1}{4}\cos(4x-1).$$

Apply the formula for integrating by parts, that is,

$$\int x\sin(4x-1)dx = -\frac{x}{4}\cos(4x-1) - \int -\frac{1}{4}\cos(4x-1)dx$$

$$= -\frac{x}{4}\cos(4x-1) + \frac{1}{4}\int \cos(4x-1)dx$$

$$= -\frac{x}{4}\cos(4x-1) + \frac{1}{4}\left[\frac{1}{4}\sin(4x-1)\right] + c$$

Substitute this result in the integral to get,

$$\int x^2\cos(4x-1)dx = \frac{x^2}{4}\sin(4x-1) - \frac{1}{2}\left[-\frac{x}{4}\cos(4x-1) + \frac{1}{16}\sin(4x-1)\right] + c.$$

Therefore, $\int x^2\cos(4x-1)dx = \frac{x^2}{4}\sin(4x-1) + \frac{x}{8}\cos(4x-1) - \frac{1}{32}\sin(4x-1) + c.$

(b) *Integrals of the form $\int x^n \ln ax\,dx$, where a is a constant*

In this case, let the logarithmic part be the function u, and differentiate it. The polynomial part in the integral is dv which will be integrated.

Example 9.16

Determine each of the following:

(a) $\int x^n \ln x dx$ (b) $\int x^2 \ln 2x\,dx$

Solution

(a) Given $\int x^n \ln x dx$.

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Let $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$.

Also, let $\frac{dv}{dx} = x^n \Rightarrow dv = x^n dx$.

Integrate both sides:

$$\int dv = \int x^n \, dx \Rightarrow v = \frac{x^{n+1}}{n+1}$$

From $\int u\,dv = uv - \int v\,du$, it implies that

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1} \times \frac{1}{x} dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int \frac{x^{n+1}}{x} dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1} \times \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

Therefore,

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c.$$

(b) Given $\int x^2 \ln 2x\,dx$.

Let $u = \ln 2x$, then

$$\frac{du}{dx} = \frac{2}{2x} \Rightarrow du = \frac{1}{x}dx$$

Also, let $\frac{dv}{dx} = x^2 \Rightarrow dv = x^2 dx$.

Integrate both sides:

$$\int dv = \int x^2 dx \Rightarrow v = \frac{x^3}{3}.$$

From $\int u\,dv = uv - \int v\,du$, it follows that

$$\int x^2 \ln 2x\,dx = \frac{x^3}{3}\ln 2x - \int \frac{x^3}{3} \times \frac{1}{x} dx$$

$$= \frac{x^3}{3}\ln 2x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\ln 2x - \frac{x^3}{9} + c$$

Therefore,

$$\int x^2 \ln 2x dx = \frac{x^3}{3}\ln 2x - \frac{x^3}{9} + c.$$

Example 9.17

Find $\int \ln x dx$.

Solution

Given $\int \ln x\,dx = \int 1 \times \ln x\,dx$.

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Leftrightarrow du = \frac{1}{x}dx$.

Also, let $\frac{dv}{dx} = 1 \Rightarrow dv = dx \Leftrightarrow v = x$.

Apply the formula $\int u\,dv = uv - \int v\,du$

Now, the integral becomes,

$$\int \ln x\,dx = x\ln x - \int x \times \frac{1}{x} dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

Therefore, $\int \ln x\,dx = x\ln x - x + c$.

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(c) *Integrals of the form* $\int x^n \sin(ax)\,dx$ *or* $\int x^n \cos(ax)\,dx$ *or* $\int x^n e^{ax}\,dx$, *where* a *is a constant*

A tabular integration can be used to integrate problems involving repeated application of integration by parts. Tabular integration are integrals in which u can be differentiated repeatedly to become zero and dv can be integrated repeatedly. However, if many repetitions are required, the calculations can be tedious. In situations like this, organize the calculations using tabular integration as illustrated in Examples 9.18 – 9.20.

Example 9.18

Find $\int x^3 \sin(10x-19)\,dx$.

Solution

Given $\int x^3 \sin(10x-19)\,dx$.

Let $u = x^3$ and $dv = \sin(10x-19)\,dx$.

Prepare a table consisting of three columns and six rows as follows.

Alternate signs	u and its derivatives	v' and its anti-derivatives
+	x^3	$\sin(10x-19)$
−	$3x^2$	$-\frac{1}{10}\cos(10x-19)$
+	$6x$	$-\frac{1}{100}\sin(10x-19)$
−	6	$\frac{1}{1000}\cos(10x-19)$
+	0	$\frac{1}{10000}\sin(10x-19)$

Differentiate u repeatedly until the derivative of the function is zero.

The solution is given by adding the signed products of the functions connected by the arrows.

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That is, $\int x^3 \sin(10x-19)\,dx = -\frac{x^3}{10}\cos(10x-19)+\frac{3x^2}{100}\sin(10x-19)+$

$$\frac{6x}{1000}\cos(10x-19)-\frac{6}{10000}\sin(10x-19)+c$$

Therefore,

$$\int x^3 \sin(10x-19)dx = -\frac{x^3}{10}\cos(10x-19)+\frac{3x^2}{100}\sin(10x-19)+\frac{3x}{500}\cos(10x-19)-$$

$$\frac{3}{5000}\sin(10x-19)+c.$$

Example 9.19

Show that $\int x^4 e^{12x}\,dx = \frac{e^{12x}}{10368}\left(864x^4-288x^3+72x^2-12x+1\right)+c$, where c is an arbitrary constant.

Solution

Given $\int x^4 e^{12x}dx$. Let $u = x^4$ and $dv = e^{12x}dx$.

Prepare a table consisting of three columns and seven rows as follows.

Alternate signs	u and its derivatives	v' and its anti-derivatives
+	x^4	e^{12x}
−	$4x^3$	$\frac{1}{12}e^{12x}$
+	$12x^2$	$\frac{1}{144}e^{12x}$
−	$24x$	$\frac{1}{1728}e^{12x}$
+	24	$\frac{1}{20736}e^{12x}$
−	0	$\frac{1}{248832}e^{12x}$

The solution is given by adding the signed products of the functions connected by the arrows. That is,

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$$\int x^4 e^{12x}dx = \frac{x^4}{12}e^{12x} - \frac{4x^3}{144}e^{12x} + \frac{12x^2}{1728}e^{12x} - \frac{24x}{20736}e^{12x} + \frac{24}{248832}e^{12x} + c$$

$$= \frac{x^4}{12}e^{12x} - \frac{x^3}{36}e^{12x} + \frac{x^2}{144}e^{12x} - \frac{x}{864}e^{12x} + \frac{24}{10368}e^{12x} + c$$

$$= \frac{864x^4e^{12x} - 288x^3e^{12x} + 72x^2e^{12x} - 12xe^{12x} + e^{12x}}{10368} + c$$

Therefore, $\int x^4 e^{12x}dx = \frac{e^{12x}}{10368}\left(864x^4 - 288x^3 + 72x^2 - 12x + 1\right) + c.$

Examples 9.20

Verify that $\int (t+2)^3 \cos t\,dt = (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c.$

Solution

Let $u = (t+2)^3$ and $dv = \cos t\,dt$.

Prepare a table consisting of three columns and six rows as follows.

Alternate signs	u and its derivatives	v' and its anti-derivatives
+	$(t+2)^3$	$\cos t$
−	$3(t+2)^2$	$\sin t$
+	$6(t+2)^1$	$-\cos t$
−	6	$-\sin t$
+	0	$\cos t$

The solution is given by adding the signed products of the functions connected by the arrows. That is,

$$\int (t+2)^3 \cos t\,dt = (t+2)^3 \sin t + 3(t+2)^2 \cos t - 6(t+2)\sin t - 6\cos t + c$$

$$= (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c$$

Therefore, $\int (t+2)^3 \cos t\,dt = (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c.$

(d) *Integrals of the form* $\int e^{ax}\sin(bx)dx$ *or* $\int e^{ax}\cos(bx)dx$, *where a and b are constants*

In this case, each part can be either u or dv. This is due to the fact that, neither e^{ax} nor $\sin bx$ decreases in power during differentiation. Therefore, to get the integral of this type, perform the integration two times and then combine the results.

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Example 9.21

Find $\int e^x \cos x\,dx$.

Solution

Given $\int e^x \cos x dx$.

In this case, let $u = e^x$ or $u = \cos x$; either way will work.

Let $u = e^x$.

Thus, $\frac{du}{dx} = e^x \Rightarrow du = e^x dx$.

Also, let $dv = \cos x dx \Rightarrow v = \sin x$.

Apply the formula $\int u dv = uv - \int v du$. It follows that

$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$ ..... (i)

The $\int e^x \sin x\,dx$ requires further integration by parts.

Use the same type of substitutions for the next integration by parts.

Let $u = e^x$, then $\frac{du}{dx} = e^x \Rightarrow du = e^x dx$.

Also, let $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$.

Thus, $\int e^x \sin x\,dx = -e^x \cos x - \left[\int -e^x \cos x\,dx\right]$

$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$ ..... (ii)

Substitute equation (ii) into equation (i) to obtain,

$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\,dx\right)$

$\Rightarrow \int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$

Collect like terms of the integral to get;

$\int e^x \cos x\,dx + \int e^x \cos x\,dx = e^x \sin x + e^x \cos x$

$\Rightarrow 2\int e^x \cos x\,dx = e^x (\sin x + \cos x)$

Therefore, $\int e^x \cos x\,dx = \frac{1}{2} e^x (\sin x + \cos x) + c$.

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Remark: Integration by parts can be applied to obtain the following integrals:

1. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$.
2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c$.

(e) ***Integrals of the form*** $\int x^n \sin^{-1} x\,dx$, $\int x^n \cos^{-1} x\,dx$ ***or*** $\int x^n \tan^{-1} x\,dx$

In this case, let the inverse trigonometric function be u, then differentiate it. Let the polynomial part of the integral be dv, then integrate it.

Example 9.22

Find each of the following:

(a) $\int \sin^{-1} x\,dx$ (b) $\int \cos^{-1} x\,dx$

Solution

(a) Given $\int \sin^{-1} x dx = \int x^0 \sin^{-1} x\,dx$.

From the formula $\int u dv = uv - \int v du$.

Let $u = \sin^{-1} x$, then $du = \frac{1}{\sqrt{1-x^2}}dx$

Also, let $dv = 1dx \Rightarrow v = x$

Substituting into the formula gives,

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int x \times \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

But $\int \frac{x}{\sqrt{1-x^2}}dx$ requires further integration.

So, let $t = \sqrt{1-x^2} \Rightarrow t^2 = 1-x^2$

$\Rightarrow 2t\,dt = -2x\,dx$

$\Rightarrow t\,dt = -x\,dx$

Thus, $\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{t}{t}dt$

$= -\int dt$

$= -t$

Substituting back $t = \sqrt{1-x^2}$ gives,

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + c$$

Therefore,

$$\int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

(b) Given $\int \cos^{-1} x\,dx$.

Let $u = \cos^{-1} x$, then $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

Also, let $v' = x^0 \Rightarrow dv = x^0 dx$.

Integrating both sides gives $v = x$.

$$\int \cos^{-1} x dx = x\cos^{-1} x - \int x \times \frac{-1}{\sqrt{1-x^2}}dx$$

$$= x\cos^{-1} + \int \frac{x}{\sqrt{1-x^2}}dx$$

But $\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}$.

Therefore,

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + c.$$

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Reduction formula

A reduction formula is an extension of integration by parts method. It helps to solve complex integration problems by relying on recurrence relations and it is used when an expression cannot be integrated directly. It can also be used for powers of elementary functions, trigonometrical functions, logarithmic functions, and exponential functions, as well as products of two or more complex functions. The formula enables one to reduce the degree of the integrand and calculate the integrals in a finite number of steps.

Consider the problem of finding $I_n = \int x^n e^{-x} dx$ for $n \geq 0$. The process of integrating by parts reduces the power of x by 1, each time. This process will give a formula for I_n. The integral is integrated as follows.

Let $u = x^n$ and $dv = e^{-x}dx$.

$\Rightarrow du = nx^{n-1}dx$ and $v = \int e^{-x}dx = -e^{-x}$.

Applying the integration by parts formula $\int udv = uv - \int vdu$ gives,

$I_n = -x^n e^{-x} + n\int x^{n-1}e^{-x}dx$

But $\int x^{n-1}e^{-x}dx = I_{n-1}$

Thus, $I_n = -x^n e^{-x} + nI_{n-1}$.

Therefore, $I_n = -x^n e^{-x} + nI_{n-1}$ is called a reduction formula for $I_n = \int x^n e^{-x}dx$ for $n \geq 1$.

Example 9.23

Use the reduction formula for $I_n = \int x^n e^{-x}dx$ to evaluate $\int x^4 e^{-x}dx$.

Solution

Given $\int x^4 e^{-x}dx$. In this case, the reduction formula for $I_n = \int x^n e^{-x}dx$ can be used. That is,

$I_n = -x^n e^{-x} + I_{n-1}$, $n \geq 1$

Thus, $I_0 = -x^0 e^{-x} + 0 = -e^{-x}$.

$\Rightarrow \quad I_1 = -x^1 e^{-x} + I_0 = -e^{-x}(x+1)$.

$I_2 = -x^2 e^{-x} + 2I_1 = -e^{-x}(x^2 + 2x + 2)$

$I_3 = -x^3 e^{-x} + 3I_2 = -e^{-x}(x^3 + 3x^2 + 6x + 6)$

$I_4 = -x^4 e^{-x} + 4I_3 = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + c$

Therefore, $\int x^4 e^{-x}dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + c$

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Example 9.24

If $I = \int \sec\ x\,dx$, show that;

(a) $I_n = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1} I_{n-2}$.

(b) $\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15}\tan x + D$.

Solution

(a) Given $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x\sec^2 x\,dx$.

Let $u = \sec^{n-2} x$ and $dv = \sec^2 x\,dx$.

$\Rightarrow du = (n-2)\sec^{n-2} x\tan x\,dx$ and $v = \tan x$.

Applying the integration by parts formula, $\int udv = uv - \int v\,du$ gives,

$$I_n = \sec^{n-2} x\tan x - \int \tan x(n-2)\sec^{n-2} x\tan x\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} x\tan^2 x\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

But $\int \sec^n x\,dx = I_n$ and $\int \sec^{n-2} x\,dx = I_{n-2}$.

Thus, $I_n = \sec^{n-2} x\tan x - (n-2)I_n + (n-2)I_{n-2}$.

Collecting like terms gives,

$$I_n + (n-2)I_n = \sec^{n-2} x\tan x + (n-2)I_{n-2}$$

$$\Rightarrow (n-1)I_n = \sec^{n-2} x\tan x + (n-2)I_{n-2}$$

Therefore, $I_n = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1} I_{n-2}$.

(b) Use the reduction formula in part (a). That is,

$$I_6 = \frac{1}{6-1}\sec^{6-2} x\tan x + \frac{6-2}{6-1} I_{6-2}$$

$$= \frac{1}{5}\sec^4 x\tan x + \frac{4}{5} I_4$$

$$= \frac{1}{5}\sec^4 x\tan x + \frac{4}{5}\left[\frac{1}{3}\sec^2 x\tan x + \frac{2}{3} I_2\right]$$

$$= \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15} I_2$$. But $I_2 = \int \sec^2 x\,dx = \tan x + c$.

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$$\Rightarrow I_6 = \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15}[\tan x + c].$$

$$\Rightarrow I_6 = \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15}\tan x + D, \text{ where } D = \frac{8}{15}c.$$

Therefore, $\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15}\tan x + D.$

Example 9.25

Derive a reduction formula for I_n in terms of I_{n-2}, where $I_n = \int \cos^n x\,dx$. Hence, show that $\int \cos^5 x\,dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + A$.

Solution

Given $I_n = \int \cos^n x\,dx = \int \cos^{n-1} x\cos x\,dx.$

Let $u = \cos^{n-1} x$ and $dv = \cos x\,dx$

$\Rightarrow du = (n-1)\cos^{n-2} x(-\sin x)dx$ and $v = \sin x$

Applying integration by parts formula, $\int u\,dv = uv - \int v\,du$ gives;

$$I_n = \cos^{n-1} x\sin x + \int (n-1)\cos^{n-2} x\sin x\sin x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

Collecting like terms gives,

$$I_n + (n-1)I_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

Thus, $I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}.$

$$\Rightarrow I_5 = \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}I_3$$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}I_1\right)$$

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$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15} I_1$$

But $I_1 = \int \cos x \, dx = \sin x + c.$

Thus, $I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + A$, where $A = \frac{8}{15}c$.

Therefore, $I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$. It follows that

$$\int \cos^5 x \, dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + A.$$

Exercise 9.5

Evaluate each of the following:

1. $\int x^2 e^{2x} dx$
2. $\int \tan^{-1} 3x \, dx$
3. $\int \cos^{-1}\left(\frac{1}{2}x\right) dx$
4. $\int x^3 \sin x \, dx$
5. $\int x \sin x \cos x dx$
6. $\int x^2 \cos x \, dx$
7. $\int x^3 \ln x \, dx$
8. $\int (x-1)\ln(2x) dx$
9. $\int \ln 3x \, dx$
10. $\int e^{2x} \cos x \, dx$
11. $\int e^{-x} \sin 2x \, dx$
12. $\int e^{\frac{1}{2}x} \cos x dx$
13. $\int e^{4x} \sin 2x \, dx$
14. $\int e^{2x} \cos 3x \, dx$
15. $\int \theta \sec^2 \theta \, d\theta$
16. $\int x^2 e^{-3x} dx$
17. $\int \sin^{-1} 2x \, dx$
18. $\int x^2 \ln x \, dx$
19. $\int \frac{\ln x}{\sqrt{x}} dx$
20. $\int \frac{\ln x}{x^7} dx$
21. $\int (x+1)^2 \ln(3x) dx$
22. Obtain a reduction formula for $I_n = \int (\ln x)^n dx$, hence use it to evaluate I_4.
23. Obtain a reduction formula for $\int x^n e^x dx$ and use it to find $\int x^4 e^x dx$.
24. Show that

$$\int x \tan^{-1} x \, dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c.$$

25. Derive a reduction formula for I_n in terms of I_{n-1}, if $I_n = \int x^3 (\ln x)^n dx$.
Hence, verify that

$$I_3 = \frac{x^4}{128}\left[32(\ln x)^3 - 24(\ln x)^2 + 12\ln x - 3\right] + c$$

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26. If $I_n = \int \tan^n \theta d\theta$, show that

$$I_n = \frac{1}{n-1}\tan^{n-1}\theta - I_{n-2}.$$

27. Prove that, if

$I_n = \int x^n (1+x^3)^7 dx$, then

$$I_n = \frac{1}{n+22}\left[x^{n-2}(1+x^3)^8 - (n-2)I_{n-3}\right].$$

Hence, determine $\int x^5 (1+x^3)^7 dx$.

28. Show that

$$\int x^n \ln x\, dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + c.$$

Integration using partial fractions

When an integrand is a proper rational function, it is resolved into partial fractions before integrating. The partial fractions are integrated separately. When an integrand is an improper rational function, divide the numerator by the denominator and express the fraction as a sum of quotient and proper fraction. The proper fraction is resolved into partial fractions. Finally, the partial fractions with quotient are integrated. Table 9.2 shows some forms of rational functions, their partial fractions, and corresponding integrals.

Table 9.2: Forms of rational functions, partial fractions, and their integrals.

Rational function	Partial fraction	Integral
$\frac{px+q}{(x-a)(x-b)}$, where $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$	$\int\left(\frac{A}{x-a}+\frac{B}{x-b}\right)dx$
$\frac{1}{x^2-a^2}$	$\frac{A}{x-a}+\frac{B}{x+a}$	$\int\left(\frac{A}{x-a}+\frac{B}{x+a}\right)dx$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$	$\int\left(\frac{A}{x-a}+\frac{B}{(x-a)^2}\right)dx$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ where $a \neq b \neq c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$	$\int\left(\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\right)dx$
$\frac{px^2+qx+r}{(x-a)^2(x-c)}$; $a \neq c$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-c}$	$\int\left(\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-c}\right)dx$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$	$\int\left(\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}\right)dx$

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Example 9.26

Find $\int \frac{2x}{x^2-4x+3}dx$.

Solution

Given

$$\int \frac{2x}{x^2-4x+3}dx = \int \frac{2x}{(x-1)(x-3)}dx.$$

Decompose the integrand into partial fractions as follows:

$$\frac{2x}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

Solving for A and B gives, $A=-1$ and $B=3$.

Thus, $\frac{2x}{(x-1)(x-3)} = \frac{-1}{x-1} + \frac{3}{x-3}$

Now the integral becomes,

$$\int \frac{2x}{x^2-4x+3}dx = \int \left(\frac{-1}{x-1} + \frac{3}{x-3}\right)dx$$

$$= \int \frac{-1}{(x-1)}dx + \int \frac{3}{(x-3)}dx$$

$$= -\ln|x-1| + 3\ln|x-3| + c$$

$$= 3\ln|x-3| - \ln|x-1| + c$$

$$= \ln|(x-3)^3| - \ln|x-1| + c$$

$$= \ln\left|\frac{(x-3)^3}{(x-1)}\right| + c$$

Therefore,

$$\int \frac{2x}{x^2-4x+3}dx = \ln\left|\frac{(x-3)^3}{(x-1)}\right| + c.$$

Example 9.27

Find $\int \frac{5x-2}{(x+3)^2}dx$.

Solution

Given $\int \frac{5x-2}{(x+3)^2}dx$.

Write the integrand in partial fractions. That is,

$$\frac{5x-2}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$$

Solving for A and B gives, $A=5$ and $B=-17$.

Thus, $\frac{5x-2}{(x+3)^2} = \frac{5}{(x+3)} - \frac{17}{(x+3)^2}$.

Now, the integral becomes;

$$\int \frac{5x-2}{(x+3)^2}dx = \int \left(\frac{5}{(x+3)} - \frac{17}{(x+3)^2}\right)dx$$

$$= \int \frac{5}{(x+3)}dx - \int \frac{17}{(x+3)^2}dx$$

$$= 5\int \frac{1}{(x-3)}dx - 17\int \frac{1}{(x-3)^2}dx$$

$$= 5\ln|x+3| + \frac{17}{x+3} + c$$

Therefore,

$$\int \frac{5x-2}{(x+3)^2}dx = 5\ln|x+3| + \frac{17}{x+3} + c.$$

Example 9.28

Find $\int \frac{-2x+4}{(x^2+1)(x-1)}dx$.

Solution

Given $\int \frac{-2x+4}{(x^2+1)(x-1)}dx$.

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Write the integrand into partial fractions.
That is,

$$\frac{-2x+4}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$$

Solving for A, B, and C gives, $A=-1$, $B=-3$, and $C=1$.

Thus, $\frac{-2x+4}{(x^2+1)(x-1)}=\frac{-x-3}{(x^2+1)}+\frac{1}{(x-1)}$.

Now, the integral becomes;

$$\int\frac{-2x+4}{(x^2+1)(x-1)}dx=\int\left(\frac{-x-3}{(x^2+1)}+\frac{1}{(x-1)}\right)dx$$

$$=\int\frac{-x-3}{(x^2+1)}dx+\int\frac{1}{(x-1)}dx$$

$$=-\int\frac{x}{x^2+1}dx-3\int\frac{dx}{x^2+1}+\int\frac{dx}{x-1}\text{ but }\int\frac{dx}{x^2+1}=\tan^{-1}x$$

$$=-\frac{1}{2}\ln\left(x^2+1\right)-3\tan^{-1}x+\ln|x-1|+c$$

Therefore, $\int\frac{-2x+4}{(x^2+1)(x-1)}dx=-\frac{1}{2}\ln\left(x^2+1\right)-3\tan^{-1}x+\ln|x-1|+c$.

Example 9.29

Integrate $\frac{x^2+x-3}{(x+1)(x-2)(x-5)}$ with respect to x.

Solution

Given $\int\frac{x^2+x-3}{(x+1)(x-2)(x-5)}dx$.

Decompose the integrand into partial fractions. That is,

$$\frac{x^2+x-3}{(x+1)(x-2)(x-5)}=\frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-5)}$$

Solving for A, B, and C gives, $A=-\frac{1}{6}$, $B=-\frac{1}{3}$, and $C=\frac{3}{2}$.

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$$\Rightarrow \frac{x^2+x-3}{(x+1)(x-2)(x-5)} = \frac{-\frac{1}{6}}{(x+1)} + \frac{-\frac{1}{3}}{(x-2)} + \frac{\frac{3}{2}}{(x-5)}$$

Now, the integral becomes,

$$\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}dx = \int \left(-\frac{1}{6(x+1)} - \frac{1}{3(x-2)} + \frac{3}{2(x-5)}\right)dx$$

$$= -\frac{1}{6}\int \frac{dx}{(x+1)} - \frac{1}{3}\int \frac{dx}{(x-2)} + \frac{3}{2}\int \frac{dx}{(x-5)}$$

$$= -\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{3}{2}\ln|x-5| + c$$

Therefore, $\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}dx = -\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{3}{2}\ln|x-5| + c.$

Example 9.30

Find $\int \frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)}dx.$

Solution

Given $\int \frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)}dx.$

Divide the numerator by the denominator and express the fraction as a sum of quotient and proper fraction to obtain,

$$\frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)} = 1 + \frac{2x^2-2x-1}{(2x+1)(x^2+2)}$$

Decompose the proper fraction into partial fractions. That is,

$$\frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+2}$$

Solving for A, B, and C gives, $A = \frac{2}{9}$, $B = \frac{8}{9}$, and $C = -\frac{13}{9}$

Thus, $\frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)} = 1 + \frac{2}{9(2x+1)} + \frac{8x-13}{9(x^2+2)}.$

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Now, the integral becomes,

$$\int \frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)}dx = \int \left(1+\frac{2}{9(2x+1)}+\frac{8x-13}{9(x^2+2)}\right)dx$$

$$= \int dx + \int \frac{2}{9(2x+1)}dx + \int \frac{8x-13}{9(x^2+2)}dx$$

$$= x+\frac{1}{9}\ln|2x+1|+\frac{4}{9}\ln|x^2+2|-\frac{13\sqrt{2}}{18}\tan^{-1}\frac{x\sqrt{2}}{18}+c$$

Therefore,

$$\int \frac{2x^3+3x^2+2x+1}{(2x+1)(x^2+2)}dx = x+\frac{1}{9}\ln|2x+1|+\frac{4}{9}\ln|x^2+2|-\frac{13\sqrt{2}}{18}\tan^{-1}\frac{x\sqrt{2}}{18}+c.$$

Exercise 9.6

Determine each of the following:

1. $\int \frac{x-9}{(x+5)(x-2)}dx$
2. $\int \frac{1}{(x+5)^2(x-1)}dx$
3. $\int \frac{10}{(x^2+9)(x-1)}dx$
4. $\int \frac{1}{s^2(s-1)^2}ds$
5. $\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx$
6. $\int \frac{5x^2+3x-2}{x^3+2x^2}dx$
7. $\int \frac{t^2+2t+18}{t(t^2+3)^2}dt$
8. $\int \frac{4y^2-7y-12}{y(y+2)(y-3)}dy$
9. $\int \frac{x-1}{x^2+3x+2}dx$
10. $\int \frac{x^2+2x-1}{x^3-x}dx$
11. $\int \frac{dt}{(t+4)(t-1)}$
12. $\int \frac{x+3}{(x-1)^3}dx$
13. $\int \frac{x-1}{(x+3)^6}dx$
14. $\int \frac{x^4}{x^4-1}dx$
15. $\int \frac{5x-17}{x^2-6x+9}dx$
16. $\int \frac{2-t}{t^2+5t}dt$
17. $\int \frac{x^2-1}{x^2-16}dx$
18. $\int \frac{x^4+x^3+x^2+1}{x^2+x-2}dx$

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19. $\int \frac{r^2+r-1}{r(r^2-1)}dr$ 20. $\int \frac{x+14}{(x+5)(x+2)}dx$ 21. $\int \frac{x^5+1}{x^3(x+2)}dx$

Integration of trigonometric functions

Integration of trigonometric functions depends on the type of the integrand.

(a) ***Integrals of the form*** $\int \sin ax \cos bx\,dx$, $\int \sin ax \sin bx\,dx$, ***or*** $\int \cos ax \cos bx\,dx$, ***where a and b are constants***

Integrals of this form are evaluated using the factor formula. The factor formula transforms the integrand which is in the form of a product of two trigonometric functions into a sum of two trigonometric functions. This makes it easier to integrate the resulting integrands. Table 9.3 lists the factor formulae and their corresponding integrals, given that P and Q are angles containing the variable x.

Table 9.3: *Factor formulae and the corresponding integrals*

Factor formulae	Integrals
$\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	$\int \sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\sin P+\sin Q)dx$
$\sin P-\sin Q=2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$	$\int \cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\sin P-\sin Q)dx$
$\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	$\int \cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\cos P+\cos Q)dx$
$\cos P-\cos Q=-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$	$\int \sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)dx=-\frac{1}{2}\int(\cos P-\cos Q)dx$

Example 9.31

Find $\int \sin 4x \cos 2x\,dx$.

Solution

Given $\int \sin 4x \cos 2x\,dx$.

The integrand is the product of sine and cosine functions. Thus, it can be expressed as the sum or difference of two sine functions. In this case, it will be expressed as a sum of two sine functions because the sine angle is greater than the cosine angle.

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In this case, the factor formula $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ is used to find the given integral.

The values of P and Q are computed as follows.

Let $\frac{P+Q}{2} = 4x$ and $\frac{P-Q}{2} = 2x$. Solving for P and Q gives, P $6x$ and $Q = 2x$.

Thus, $2\sin 4x\cos 2x = \sin 6x + \sin 2x$

$\Rightarrow \sin 4x\cos 2x = \frac{1}{2}(\sin 6x + \sin 2x)$

Now,

$$\int \sin 4x\cos 2x\,dx = \int \frac{1}{2}(\sin 6x + \sin 2x)\,dx$$

$$= \frac{1}{2}\int \sin 6x\,dx + \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\left(-\frac{\cos 6x}{6}\right) + \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + c$$

Therefore, $\int \sin 4x\cos 2x\,dx = -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + c.$

Example 9.32

Find $\int \cos 7x \sin 5x\,dx$.

Solution

Given $\int \cos 7x \sin 5x\,dx$.

In this case, the cosine angle is greater than the sine angle. Thus, the integrand can be expressed as the difference of two sine functions using the following factor formula.

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Compute the values of P and Q as follows:

Let $\frac{P+Q}{2} = 7x$ and $\frac{P-Q}{2} = 5x$. Solving for P and Q gives, $P = 12x$ and $Q = 2x$.

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Thus, $2\cos 7x\sin 5x = \sin 12x - \sin 2x$.

$\Rightarrow \cos 7x\sin 5x = \frac{1}{2}(\sin 12x - \sin 2x)$

Now, $\int \cos 7x\sin 5x\,dx = \frac{1}{2}\int(\sin 12x - \sin 2x)\,dx$

$$= \frac{1}{2}\int \sin 12x\,dx - \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\left(-\frac{\cos 12x}{12}\right) - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= \frac{1}{4}\left(\cos 2x - \frac{1}{6}\cos 12x\right) + c$$

Therefore, $\int \cos 7x\sin 5x\,dx = \frac{1}{4}\left(\cos 2x - \frac{1}{6}\cos 12x\right) + c.$

Example 9.33

Integrate $\sin 4x\sin x$ with respect to x.

Solution

Given $\int \sin 4x\sin x\,dx$.

The integrand is a product of sine functions. It can be expressed as the difference of two cosine functions using the following factor formula:

$$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Compute the values of P and Q as follows:

Let $\frac{P+Q}{2} = 4x$ and $\frac{P-Q}{2} = x$. Solving for P and Q gives, $P = 5x$ and $Q = 3x$.

Thus, $-2\sin 4x\sin x = \cos 5x - \cos 3x$.

$\Rightarrow \sin 4x\sin x = -\frac{1}{2}(\cos 5x - \cos 3x)$

$\Rightarrow \int \sin 4x\sin x\,dx = -\frac{1}{2}\int(\cos 5x - \cos 3x)dx$

$$= -\frac{1}{2}\int \cos 5x\,dx + \frac{1}{2}\int \cos 3x\,dx$$

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$$= -\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + c$$

Therefore, $\int \sin 4x \sin x\,dx = -\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + c.$

(b) ***Integrals of the form*** $\int \sin^n x\,dx$ ***or*** $\int \cos^n x\,dx$, ***where*** n ***is an odd or even integer***

In this form of integrals, the exponent of trigonometric functions determines the approach to be used. Integrals of even exponents of $\sin x$ or $\cos x$ are integrated with the help of the double angle formula of cosine which reduce the exponent of the given trigonometric function. If the integrand contains odd exponent of $\sin x$ or $\cos x$, the exponent is split, and the remaining part is expressed as an even exponent of $\sin x$ or $\cos x$. For instance, $\sin^3 x = \sin^2 x \sin x$.

Example 9.34

Find $\int \sin^2 x\,dx$.

Solution

Given $\int \sin^2 x\,dx$.

From $\cos 2x = 1 - 2\sin^2 x$, it implies that,

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Now, the integral becomes,

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Therefore,

$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$

Example 9.35

Find $\int \sin^3 x\,dx$.

Solution

Given $\int \sin^3 x\,dx$, then

$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$

But $\sin^2 x = 1 - \cos^2 x$.

$$\Rightarrow \int \sin^3 x\,dx = \int (1 - \cos^2 x)\sin x\,dx$$

$$= \int \sin x\,dx - \int \sin x \cos^2 x\,dx$$

$$= -\cos x - \int \sin x \cos^2 x\,dx$$

$\int \sin x \cos^2 x\,dx$ can be integrated by substitution method as follows:

Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$.

Thus, $dx = -\frac{1}{\sin x}du$.

Substitute these values into $\int \sin x \cos^2 x\,dx$ as follows,

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$$\int u^2 \sin x\left(-\frac{1}{\sin x}du\right)$$
$$= \int -u^2 du$$
$$= -\frac{1}{3}u^3 + c$$

Hence,
$$\int \sin x \cos^2 x\, dx = -\frac{1}{3}\cos^3 x + c.$$

Therefore,
$$\int \sin^3 x\, dx = -\cos x + \frac{1}{3}\cos^3 x + c.$$

Example 9.36

Find $\int \cos^4 x\, dx$.

Solution

Given $\int \cos^4 x\, dx$.

$$\Rightarrow \int \cos^4 x\, dx = \int (\cos^2 x)^2 dx$$

But $\cos x = \frac{1}{2}(1+\cos 2x)$.

$$\Rightarrow \int \cos^4 x\, dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$
$$= \frac{1}{4}\int (1+\cos 2x)^2 dx$$
$$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx$$
$$= \frac{1}{4}\int dx + \frac{2}{4}\int \cos 2x dx + \frac{1}{4}\int \cos^2 2x\, dx$$
$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \cos^2 2x\, dx$$

But $\int \cos^2 2x\, dx$ requires further integration as follows:

From $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos 4x = 2\cos^2 2x - 1$

$$\Rightarrow \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$

Now, $\int \cos^2 2x\, dx = \int \frac{1}{2}(1+\cos 4x)\, dx$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4x\, dx$$
$$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$$

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Hence, $\int \cos^4 x\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{8}\sin 4x + c\right)$.

Therefore, $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + B$, where $B = \frac{1}{4}c$.

Example 9.37

Find $\int \sin^2 x \cos^5 x\,dx$.

Solution

Given $\int \sin^2 x \cos^5 x\,dx$.

$$\Rightarrow \int \sin^2 x \cos^5 x\,dx = \int \sin^2 x \cos^4 x \cos x\,dx$$
$$= \int \sin^2 x \left(\cos^2 x\right)^2 \cos x\,dx$$
$$= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x\,dx$$
$$= \int \sin^2 x \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x\,dx$$
$$= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x\,dx$$

Let $u = \sin x \Rightarrow du = \cos x\,dx$.

Thus, $\int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x\,dx = \int (u^2 - 2u^4 + u^6)\,du$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + c$$
$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$$

Therefore, $\int \sin^2 x \cos^5 x\,dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$.

Remark,

1. If the power of sine is an odd positive integer, save one sine factor and convert the remaining factors to cosine. Next, expand and integrate.
2. If the power of cosine is an odd positive integer, save one cosine factor and convert the remaining factors to sine. Next, expand and integrate.
3. If the powers of both sine and cosine terms are even positive integers, make repeated use of the identities, $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$.

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(c) *Integrals of the form* $\int \tan^n x\,dx$ *or* $\int \sec^n x\,dx$, *where n is an odd or even integer*

For these forms of integrals, the following approaches can be used:

1. If the power of the tangent is an odd positive integer, save one tangent factor and convert the remaining factors to secant. Next, expand and integrate the resulting function.
2. If the power of the secant is an even positive integer, save one secant-squared factor and convert the remaining factors to tangent. Next, expand and integrate the resulting function.
3. If there is no secant term and the power of the tangent term is an even positive integer, convert one tangent-squared factor to secant-squared factor. Next, expand and integrate the resulting function.
4. If the integral is of the form $\int \sec^n x\,dx$, where n is an odd positive integer, then use integration by parts.

Example 9.38

Find $\int \tan^4\theta\,d\theta$

Solution

Given $\int \tan^4\theta\,d\theta$.

$$\Rightarrow \int \tan^4\theta\,d\theta = \int \tan^2\theta\left(\tan^2\theta\right)d\theta$$

$$= \int \tan^2\theta\left(\sec^2\theta - 1\right)d\theta$$

$$= \int \tan^2\theta\sec^2\theta\,d\theta - \int \tan^2\theta\,d\theta$$

$$= \int \tan^2\theta\sec^2\theta\,d\theta - \int\left(\sec^2\theta - 1\right)d\theta$$

Let $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$.

$$\Rightarrow \int \tan^4\theta\,d\theta = \int u^2 du - \int \sec^2\theta\,d\theta + \int d\theta$$

$$= \frac{u^3}{3} - \tan\theta + \theta + c$$

$$= \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c$$

Therefore, $\int \tan^4\theta d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c.$

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Example 9.39

Determine: (a) $\int \sec^4 6x \tan^3 6x\,dx$ (b) $\int \sec^3 x\,dx$

Solution

(a) Given $\int \sec^4 6x \tan^3 6x\,dx$.

$$\Rightarrow \int \sec^4 6x \tan^3 6x\,dx = \int \sec^2 6x \tan^3 6x\left(\sec^2 6x\right)dx$$

$$= \int \left(1+\tan^2 6x\right)\tan^3 6x \sec^2 6x\,dx$$

$$= \int \tan^3 6x \sec^2 6x\,dx + \int \tan^5 6x \sec^2 6x\,dx$$

Let $u = \tan 6x \Rightarrow \frac{1}{6}du = \sec^2 6x\,dx$.

$$\Rightarrow \int \sec^4 6x \tan^3 6x\,dx = \int u^3 \cdot \frac{1}{6}du + \int u^5 \cdot \frac{1}{6}du$$

$$= \frac{1}{6}\left[\frac{u^4}{4} + \frac{u^6}{6}\right] + c$$

Therefore, $\int \sec^4 6x \tan^3 6x\,dx = \frac{1}{36}\tan^6 6x + \frac{1}{24}\tan^4 6x + c$.

(b) Given $\int \sec^3 x\,dx$.

$$\Rightarrow \int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx$$

Let $u = \sec x$, then $du = \sec x \tan x\,dx$.

Also; $dv = \sec^2 x\,dx \Rightarrow v = \tan x$.

Substituting into the integral gives,

$$\int \sec x \sec^2 x\,dx = \sec x \tan x - \int \tan x(\sec x \tan x)dx$$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow \int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$$

Therefore, $\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$.

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Exercise 9.7

Evaluate each of the following:

1. $\int \sin 4x \sin 3x\, dx$
2. $\int \sin 5x \cos 3x\, dx$
3. $\int \cos 6x \cos 4x\, dx$
4. $\int \cos 4x \sin 2x\, dx$
5. $\int \sin\theta \sin 2\theta \sin 3\theta\ d\theta$
6. $\int \sin 2\theta \sin 4\theta\, d\theta$
7. $\int \sin^5 2x\, dx$
8. $\int \sin^2(2x-1)\, dx$
9. $\int \cos^4(3x+1)\, dx$
10. $\int \sin^4 3x\, dx$
11. $\int \cos^3 2x\, dx$
12. $\int 2\sin^4(x-1)\, dx$
13. $\int \cos^3 x\, dx$
14. $\int \sin^5 x\, dx$
15. $\int \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta\, d\theta$
16. $\int \sin^3 x \cos^2 x\, dx$
17. $\int \sin^4 x \cos^3 x\, dx$
18. $\int \sin^2 2x \cos^2 2x\, dx$
19. $\int$ ü ^{5} ü
20. $\int \sin^5 x \cos^2 x\, dx$
21. $\int \tan^3(3\theta)\sec^4(3\theta)\, d\theta$
22. $\int \sec^3 2x\, dx$

Trigonometric substitutions

Integrals of rational functions whose denominators cannot be factorized, require trigonometric substitutions. The common useful trigonometric substitutions include tangent, sine, and cosine.

(a) ***Integrals of the form*** $\int \frac{1}{x^2+a^2}dx$, ***where a is a constant***

The suitable trigonometric substitution for integrals of this kind is the tangent function.

Let $x = a\tan\theta \Rightarrow \frac{dx}{d\theta} = a\sec^2\theta$.

Thus, $dx = a\sec^2\theta d\theta$.

Substitute these values into the integral as follows,

$$\int \frac{1}{x^2+a^2}dx = \int \frac{1}{(a\tan\theta)^2+a^2} \times a\sec^2\theta d\theta$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2(\tan^2\theta+1)}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

But $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$.

Therefore,

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

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Example 9.40

Find $\int \frac{1}{x^2+25}dx$.

Solution

Given $\int \frac{1}{x^2+25}dx = \int \frac{1}{x^2+5^2}dx$.

Let $x = 5\tan\theta \Rightarrow \frac{dx}{d\theta} = 5\sec^2\theta$.

Thus, $dx = 5\sec^2\theta d\theta$.

Substitute these values into the integral to get;

$$\int \frac{1}{(5\tan\theta)^2+5^2} \times 5\sec^2\theta\, d\theta$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2\tan^2\theta+5^2}$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2(\tan^2\theta+1)}$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2\sec^2\theta}$$

$$= \frac{1}{5}\int d\theta$$

$$= \frac{1}{5}\theta + c$$

But

$x = 5\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{5}\right)$.

Therefore,

$$\int \frac{1}{x^2+25}dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+c.$$

Example 9.41

Find $\int \frac{1}{x^2\quad 2}dx$.

Solution

Given

$$\int \frac{1}{x^2+2}dx = \int \frac{1}{x^2+\left(\sqrt{2}\right)^2}dx.$$

Let $x = \sqrt{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{2}\sec^2\theta$.

Thus, $dx = \sqrt{2}\sec^2\theta d\theta$.

Substituting these values, the integral becomes,

$$\int \frac{1}{\left(\sqrt{2}\tan\theta\right)^2+\left(\sqrt{2}\right)^2} \times \sqrt{2}\sec^2\theta d\theta$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\tan^2\theta+\left(\sqrt{2}\right)^2}$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\left(\tan^2\theta+1\right)}$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\sec^2\theta}$$

$$= \frac{1}{\sqrt{2}}\int d\theta$$

$$= \frac{\sqrt{2}}{2}\theta + c$$

But $x = \sqrt{2}\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{2}x}{2}\right)$.

Therefore,

$$\int \frac{1}{x^2+2}dx = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{2}\right)+c.$$

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Example 9.42

Show that $\int \frac{1}{x^2+6x+16}dx = \frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)+c.$

Solution

Given $\int \frac{1}{x^2+6x+16}dx$.

By completing the square, the denominator $x^2+6x+16$ becomes,

$$\begin{aligned} x^2+6x+16 &= x^2+6x+3^2+16-3^2 \\ &= x^2+6x+3^2+7 \\ &= (x+3)^2+7 \\ &= (x+3)^2+7 \end{aligned}$$

Thus, $\int \frac{1}{x^2+6x+16}dx = \int \frac{1}{(x+3)^2+7}dx = \int \frac{1}{(x+3)^2+\left(\sqrt{7}\right)^2}dx$

Let $x+3=\sqrt{7}\tan\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{7}\sec^2\theta$.

Thus, $dx = \sqrt{7}\sec^2\theta d\theta$.

Substitute these values into the integral to obtain,

$$\begin{aligned} \int \frac{1}{(x+3)^2+\left(\sqrt{7}\right)^2}dx &= \int \frac{1}{\left(\sqrt{7}\tan\theta\right)^2+\left(\sqrt{7}\right)^2}\times\sqrt{7}\sec^2\theta d\theta \\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2\tan^2\theta+\left(\sqrt{7}\right)^2} \\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2(\tan^2\theta+1)} \\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2\sec^2\theta} \\ &= \frac{1}{\sqrt{7}}\int d\theta \\ &= \frac{\sqrt{7}}{7}\theta+c \end{aligned}$$

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But $x+3=\sqrt{7}\tan\theta \Rightarrow \theta=\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)$.

Therefore, $\int \frac{1}{x^2+6x+16}dx=\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)+c$.

(b) ***Integrals of the form*** $\int \frac{1}{\sqrt{a^2-x^2}}dx$, ***where a is a constant***

In this case, sine or cosine trigonometric substitutions are suitable for this kind of integrals.

Let $x=a\sin\theta \Rightarrow \frac{dx}{d\theta}=a\cos\theta$.

Thus, $dx=a\cos\theta d\theta$.

Substitute these values into the integral as follows;

$$\int \frac{1}{\sqrt{a^2-x^2}}dx=\int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}}\times a\cos\theta d\theta$$

$$=\int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\times a\cos\theta d\theta$$

$$=\int \frac{1}{\sqrt{a^2(1-\sin^2\theta)}}\times a\cos\theta d\theta$$

$$=\int \frac{1}{\sqrt{a^2\cos^2\theta}}\times a\cos\theta d\theta$$

$$=\int \frac{a\cos\theta}{a\cos\theta}d\theta$$

$$=\int d\theta$$

$$=\theta+c$$

But $x=a\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$.

Therefore, $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$.

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Example 9.43

Determine $\int \frac{1}{\sqrt{4-x^2}}dx$.

Solution

Given $\int \frac{1}{\sqrt{4-x^2}}dx = \int \frac{1}{\sqrt{2^2-x^2}}dx$.

Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$.

Thus, $dx = 2\cos\theta d\theta$.

Substitute these values into the integral to get,

$$\int \frac{1}{\sqrt{2^2-x^2}}dx$$

$$= \int \frac{1}{\sqrt{2^2-(2\sin\theta)^2}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{2^2-2^2\sin^2\theta}} \times 2\cos\theta \ \theta$$

$$= \int \frac{1}{\sqrt{2^2(1-\sin^2\theta)}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{2\sqrt{1-\sin^2\theta}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{\cos^2\theta}} \times \cos\theta d\theta$$

$$= \int \frac{1}{\cos\theta} \times \cos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

But $x \aleph 2\sin\theta \quad \theta \quad \sin^{-1}\left(\frac{x}{2}\right)$.

Therefore,

$$\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\left(\frac{x}{2}\right) + c.$$

Example 9.44

Find $\int \frac{1}{\sqrt{7-6x-x^2}}dx$.

Solution

Given $\int \frac{1}{\sqrt{7-6x-x^2}}dx$.

By completing the square, the quadratic equation $7-6x-x^2$ becomes,

$$7-6x-x^2 = 7-(x^2+6x)$$
$$= 7+9-(x^2+6x+9)$$
$$= 16-(x+3)^2$$

Thus,

$$\int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{16-(x+3)^2}}dx$$
$$= \int \frac{1}{\sqrt{4^2-(x+3)^2}}dx$$

Let $x+3 = 4\sin\theta \Rightarrow \frac{dx}{d\theta} = 4\cos\theta$.

Thus, $dx = 4\cos\theta d\theta$.

Substitute these values into the integral as follows,

$$\int \frac{1}{\sqrt{4^2-(x+3)^2}}dx$$

$$= \int \frac{1}{\sqrt{4^2-(4\sin\theta)^2}} \times 4\cos\theta d\theta$$

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$$=\int \frac{1}{\sqrt{4^2-4^2\sin^2\theta}}\times 4\cos\theta d\theta$$

$$=\int \frac{1}{\sqrt{4^2(1-\sin^2\theta)}}\times 4\cos\theta d\theta$$

$$=\int \frac{}{4\sqrt{1-\sin\ \theta}}\times 4\cos\theta d\theta$$

$$=\int \frac{1}{\sqrt{\cos^2\theta}}\times \cos\theta d\theta$$

$$=\int \frac{1}{\cos\theta}\times \cos\theta d\theta$$

$$=\int d\theta$$

$$=\theta+c$$

But

$$(x+3)=4\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x+3}{4}\right).$$

Therefore,

$$\int \frac{1}{\sqrt{7-6x-x^2}}dx=\sin^{-1}\left(\frac{x+3}{4}\right)+c.$$

(c) ***Integrals of the form*** $\int\sqrt{a^2-x^2}dx$, ***where a is a constant***

In this case, sine or cosine trigonometric substitution is suitable for this kind of integrals.

Let $x=a\sin\theta \Rightarrow \frac{dx}{d\theta}=a\cos\theta$.

Thus, $dx=a\cos\theta d\theta$.

Substitute these values into the integral to get,

$$\int\sqrt{a^2-x^2}dx=\int\sqrt{a^2-(a\sin\theta)^2}\times a\cos\theta d\theta$$

$$=\int\sqrt{a^2-a^2\sin^2\theta}\times a\cos\theta d\theta$$

$$=\int\sqrt{a^2(1-\sin^2\theta)}\times a\cos\theta d\theta$$

$$=\int a\sqrt{\cos^2\theta}\times a\cos\theta d\theta$$

$$=a\int\cos\theta\times\cos\theta d\theta$$

$$=a^2\int\frac{1}{2}(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}a^2\left[\int d\theta+\int\cos 2\theta d\theta\right]$$

$$=\frac{1}{2}a^2\theta+\frac{1}{4}a^2\sin 2\theta+c$$

But $x=a\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)$.

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$$\Rightarrow \int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}a^2 \sin 2\left(\sin^{-1}\left(\frac{x}{a}\right)\right) + c$$

From the relation $\sin 2\theta = 2\sin\theta\cos\theta$, it implies that,

$$\sin 2\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = 2\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)$$

Thus, $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{2}{4}a^2 \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right) + c$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^2 \times \left(\frac{x}{a}\right)\sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x}{a}\right)\right)} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}ax\sqrt{1 - \left(\frac{x}{a}\right)^2} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}ax\sqrt{\frac{a^2 - x^2}{a^2}} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2} + c$$

Therefore, $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2} + c.$

Example 9.45

Find $\int \sqrt{36 - x^2}\,dx$.

Solution

Given $\int \sqrt{36 - x^2}\,dx = \int \sqrt{6^2 - x^2}\,dx$.

Let $x = 6\sin\theta \Rightarrow \frac{dx}{d\theta} = 6\cos\theta$.

Thus, $dx = 6\cos\theta d\theta$.

Substituting these values into the integral gives,

$$\int \sqrt{6^2 - x^2}\,dx = \int \sqrt{6^2 - (6\sin\theta)^2} \times 6\cos\theta d\theta$$

$$= \int \sqrt{6^2 - 6^2\sin^2\theta} \times 6\cos\theta d\theta$$

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$$= \int \sqrt{6^2\left(1-\sin^2\theta\right)} \times 6\cos\theta d\theta$$

$$= \int 6\sqrt{1-\sin^2\theta} \times 6\cos\theta d\theta$$

$$= \int 6\sqrt{\cos^2\theta} \times 6\cos\theta d\theta$$

$$= 36\int \cos\theta \times \cos\theta d\theta$$

$$= 36\int \frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= 18\left[\int d\theta + \int \cos 2\theta d\theta\right]$$

$$= 18\theta + 9\sin 2\theta + c$$

But $x = 6\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{6}\right)$.

Thus, $\int \sqrt{36-x^2}dx = 18\sin^{-1}\left(\frac{x}{6}\right) + 9\sin 2\left(\sin^{-1}\left(\frac{x}{6}\right)\right) + c$.

Further simplification gives $\int \sqrt{36-x^2}dx = 18\sin^{-1}\left(\frac{x}{6}\right) + \frac{1}{2}x\sqrt{36-x^2} + c$.

Therefore, $\int \sqrt{36-x^2}dx = 18\sin^{-1}\left(\frac{x}{6}\right) + \frac{1}{2}x\sqrt{36-x^2} + c$.

Example 9.46

Show that $\int \sqrt{3-2x-x^2}dx = 2\sin^{-1}\left(\frac{x+1}{3}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + c$.

Solution

Given $\int \sqrt{3-2x-x^2}dx = 2\sin^{-1}\left(\frac{x+1}{3}\right) + \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + c$.

Consider the left-hand side:

By completing the square, $3-2x-x^2$ gives, $3-2x-x^2 = 2^2-(x+1)^2$.

Now, the integral becomes, $\int \sqrt{3-2x-x^2}dx = \int \sqrt{2^2-(x+1)^2}dx$

Let $x+1 = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$.

Thus, $dx = 2\cos\theta d\theta$.

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Substituting these values into the integral gives,

$$\int\sqrt{2^2-(x+1)^2}\,dx=\int\sqrt{2^2-(2\sin\theta)^2}\times 2\cos\theta d\theta$$

$$=\int\sqrt{2^2-2^2\sin^2\theta}\times 2\cos\theta d\theta$$

$$=\int\sqrt{2^2\left(1-\sin^2\theta\right)}\times 2\cos\theta d\theta$$

$$=\int 2\sqrt{\cos^2\theta}\times 2\cos\theta d\theta$$

$$=4\int\cos\theta\times\cos\theta d\theta$$

$$=4\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=2\left[\int d\theta+\int\cos 2\theta d\theta\right]$$

$$=2\theta+\sin 2\theta+c$$

But $x+1=2\sin\theta\Rightarrow\theta=\sin^{-1}\left(\frac{x+1}{2}\right)$.

$$\Rightarrow\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\sin 2\left(\sin^{-1}\left(\frac{x+1}{2}\right)\right)+c$$

Further simplification gives,

$$\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+c.$$

Therefore, $\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+c.$

Exercise 9.8

Evaluate each of the following:

1. $\int\frac{1}{x^2+49}dx$
2. $\int\frac{1}{5x^2+81}dx$
3. $\int\frac{9}{3+x^2}dx$
4. $\int\frac{1}{z^2+2z+5}dz$
5. $\int\frac{dy}{y\sqrt{9+4y^2}}$
6. $\int\frac{dz}{z^2\sqrt{4+z^2}}$

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7. $\int \frac{1}{1+x^2}dx$

8. $\int \frac{y+3}{y^2+9}dy$

9. $\int \frac{14}{49+16y^2}dy$

10. $\int \frac{1}{\sqrt{1-2x^2}}dx$

11. $\int \frac{12}{\sqrt{9-4x^2}}dx$

12. $\int \frac{1}{\sqrt{5-x^2}}dx$

13. $\int 2\sqrt{1-4x^2}\,dx$

14. $\int \sqrt{6-4x-x^2}dx$

15. $\int \frac{1}{x^2\sqrt{9-x^2}}dx$

16. $\int \sqrt{-2x-x^2}dx$

17. $\int \sqrt{5+4x-x^2}\,dx$

18. $\int \frac{t^3}{\sqrt{t^2+9}}dt$

19. $\int \frac{x^2}{\sqrt{4-x^2}}dx$

20. $\int x^2\sqrt{1-x^2}dx$

21. $\int \frac{1}{x^2+4x+8}dx$

22. $\int \frac{2}{x^2+10x+30}dx$

23. $\int \frac{1}{\sqrt{4-(x+2)^2}}dx$

24. $\int \frac{2x-7}{\sqrt{9-x^2}}dx$

25. $\int \frac{2}{13-4x+x^2}dx$

26. $\int \frac{s^3}{(4s^2+9)^{\frac{3}{2}}}ds$

27. $\int \left(\frac{7}{\sqrt{1+x^2}}+\frac{\sqrt{1-2x^2}}{4}\right)dx$

28. Show that

$$\int \frac{1}{a^2x^2+b^2}dx=\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}x\right)+c,$$

where a, b, and c are constants.

(d) *Integrals of the form* $\int \frac{1}{a+b\sin^2 x+c\cos^2 x}dx$, *where* a, b, *and* c *are constants*

Integrals of this type are evaluated by applying the trigonometric substitution $t=\tan x$. Consider a right-angled triangle with $t=\tan x$ as shown in Figure 9.1.

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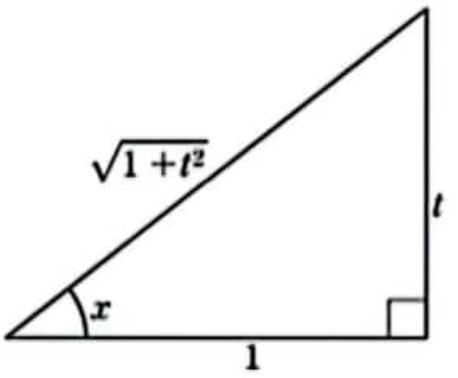

Figure 9.1: *A right-angled triangle*

From Figure 9.1, it can be observed that, if $t = \tan x \Rightarrow \sin x = \frac{t}{\sqrt{1+t^2}}$ and $\cos x = \frac{}{\sqrt{1+t}}$.

Differentiate $t = \tan x$ with respect to x, and re-write the integral in terms of t as follows:

$dt = \sec^2 x dx$

But $\sec^2 x = 1 + \tan^2 x$.

Now,

$dt = (1+\tan^2 x)dx$, $dt = (1+t^2)dx$

Thus, $dx = \frac{1}{(1+t^2)}dt$.

Substituting these values in the given integral and proceeding with integration leads to the required solution.

Example 9.47

Find $\int \frac{1}{1+\sin^2 x}dx$.

Solution

Given $\int \frac{1}{1+\sin^2 x}dx$. In this case, the substitution of $t = \tan x$ can be used.

Let $t = \tan x \Rightarrow \sin x = \frac{t}{\sqrt{1+t^2}}$ and $dx = \frac{1}{1+t^2}dt$.

Thus, $\int \frac{1}{1+\sin^2 x}dx = \int \frac{1}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2} \times \frac{1}{1+t^2}dt$

$= \int \frac{1}{1+\frac{t^2}{1+t^2}} \times \frac{1}{1+t^2}dt$

$= \int \frac{1+t^2}{1+2t^2} \times \frac{1}{1+t^2}dt$

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$$=\int \frac{1}{1+2t^2}dt$$

$$=\int \frac{1}{1+\left(\sqrt{2}t\right)^2}\,dt$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}t\right)+c$$

But $t=\tan x$.

$$\Rightarrow\int \frac{dx}{1+\sin^2 x}=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c$$

Therefore, $\int \frac{dx}{1+\sin^2 x}=\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\tan x\right)+c.$

Example 9.48

Show that $\int \frac{4}{\cos^2 x+9\sin^2 x}dx=\frac{4}{3}\tan^{-1}(3\tan x)+c.$

Solution

Given $\int \frac{4}{\cos^2 x+9\sin^2 x}dx$. In this case, the substitution of $t=\tan x$ can be used.

Let $t=\tan x \Rightarrow \sin x=\frac{t}{\sqrt{1+t^2}}$, $\cos x=\frac{1}{\sqrt{1+t^2}}$, and $dx=\frac{1}{1+t^2}dt$.

Thus, $\int \frac{4}{\cos^2 x+9\sin^2 x}dx=\int \frac{4}{\frac{1}{1+t^2}+9\left(\frac{t^2}{1+t^2}\right)}\times\frac{1}{1+t^2}dt$

$$=\int \frac{4\left(1+t^2\right)}{1+9t^2}\times\frac{1}{1+t^2}\,dt$$

$$=4\int \frac{dt}{1+9t^2}$$

Let $3t=\tan\theta \Rightarrow dt=\frac{1}{3}\sec^2\theta\, d\theta$.

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$$\Rightarrow \int \frac{4}{\cos^2 x + 9\sin^2 x}\,dx = 4\int \frac{\frac{1}{3}\sec^2\theta\,d\theta}{1+\tan^2\theta}$$

$$\frac{4}{3}\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$$

$$= \frac{4}{3}\int d\theta = \frac{4}{3}\theta + c$$

But $\theta = \tan^{-1}(3t)$ and $t = \tan x$.

$$\Rightarrow \int \frac{4}{\cos^2 x + 9\sin^2 x}\,dx = \frac{4}{3}\tan^{-1}(3\tan x) + c$$

Therefore, $\int \frac{4}{\cos^2 x + 9\sin^2 x}\,dx = \frac{4}{3}\tan^{-1}(3\tan x) + c$.

(e) ***Integrals of the form*** $\int \frac{1}{a + b\sin x + c\cos x}\,dx$, ***where a, b, and c are constants***

Integrals of this type are also evaluated using the trigonometric substitution of $t = \tan\left(\frac{x}{2}\right)$ and the double angle formula of tangent of an angle, that is

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

For instance, if $A = \frac{x}{2}$, then the double angle formula of tangent will be as follows;

$$\tan 2\left(\frac{x}{2}\right) = \tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}.$$

But $\tan\left(\frac{x}{2}\right) = t \Rightarrow \tan x = \frac{2t}{1-t^2}$.

The relation $\tan x = \frac{2t}{1-t^2}$ can be represented in a right-angled triangle ABC as shown in Figure 9.2.

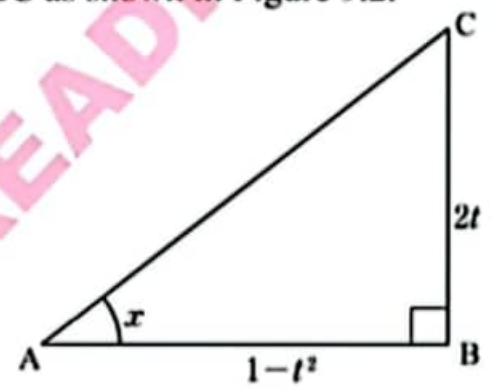

Figure 9.2: *Right-angled triangle*

In Figure 9.2, the hypotenuse is obtained by Pythagoras' theorem. That is,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$= (1-t^2)^2 + (2t)^2$$

$$\Rightarrow \overline{AC}^2 = 1 - 2t^2 + t^4 + 4t^4$$

$$= 1 + 2t^2 + t^4$$

$$= (1+t^2)^2$$

Therefore, $\overline{AC} = 1 + t^2$.

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Now, the sides of the triangle can be represented as shown in Figure 9.3.

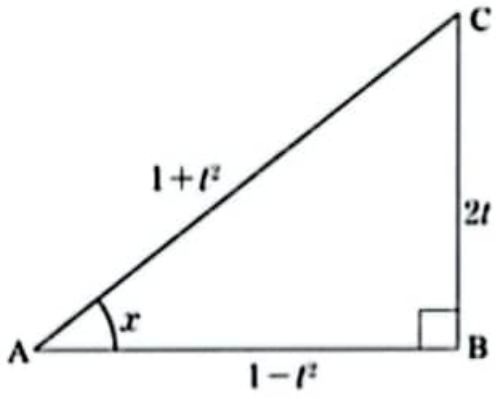

Figure 9.3: *Right-angled triangle*

In Figure 9.3, the ratios of sine and cosine will be as follows;

$\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$.

Also, if $t = \tan\frac{x}{2}$, then $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$.

But $\sec^2\frac{x}{2} = 1+\tan^2\frac{x}{2}$.

Now, $\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$.

Substituting $t = \tan\left(\frac{x}{2}\right)$ gives,

$\frac{dt}{dx} = \frac{1}{2}\left(1+t^2\right) \Rightarrow dx = \frac{2}{1+t^2}dt$.

Example 9.49

Show that

$\int \sec x\,dx = \ln|\sec x + \tan x| + c$.

Solution

Given $\int \sec x\,dx = \int \frac{1}{\cos x}dx$.

Substituting $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, the integral becomes,

$$\int \frac{1+t^2}{1-t^2}\times\frac{2}{1+t^2}dt = \int \frac{2}{1-t^2}dt$$

$$= 2\int \frac{1}{1-t^2}dt$$

But $\frac{1}{1-t^2}$ $\frac{1}{(1-t)(1+t)}$.

Decompose the fraction into partial fractions. That is,

$$\frac{1}{(1-t)(1+t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)}$$

Solving for A and B gives,

$A = \frac{1}{2}$ and $B = \frac{1}{2}$.

Thus, $\frac{1}{(1-t)(1+t)} = \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t}$

$$= \frac{1}{2(1-t)} + \frac{1}{2(1+t)}$$

Now, the integral becomes;

$$\int \sec x\,dx = 2\int\left(\frac{1}{2(1-t)} + \frac{1}{2(1+t)}\right)dt$$

$$= \int \frac{1}{1-t}dt + \int \frac{1}{1+t}dt$$

$$= -\ln|1-t| + \ln|1+t| + c$$

$$= \ln|1+t| - \ln|1-t| + c$$

$$= \ln\left|\frac{1+t}{1-t}\right| + c$$

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But $t = \tan\frac{x}{2}$.

Hence, $\int \sec x\,dx = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + c.$

Further simplification gives,

$$\int \sec x\,dx = \ln\left|\frac{1+\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}{1-\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}\right| + c$$

$$= \ln\left|\frac{\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\frac{x}{2}}}{\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}\right| + c$$

$$= \ln\left|\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right| + c$$

Multiplying the numerator and denominator by $\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)$ gives,

$$\int \sec x\,dx = \ln\left|\frac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}{\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}\right| + c$$

$$= \ln\left|\frac{\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + c$$

$$= \ln\left|\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right| + c$$

Therefore, $\int \sec x dx = \ln|\sec x + \tan x| + c.$

Alternatively,

Given $\int \sec x\,dx.$

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Mutliplying by $\sec x + \tan x$ both the numerator and denominator gives,

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx$$

$$= \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$$

Let $u = \sec x + \tan x \Rightarrow du = (\sec x\tan x + \sec^2 x)dx$.

$$\Rightarrow dx = \frac{du}{\sec^2 x+\sec x\tan x}$$

Thus, $\int \sec x\,dx = \int \frac{\sec^2 x+\sec x\tan x}{u}\times\frac{du}{\sec^2 x+\sec x\tan x}$

$$= \int \frac{1}{u}du$$

$$= \ln|u| + c$$

But $u = \sec x + \tan x$.

$\Rightarrow \int \sec x\,dx = \ln|\sec x + \tan x| + c$.

Therefore, $\int \sec x\,dx = \ln|\sec x + \tan x| + c$.

Example 9.50

Verify that

$\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$.

Solution

Given $\int \operatorname{cosec} x\,dx = \int \frac{1}{\sin x}dx$.

Substituting $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, into the integral gives,

$$\int \operatorname{cosec} x\,dx = \int \frac{1+t^2}{2t}\times\frac{2}{1+t^2}dt$$

$$= \int \frac{1}{t}dt$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\left(\frac{x}{2}\right)\right| +$$

$$= \ln\left|\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\times\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{\sin^2\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}\right| + c$$

$$= \ln\left|\frac{1-\cos x}{\sin x}\right| + c$$

$$= \ln|\operatorname{cosec} x - \cot x| + c$$

Therefore,

$\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$.

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Example 9.51

Find $\int \frac{3dx}{15+9\cos x}$.

Solution

Given $\int \frac{3dx}{15+9\cos x}$.

Substituting $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, the integral becomes,

$$\int \frac{3dx}{15+9\cos x} = \int \frac{3}{15+9\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= 6\int \frac{1+t^2}{15(1+t^2)+9(1-t^2)} \times \frac{dt}{1+t^2}$$

$$= 6\int \frac{dt}{6t^2+24} = \int \frac{dt}{t^2+4}$$

Let $t = 2\tan\theta \Rightarrow dt = 2\sec^2\theta\, d\theta$.

Thus, $\int \frac{3dx}{15+9\cos x} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta+4}$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta = \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + c$$

But $\theta = \tan^{-1}\left(\frac{t}{2}\right)$.

$$\Rightarrow \int \frac{3dx}{15+9\cos x} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c.$$

But $t = \tan\left(\frac{x}{2}\right)$.

Therefore, $\int \frac{3dx}{15+9\cos x} = \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c.$

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Exercise 9.9

Evaluate each of the following:

1. $\int \frac{1}{1+\cos x}\,dx$
2. $\int \frac{1}{5+4\cos\theta}\,d\theta$
3. $\int \frac{1}{1+\sin 2x}\,dx$
4. $\int \frac{1}{1+\cos^2 x}\,dx$
5. $\int \frac{\sin x}{1-\cos^2 x}\,dx$
6. $\int \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx$
7. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$
8. $\int \frac{1}{\cos 2x+3\sin^2 x}\,dx$
9. $\int \frac{1}{1-10\sin^2 x}\,dx$
10. $\int \frac{1}{1+\cos 2\theta}\,d\theta$
11. $\int \frac{\sec x \tan x}{\sec x+\sec^2 x}\,dx$
12. $\int \frac{\sec x}{\tan x}\,dx$
13. $\int \frac{1}{\cos^2 2\theta-\sin^2 2\theta}\,d\theta$
14. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$
15. $\int \frac{\cos 2x}{\cos x}\,dx$
16. $\int \frac{1}{3+5\cos\left(\frac{x}{2}\right)}\,dx$
17. $\int \frac{1}{4+5\cos x}\,dx$
18. $\int \operatorname{cosec}\frac{1}{2}x\,dx$
19. $\int \frac{dy}{1+\cos 3y}$
20. $\int \frac{dx}{2-\sin x}$
21. $\int \frac{dx}{2+\sin x}$
22. $\int \frac{1}{5+3\cos\left(\frac{\theta}{2}\right)}\,d\theta$
23. $\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx$
24. $\int \frac{1}{2+2\sin\theta+\cos\theta}\,d\theta$
25. $\int \frac{dx}{2+\cos x}$
26. $\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$
27. $\int \frac{\sec^2 x}{\tan^2 x+2\tan x+2}\,dx$

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28. $\int \frac{3+\cos\theta}{2-\cos\theta}\,d\theta$

29. $\int \frac{1}{2+2\cos x+\sin x}\,dx$

30. $\int \frac{\sqrt{1+\sin x}}{\sec x}\,dx$

Integration by splitting the numerator

If an integrand is a rational function with a quadratic denominator which cannot be expressed in simple partial fractions, then it can be expressed in terms of two fractions by splitting its numerator in the following form:

$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B,$

where A and B are constants which split the numerator and $\frac{d}{dx}(\text{denominator})$ is the derivative of the denominator with respect to the given variable.

Example 9.52

Determine $\int \frac{2+x}{4+x^2}dx.$

Solution

Given $\int \frac{2+x}{4+x^2}dx.$

Split the numerator of the integrand. That is,

$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B$

$\Rightarrow 2+x = A\frac{d}{dx}(4+x^2) + B$

$\Rightarrow 2+x = 2Ax + B$

Comparing the coefficients of x and the constants of the two sides gives

$A = \frac{1}{2}$ and $B = 2.$

Now, the integral becomes,

$$\int \frac{2+x}{4+x^2}dx = \int \frac{\frac{1}{2}(2x)+2}{4+x^2}dx$$

$$= \int \frac{x}{4+x^2}dx + \int \frac{2}{4+x^2}dx$$

$$= \frac{1}{2}\ln(4+x^2) + \tan^{-1}\left(\frac{x}{2}\right) + c$$

Therefore,

$$\int \frac{2+x}{4+x^2}dx = \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\ln(4+x^2) + c.$$

Example 9.53

Find $\int \frac{5x+7}{x^2+4x+8}dx.$

Solution

Given $\int \frac{5x+7}{x^2+4x+8}dx.$

Split the numerator of the integrand. That is,

$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B$

$\Rightarrow 5x+7 = A\frac{d}{dx}(x^2+4x+8) + B$

$\Rightarrow 5x+7 = (2x+4)A + B$

$\Rightarrow 5x+7 = 2Ax + 4A + B$

Comparing the coefficients of x and of the constants on both sides gives,

$A = \frac{5}{2}$ and $B = -3.$

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Now, the integral becomes,

$$\int \frac{5x+7}{x^2+4x+8}dx = \int \left(\frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}\right)dx$$

$$= \int \frac{\frac{5}{2}(2x+4)}{x^2+4x+8}dx - \int \frac{3}{x^2+4x+8}dx$$

$$= \frac{5}{2}\ln\left|x^2+4x+8\right| - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c$$

Therefore, $\int \frac{5x+7}{x^2+4x+8}dx = \frac{5}{2}\ln\left|x^2+4x+8\right| - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c.$

Integrals of the form $\int \frac{a\cos x \pm b\sin x}{p\cos x \pm q\sin x}dx$, ***where a, b, p, and q are constants***

The integral in this form can be easily integrated by splitting the numerator of the integrand. In this case, the numerator is expressed in the following form:

$$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B(\text{denominator})$$

The constants A and B are determined by comparing the coefficients of cosines and sines on both sides of the equation.

Note that, if the integral has the form

$$\int \frac{a\cos x \pm b\sin x + c}{p\cos x \pm q\sin x}dx \quad \text{or} \quad \int \frac{a\cos x \pm b\sin x}{p\cos x \pm q\sin x + r}dx$$

where *a, b, c, p, q,* and *r* are constants, then the numerator of the integrand is expressed in the following form.

$$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B(\text{denominator}) + C$$

The constants A, B, and C are determined by comparing the terms of both sides of the equation.

Example 9.54

Determine $\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}dx.$

Solution

Given $\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}dx.$

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Split the numerator of the integrand as follows:

$$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B(\text{denominator}).$$

Thus, $2\cos x + 3\sin x = A\frac{d}{dx}(\cos x + \sin x) + B(\cos x + \sin x)$

$\Rightarrow 2\cos x + 3\sin x = A(-\sin x + \cos x) + B(\cos x + \sin x)$

Comparing the coefficients of cosines and sines on both sides gives,

$2 = A + B$.....(i)

$3 = -A + B$.....(ii)

Solving equations (i) and (ii) simultaneously gives,

$A = -\frac{1}{2}$ and $B = \frac{5}{2}$.

Now, the integral becomes,

$$\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}dx = \int \frac{-\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x}dx + \int \frac{\frac{5}{2}(\cos x + \sin x)}{\cos x + \sin x}dx$$

$$= -\frac{1}{2}\ln|\cos x + \sin x| + \frac{5}{2}x + c$$

Therefore, $\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}dx = -\frac{1}{2}\ln|\cos x + \sin x| + \frac{5}{2}x + c.$

Example 9.55

Show that $\int \frac{2\cos x + 5\sin x}{3\cos x - \sin x}dx = -\frac{17}{10}\ln|3\cos x - \sin x| + \frac{1}{10}x + c.$

Solution

Given $\int \frac{2\cos x + 5\sin x}{3\cos x - \sin x}dx.$

Split the numerator of the integrand as follows:

$$2\cos x + 5\sin x = A\frac{d}{dx}(3\cos x - \sin x) + B(3\cos x - \sin x)$$

$\Rightarrow 2\cos x + 5\sin x = A(-3\sin x - \cos x) + B(3\cos x - \sin x)$

comparing the coefficients of cosines and sines on both sides gives,

$-A + 3B = 2$..... (i)

$-3A - B = 5$..... (ii)

Solving equations (i) and (ii) simultaneously gives, $A = -\frac{17}{10}$ and $B = \frac{1}{10}$.

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Thus, the integral becomes,

$$\int \frac{2\cos x+5\sin x}{3\cos x-\sin x}dx=-\frac{17}{10}\int \frac{-3\sin x-\cos x}{3\cos x-\sin x}dx+\frac{1}{10}\int \frac{3\cos x-\sin x}{3\cos x-\sin x}dx$$

$$=-\frac{17}{10}\ln|3\cos x-\sin x|+\frac{1}{10}x+c$$

Therefore, $\int \frac{2\cos x+5\sin x}{3\cos x-\sin x}dx=-\frac{17}{10}\ln|3\cos x-\sin x|+\frac{1}{10}x+c.$

Exercise 9.10

Find each of the following:

1. $\int \frac{x+7}{x^2+5}dx$
2. $\int \frac{x+2}{2x^2+6x+5}dx$
3. $\int \frac{\sin\theta}{\cos\theta+\sin\theta}d\theta$
4. $\int \frac{2\sin\theta-\cos\theta}{\sin\theta+3\cos\theta}d\theta$
5. $\int \frac{2x+3}{x^2+2x+10}dx$
6. $\int \frac{y+4}{y^2+6y+9}dy$
7. $\int \frac{3u+8}{u^2+2u+5}du$
8. $\int \frac{2\cos\theta-3\sin\theta}{\cos\theta+4\sin\theta}d\theta$
9. $\int \frac{\sin x+2\cos x}{3+\cos x+2\sin x}dx.$
10. $\int \frac{3\cos x-2\sin x}{\cos x+\sin x}dx$
11. $\int \frac{1}{2+3\tan\theta}d\theta$
12. $\int \frac{\sin 2x-5\cos 2x}{7\cos 2x-2\sin 2x}dx$

Integrals of exponential functions of the form $\int a^x dx$

Integrals of this type, where x is a variable and a is a constant are easily evaluated by the substitution method, with the application of natural logarithms.

Given $\int a^x dx$. Let $u=a^x$.

Applying natural logarithm on both sides gives,

$\ln u=\ln a^x$

$\Rightarrow \ln u=x\ln a$

Differentiating with respect to x both sides of the equation gives,

$$\frac{1}{u}\frac{du}{dx}=\ln a \Rightarrow dx=\frac{1}{u\ln a}du$$

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Substituting these values in the integral as follows,

$$\int a^x dx = \int u \times \frac{1}{u \ln a} du$$

$$= \frac{1}{\ln a} \int du$$

$$= \frac{1}{\ln a} u + c$$

But $u = a^x$.

Therefore, $\int a^x dx = \frac{1}{\ln a} a^x + c$.

Example 9.56

Determine $\int 3^x dx$.

Solution

Given $\int 3^x dx$. Let $u = 3^x$.

Apply natural logarithm on both sides of the equation to get,

$$\ln u = \ln 3^x$$

$$\Rightarrow \ln u = x \ln 3$$

Differentiating with respect to x on both sides of the equation gives,

$$\frac{1}{u}\frac{du}{dx} = \ln 3 \Rightarrow dx = \frac{1}{u \ln 3} du$$

Now, the integral becomes,

$$\int 3^x dx = \int u \times \frac{1}{u \ln 3} du$$

$$= \frac{1}{\ln 3} \int du$$

$$= \frac{1}{\ln 3} u + c$$

But $u = 3^x$.

Therefore, $\int 3^x dx = \frac{3^x}{\ln 3} + c$.

Example 9.57

Find $\int 2^{4x+3} dx$.

Solution

Given $\int 2^{4x\ 3} dx$.

Let $u = 2^{4x+3}$.

Apply natural logarithm on both sides of the equation to get,

$$\ln u = \ln 2^{4x+3}$$

$$\ln u = (4x+3) \ln 2$$

Differentiate with respect to x on both sides of the equation to get,

$$\frac{1}{u}\frac{du}{dx} = 4 \ln 2 \Rightarrow dx = \frac{1}{4u \ln 2} du$$

Substitute these values into the integral as follows,

$$\int 2^{4x+3} dx = \int u \times \frac{1}{4u \ln 2} du$$

$$= \frac{1}{4 \ln 2} \int du$$

$$= \frac{1}{4 \ln 2} u + c$$

But $u = 2^{4x+3}$.

Therefore, $\int 2^{4x+3} dx = \frac{2^{(4x+3)}}{4 \ln 2} + c$.

Integrals of logarithmic functions of the form $\int \log ax\, dx$, where a is a constant

Integrals of logarithmic functions of the form $\int \log ax\, dx$ are determined by the method of integration by parts. This is achieved after converting the function from logarithm base 10

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to natural logarithm. Integration of $\int \log ax\,dx$ can be done as follows.

Convert $\int \log ax\,dx$ to natural logarithmic form as follows:

$$\int \log ax\,dx = \int \frac{\ln ax}{\ln 10}dx$$

$$= \frac{1}{\ln 10}\int \ln ax\,dx$$

Let $u = ax \Rightarrow du = a\,dx$.

$$\Rightarrow dx = \frac{1}{a}du$$

Now, the integral becomes,

$$\frac{1}{\ln 10}\int (\ln u)\left(\frac{1}{a}du\right)$$

$$= \frac{1}{a\ln 10}\int \ln u\,du$$

But $\int \ln u\,du = u\ln|u| - u + c$

$$\Rightarrow \int \log ax\,dx = \frac{1}{a\ln 10}u(\ln|u| - 1) + c$$

But $u = ax$.

Thus,

$$\int \log ax\,dx = \frac{1}{a\ln 10} \times ax\left(\ln|ax| - 1\right) + c.$$

Therefore,

$$\int \log ax\,dx = \frac{1}{\ln 10}x\left(\ln|ax| - 1\right) + c.$$

Example 9.58

Find $\int \log 4x\,dx$.

Solution

Given $\int \log 4x\,dx$.

Convert the integral into natural logarithmic form as follows:

$$\int \log 4x\,dx = \frac{1}{\ln 10}\int \ln 4x\,dx$$

Let $u = 4x \Rightarrow du = 4dx$.

$$\Rightarrow dx = \frac{1}{4}du$$

Substitute these values into the integral as follows,

$$\int \log 4x\,dx = \frac{1}{\ln 10}\int \ln u\left(\frac{1}{4}du\right)$$

$$= \frac{1}{4\ln 10}\int \ln u\,du$$

$$= \frac{1}{4\ln 10}\left(u\ln|u| - u\right) + c$$

$$= \frac{1}{4\ln 10}u\left(\ln|u| - 1\right) + c$$

But $u = 4x$.

Thus,

$$\int \log 4x\,dx = \frac{1}{4\ln 10} \times 4x\left(\ln|4x| - 1\right) + c$$

Therefore,

$$\int \log 4x\,dx = \frac{x}{\ln 10}\left(\ln|4x| - 1\right) + c.$$

Example 9.59

Determine $\int \frac{\log_3 2x}{5x}dx$.

Solution

Given $\int \frac{\log_3 2x}{5x}dx$.

Convert the integral into natural logarithmic form as follows,

$$\int \frac{\log_3 2x}{5x}dx = \int \frac{\ln 2x}{5x\ln 3}dx$$

$$= \frac{1}{5\ln 3}\int \frac{\ln 2x}{x}dx$$

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Let $u = \ln 2x \Rightarrow du = \frac{1}{x}dx$.

Thus, $\int \frac{\log_3(2x)}{5x}dx = \frac{1}{5\ln 3}\int u\,du$

$= \frac{1}{5\ln 3}\left(\frac{u^2}{2}\right)+c$

$= \frac{1}{10\ln 3}u^2 + c$

But $u = \ln 2x$.

Therefore,

$\int \frac{\log_3(2x)}{5x}dx = \frac{1}{10\ln 3}(\ln 2x)^2 + c.$

Exercise 9.11

Determine each of the following:

1. $\int 7^x dx$
2. $\int \frac{\log x}{x}dx$
3. $\int 3(8^x)dx$
4. $\int \log(e^{-2x})dx$
5. $\int 3^{2x}dx$
6. $\int 6^{1-2y}dy$
7. $\int 4^{-x}dx$
8. $\int 9^{3-7x}dx$
9. $\int 10^{x-1}dx$
10. $\int \log 3x\,dx$
11. $\int \frac{\log_2 x}{x}dx$
12. $\int (2^x + \log 5x)dx$
13. $\int \log(2x+1)dx$
14. $\int (2x+1)\log(x^2+x-1)dx$
15. $\int \frac{\log 2x}{6}dx$
16. $\int \log\frac{3}{4}x\,dx$
17. $\int \left(\frac{1}{3}\right)^x dx$
18. $\int \frac{\log(\tan x)}{\cos^2 x}dx$
19. $\int \log_2 x\,dx$
20. $\int \frac{\log_5 3x}{2x}dx$

Definite integrals

When a function is integrated over a specified interval, then the integral is referred to as a definite integral.

Suppose that a function $f(x)$ is integrated over $a \le x \le b$ or $[a, b]$, then the integral is represented as $\int_a^b f(x)dx$. In this integral, a and b are called the boundaries or limits of integration, where a is the lower limit and b is the upper limit.

Definite integrals of polynomial, trigonometric, exponential, logarithmic, and rational functions

A definite integral defines the area of the region bounded by a curve or curves, the lines $x = a$, $x = b$, and the x-axis. Figure 9.4 shows the bounded region of the function $y = f(x)$ over the interval $[a,b]$ on the xy-plane.

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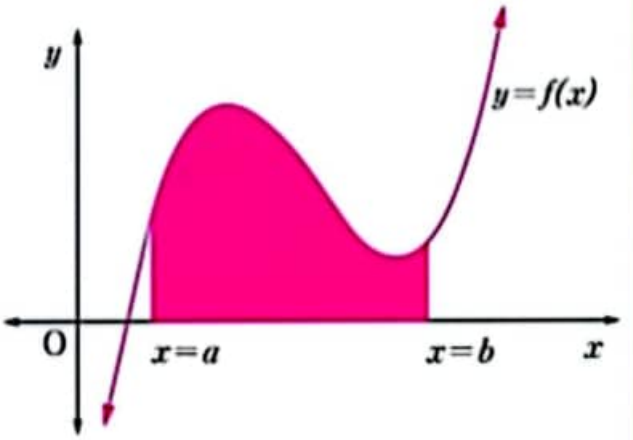

Figure 9.4: *Bounded region of the function*

The area of the region defined by the integral, $\int_a^b f(x)\,dx$ over the interval $a \le x \le b$ is evaluated as follows:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

That is, the value of the integral is obtained by taking the difference between the values of the anti-derivative $F(x)$ of the given function $f(x)$ at the upper limit b and the lower limit a of the variable x.

Example 9.60

Evaluate $\int_0^2 \left(x^3 + 2x^2 + x - 1\right)dx$.

Solution

Given $\int_0^2 \left(x^3 + 2x^2 + x - 1\right)dx$.

Integrate the function and substitute the given limits,

$$\int_0^2 \left(x^3 + 2x^2 + x - 1\right)dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - x\right]_0^2$$

$$= \left[\left(\frac{2^4}{4} + \frac{2(2)^3}{3} + \frac{(2)^2}{2} - 2\right) - \left(\frac{(0)^4}{4} + \frac{2(0)^3}{3} + \frac{(0)^2}{2} - 0\right)\right]$$

$$= \frac{28}{3}$$

Therefore, $\int_0^2 \left(x^3 + 2x^2 + x - 1\right)dx = \frac{28}{3}$.

Example 9.61

Show that $\int_1^2 \frac{x^3 + 4x^2 + 3x - 2}{x^2 + 4x + 3}dx = \frac{3}{2} + \ln\left(\frac{5}{6}\right)$.

Solution

Given $\int_1^2 \frac{x^3 + 4x^2 + 3x - 2}{x^2 + 4x + 3}dx$.

The integrand is an improper fraction.

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By division,

$$\frac{x^3+4x^2+3x-2}{x^2+4x+3} = x - \frac{2}{x^2+4x+3}$$

$$= x - \frac{2}{(x+1)(x+3)}$$

Let $\frac{2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$.

Solving for A and B gives, $A = 1$ and $B = -1$.

$$\Rightarrow \frac{2}{(x+1)(x+3)} = \frac{1}{x+1} - \frac{1}{x+3}$$

Thus,

$$\int_1^2 \frac{x^3+4x^2+3x-2}{x^2+4x+3}\,dx = \int_1^2 x\,dx - \int_1^2 \frac{1}{x+1}\,dx + \int_1^2 \frac{1}{x+3}\,dx$$

$$= \left[\frac{1}{2}x^2 - \ln|x+1| + \ln|x+3|\right]_1^2$$

$$= \left[\frac{1}{2}x^2 + \ln\left|\frac{x+3}{x+1}\right|\right]_1^2$$

$$= \left[\frac{1}{2}(4) + \ln\left(\frac{5}{3}\right)\right] - \left[\frac{1}{2}(1) + \ln\left(\frac{4}{2}\right)\right]$$

$$= 2 + \ln\frac{5}{3} - \ln 2 - \frac{1}{2}$$

$$= \frac{3}{2} + \ln\left(\frac{5}{3} \div 2\right)$$

$$= \frac{3}{2} + \ln\left(\frac{5}{6}\right)$$

Therefore, $\int_1^2 \frac{x^3+4x^2+3x-2}{x^2+4x+3}\,dx = \frac{3}{2} + \ln\left(\frac{5}{6}\right)$.

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Example 9.62

Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx$.

Solution

Given $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx$.

Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$.

$\Rightarrow dx = \frac{du}{\cos x}$

Change the limits of integration with respect to the variable u as follows:

Since the lower limit in terms of x is $\frac{\pi}{4}$, then the limit in terms of u will be $u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Similarly, since the upper limit in term of x is $\frac{\pi}{2}$, then the limit in terms of u will be $u = \sin\frac{\pi}{2} = 1$.

Now, the integral becomes,

$$\int_{\frac{\sqrt{2}}{2}}^{1} \frac{\cos x}{u^2} \times \frac{1}{\cos x} du = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{u^2} du$$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} u^{-2} du$$

$$= \left[-\frac{1}{u}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \left[\left(-\frac{1}{1}\right) - \left(-\frac{1}{\frac{\sqrt{2}}{2}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Therefore, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \sqrt{2} - 1$.

Alternatively, the integral is evaluated as an indefinite integral and the limits are applied after integration where the constant term is not considered. That is;

Given $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx$.

Let $u = \sin x \Rightarrow du = \cos x\, dx$.

Thus, $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{du}{u^2}$

$= -\frac{1}{u}$.

Substitute back $u = \sin x$.

Thus, $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}$.

Now, introduce the limits:

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \left[-\frac{1}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[-\frac{1}{\sin\frac{\pi}{2}} - \left(\frac{-1}{\sin\frac{\pi}{4}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Therefore, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \sqrt{2} - 1$.

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Example 9.63

Show that

$$\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx = 2\ln\left(\frac{4}{3}\right).$$

Solution

Given $\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx$.

Let $u = \sqrt{e^x} \Rightarrow u^2 = e^x$.

$$\Rightarrow 2u\,du = e^x\,dx \Leftrightarrow dx = \frac{2u}{e^x}du$$

$$\Rightarrow dx = \frac{2u}{u^2}du \Rightarrow dx = \frac{2}{u}du$$

Change the limits of integration for x into those for the variable u as follows:

x	$\ln 4$	$\ln 9$
u	2	3

Now, the integral becomes,

$$\int_2^3 \frac{u}{1+u}\times\frac{2}{u}du = 2\int_2^3 \frac{1}{1+u}du$$

$$= 2\left[\ln|1+u|\right]_2^3$$

$$= 2[\ln 4 - \ln 3]$$

$$= 2\ln\left(\tfrac{4}{3}\right)$$

Therefore,

$$\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}}\,dx = 2\ln\left(\frac{4}{3}\right).$$

Example 9.64

Evaluate $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx$ correct to five significant figures.

Solution

Given $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx$.

Let $u = \ln 2x \Rightarrow du = \frac{1}{x}dx$ and $dv = x^4dx \Rightarrow v = \frac{x^5}{5}$.

Using the method of integration by parts, the integral becomes,

$$\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx = \left[\left(\frac{x^5}{5}\right)\ln 2x\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1}\left(\frac{1}{x}\right)\left(\frac{x^5}{5}\right)dx$$

$$\Rightarrow \int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx = \left[\left(\frac{x^5}{5}\right)\ln 2x\right]_{\frac{1}{2}}^{1} - \frac{1}{5}\left[\left(\frac{x^5}{5}\right)\right]_{\frac{1}{2}}^{1}$$

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$$\Rightarrow \int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx = \frac{1}{5}\left[1\times \ln 2 - \left(\frac{1}{2}\right)^5 \ln 1\right]_{\frac{1}{2}}^{1} - \frac{1}{25}\left[1-\left(\frac{1}{2}\right)^5\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{5}[1\times \ln 2] - \frac{1}{25}\left[1-\left(\frac{1}{2}\right)^5\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{5}\ln 2 - \frac{1}{25} + \frac{1}{800}$$

$$= \frac{1}{5}\ln 2 - \frac{31}{800}$$

$$= 0.099879436$$

Therefore, $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx = 0.099879.$

Exercise 9.12

Evaluate each of the following:

1. $\int_{-1}^{3}(6-2x)dx$
2. $\int_{-4}^{0}\left(x^2+x+1\right)dx$
3. $\int_{0}^{4}(x^3-2x-3\sqrt{x})dx$
4. $\int_{2}^{6}(3t^2-2t+1)dt$
5. $\int_{0}^{\ln 2}(7e^{-x}+4e^{x})dx$
6. $\int_{1}^{2}\frac{y^3+1}{y^2}dy$
7. $\int_{2}^{4}\frac{dx}{x}$
8. $\int_{0}^{\frac{\pi}{4}}\frac{4\sin x}{3\tan x}dx$
9. $\int_{-1}^{1}\left(x^3+7x-3\cos\frac{\pi}{2}x\right)dx$
10. $\int_{1}^{4}\left(\frac{x^4-x^3+\sqrt{x}-1}{x^2}\right)dx$
11. $\int_{1}^{2}\frac{t^2+5t^6-4}{t^3}dt$
12. $\int_{1}^{2}\left(3r+\frac{1}{r^2}-\frac{1}{r^4}\right)dr$
13. $\int_{0}^{1}(6x-9)^8dx$
14. $\int_{-1}^{0}(1-2x)^{\frac{3}{2}}dx$

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15. $\int_1^3 (3t-1)^{-3}dt$

16. $\int_{-3}^{-1} 3(1-x)^{-1}dx$

17. $\int_0^3 e^{\frac{x}{3}}dx$

18. $\int_0^{\frac{1}{3}} \frac{dt}{4-3t}$

19. $\int_0^{\frac{\pi}{2}} 6\cos(\pi-3x)dx$

20. $\int_1^3 \frac{dx}{5-7x}$

21. $\int_{\frac{\pi}{2}}^{\pi} \tan\left(\theta+\frac{\pi}{4}\right)d\theta$

22. $\int_1^4 \frac{dx}{2x-1}$

23. $\int_{-2}^1 e^{3-4x}dx$

24. $\int_0^{\frac{\pi}{2}} 5\sin\left(\frac{1}{2}x-1\right)dx$

25. $\int_{\ln 2}^{\ln 4} e^x dx$

26. $\int_0^1 e^{2x-2}dx$

27. $\int_2^4 \frac{1}{4}\sqrt[4]{x^5}dx$

28. $\int_0^1 x^3\sqrt{1-x^2}\,dx$

29. $\int_{-2}^4 \frac{2t+1}{t^2+t+3}dt$

30. $\int_0^2 \frac{1}{t^2-6t+9}dt$

31. $\int_0^{\frac{\pi}{2}} \cos\theta d\theta$

32. $\int_0^{\pi} 3\sin\theta\, d\theta$

33. $\int_0^{\frac{\pi}{2}} (\cos 2\theta+\sin 2\theta)d\theta$

34. $\int_1^e \frac{1+\ln x}{x}dx$

35. $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}dx$

36. $\int_{-1}^0 \frac{\ln(x+2)}{2x+4}dx$

37. $\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta$

38. $\int_0^9 \frac{\sqrt{x}}{1+\sqrt{x}}dx$

39. $\int_1^{\sqrt{10}} x\sqrt{x^2-1}\,dx$

40. $\int_{\sqrt{\ln\frac{4}{3}}}^{\sqrt{\ln\pi}} xe^{x^2}\cos(3e^{x^2})dx$

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41. $\int_0^1 (x+2)e^{x^2+4x}dx$

42. $\int_{\ln 2}^{\ln 8} \frac{1-e^t}{1+e^t}dt$

43. $\int_{e^2}^{e^5} \frac{dx}{x(\ln x)^2}$

44. $\int_1^9 \frac{\ln x}{\sqrt{x}}dx$

45. $\int_1^e \frac{\ln x}{x^7}dx$

46. $\int_1^2 (x+1)^2 \ln x\,dx$

47. $\int_2^3 \frac{1}{x^2-1}dx$

48. $\int_0^1 \frac{x-1}{x^2+3x+2}dx$

49. $\int_0^{\frac{2\pi}{3}} \cos^3 x\,dx$

50. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x\,dx$

51. $\int_0^{\pi} \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta d\theta$

52. $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^3 x\,dx$

53. $\int_0^{\frac{\pi}{2}} \sin^3 x\cos^2 x\,dx$

54. $\int_0^{2\pi} \cos^2 3x\,dx$

55. $\int_0^1 \frac{5}{x^2+2x+1}dx$

56. $\int_0^1 \frac{1}{1+x^2}dx$

57. $\int_0^1 \frac{1}{x^2+4x+8}dx$

58. $\int_2^3 \frac{4}{13-4x+x^2}dx$

59. $\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}d\theta$

60. $\int_0^1 \frac{2x-7}{\sqrt{9-x^2}}dx$

Applications of integration

Definite integrals are used to find the area under a curve, area enclosed between two curves, volume of a solid of revolution about the x-axis, about the y-axis, and about any line, length of an arc, and area of a sector.

Area under a curve and area enclosed between two curves

Integration can be used to find the area under a curve and the area enclosed between two curves.

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Area under a curve and ignore the *x*-axis

When calculating the area under a curve, the first step is to make a sketch of the curve. Remember that the area lying above the *x*-axis will have a positive value, whereas the area lying below the *x*-axis will have a negative value. In some cases, the required area may lie above and below the *x*-axis.

Consider Figure 9.5 which shows the curve $y = f(x)$, enclosing the shaded area with the positive *x*-axis. If the area is denoted by A, then it is given by,

$$A = \left| \int_a^b f(x)dx - \int_b^c f(x)dx \right| \text{ or}$$

$$A = \left| \int_a^b f(x)dx \right| + \left| \int_b^c f(x)dx \right|$$

where *a*, *b*, and *c* are values of *x*.

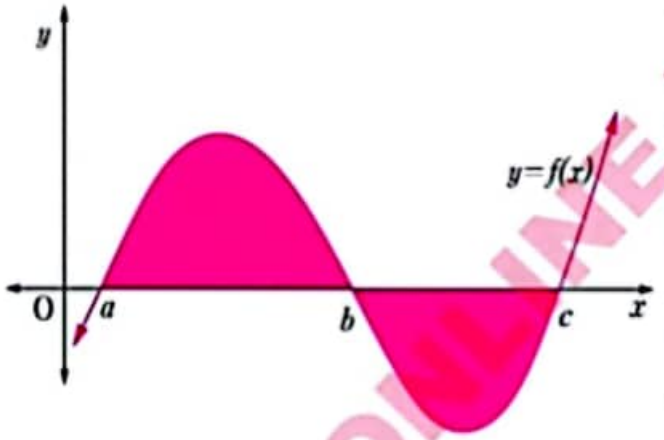


Figure 9.5: *Area under a curve*

Note that, the area under the curve is taken as the magnitude of A. When calculating the area under the curve $y = f(x)$, follow the following steps:

1. Sketch the curve and identify the area enclosed.
2. Determine the limits of integration if not given.
3. Set up the definite integral.
4. Integrate to get the area.

Example 9.65

Calculate the area enclosed between the curve $f(x) = -x^2 - 2x$ and the *x*-axis.

Solution

The curve intersects the *x*-axis where $f(x) = 0$.

That is, $f(x) = -x^2 - 2x$

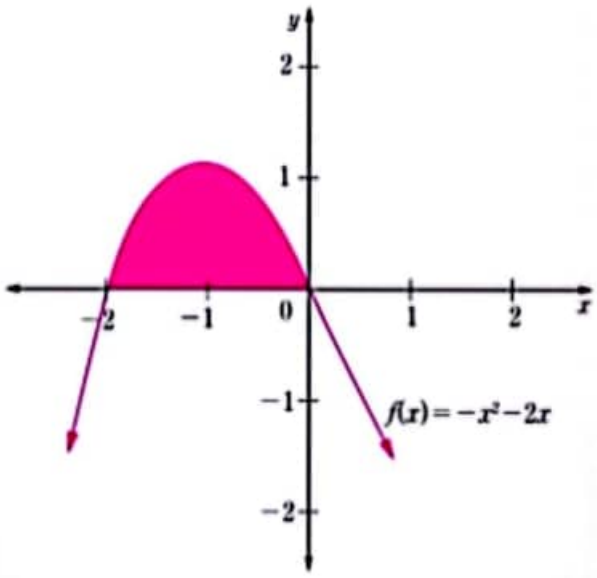
$\Rightarrow -x(x+2) = 0$

Either $-x = 0$ or $x + 2 = 0$

$\Rightarrow x = 0$ or $x = -2$

Hence, $f(x)$ intersects the *x*-axis at $x = 0$ and $x = -2$.

The coefficient of x^2 is negative. Thus, the curve for $f(x)$ opens downwards as shown in the following figure.



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From the figure, the area between the curve and the x-axis is shaded and is given by;

From $A = \int_a^b f(x)dx$, it implies that

$$A = \int_{-2}^{0}(-x^2 - 2x)dx$$

$$= \left[\frac{-1}{3}x^3 - x^2\right]_{-2}^{0}$$

$$= \left[-\frac{1}{3}(0) - (0)\right] - \left[-\frac{1}{3}(-2)^3 - (-2)^2\right]$$

$$= -\left(\frac{8}{3} - 4\right)$$

$$= -\left(-\frac{4}{3}\right)$$

$$= \frac{4}{3} \text{ square units.}$$

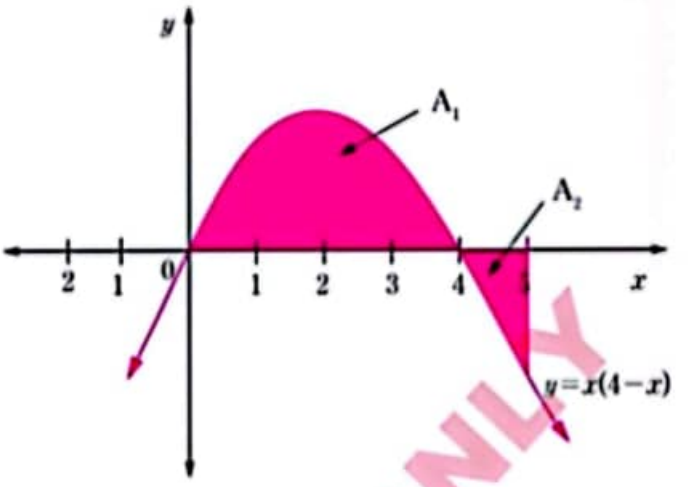
Therefore, the area is $\frac{4}{3}$ square units.

Example 9.66

Find the area enclosed between the curve $y = x(4-x)$ and the x-axis from $x = 0$ to $x = 5$.

Solution

The following is a sketch of the curve $y = x(4-x)$.



The sketch shows that the required area is divided into two parts. One part lies above the x-axis and hence it has a positive area. The other part lies below the x-axis and it has negative area.

Using $A = \int_a^b y\,dx$ and calculating the two areas separately gives,

$$A_1 = \int_0^4 x(4-x)dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= \left(32 - \frac{64}{3}\right) - 0$$

$$= \frac{32}{3}$$

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Thus, $A_1 = \frac{32}{3}$ square units.

Similarly,

$$A_2 = \int_4^5 x(4-x)\,dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_4^5$$

$$= \left(50 - \frac{125}{3}\right) - \left(32 - \frac{64}{3}\right)$$

$$= (50-32) + \left(\frac{64}{3} - \frac{125}{3}\right)$$

$$= 18 + \left(-\frac{61}{3}\right)$$

$$= \frac{-7}{3}$$

Thus,

$$A_2 = \left|\frac{-7}{3}\right| = \frac{7}{3} \text{ square units.}$$

The total area under the curve between $x = 0$ and $x = 5$ is the sum of the numerical values of the two areas, A_1 and A_2. That is,

$$A = \frac{32}{3} + \frac{7}{3} \text{ square units}$$

$$= 13 \text{ square units}$$

Therefore, the area is 13 square units.

Alternatively:

From $A = \left|\int_a^b f(x)\,dx - \int_b^c f(x)\,dx\right|$, it follows that

$$= \left|\int_0^4 x(4-x)\,dx - \int_4^5 x(4-x)\,dx\right|$$

$$= \left|\int_0^4 (4x-x^2)\,dx - \int_4^5 (4x-x^2)\,dx\right|$$

$$= \left|\left[2x^2 - \frac{x^3}{3}\right]_0^4 - \left[2x^2 - \frac{x^3}{3}\right]_4^5\right|$$

$$= \left|\left[\left(32 - \frac{64}{3}\right) - 0\right] - \left[(50-32) - \left(32 - \frac{64}{3}\right)\right]\right|$$

$$= \left|\frac{32}{3} - \left(\frac{-7}{3}\right)\right|$$

Thus, $A = |13| = 13$ square units.

Therefore, the area is 13 square units.

Note that, it is possible to evaluate the definite integral $A = \int_0^5 x(4-x)\,dx$, but this would not give the correct answer for the area under the curve.

Example 9.67

A curve is defined by the parametric equations $x = at^2$ and $y = 2at$. Find the area bounded by the curve, the x-axis and the ordinates at $t = 1$ and $t = 2$.

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Solution

Given that $A = \left|\int_a^b y\,dx\right|$, where a and b are the limits or bounding values of the variable. Replacing y by $2at$ gives, $A = \left|\int_a^b 2at\,dx\right|$.

But, it is not possible to integrate a function of t with respect to x directly. Thus, change the variable of the integral.

Given $x = at^2$.

$\Rightarrow \frac{dx}{dt} = 2at$, so $dx = 2at\,dt$

Now, the integral becomes,

$$A = \left|\int_1^2 (2at).(2at)\,dt\right|$$

$$= \left|\int_1^2 4a^2t^2\,dt\right|$$

$$= \left|4a^2\left[\frac{t^3}{3}\right]_1^2\right|$$

$$= 4a^2\left|\left(\frac{8}{3} - \frac{1}{3}\right)\right| = \frac{28a^2}{3}$$

Therefore, the area bounded by the curve is $\frac{28a^2}{3}$ square units.

Area under a curve and the y-axis

Suppose it is required to find the area between a curve $y = f(x)$ and the y-axis, from $y_1 = a$ to $y_2 = b$.

Consider a sketch of $y = f(x)$ as shown in Figure 9.6.

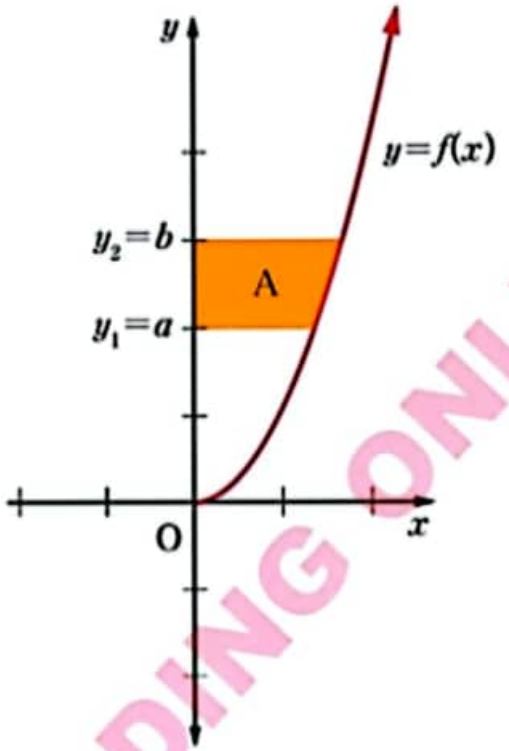

Figure 9.6: *Area between a curve $y = f(x)$ and the y-axis*

The enclosed area between the curve $y = f(x)$ and the y-axis, from $y_1 = a$ to y_2 b is given by:

$A = \int_a^b x\,dy$, where x is a function of y.

Example 9.68

Find the area enclosed between the curve $y^2 = 4 - x$ and the y-axis.

Solution

Sketch the curve $y^2 = 4 - x$.

The curve crosses the y-axis at $(0, 2)$ and $(0, -2)$, and crosses the x- axis at $(4, 0)$ as shown in the following figure.

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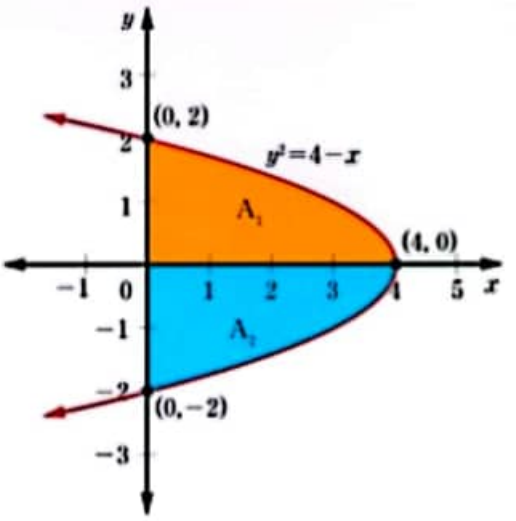


From the figure, the area A enclosed between the curve and the y-axis is given by:

$$A = \int_{y=-2}^{y=2} x\,dy$$

In this case, the required area is lying below the curve $x = 4 - y^2$ and above the x-axis. Thus, the required area A is divided into two areas, $A_1 = \int_{-2}^{0} x\,dy$ and $A_2 = \int_{0}^{2} x\,dy$.

From $y^2 = 4 - x$, $x = 4 - y^2$.

Thus, $A_1 = \int_{-2}^{0} (4 - y^2)\,dy$

$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{0}$$

$$= 0 - \left(-8 + \frac{8}{3}\right)$$

$$= -\left(-\frac{16}{3}\right)$$

$$= \frac{16}{3} \text{ square units}$$

$$A_2 = \int_{0}^{2} (4 - y^2)\,dy$$

$$= \left[4y - \frac{y^3}{3}\right]_{0}^{2}$$

$$= 4(2) - \frac{8}{3}$$

$$= \frac{16}{3} \text{ square units.}$$

Hence, $A = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$ square units.

Therefore, the area enclosed is $\frac{32}{3}$ square units.

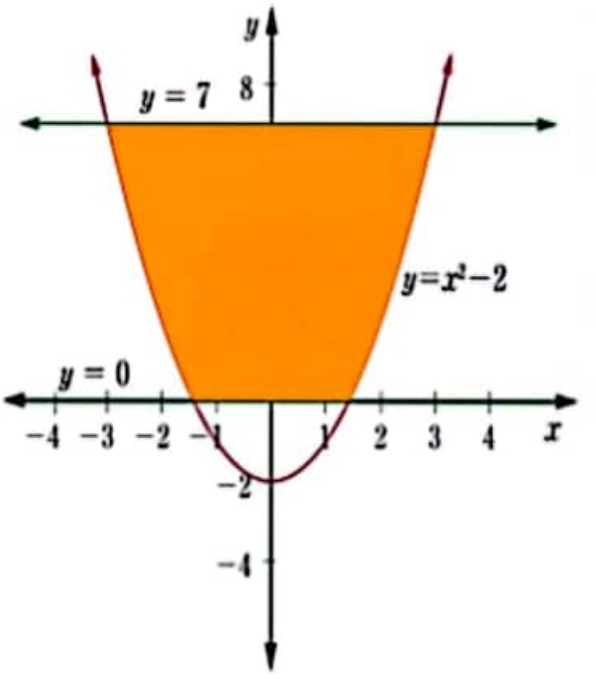
Example 9.69

Find the area enclosed by $y = x^2 - 2$, x-axis and the line $y = 7$.

Solution

Sketch the curve $y = x^2 - 2$.

The curve is symmetrical about the y-axis, crosses x-axis at $(-2,0)$ and $(2,0)$, and crosses y-axis at $(0,-2)$ as shown in the following figure.



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From the figure, the required area A is given by;

$$A = \int_{y=0}^{y=7} x\,dy$$

From $y = x^2 - 2$.

$$\Rightarrow x = \sqrt{y+2}$$

Thus, $A = \int_{y=0}^{y=7} \sqrt{y+2}\,dy$.

Let $u = y+2 \Rightarrow du = dy$.

$$\Rightarrow A = \int_{u=2}^{u=9} u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=2}^{u=9}$$

$$= \frac{2}{3}\left[9^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= 16.1144 \text{ Square units.}$$

Therefore, the area enclosed is 16.1144 square units.

Area of a region enclosed between two curves

The area of a region enclosed between two curves is obtained by determining the integral of the difference between the two functions using common limits. The common limits are obtained from the points of intersection of the two curves.

In Figure 9.7, the area enclosed between the two curves $f(x)$ and $g(x)$ is given by:

$$A = \left|\int_a^b [f(x) - g(x)]dx\right| \quad \text{or}$$

$$A = \int_a^b [(\text{upper function}) - (\text{lower function})]\,dx,$$

where a and b are the values of x at which $f(x)$ and $g(x)$ intersect.

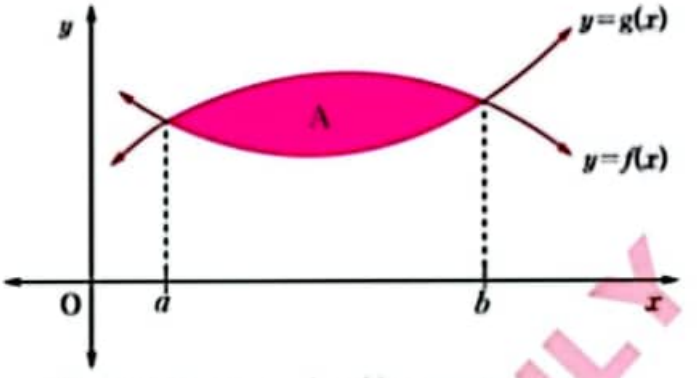

Figure 9.7: *Area enclosed between two curves*

Example 9.70

Determine the area enclosed between the curve $y = (x-1)^2$ and the straight line $y = x+1$.

Solution

The limits of the enclosed area are given by the x-values at the points of intersection. These can be obtained by solving the two equations simultaneously. That is,

$$(x-1)^2 = x+1$$

$$\Rightarrow x^2 - 2x + 1 = x + 1$$

$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x-3) = 0$$

$$\Rightarrow x = 0 \text{ or } x - 3 = 0$$

Hence, the curve and the line meet at $x = 0$ and $x =$.

The area enclosed between the curve and the straight line at $x = 0$ and $x = 3$ is the shaded region shown in the following figure.

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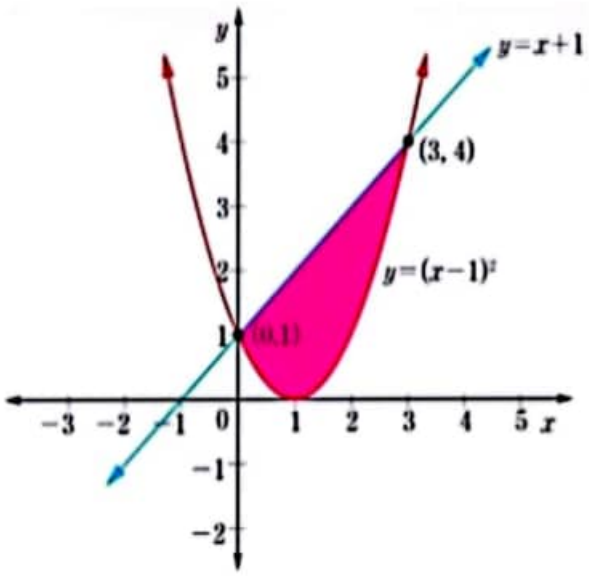


From the figure, the area A enclosed between the curve and the straight line is given by,

$$A = \int_0^3 \left((x+1)-(x-1)^2\right)dx$$

$$= \int_0^3 \left(x+1-x^2-1+2x\right)dx$$

$$= \int_0^3 \left(3x-x^2\right)dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{3}{2}(3)^2 - \frac{(3)^3}{3}\right] - [0-0]$$

$$= \left(\frac{3}{2}\right)(9) - \frac{27}{3} - 0$$

$$= \frac{9}{2}$$

= 4.5 square units

Therefore, the area enclosed between the curve and the straight line is 4.5 square units.

Example 9.71

Find the area enclosed between the curves representing the functions $f(x) = x^2 + 2x + 2$ and $g(x) = -x^2 + 2x + 10$.

Solution

The limits of the enclosed area are given by the x-values at the points of intersection. These can be obtained by solving the two equations simultaneously. That is,

$$x^2 + 2x + 2 = -x^2 + 2x + 10$$

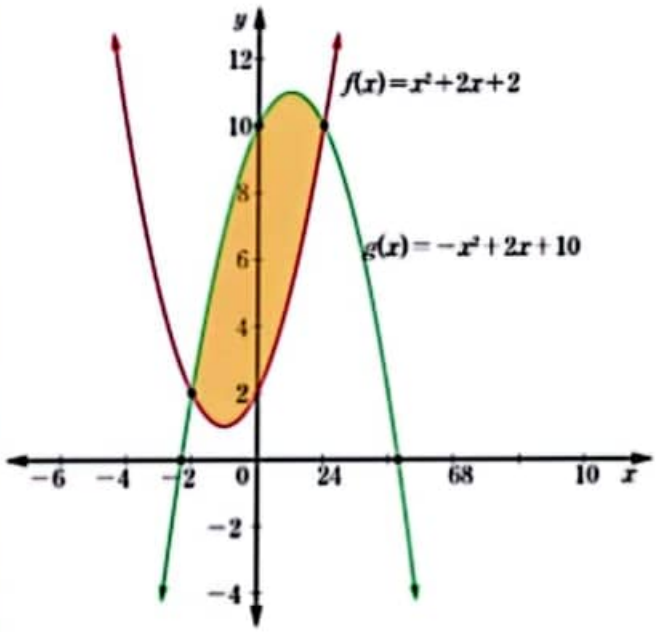
$$\Rightarrow 2x^2 - 8 = 0$$

$$\Rightarrow (x-2)(x+2) = 0$$

$$\Rightarrow x = 2 \text{ and } x = -2$$

Hence, the two curves intersect at $x = 2$ and $x = -2$.

The area enclosed between the two curves within $x = 2$ and $x = -2$ is the shaded region as shown in the following figure.



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From the figure, the area A enclosed between the two curves given by;

$$A = \int_{-2}^{2}\left[\left(-x^2+2x+10\right)-\left(x^2+2x+2\right)\right]dx$$

$$= \int_{-2}^{2}\left(-2x^2+8\right)dx = \left[-\frac{2x^3}{3}+8x\right]_{-2}^{2}$$

$$= \left(-\frac{16}{3}+16\right)-\left(\frac{16}{3}-16\right) = \frac{32}{3}+\frac{32}{3}$$

$$= \frac{64}{3} \text{ square units}$$

Therefore, the area of the region enclosed between the two curves is $\frac{64}{3}$ square units.

Length of an arc

The length of a curve can be obtained by taking the sum of the lengths of small chords $\overline{P_1P_2}$, $\overline{P_2P_3}$,... on the curve as shown in Figure 9.8.

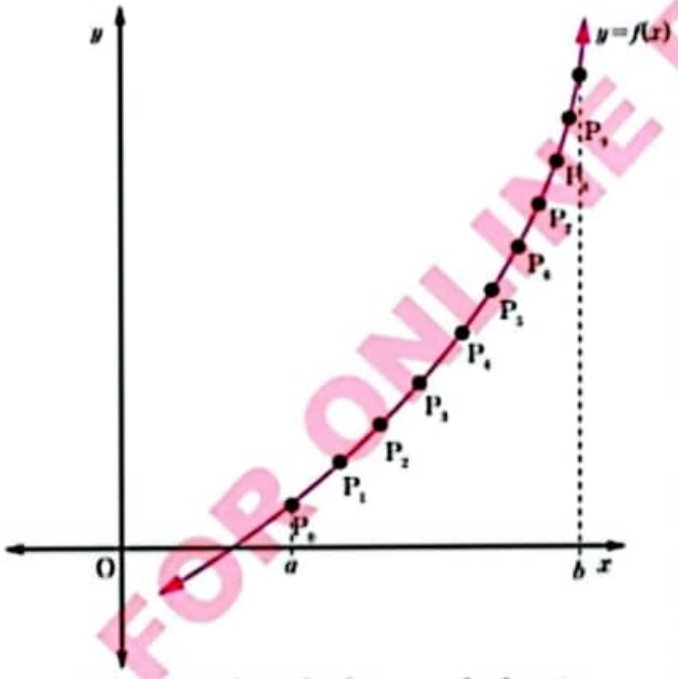

Figure 9.8: *Length of an arc of a function*

Let δx and δy be small increments in x and y, respectively, from P_{r-1} to P_r as shown in Figure 9.9.

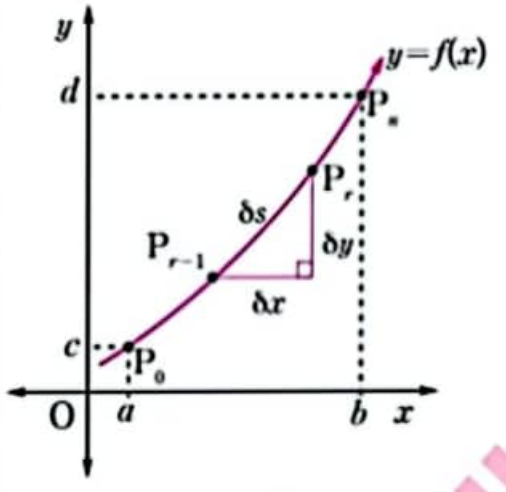

Figure 9.9: *Length of an arc*

From Figure 9.9, apply the Pythagoras' theorem,

$$\left(\overline{P_{r-1}P_r}\right)^2 = (\delta x)^2+(\delta y)^2.$$

But $\overline{P_{r-1}P_r} = \delta s$.

Factorize $(\delta x)^2$ to obtain,

$$(\delta s)^2 = \left[1+\left(\frac{\delta y}{\delta x}\right)^2\right](\delta x)^2 \qquad (9.1)$$

For an equation of the form $y = f(x)$, it is convenient to use the variable x. Equation (9.1) can be written as,

$$\delta s = \sqrt{\left(1+\left(\frac{\delta y}{\delta x}\right)^2\right)(\delta x)^2}$$

Hence, $\delta s = \sqrt{\left(1+\left(\frac{\delta y}{\delta x}\right)^2\right)}\delta x.$

The sum of the δs of all the chords gives the length of an arc. That is,

$$s = \int_a^b \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}dx,$$

where a and b are values of x at the end points of the arc.

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Therefore, the arc length from $x = a$ to $x = b$ is given by,

$$s = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\,dx \text{ if}$$

$y = f(x)$, $a \le x \le b$, and

$$s = \int_c^d \sqrt{\left(1 + \left(\frac{dx}{dy}\right)^2\right)}\,dy$$

if $x = h(y)$, $c \le y \le d$.

Length of an arc described by parametric equations

Suppose that the curve is in parametric form, $x = f(t)$ and $y = g(t)$. It is more convenient to find the length of the arc when the integral is evaluated with respect to t.

From equation (9.1) in the previous section, the variable t is introduced in the equation as follows:

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$= \left(\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right) \times (\delta t)^2$$

Thus, $\delta s = \sqrt{\left(\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right)}\,\delta t.$

The sum of the δs for all the chords gives the length of the arc. That is,

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

where t_1 and t_2 are the values of t corresponding to the end points of the arc.

Length of an arc in polar coordinates

If $r = f(\theta)$ describes a polar curve, the arc of this curve swept out as the angle θ varies from θ_1 to θ_2 can be derived as follows:

Suppose $r = f(\theta)$ is a given polar curve, where the angle θ varies over the interval $\theta_1 < \theta < \theta_2$. By converting to cartesion coordinates, the following parametric form is obtained:

$$x(\theta) = r\cos\theta = f(\theta)\cos\theta \quad (9.2)$$

$$y(\theta) = r\sin\theta = f(\theta)\sin\theta \quad (9.3)$$

Differentiating equations (9.2) and (9.3) with respect to θ gives;

$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad (9.4)$$

$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta \quad (9.5)$$

Squaring and adding equations (9.4) and (9.5) give,

$$x'(\theta)^2 + y'(\theta)^2 = \left[f'(\theta)\cos\theta - f(\theta)\sin\theta\right]^2 + \left[f'(\theta)\sin\theta + f(\theta)\cos\theta\right]^2$$

$$= f'(\theta)^2\cos^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta +$$

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$$2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\cos^2\theta$$
$$= \left(f'(\theta)^2 + f(\theta)^2\right)\sin^2\theta + \left[f'(\theta)^2 + f(\theta)^2\right]\cos^2\theta$$
$$= \left(f'(\theta)^2 + f(\theta)^2\right)\left(\sin^2\theta + \cos^2\theta\right)$$
$$\Rightarrow x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2$$

Using the formula for parametric arc length, the arc length of this curve is obtained as,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + r^2}\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of the arc is given as $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Example 9.72

Find the length of the arc in the first quadrant of the curve $y = 2x^{\frac{3}{2}}$, from $x = 0$ to $x = \frac{1}{3}$.

Solution

The length of an arc is given by,

$$S = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$$

Given $y = 2x^{\frac{3}{2}}$, then $\frac{dy}{dx} = 2 \times \frac{3}{2} x^{\frac{3}{2}-1}$.

$\frac{dy}{dx} = 3x^{\frac{1}{2}}$

Thus, $S = \int_0^{\frac{1}{3}} \sqrt{1 + \left(3x^{\frac{1}{2}}\right)^2}\, dx$

$= \int_0^{\frac{1}{3}} \sqrt{1 + 9x}\, dx$

$= \left[\frac{2(1+9x)^{\frac{3}{2}}}{27}\right]_0^{\frac{1}{3}}$

$= \frac{2}{27}\left[\left(1 + 9\left(\frac{1}{3}\right)\right)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right]$

$= \frac{2}{27}(8-1)$

$= \frac{14}{27}$.

Therefore, the length of the arc is $\frac{14}{27}$ units.

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Example 9.73

Find the length of the arc of the cardioid $r = 1 + \sin\theta$ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

Solution

The length of the arc is given by

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Given $r = 1 + \sin\theta$ then $\frac{dr}{d\theta} = \cos\theta$

Thus,

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1+\sin\theta)}\, d\theta$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{1+\sin\theta}\cdot\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta$$

Let $u = \sqrt{1-\sin\theta} \Rightarrow u^2 = 1 - \sin\theta$.

$\Rightarrow 2u\,du = -\cos\theta\, d\theta$

$\Rightarrow -2u\,du = \cos\theta\, d\theta$

By changing limits;

θ	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
u	$\sqrt{2}$	0

$$\Rightarrow S = \sqrt{2}\int_{\sqrt{2}}^{0} \frac{-2u}{u}\, du$$

$$= -2\sqrt{2}\int_{\sqrt{2}}^{0} du = -2\sqrt{2}[u]_{\sqrt{2}}^{0}$$

$$= -2\sqrt{2}\left[0 - \sqrt{2}\right]$$

$$= 4$$

Therefore, the length of the arc is 4 units.

Example 9.74

Determine the length of the curve given by the parametric equations $x = 3\sin(3t)$ and $y = 3\cos(3t)$, $0 \le t \le 2\pi$.

Solution

Given

$x = 3\sin(3t)$ and $y = 3\cos(3t)$.

$$\Rightarrow \frac{dx}{dt} = 9\cos(3t)$$

and $\frac{dy}{dt} = -9\sin(3t)$.

The length of the curve is given by;

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$$

$$= \int_0^{2\pi} \sqrt{(9\cos 3t)^2 + (-9\sin 3t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9^2\left(\cos^2 3t + \sin^2 3t\right)}\, dt$$

$$= \int_0^{2\pi} 9\, dt$$

$$= [9t]_0^{2\pi}$$

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$=[9\times 2\pi]-[9(0)]$

$=18\pi.$

Therefore, the length of the curve is 18π units.

Example 9.75

Find the length of the arc of the curve $6xy=3+x^4$ between the points whose abscissae are 1 and 4.

Solution

Given $6xy=3+x^4 \Rightarrow y=\frac{1}{2x}+\frac{x^3}{6}$.

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2x^2}+\frac{x^2}{2}=\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\left[\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)\right]^2_]$$

$$=1+\frac{1}{4}\left(x^4-2+\frac{1}{x^4}\right)$$

$$=\left[\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\right]^2_]$$

The length of the curve is given by;

$$S=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_1^4\sqrt{\left[\tfrac{1}{2}\left(x^2+\tfrac{1}{x^2}\right)\right]^2}\,dx$$

$$=\int_1^4\tfrac{1}{2}\left(x^2+\tfrac{1}{x^2}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_1^4$$

$$=\frac{1}{2}\left[\left(\frac{64}{3}-\frac{1}{4}\right)-\left(\frac{1}{3}-1\right)\right]_]$$

$=10\frac{7}{8}$ units

Therefore, the length of the arc is $10\frac{7}{8}$ units.

Exercise 9.13

1. Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$.
2. If $x=a\sin\theta$, $y=b\cos\theta$, find the area under the curve between $\theta=0$ and $\theta=\pi$.
3. If $x=-\sin\theta, y=1-\cos\theta$, find the area under the curve between $\theta=0$ and $\theta=\pi$.
4. Find the area bounded by the curves $y=3e^{2x}$ and $y=3e^{-x}$, and the ordinates at $x=-1$ and $x=2$.
5. Determine the area of a region enclosed between the curve $x=\frac{1}{2}y^2-3$ and the straight line $y=x-1$.
6. The parametric equations of a curve are $x=2+2t$ and $y=2\sin\frac{\pi t}{10}$. Find the area under the curve between $t=0$ and $t=10$.

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7. Determine the area enclosed by the arc of the cycloid, $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ from $\theta = 0$ and $\theta = 2\pi$.

8. Find the area enclosed by the curves $y = \sin x$ and $y = \sin^2 x$, between $x = 0$ and $x = \frac{\pi}{3}$.

9. Find the area of the region enclosed between the two curves, $y = x^3 - 6x^2 + 8x$ and $y = x^2 - 4x$.

10. Show that the area enclosed by the curve, $x = a(2t - \sin 2t)$, $y = 2a\sin^2 t$ and the x-axis between $t = 0$ and $t = \pi$ is $3\pi a^2$ square units.

11. Find the length of the curve $y = 16x^{\frac{3}{2}}$ over the interval $0 \le x \le 1$.

12. Find the length of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, between the lines $x = 4$ and $x = 8$.

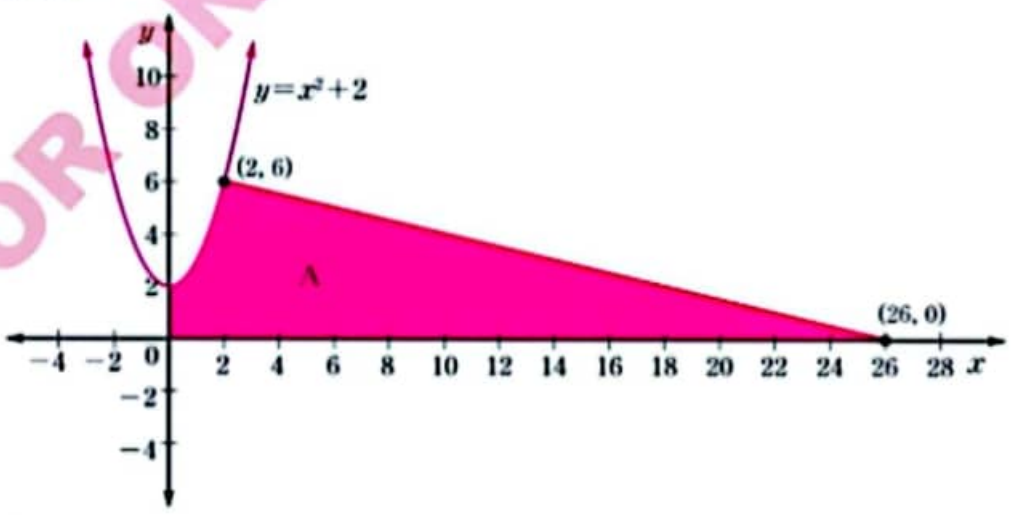
13. Find the length of the arc of the curve $y^2 = 8x^3$, between the lines $x = 1$ and $x = 3$.

14. Find the length of the arc of the curve $x^2 + y^2 = r^2$, between the points whose abscissae are $x = 0$ and $x = r$.

15. Determine the length of the curve with parametric equations $x = t\sin t$, $y = t\cos t$; $0 \le t \le 2\pi$, correct to 4 decimal places.

16. Compute the length of the curve defined by $x = 2\cos^2\theta$ and $y = 2\cos\theta\sin\theta$, between 0 and π.

17. The region A is bounded by the curve $y = x^2 + 2$, the x, y-axes, and the line joining the points (2, 6) and (26, 0) as shown in the figure. Find the area A.



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18. Find the length of the curve $r = a\sin^3\frac{\theta}{3}$ from $\theta = 0$ to $\theta = 3\pi$.

19. Prove that the length of the curve $r = ae^{k\theta}$ from $\theta = 0$ to $\theta = 2\pi$ is $\frac{a\sqrt{1+k^2}}{k}\left(e^{2\pi k} - 1\right)$.

20. Verify that the length of the curve $x = 2\cos^3\theta$ and $= 2\sin^3\theta$ between the point corresponding to $\theta = 0$ to $\theta = \frac{\pi}{2}$ is 3 units.

Volumes of a solid of revolution

If the plane in Figure 9.10(a) bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated through a complete revolution about the x-axis, it will generate a solid symmetric about the x-axis.

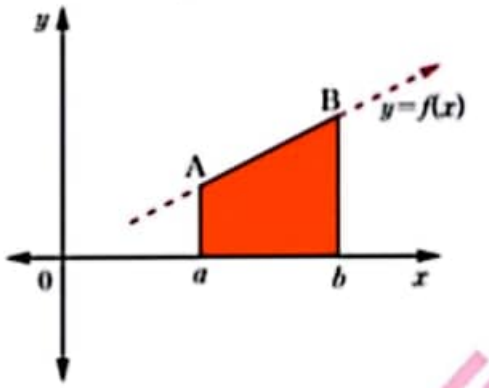

Figure 9.10 (a): *The curve and its boundaries*

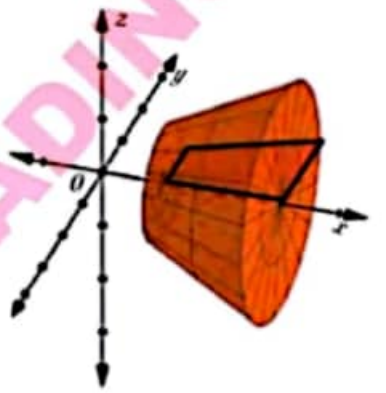

Figure 9.10 (b): *Solid rotated through a complete revolution about the x-axis*

Let V be the volume of the solid of revolution obtained after making a complete rotation about the x-axis. To find V in Figure 9.10(b), consider a thin strip of the original plane as in Figures 9.11(a) and 9.11(b).

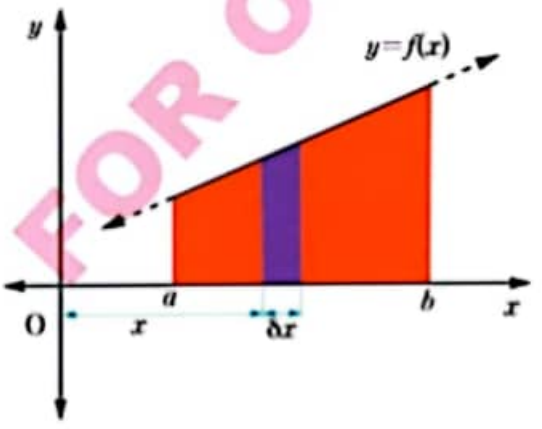

Figure 9.11(a): *Thin strip of the plane*

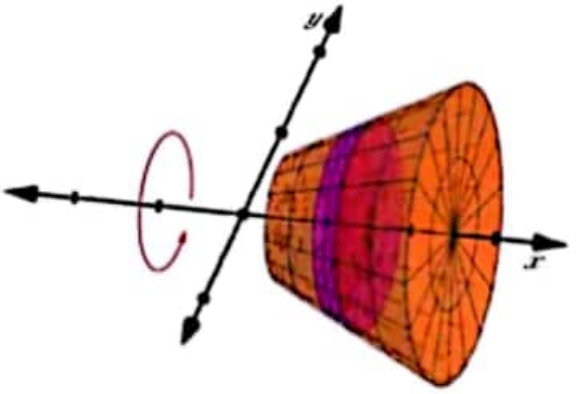

Figure 9.11(b): *Solid sliced into small cylinders*

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The volume of the solid of revolution generated when the strip is rotated completely about the x-axis is approximately equal to the volume of the cylinder, that is $\pi y^2 \delta x$. Since the solid generated is a flat cylinder, divide the plane into a number of such strips. Each strip will contribute its own flat disc with volume $\pi y^2 \delta x$ as shown in Figure 9.12 (a).

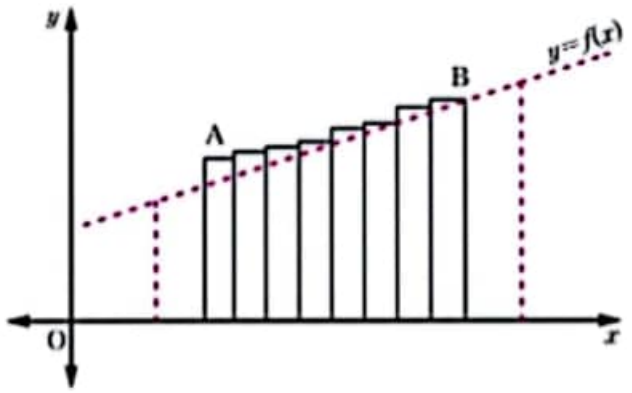

Figure 9.12(a): *Slices of a rotated solid*

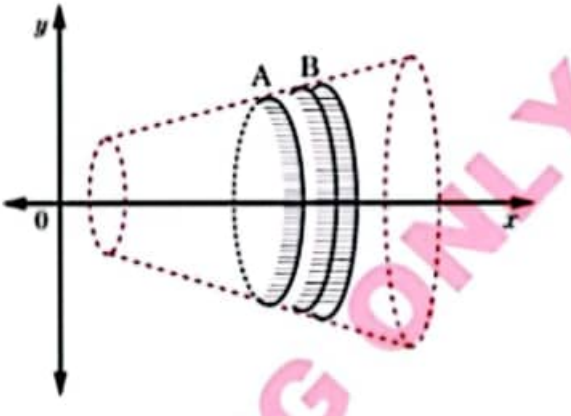

Figure 9.12(b): *A rotated solid*

The total volume, V of the slices of the solid generated is approximately given by;

$$V \simeq \sum_{x=a}^{x=b} \pi y^2 \delta x \qquad (9.6)$$

The error in approximation is due to the areas of the rectangles in Figure 9.12(a) causes the step formation in the solid as shown in Figure 9.12(b). The error disappears when $\delta x \to 0$.

Therefore, the volume of revolution of the solid is $V = \int_a^b \pi y^2 \, dx$.

The volume of the solid of revolution formed by rotating an area through one revolution about the y-axis can be found in a similar way. In this case, the volume is $V = \int_a^b \pi x^2 dy$ (9.7)

Note that, the integration in (9.7) will be with respect to y, and hence the limits of integration will be values of y.

Example 9.76

Find the volume of the solid generated by rotating about the y-axis the area in the first quadrant enclosed by $y = x^2$, $y = 1$, $y = 4$ and the y-axis.

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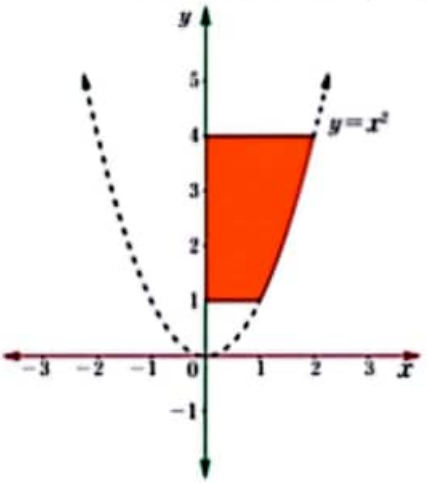
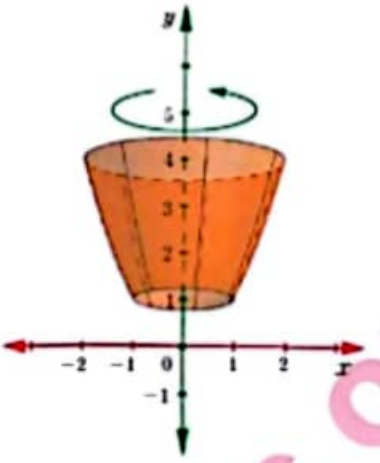
Solution

The volume generated will be obtained by rotating the shaded area about the y-axis as shown in the following figure.





The element of volume $= \pi x^2 \delta y = \pi y \delta y$. It follows that

$$V = \int_1^4 \pi x^2 dy \text{ but } x^2 = y$$

$$= \int_1^4 \pi y dy$$

$$= \left[\frac{\pi y^2}{2}\right]_1^4$$

$$= \frac{\pi}{2}\left(4^2 - 1^2\right)$$

$$= \frac{15\pi}{2} \text{ cubic units.}$$

Therefore, the volume is $\frac{15\pi}{2}$ cubic units.

Example 9.77

Find the volume generated when the plane bounded by $y = 5\cos 2x$, the x-axis, and ordinates at $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis through a complete revolution.

Solution

Given the curve $y = 5\cos 2x$. The solid of revolution formed is shown in the following figure.

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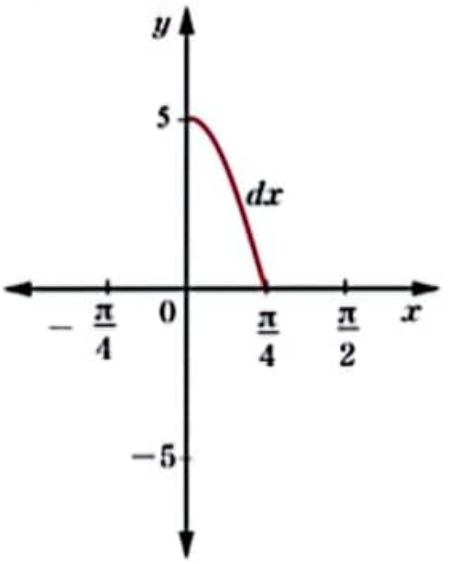

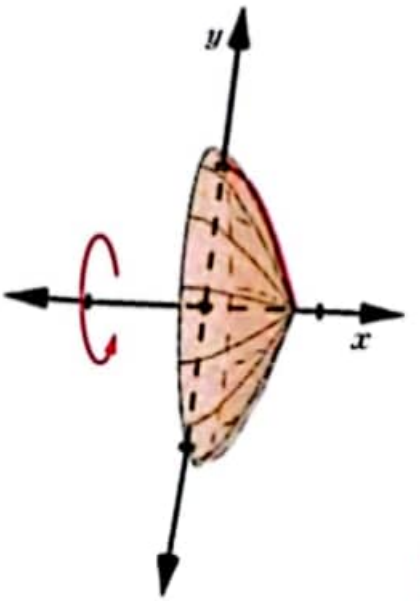

The volume generated is given by;

$$V = \int_a^b \pi y^2 dx$$

$$= \int_0^{\frac{\pi}{4}} \pi y^2\, dx = 25\pi \int_0^{\frac{\pi}{4}} \cos^2 2x\, dx$$

Express $\cos^2 2x$ in terms of double angle. That is,

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

Thus,

$$V = 25\pi \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{25\pi}{2} \int_0^{\frac{\pi}{4}} (1+\cos 4x)\, dx$$

$$= \frac{25\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{25\pi}{2}\left(\left(\frac{\pi}{4}+0\right)-(0+0)\right)$$

$$= \frac{25\pi^2}{8} \text{ cubic units.}$$

Therefore, the volume is $\frac{25\pi^2}{8}$ cubic units.

Example 9.78

The parametric equations of a curve are $x = 3t^2$ and $y = 3t - t^2$. Find the volume generated when the plane bounded by the curve, the x-axis and the ordinates corresponding to $t = 0$ and $t = 2$ is rotated about the x-axis (give your answer correct to 3 significant figures).

Solution

The volume generated is given by;

$$V = \int_a^b \pi y^2 dx$$

But $x = 3t^2$ and $y = 3t - t^2$.

$$\Rightarrow \frac{dx}{dt} = 6t \Leftrightarrow dx = 6t\, dt$$

Thus,

$$V = \int_{t=0}^{t=2} \pi\left(3t - t^2\right)^2 6t\, dt$$

$$= \pi \int_0^2 \left(9t^2 - 6t^3 + t^4\right)^2 6t\, dt$$

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$$= 6\pi\int_0^2 \left(9t^3 - 6t^4 + t^5\right) dt$$

$$= 6\pi\left[\frac{9t^4}{4} - \frac{6t^5}{5} + \frac{t^6}{6}\right]_0^2$$

$$= 6\pi\left[36 - \frac{192}{5} + \frac{64}{6}\right]$$

$$= \frac{248}{5}\pi = 155.8229956 \approx 156$$

Therefore, the volume is approximately 156 cubic units.

In case the limits are given, it is possible to evaluate the volume generated when the plane bounded by the curve, $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ rotates completely about the y-axis by evaluating the definite integral $V = 2\pi\int_a^b xydx$.

Example 9.79

Find the volume generated when the plane bounded by the curve, $y = x^2 + 5$, the x-axis and the ordinates $x = 1$ and $x = 3$ is rotated about the y-axis through a complete revolution.

Solution

Given the curve $y = x^2 + 5$.

From $V = 2\pi\int_a^b xydx$.

$$\Rightarrow V = 2\pi\int_1^3 x\left(x^2 + 5\right)dx$$

$$= 2\pi\int_1^3 (x^3 + 5x)dx$$

$$= 2\pi\left[\frac{x^4}{4} + \frac{5x^2}{2}\right]_1^3$$

Thus,

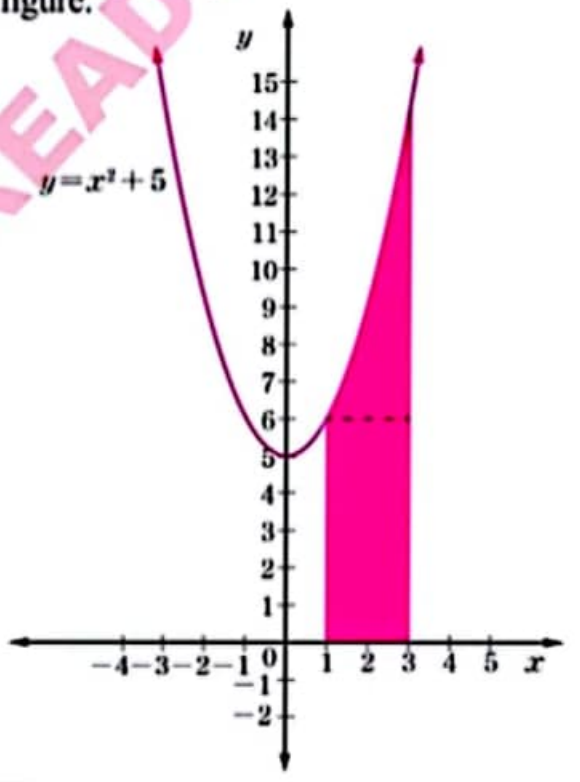
$$V = 2\pi\left[\left(\frac{3^4}{4} + \frac{5(3^2)}{2}\right) - \left(\frac{1^4}{4} + \frac{5(1)^2}{2}\right)\right]$$

$$= 2\pi\left[\frac{81}{4} + \frac{45}{2} - \frac{1}{4} - \frac{5}{2}\right]$$

$$= 80\pi$$

Therefore, the volume generated is 80π cubic units.

Alternatively, the volume can also be obtained by rotating about the y-axis the shaded region as shown in the following figure.



Thus,

$$V_A = \int_6^{14}\left[\pi(3)^2 - \pi\left(\sqrt{y-5}\right)^2\right]dy$$

$$= \pi\int_6^{14}(9 - y + 5)dy$$

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$$= \pi\int_6^{14}(14-y)\,dy$$

$$= \pi\left[14y - \frac{y^2}{2}\right]_6^{14}$$

$$= \pi[(196-98)-(84-18)]$$

$$= 32\pi$$

Similarly,

$$V_B = \int_0^6\left[\pi(3)^2 - \pi(1)^2\right]dy$$

$$= \int_0^6(9\pi-\pi)\,dy$$

$$= 8\pi\int_0^6 dy = 8\pi[y]_0^6$$

$$= 48\pi$$

Thus, the total volume is,

$V = V_A + V_B = 32\pi + 48\pi$

$= 80\pi$

Therefore, the total volume is 80π cubic units.

Volume of solid of revolution about any line

(a) *Disk method*

In this case, the volume of the solid of revolution is given by,

$$V = \pi\int_a^b\left[f(x) - (\text{axis of rotation})\right]^2 dx$$

or

$$V = \pi\int_a^b\left[f(y) - (\text{axis of rotation})\right]^2 dy$$

Example 9.80

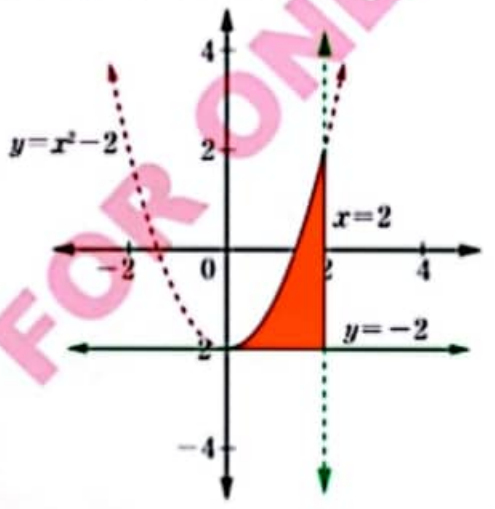
The area enclosed by the curves $y = x^2 - 2$, $y = -2$, and $x = 2$ is rotated about the line $y = -2$. Find the volume of the resulting solid.

Solution

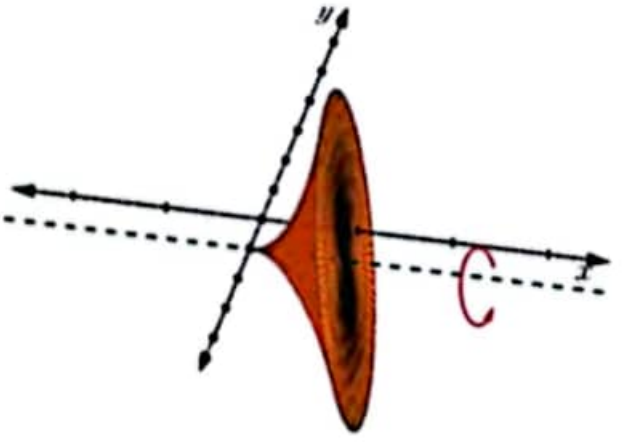
Given $y = x^2 - 2$ and $y = -2$. The point of intersection is given by,

$x^2 - 2 = -2 \Rightarrow x = 0$

Since the rotation is about the horizontal axis, then the cross-sectional area will be a function of x.



Before rotation



After rotation

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Using disk method, it follows that

$$V = \pi\int_a^b \left[f(x) - \text{axis of rotation}\right]^2 dx.$$

$$\Rightarrow V = \pi\int_0^2 \left(f(x) - \text{axis of rotation}\right)^2 dx$$

$$= \pi\int_0^2 \left[\left(x^2 - 2\right) - (-2)\right]^2 dx$$

$$= \pi\int_0^2 \left(x^2\right)^2 dx$$

$$= \pi\int_0^2 x^4 dx$$

$$= \left[\frac{x^5}{5}\pi\right]_0^2$$

$$= \frac{32}{5}\pi \text{ cubic units}$$

Therefore, the volume is $\frac{32}{5}\pi$ cubic units.

(b) *Washer method*

In this case, the volume of the solid of revolution is given by;

$$V = \pi\int_a^b \left[\left(\text{outer curve} - \text{axis of rotation}\right)^2 - \left(\text{inner curve} - \text{axis of rotation}\right)^2\right] dx$$

A similar formula holds when the region lying between the two curves is revolved about the y-axis, that is,

$$V = \pi\int_a^b \left[\left(\text{outer curve} - \text{axis of rotation}\right)^2 - \left(\text{inner curve} - \text{axis of rotation}\right)^2\right] dy$$

Example 9.81

The region enclosed by $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the resulting solid.

Solution

The volume generated will be obtained by rotating the shaded region as shown in the following figure.

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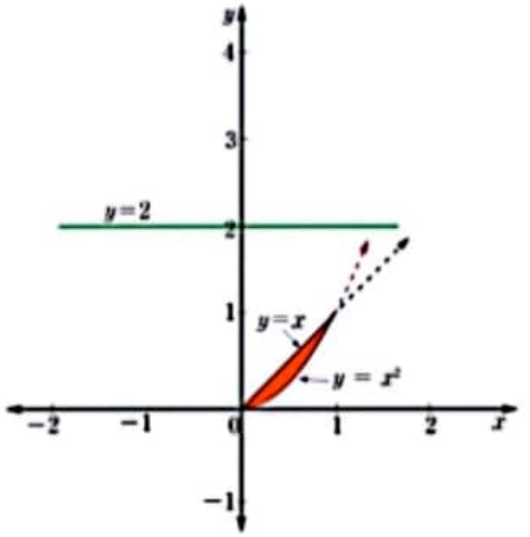

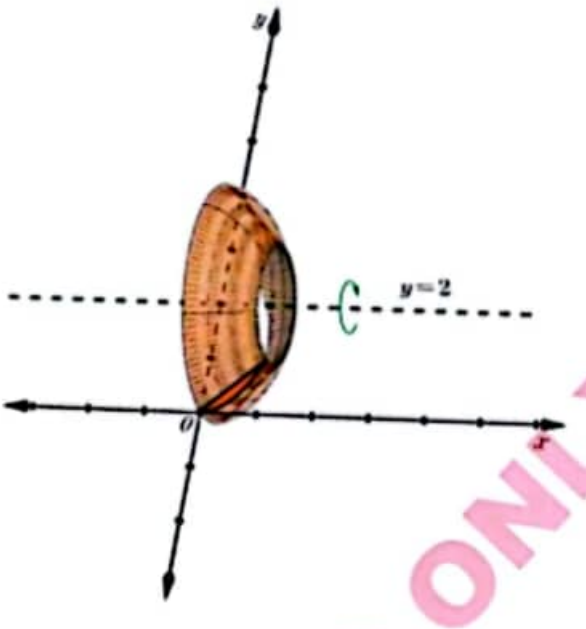

Given $y = x$ and $y = x^2$.

The points of intersection are given by,

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

But the outer radius is $x^2 - 2$ and inner radius is $x - 2$.

$$\Rightarrow V = \pi\int_a^b \left[\begin{array}{l}(\text{outer curve-axis of rotation})^2 \\ -(\text{inner curve-axis of rotation})^2\end{array}\right] dx.$$

Thus, $V = \pi\int_0^1 \left[(x^2-2)^2 - (x-2)^2\right] dx$

$$= \pi\int_0^1 \left[(x^4 - 4x^2 + 4) - (x^2 - 4x + 4)\right] dx$$

$$= \pi\int_0^1 (x^4 - 5x^2 + 4x)\, dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2\right]_0^1$$

$$= \pi\left[\left(\frac{1^5}{5} - \frac{5(1)^3}{3} + 2(1)^2\right) - \left(\frac{0^5}{5} - \frac{5(0)^3}{3} + 2(0)^2\right)\right]$$

$$= \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8}{15}\pi \text{ cubic units}$$

Therefore, the volume of the solid is $\frac{}{15}\pi$ cubic units.

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Example 9.82

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2\sqrt{x-1}$ and the straight line $y = x-1$ about the line $x = -1$.

Solution

Since the rotation is made about a vertical axis, the cross-sectional area will be a function of y.

Given $y = 2\sqrt{x-1} \Rightarrow x = \frac{y^2}{4}+1$ and $y = x-1 \Rightarrow x = y+1$

Points of intersection are given by,

$$\frac{y^2}{4}+1 = y+1$$

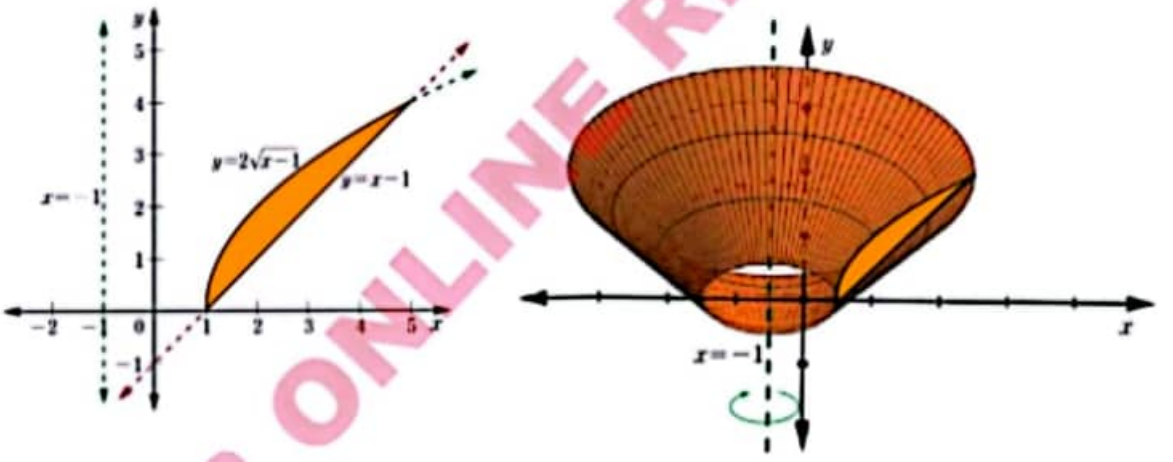
$$\Rightarrow y^2 = 4y$$

$$\Rightarrow y^2 - 4y = 0$$

$$\Rightarrow y(y-4) = 0$$

$$\Rightarrow y = 0,\ y = 4$$

The volume of the solid region obtained after the rotation is shown in the following figure.



$$\Rightarrow V = \pi\int_a^b \left[(\text{outer curve} - \text{axis of rotation})^2 - (\text{inner curve} - \text{axis of rotation})^2\right] dy$$

$$= \pi\int_0^4 \left[(y+1+1)^2 - \left(\frac{y^2}{4}+1+1\right)^2\right] dy$$

$$= \pi\int_0^4 \left[(y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right] dy$$

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$$= \pi \int_0^4 \left(4y - \frac{y^4}{16}\right) dy$$

$$= \pi \left[2y^2 - \frac{y^5}{80}\right]_0^4$$

$$= \pi \left[\left(2(4)^2 - \frac{(4)^5}{80}\right) - \left(2(0)^2 - \frac{(0)^5}{80}\right)\right]$$

$$= \frac{96}{5}\pi \text{ cubic units}$$

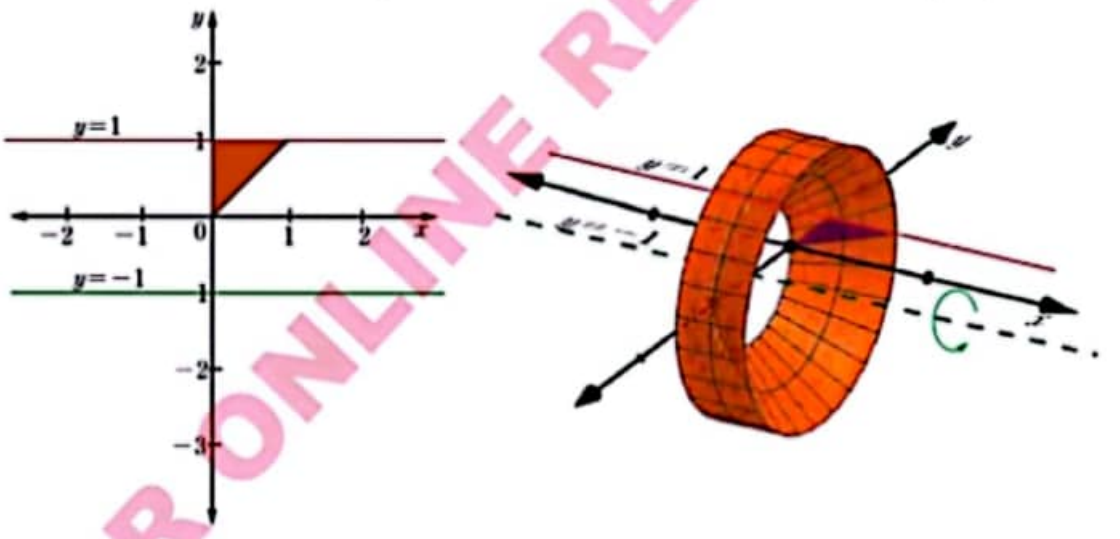
Therefore, the volume is $\frac{96}{5}\pi$ cubic units.

Example 9.83

Find the volume of the solid of revolution generated by revolving the region bounded by the lines $y = x$, $y = 1$, and $x = 0$ about the line $y = -1$.

Solution

The volume of the solid generated is as shown in the following figure.



The limits of integration are $x = 0$ and $x = 1$. The outer radius is $R(x) = 1 - (-1) = 2$ and the inner radius is $r(x) = x - (-1) = x + 1$.

Thus,

$$V = \pi \int_0^1 (R^2 - r^2)\, dx$$

$$= \pi \int_0^1 (2^2 - (x+1)^2)\, dx$$

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$$= \pi \int_0^1 (3 - 2x - x^2)\, dx$$

$$= \pi \left[3x - x^2 - \frac{x^3}{3} \right]_0^1$$

$$= \pi \left[\left(3 - 1^2 - \frac{1^3}{3} \right) - (0) \right]$$

$$= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Therefore, the volume is $\frac{5\pi}{3}$ cubic units.

(c) *Shell method*

In this case, the volume of the solid of revolution is given by,

$$V = 2\pi \int_a^b [(\text{shell radius}) \times (\text{heigh of the shell})]\, dx$$

$$\Rightarrow V = 2\pi \int_a^b [(x - \text{axis of rotation}) \times (\text{top curve} - \text{bottom curve})]\, dx$$

A similar formula holds when the region R is rotated in the first quadrant between the y-axis and the curve $x \quad f(y)$ lying between $y = a$ and $y = b$. That is,

$$V = 2\pi \int_a^b [(\text{shell radius}) \times (\text{heigh of the shell})]\, dx$$

$$V = 2\pi \int_a^b [(y - \text{axis of rotation}) \times (\text{top curve} - \text{bottom curve})]\, dy$$

Example 9.84

The region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $x = -1$. Find the volume of the resulting solid.

Solution

The volume of the solid generated is obtained by rotating the shaded region as shown in the following figure.

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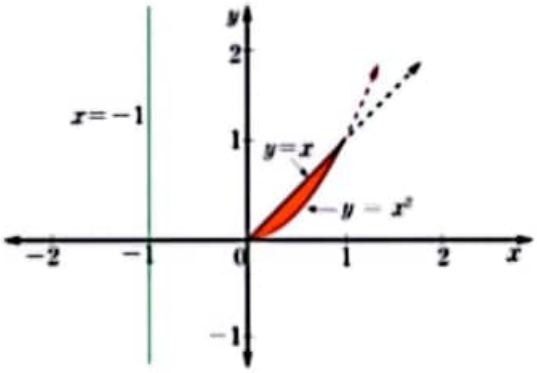

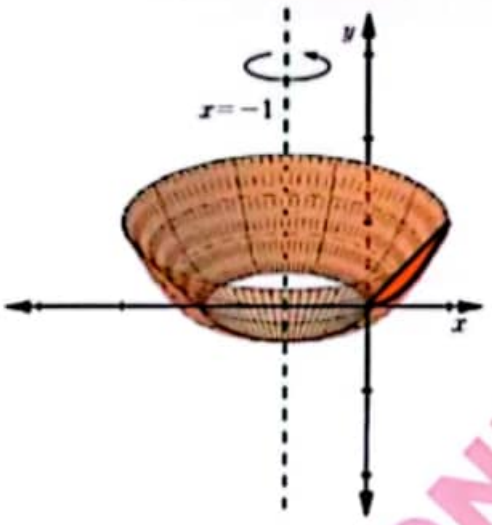

Given $y = x$ and $y = x^2$.

The points of intersection of the curve and the line are obtained as follows.

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

Thus, $V = 2\pi\int$ (x − axis of rotation)(top curve − bottom curve)dx

$$\Rightarrow V = 2\pi\int_0^1 (x-(-1))(x-x^2)\,dx$$

$$= 2\int_0^1 (x+1)(x-x^2)\,dx$$

$$= 2\pi\int_0^1 (x^2 - x^3 + x - x^2)\,dx$$

$$= 2\pi\int_0^1 (x - x^3)\,dx$$

$$= 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$V = 2\pi\left[\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0^2}{2} - \frac{0^4}{4}\right)\right]$$

$$= 2\pi\left(\frac{2-1}{4}\right)$$

$$= \frac{\pi}{2} \text{ cubic units}$$

Therefore, the volume is $\frac{\pi}{2}$ cubic units.

Area of a sector

Consider a sector with a central angle θ and radius r. Suppose that a small sector OPQ with a central angle $\delta\theta$ and radius $r + \delta r$ is partitioned as in Figure 9.13.

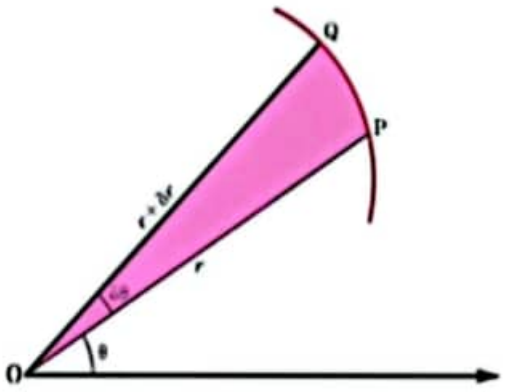

Figure 9.13: *A sector OPQ*

In Figure 9.13, the radius vectors (Position vectors) $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are r and $r\ \ \delta r$, respectively. The angles between the radius vectors and the fixed

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line $\overline{OX}$ are θ and $\theta+\delta\theta$, respectively. If $\delta\theta$ is small, the area of the sector OPQ is approximately equal to the area of the triangle.

Area of the sector

$$OPQ \approx \frac{1}{2}r(r+\delta r)\sin\delta\theta$$

$$\approx \frac{1}{2}r^2\sin\delta\theta + \frac{1}{2}r\delta r\sin\delta\theta$$

By Maclaurin's series,

$$\sin\delta\theta = \delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots$$

and the product $\delta r\sin\delta\theta$ is small compared with $\delta\theta$.

Thus, the area of the sector OPQ up to terms in $\delta\theta$ is given by;

Area of the sector

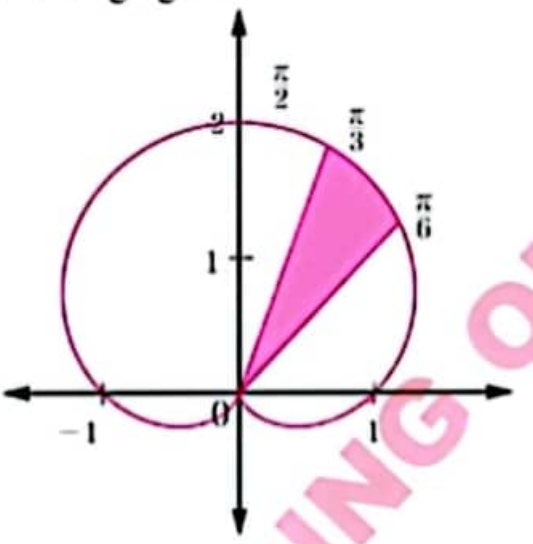
$$OPQ = \frac{1}{2}r^2\delta\theta$$

Here, it is assumed that the difference between the sector OPQ and the triangle OPQ is small compared to $\delta\theta$.

Summing all elements in the sector and proceeding to the limit, the area A of a sector becomes, $A = \int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 d\theta$, where θ_1 and θ_2 are the values of θ corresponding to the bounding radius vectors of the sector.

Example 9.85

Find the area of the sector bounded by the cardioid $r = 1+\sin\theta$ between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$, as shown in the following figure.



Solution

The area of the sector is given by;

$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,\delta\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+\sin\theta)^2 d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+2\sin\theta+\sin^2\theta)d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\frac{3\theta}{2} - \frac{1}{4}\sin 2\theta - 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - (1)\right] - \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\left(\sqrt{3}+\frac{\pi}{4}-1\right)$$

Therefore, the area of the cardioid is $\frac{1}{2}\left(\sqrt{3}+\frac{\pi}{4}-1\right)$ square units.

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Example 9.86

Find the area of the sector bounded by the cardioid $r = 3 + 4\cos 2\theta$ between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Solution

The formula for the area A of a sector is given by,

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (3 + 4\cos 2\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (9 + 24\cos 2\theta + 16\cos^2 2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[9 + 24\cos 2\theta + 16\left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (17 + 24\cos 2\theta + 8\cos 4\theta)\, d\theta$$

$$= \frac{1}{2}[17\theta + 12\sin 2\theta + 2\sin 4\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{51\pi}{4} - 12 + 0\right) - \left(\frac{17\pi}{4} + 12 + 0\right)\right]$$

$$= \frac{17\pi}{4} - 12$$

Therefore, the area of the cardioid is $\left(\frac{17\pi}{4} - 12\right)$ square units.

Exercise 9.14

1. The parametric equations of a curve are $x = 3t^2$ and $y = 3t - t^2$. Find the volume of the solid generated when the plane bounded by the curve, the x-axis and the ordinates corresponding to $t = 0$ and $t = 2$ rotates about the y-axis.
2. Part of the curve $y = x^3$ from $x = 1$ to $x = 2$ is rotated completely around the y-axis. Find the volume of the solid generated.
3. Find the volume of the solid generated when the plane figure bounded by $y = 5\cos 2x$, the x-axis and abscissae $x = 0$ and $x = \frac{\pi}{4}$ rotates about the x-axis through 2π radians.
4. Find the volume of the solid generated by revolving the area bounded by the parabola $y^2 = 8x$ and the line $x = 2$ in the first quadrant about the x-axis.
5. Find the volume of the solid generated when the area enclosed between the line $y = x + 2$, the x-axis, from $x = -2$ to $x = 2$ is rotated about the x-axis.
6. The region bounded by the y-axis, the line $y = 27$ and the curve $y = \frac{x^3}{8}$ is rotated completely about the y-axis. Find the volume of the generated.
7. Find the volume of the solid generated when the region bounded by the curve, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (an ellipse of length $2a$ and height $2b$) rotates through 180° about,
   (a) x-axis (b) y-axis
8. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = 2x^2 - x^3$ and the x-axis.

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9. Find the volume of the solid generated by revolving the area bounded by the curve, $y = 2x^2$ from $y = 0$ and $x = 5$ about the line $y = 0$.
10. Find the volume of the solid generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the x-axis about the line $y = 7$.
11. The region R is bounded by the curve $4y = x^2$ and the lines, $x = 4, y = 1$. Find the volume of the solid formed when R is rotated completely about the
    (a) x-axis (c) line $y = 1$
    (b) y-axis (d) line $x = 4$
12. Find the volume of the solid generated when the area bounded by the x and y-axes, the line $x = 1$, and the curve $y = e^x$ is rotated through one revolution about the x-axis.
13. Find the volume of the solid formed by revolving the region bounded by the graph of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.
14. Find the volume of the solid generated by rotating about the x-axis the region bounded by the curve $y = \frac{4}{x}$ and the line $y = 5 - x$.
15. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.
16. Find the area of the cardioid $r = \cos\theta$ bounded between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$.
17. Find the area under the curve $r = \sin 4\theta$, over the interved $\frac{\pi}{2} \le \theta \le \pi$.
18. Determine the area of a sector bounded by $r = 1 - \cos\theta$, where $0 \le \theta \le \frac{\pi}{3}$.

Chapter summary

1. The reverse process of differentiation is called integration.
2. An indefinite integral of a function $f(x)$ is a differentiable function $F(x)$ such that its derivative gives the class of all functions whose derivative is $f(x)$.
3. Integration by the substitution method is used in reversing the chain rule. It is sometimes called chain rule backward.
4. If $u(x)$ and $v(x)$ are any two differentiable functions, then $\int udv = uv - \int vdu$ is the formula for integrating by parts.
5. The area enclosed between two curves $f(x)$ and $g(x)$ is given by $A = \left|\int_a^b [f(x) - g(x)]dx\right|$.
6. Integrals of the form $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$, where n is odd or even, the power of the trigonometric function determines the approach to be used.
7. The formula for calculating the length of an arc are $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$,

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$$s=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,$$

$$s=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \text{ and}$$

$$s=\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

8. Integrals of the form $\int \frac{1}{a+b\cos^2 x+c\sin^2 x}dx$ are evaluated by applying the trigonometric substitution $t=\tan x$.

9. Integration by splitting the numerator of the integrand expresses the numerator in the form:
Numerator $= A\frac{d}{dx}(\text{denominator})+B$,
for rational polynomial function.
Numerator
$=A\frac{d}{dx}(\text{denominator})+B(\text{denominator})$,
for rational trigonometric function.

10. The volume of the solid of revolution formed by rotating an area through one revolution about the x- axis is $V=\pi\int_a^b y^2dx$ and y-axis is $V=\pi\int_a^b x^2\,dy$.

11. The area of a sector is $A=\int_{\theta_1}^{\theta_2}\frac{1}{2}r^2\,d\theta$ where θ_1 and θ_2 are the values of θ corresponding to the bounding radius vectors of the sector.

12. If a function is integrated over a specified interval, its integral is referred to as a definite integral and it is given by $\int_a^b f(x)dx$.

Revision exercise 9

1. Determine each of the following:
   (a) $\int\sqrt{3x-1}\,dx$
   (b) $\int(x^2+3)x^2dx$
   (c) $\int\frac{e^x}{e^x+e^{-x}}dx$
   (d) $\int x(x+4)^3\,dx$
   (e) $\int x\sqrt{x-1}\,dx$
   (f) $\int x^2\tan^{-1}x\,dx$

2. Use the substitution $x=\sec\theta$ to evaluate $\int_{\sqrt{2}}^{2}\frac{dx}{x\sqrt{x^2-1}}$.

3. Use the substitution $u=\sin x$ to evaluate $\int_0^{\frac{\pi}{2}}\frac{\cos x}{3+\cos^2 x}dx$.

4. Find each of the following:
   (a) $\int_1^4(1-x)\sqrt{x}\,dx$
   (b) $\int\frac{d\theta}{5+4\cos\theta}$
   (c) $\int\theta\sqrt{2\theta-3}\,d\theta$
   (d) $\int x^2e^x\,dx$

5. By using the substitution $x^2=\frac{1}{u}$, evaluate $\int_1^2\frac{dx}{x^2\sqrt{(5x^2-1)}}$ to 4 decimal places.

6. Show that
$\int_0^{\pi}\frac{2\cos\theta+11\sin\theta}{3\cos\theta+4\sin\theta}d\theta=2\pi$.

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7. Evaluate each of the following:

(a) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5\sin\theta+3}$

(b) $\int \frac{dx}{x^2\sqrt{-x^2+4}}$

(c) $\int \sin 4x \sin 3x\, dx$

(d) $\int e^x \sin x\, dx$

(e) $\int \frac{d\theta}{5+\sin\theta+\cos\theta}$

(f) $\int \frac{d\theta}{\theta^2-6\theta+13}$

(g) $\int_{-1}^{1} \frac{d\theta}{1+\sin\theta\cos\theta}$

(h) $\int 24^{6x}\, dx$

8. Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2x\, dx = \frac{1}{2}\ln 3$.

9. Evaluate each of the following:

(a) $\int 2x\cos(x^2-5)\, dx$

(b) $\int (3x^2-4x)\sin(x^3-2x^2+1)\, dx$

(c) $\int \sqrt{1-\cos x}\, dx$

(d) $\int \sin^8 3z\cos^5 3z\, dz$

(e) $\int \cos^4 2t\, dt$

(f) $\int \sin 3x \sin(\cos 3x) dx$

(g) $\int \sec^6 3y \tan^2 3y\, dy$

(h) $\int \cos 3t \sin 8t\, dt$

(i) $\int \tan^5 x\, dx$

(j) $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$

(k) $\int (\cos x)e^{4+\sin x}\, dx$

(l) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin 8x \sin x\, dx$

(m) $\int_0^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)dw$

(n) $\int (\sin x+\cos x)(\sin x-\cos x)^5\, dx$

10. Derive the reduction formula for $I_n = \int \sin^n x\, dx\, (n \geq 2)$ and use it to find I_6 and I_7.

11. Evaluate each of the following:

(a) $\int \frac{dx}{(1+x^2)^2}$

(b) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

(c) $\int \frac{\sqrt{1+\ln x}}{x}dx$

(d) $\int \frac{xdx}{\sqrt{x-1}}$

(e) $\int \frac{x+1}{x^2+4x+8}dx$

(f) $\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos x}dx$

12. Evaluate each of the following:

(a) $\int_2^e \frac{1}{x^2\ln x}\left(1+\frac{1}{\ln x}\right)dx$

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{\tan x-\sin x}$

(c) $\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}dx$

(d) $\int_0^2 \frac{xdx}{\sqrt{4x^2+9}}$

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13. Show that $\int_0^{\frac{1}{3}} \frac{dx}{3-\sin x+\cos x} = \ln\frac{1}{4}\left(5+\sqrt{3}\right)$

14. Determine each of the following:

(a) $\int \frac{\sqrt{x^2-25}}{x}dx,\ x>5$

(b) $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

(c) $\int_{10}^{100} \frac{dx}{x(\ln x)(\ln x)}$

(d) $\int_{-7}^{-3} \frac{d\theta}{\theta^2+10\theta+29}$

15. Determine each of the following:

(a) $\int \cos x\sin x\sqrt{1-\sin^4 x}\,dx$

(b) $\int \frac{e^{\sqrt{x-2}}}{\sqrt{x-2}}dx$

(c) $\int \frac{dx}{3^{\sqrt{x}}\sqrt{x}}$

(d) $\int \frac{x+1}{x^2+2x+5}dx$

(e) $\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}dx$

(f) $\int \frac{x^3-4x-10}{x^2-x-6}dx$

(g) $\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$

(h) $\int \frac{x^2+x+1}{(x^2+1)^2}dx$

(i) $\int \frac{dx}{\sqrt{14-12x-2x^2}}$

(j) $\int \frac{\ln x}{x(1+\ln x)}dx$

(k) $\int \frac{2x^2}{(x^3+1)^2}dx$

(l) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x-1}}$

(m) $\int 7^{2x}dx$

16. Evaluate each of the following:

(a) $\int_{\frac{\pi}{2}}^{\pi} \frac{2\cos\theta+5\sin\theta}{3\cos\theta-\sin\theta}d\theta$

(b) $\int \frac{\theta+8}{16+\theta^2}d\theta$

(c) $\int \sec 4\theta\,d\theta$

(d) $\int \frac{2\cos x-\sin x+8}{\cos x+\sin x}dx$

17. Show that $\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{n-1}{n}I_{n-2}$. Hence, evaluate the following:

(a) $\int_0^{\frac{\pi}{2}} \cos^8\theta\,d\theta$ (b) $\int_0^{\frac{\pi}{2}} \cos^9\theta\,d\theta$

18. Verify that:

(a) $\int \frac{1}{2\sin^2\theta+\cos^2\theta}d = \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{\sqrt{2}\tan}{2}\right)+c$

(b) $\int_{-7}^{1} \frac{dx}{\sqrt{2\left(7-6x-x^2\right)}} = \frac{\sqrt{2}}{2}$

(c) $\int_0^{\frac{\pi}{2}} e^{-\theta}\cos^6\theta\,d\theta = \frac{263-144e^{-\frac{\pi}{2}}}{629}$

(d) $\int_0^{\frac{\pi}{2}} \frac{1-2\tan\theta}{3+4\tan\theta}d\theta = \frac{2}{5}\ln\left(\frac{4}{3}\right)-\frac{\pi}{10}$

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19. Use the reduction formula for $\int_0^{\frac{\pi}{6}} \sec^n\theta\, d\theta$ to evaluate each of the following:

(a) $\int_0^{\frac{\pi}{6}} \sec^8\theta\, d\theta$ (b) $\int_0^{\frac{\pi}{6}} \sec^5\theta\, d\theta$

20. Show that

(a) $\int \frac{5}{16\sin^2 x+\cos^2 x}dx = \frac{5}{4}\tan^{-1}(4\tan x)+c$

(b) $\int \frac{1}{49x^2+81}dx = \frac{1}{63}\tan^{-1}\left(\frac{7}{9}x\right)+c$

21. (a) Show that $\int_0^{\pi} \frac{y\sin y}{1+\cos^2 y}dy = \left(\frac{\pi}{2}\right)^2$.

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\cos\theta\sin\theta)^2}d\theta$.

22. Find the arc length of the curve $y=\ln(\cos x)$ over the interval $0\le x\le\frac{\pi}{4}$.

23. (a) Find the area of the region enclosed by the curve $y=\ln x$, the lines $x=1$ and $x=e$, and the x-axis.

(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

24. A particle moving along the x-axis has a velocity function, $v(t)=t^2e^{-t}$. How far does the particle travel from $t=0$ to $t=5$?

25. Show that the volume of the solid generated by rotating about the x-axis the area inscribed between the x-axis and the curve $cy=(x-a)(x-b)$ is $\frac{\pi(a-b)^5}{30c^2}$, where a and b are the limits and c is a constant.

26. The trend of sales of a certain bookshop is given by $s(t)=1500\sin\frac{\pi}{6}(t-7)+2000$, where t is the time in months and $t=0$ represents 1st January. Estimate the total sales over the four month period beginning 1st March.

27. Find the area bounded by of the cardioid $r=1+\sin 2\theta$ between $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$.

28. Find the length of the curve $24xy=x^4+48$ from $x=2$ to $x=4$.

29. Prove that the arc length of the curve $(y-1)^3=x^2$ over the interval $0\le x\le 8$ is 9.0734 units.

30. The line $y=4x$ meets the curve $y=\frac{9x^2}{25-x^2}$ at the origin and at points P and Q. Find:

(a) The coordinates of the points P and Q.

(b) The area enclosed by the curve and the line in the first quadrant.

31. (a) Determine the area enclosed between the curves $x^2=10-y$ and $x^2=y-2$.

(b) Show that the area enclosed by the curve $x^2=4x+y-20$, the line $3x-y=0$, and the ordinates, $=1$ and $x=4$ is 28.5 square units.

32. (a) Determine the area of a sector bounded by $r+\cos 4\theta=1$, where $0\le\theta\le\frac{\pi}{6}$.

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(b) Find the area of the cardioid given by the curve $r = 4 + \sin 3\theta$ bounded between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

33. Find the volume of the solid generated when the plane bounded by the curve $y = x^2 + 3$, the x-axis and the coordinates $x = 2$ and $x = 4$ is rotated about y-axis through 2π radians.

34. The portion of the curve $y = x^2$ between $x = 0$ and $x = 2$ is rotated completely about the x-axis. Find the volume of the solid created.

35. Find the volume of the solid generated by rotating the straight line $y = x + 1$ from $x = 1$ to $x = 2$ completely around the x-axis.

36. Show that the area enclosed by the curves $x = 2(2t - \sin 2t)$, $y = 2a\sin^2 t$ and the x-axis between $t = 0$ and $t = \pi$ is $3\pi a^2$ sq. units.

37. Find the area swept out by the radius vector of the equiangular spiral $r = ae^{k\theta}$ as θ increases from $-\pi$ to π, where a and k are constants.

38. A curve has the parametric equations $x = t^2 - 2t$ and $y = t + 1$. Find the area bounded by the curve, the x-axis, and the ordinates at $t = -1$ and $t = 3$.

39. Show that the area of a sector in polar form is given by $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$, where r is the radius and θ is the measure of the central angle in radians.

40. (a) Find the volume of the solid obtained by rotating the region bounded by the curve $f(x) = 4\sqrt{x-3}$ and the straight line $g(x) = x - 3$ about the line $x + 1 = 0$.

(b) Determine the area enclosed by the curve $f(x) = \frac{4}{x^2}$ and the lines $y = 1$, $y = 4$. If this area is rotated about the x-axis through 2π radians, show that the volume generated is $\frac{112}{3}\pi$ cubic units.

Project work

Select an area of your interest related to Mathematics, then design and conduct a project.

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Answers

Exercise 1.1

1. (a) A = {Cubes of all positive integers} or {Cubes of all natural numbers}
   (b) B = {Multiples of 3}
   (c) C = {All integers from −4 to 4 inclusive}
   (d) D = {Domestic animals}
2. (a) $A = \{x : x^2, x \in \mathbb{N}\}$ (b) $B = \{x : x^3, x \in \mathbb{N}\}$ (c) $C = \{x : x = -3 \text{ and } x = 3\}$.
   (d) $D = \{x : x \in \mathbb{N}\}$ (e) $E = \{x : x \text{ is a positive even number}\}$
   (f) $F = \{x : x \in W\}$ (g) $\left\{x : x = \frac{n}{n+1}, \text{ for } 1 \leq n \leq 7, n \in \mathbb{N}\right\}$
3. (a) $A = \{-1, 1\}$ (b) $B = \{3, 4, 5, 6, ...\}$ (c) $C = \{1, 3\}$

Exercise 1.2

1. (a) Infinite set – Number of all plants on the earth is uncountable
   (b) Infinite – Real numbers between 10 and 30 are uncountable
   (c) Infinite – Non odd numbers between 10 and 20 inclusive are uncountable
2. T and R are equivalent
   T and R are equal
   H and T are unequal
   H and R are unequal
3. (a) True (b) True (c) True (d) False (e) False
4. (a){ }, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d},
   {a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}, {a, b, c, d}.
   (b) $n(K) = 16$.
5. (a) P(J) = {{ }, {Dog}, {Cat}, {Lion}, {Zebra}, {Dog, Cat}, {Dog, Lion}
   {Cat, zebra}, {Dog, zebra}, {Cat, Lion}, {Lion, zebra}, {Dog, Cat, Lion},
   {Dog, Cat, Zebra}, { Cat, Lion, Zebra}, {Dog, Lion, Zebra}
   {Dog, Cat, Lion, Zebra}}
   (b) n[P(J)] = 16
6. (a) U = {2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 17, 19} (b) U = {x : x is an integer} = $\mathbb{Z}$.
7. (a) A = { } (b) B = {−5}

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Exercise 1.3

1. $A \cup B = \{a, b, c, d, e, f, g, h\}$
   $A \cap B = \{a, b, c, d, e\}$
2. $A \cup B = \{\text{Counting numbers}\}$
   $A \cap B = \{\text{Even numbers}\}$
3. $G \cup H = \{20, 25, 30, 45\}$
   $G \cap H = \{25\}$
4. $J \cup K = \{0, \Delta, 3\}$
   $J \cap K = \{\Delta\}$
5. $A \cup B = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 48, 54\}$
   $A \cap B = \{12, 24, 36\}$
6. $W \cup Z = \{14, 16, 18, 20\}$
   $W \cap Z = \{\ \}$ or $\varnothing$
7. $A \cup B = \{94, 110, 120, 131, 140, 265\}$
   $A \cap B = \{94, 110\}$
8. $A \cup B = \{2, 3, 5\}$
   $A \cap B = \{3\}$
9. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26\}$
   $A \cap B = \{6, 12, 18, 24\}$
10. $A \cup B \cup C = \{a, b, c, d, e\}$
    $A \cap B \cap C = \{\ \}$
11. $M \cap N = \{\ \}$
12. Disjoint sets
13. $M' = \{5, 6, 7, 8, 9, 10, 11, 12\}$
    $N' = \{2, 4, 5, 6, 8, 10, 12\}$
14. $A = \{\text{Orange, cabbage, pineapple}\}$
15. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
16. (a) $A \cap B' = \{3, 4\}$
    (b) $B \cup A' = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$
    (c) $B' \cap A' = \{9, 10, 11, 12, 13\}$
    (d) $A' \cap B = \{7, 8\}$
17. (a) F (b) T (c) F (d) F
18. $A' = \{\text{magenta, red, blue}\}$
19. (a) $A \cap B = \{a, d, e\}$
    (b) $A \cup B = \{a, b, c, d, e, f\}$
    (c) $A' \cap B = \{\ \}$
    (d) $A \cap B' = \{b, c, f\}$
    (e) $A' \cap B' = \{g, h, i, j\}$
    (f) $A' \cup B' = \{b, c, f, g, h, i, j\}$
20. (a) $A \cup F = \{22, 24, 26, 28, 30\}$
    $A \cap F = \{24, 26\}$
    (b) $Y \cup W = \{2 < x < 23\}$
    $Y \cap W = \{7 \leq x \leq 18\}$
21. (a) $A - B = \{13\}$
    (b) $B - A = \{16, 17, 18\}$
22. (a) $M - K = \{32, 42\}$
    (b) $K - M = \{33, 35, 37\}$
    (c) $(M - K) \cup (K - M) = \{32, 33, 35, 37, 42\}$
23. (a) $P - R = \{8, 10, 12\}$
    (b) $R - P = \{14, 16, 18\}$
    (c) $Q - P = \{14, 16, 18, 20, 22\}$

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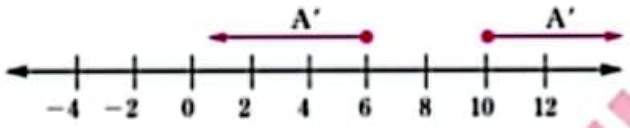
24. (a) $B-A=\{3,5,9\}$ (b) $(A-B)'=\{1,3,5,9,11,17,19\}$

25. $X-Y=\{2,4,6,8,10\}$

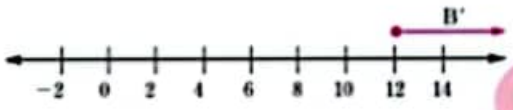
26. $x=9,\ y=12$ or $x=12,\ y=9$.

Exercise 1.4

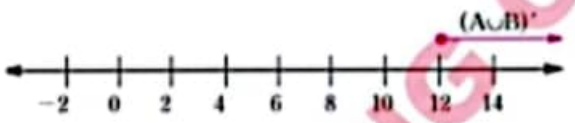
1. (a) A'



(b) B'



(c) $(A\cup B)'$



(d) $(A\cap B)'$

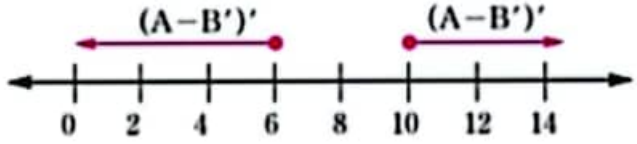
(e) $A-B$

$A-B=A\cap B'=\emptyset$

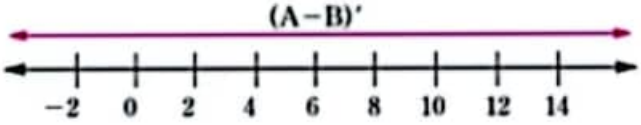
(f) $B-A$

(g) $A-B'$

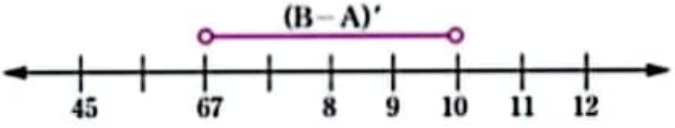
(h) $(A-B')'$



(i) $(A-B)'$



(j) $(B-A)'$



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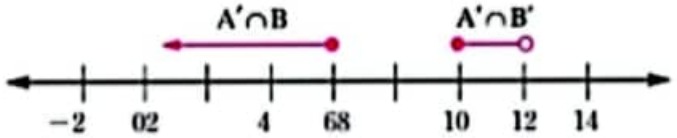
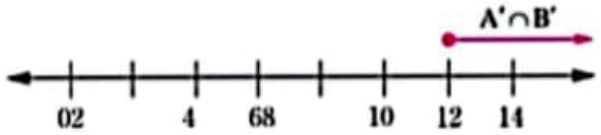
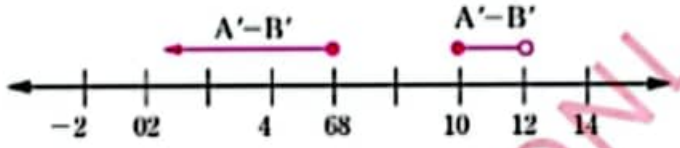
(k) $A' \cap B$



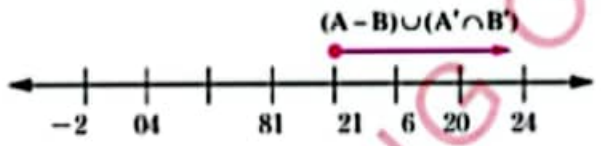
(l) $A' \cap B'$



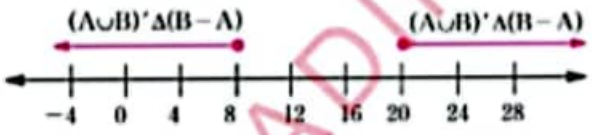
(m) $A' - B'$



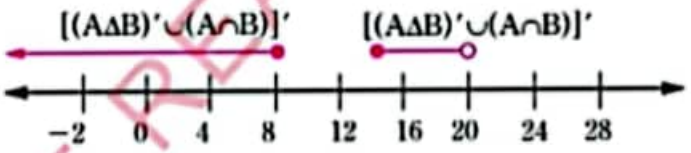
(n) $(A - B) \cup (A' \cap B')$



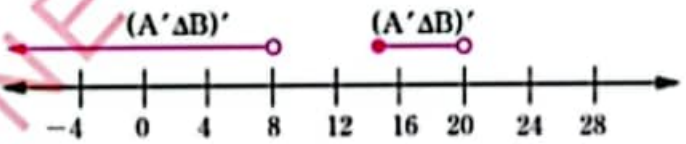
2. (a) $(A \cup B)' \Delta (B - A)$



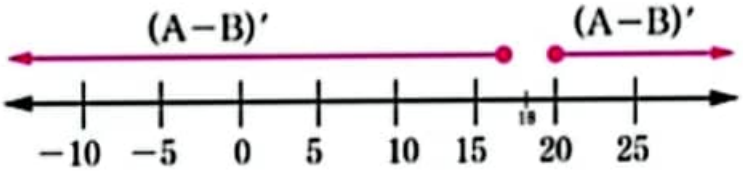
(b) $\left[(A \Delta B)' \cup (A \cap B)\right]'$



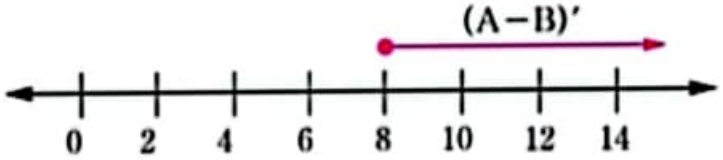
(c) $(A' \Delta B)'$



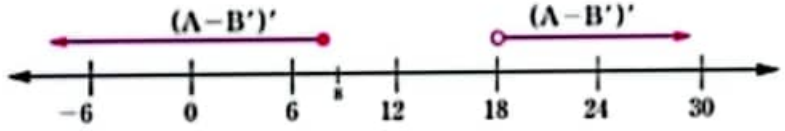
3. (a) $(A - B)'$



(b) $(B - A)'$



(c) $(A - B')'$



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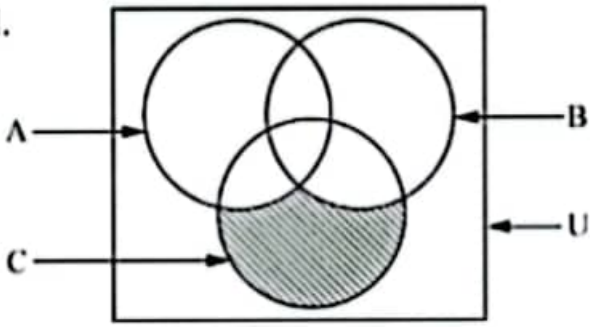
Exercise 1.5

3. (a) $A \cup (B \cap C)$ (b) $(A \cup B)'$ (c) A (d) $\varnothing$
(e) $A' \cup B$ (f) $\varnothing$ (g) $A \cup B'$
(h) $\varnothing$ (i) B (j) $\varnothing$

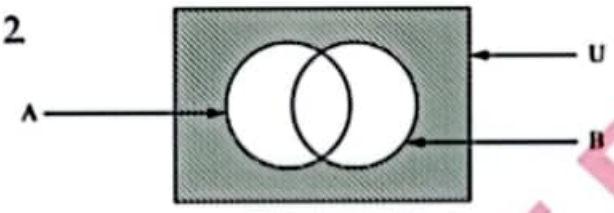
Exercise 1.6

1.



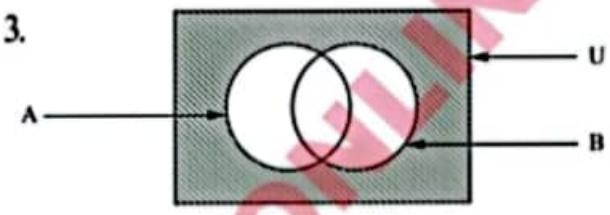
$(A \cup B)' \cap (A \cup C)$

2.



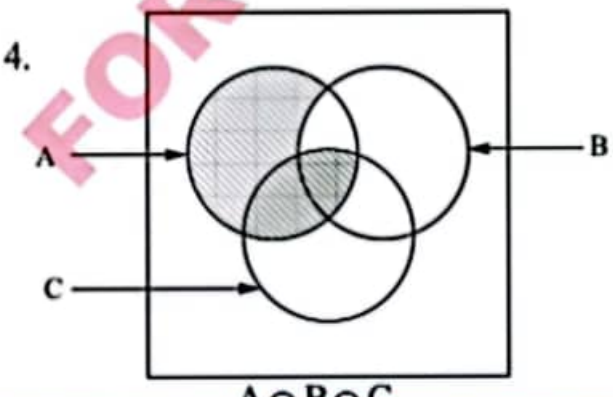
$(A \cup B)'$

3.



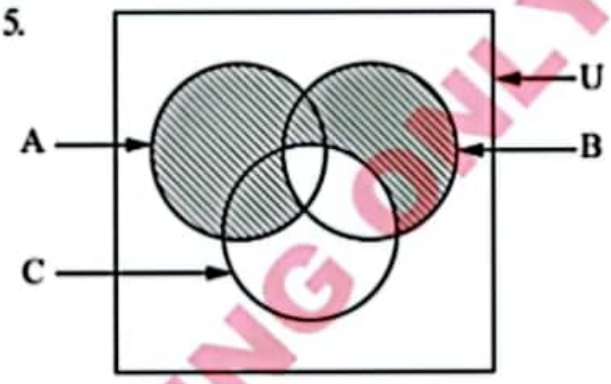
$A' \cap B'$

4.



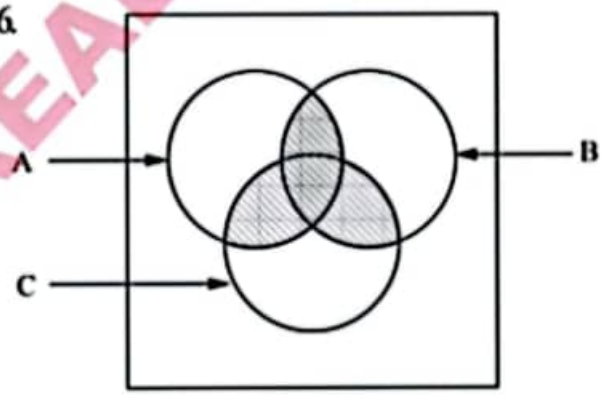
$A \cap B \cap C$

5.



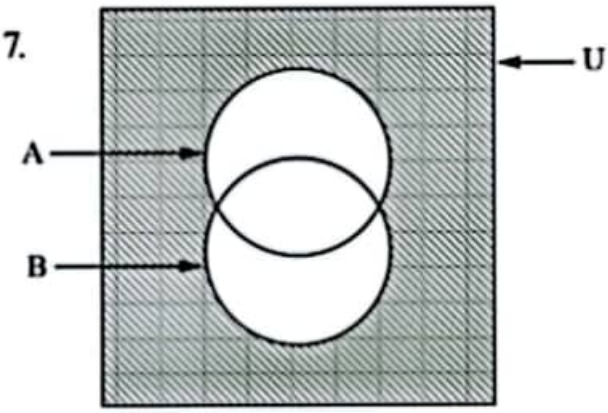
$(A - B) \cup (A - C) \cup (B - C)$

6.



$(A \cap B) \cup (A \cap C) \cup (B \cap C)$
is shaded

7.



$(A \cup B)' \cap (A' \cap B')$

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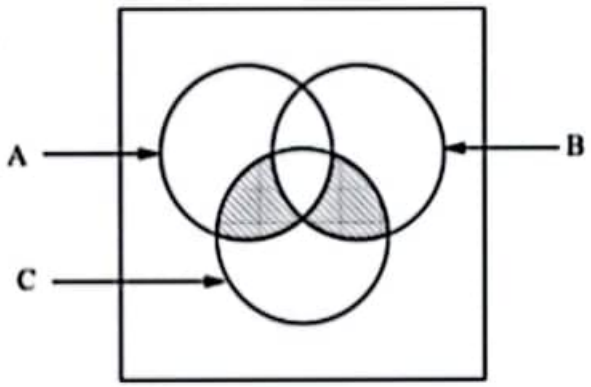
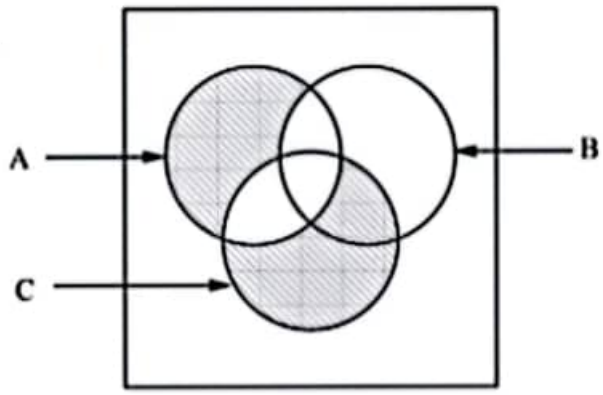
8.



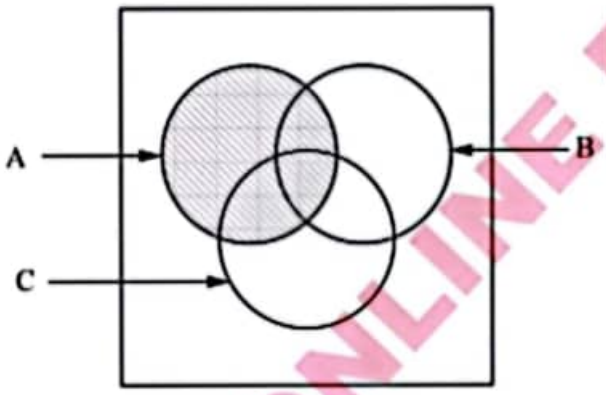
$(A \Delta B) \cap C$

9.



$(A - B) \Delta C$

10.



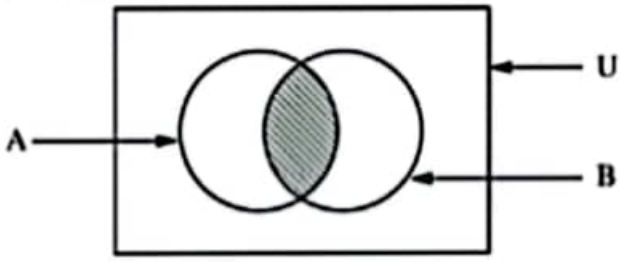
$A - (B \cap C)$

Exercise 1.7

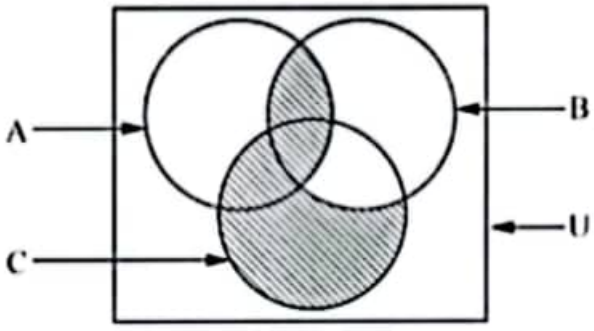
1. (a) $n(A \cap B) = 3$ (c) $n(A \cup B) = 5$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 1$
2. (a) $n(A \cap B) = 4$ (c) $n(A \cup B) = 10$
   (b) $n(A \Delta B) = 6$ (d) $n(A - B) = 4$
3. (a) $n(A \cap B) = 0$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 3$ (d) $n(A - B) = 1$
4. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 0$
5. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 2$
   (b) $n(A \Delta B) = 1$ (d) $n(A - B) = 1$
6. $n(A \cap B)' = 7$
7. (a) $n(A) = 150$ (b) $n(B) = 100$
8. Who had three diseases = 20 patient
9. (a) 2 (b) 119 (c) 126 (d) 129

Revision exercise 1

1. (a) $A \cup B \cup C = \{0, 2, 4, 6, 8, 12, 16, 18\}$
   (b) $A \cap B \cap C = \{0\}$
   (c) (i) $n(A \cup B \cup C) = 8$
   (ii) $n(A \cap B \cap C) = 1$
3. (a) 8 (b) 16 (c) 64
4. (a) (i) $B - A'$



(ii) $(A \cap B) \cup (B' \cap C)$



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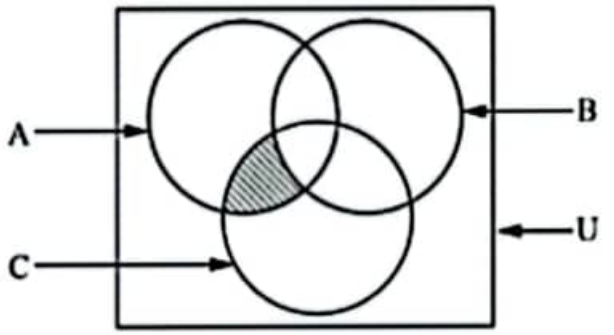
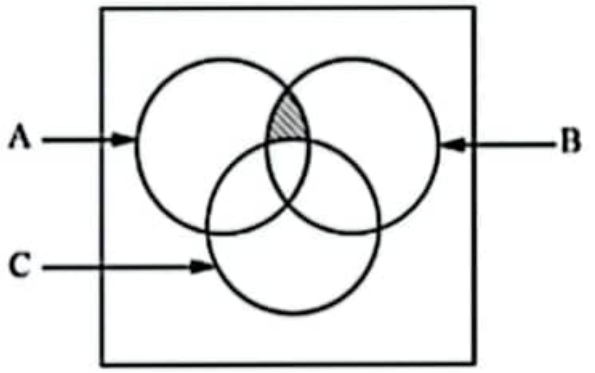
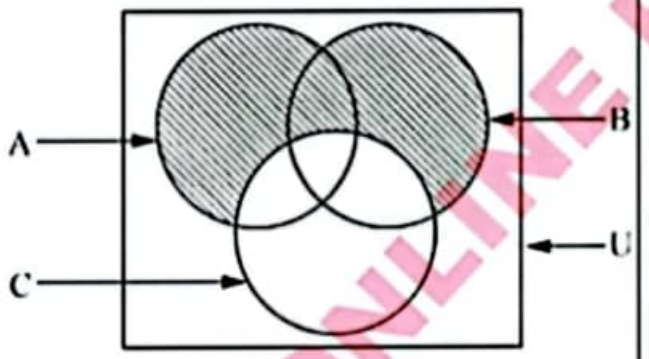
(iii) $A \cap B' \cap C$



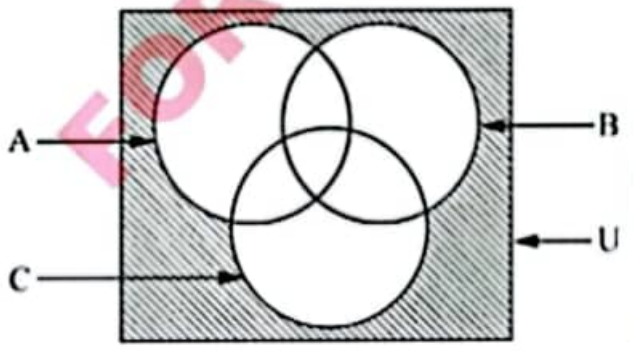
(iv) $A \cap B \cap C'$



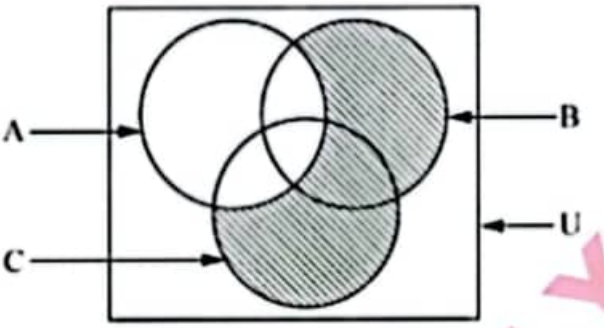
(v) $(A \cup B) \cap C'$



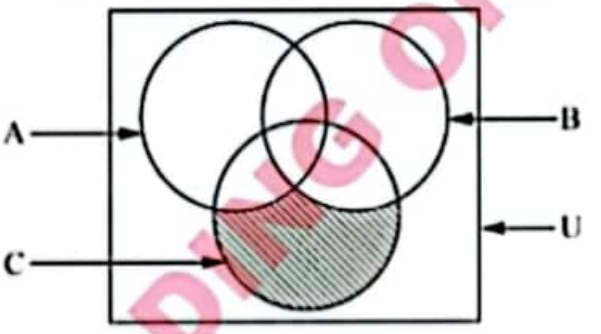
(vi) $A' \cap B' \cap C'$



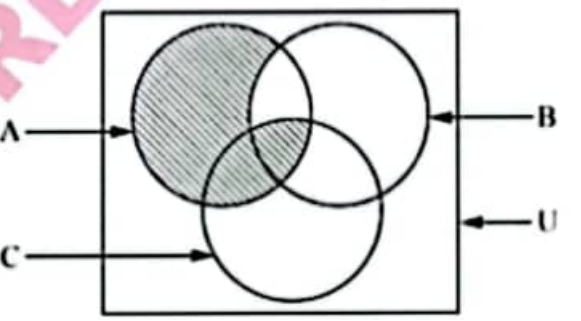
(vii) $A' \cap (B \cup C)$



(viii) $A' \cap B' \cap C$



(ix) $A \cap (B' \cup C)$



5. (a) $(B-C)=\{b,c\}$

(b) $(A-C)=\{a, b, c\}$

(c) $(A-B)=\{a\}$

(d) $(C-B)=\{e, f, g\}$

(e) $(B-A)=\{d\}$

(f) $(C-A)=\{d, e, f, g\}$

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7. (a) True (b) True (c) True (d) True
   (e) False (f) False (g) False (h) False

8. (a) (i) $A=\{5, 10, 15, 20, 25, 30, 35, 40, 45\}$
   $B=\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\}$ $A\cap B=\{20, 40\}$
   $A\cup B=\{4, 5, 8, 10, 12, 15, 16, 20, 24, 25, 28, 30, 32, 35, 36, 40, 44, 45, 48\}$
   (ii) $A\cap B=\{$positive integers less than 50, which are both multiples of 4 and 5$\}$
   (iii) $n(A) = 9$, $n(B) = 12$, $n(A\cap B)=2$
   (b) $A=\{q,r,s\}$, $B=\{q,r,p\}$, $C=\{s,t,q\}$

10. (a) U (b) $\varnothing$
    (c) A (d) B
    (e) $A'\cup B$ (f) $A\cap B'$
    (g) C (h) $X\cup Y'$
    (i) $\varnothing$ (j) $A\cup B$

11. (a) $\varnothing$

14. (a) 240 (b) 504
    (c) 304 (d) 40

15. (a) 11 (b) 3
    (c) 12 (d) 7

16. (a) 5 (b) 50 (c) 95

17. (a) 5% (b) 35% (c) 60%

18. (a) 83.8% (b) 16.2% (c) 95.6%.

19. 19. (a) 6 (b) 18 (c) 33

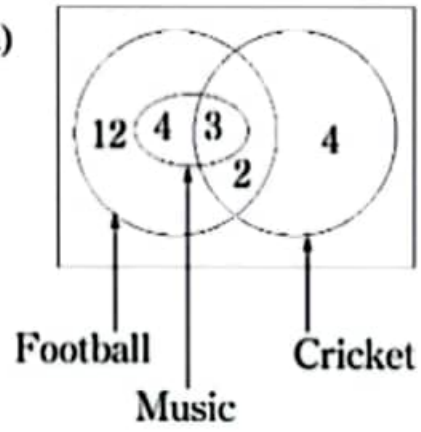
20. (a) Maximum 25, Minimum 15
    (b) Maximum 10, Minimum 0.

21. (a) 5 (b) 12
    (c) 43 (d) 22

22. (a) B (b) $A'\cap B'$
    (c) $A\cap B$ (d) B

25. (a) 15 (b) 10
    (c) 45 (d) 40.

26. (a)



(b) 25 (c) 4 (d) 5

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Answers

Chapter Two

Exercise 2.1

1. a, b, c, e.
2. (a) Simple statement
   (b) Compound statement
   (c) Simple statement
   (d) Compound statement
   (e) Compound statement
3. (a)

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

(b)

p	q	r	s
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

4. (a) Pili is a woman.
   (b) It is not raining now.
   (c) Tanzanian's do not maintain their good culture.
   (d) Summer does not come after spring.
   (e) Industries are not friendly to the environment.
   (f) Tomorrow is not Saturday.

Exercise 2.2

1. (a) False (b) False (c) True
2. (a) $p \wedge q$ (d) $\sim q \wedge \sim p$
   (b) $p \wedge \sim q$ (e) $p \vee (\sim p \wedge q)$
   (c) $\sim (\sim p \vee q)$
3. Let $a \equiv \sim p \leftrightarrow \sim q$ and $b \equiv \sim (p \rightarrow q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$	$p \rightarrow q$	$\sim (p \rightarrow q)$	$a \leftrightarrow b$
T	T	F	F	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	T
F	F	T	T	T	T	F	F

4. (a) $p \rightarrow q$
   (b) $q \rightarrow p$
   (c) $\sim (\sim p \leftrightarrow \sim q)$
5. (a) You like Physics or Chemistry but not Biology
   (b) You like Physics and Chemistry or you do not like Physics and Biology
   (c) It is not true that you like Physics but not Biology.

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6.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

7. (a) $\sim (p \wedge \sim q)$ (b) $\sim p \vee \sim q$

8.

p	q	r	$\sim p$	$q \rightarrow r$	$\sim p \leftrightarrow (q \rightarrow r)$
T	T	T	F	T	F
T	T	F	F	F	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

9. (a) False (b) False (c) True

10. (a) $(\sim P \wedge Q) \vee (P \wedge \sim Q)$
(b) $[(P \rightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (\sim R \wedge \sim Q)$

11. (a) Halima works hard or she is poor.
(b) It is not true that Halima is poor and she works hard.
(c) If Halima is poor, then she works hard.
(d) Halima is poor and she works hard.
(e) If Halima is not poor, then she doesn't work hard.
(f) Halima is not poor or she doesn't work hard.
(g) Halima works hard if and only if she is poor.
(h) Halima does not work hard or she is poor and she works hard.

12. (a) Swimming in the poor is dangerous.
(b) Few people have not been drawn in the swimming pool if and only if Swimming in the pool is not dangerous.
(c) If swimming in the pool is dangerous, then few people have been drawn in the Swimming pool.

13. (a)

p	q	$p \wedge q$	$\sim (p \wedge q)$	$(p \wedge q) \wedge \sim (p \wedge q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

(b)

p	q	$q \rightarrow p$	$p \rightarrow q$	$(q \rightarrow p) \leftrightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

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(c)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(d)

p	q	r	$\sim r$	$p \vee q$	$(p \vee q) \rightarrow \sim r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

(e)

p	q	r	t	$\sim p$	$\sim p \leftrightarrow q$	$(\sim p \leftrightarrow q) \wedge r$	$[(\sim p \leftrightarrow q)] \wedge r \rightarrow t$
T	T	T	T	F	F	F	T
T	T	T	F	F	F	F	T
T	T	F	T	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	T	F	T	T	T
T	F	T	F	F	T	T	F
T	F	F	T	F	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F
F	T	F	T	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	T	F	F	T
F	F	T	F	T	F	F	T
F	F	F	T	T	F	F	T
F	F	F	F	T	F	F	T

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Exercise 2.3

1.(a) *Inverse*:If tomorrow is not Saturday, then Paul will not go to the beach.
Converse: If Paul will go to the beach, then tomorrow is Saturday.
Contrapositive: If Paul will not go to the beach, then tomorrow is not Saturday.

(b) *Inverse*: If it is not raining, then the shop is not closed.
Converse: If the shop is closed, then it is raining.
Contrapositive: If the shop is not closed, then it is not raining.

(c) *Inverse*: If industries are not in harmony with the surroundings, then they are not environmentally friendly.
Converse: If the industries are environmentally friendly, then they are in harmony with the surroundings.
Contrapositive: If the industries are not environmentally friendly, then they are not in harmony with the surroundings.

(d) *Inverse*: If ABC is not an equilateral triangle, then it is not a right-angled triangle.
Converse: If ABC is a right-angled triangle, then it is an equilateral triangle.
Contrapositive: If ABC is not a right-angled triangle, then it is not an equilateral triangle.

(e) *Inverse*: If $f(x)$ is not a rational function, then it has no asymptotes.
Converse: If $f(x)$ has asymptotes, then it is a rational function.
Contrapositive: If $f(x)$ has no asymptotes, then it is not a rational function.

2. (a)

p	q	$\sim p$	$q \rightarrow \sim p$	$(q \rightarrow \sim p) \rightarrow p$
T	T	F	F	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

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(b)

p	q	$\sim p$	$q \rightarrow \sim p$	$\sim(q \rightarrow \sim p)$	$\sim(q \rightarrow \sim p) \rightarrow \sim p$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

3. (a) If two vectors are not orthogonal, then their dot product is not zero.
   (b) If the dot product of two vectors is zero, then they are orthogonal.
   (c) If the dot product of two vectors is not zero, then the two vectors are not orthogonal.
4. (a) If Halima wont win, then she has no courage.
   (b) One cannot be a sailor if is not strong.
   (c) If a geometrical figure is not a rectangle, then it is not a square.
5. Converse $(q \wedge r) \rightarrow p$
   Inverse $\sim p \rightarrow \sim(q \wedge r)$
   Contrapositive $\sim(q \wedge r) \rightarrow \sim p$
6. (a) $q \rightarrow p$
   (b) $\sim(q \rightarrow \sim r) \rightarrow p$
   (c) $\sim(p \leftrightarrow q) \rightarrow \sim(\sim p \leftrightarrow \sim r)$

Exercise 2.4

1. Equivalent
2. Not equivalent
3. Equivalent
4. Equivalent
5. Equivalent
6. Equivalent
7. Equivalent
8. Not equivalent
9. Equivalent
10. Not equivalent

Exercise 2.5

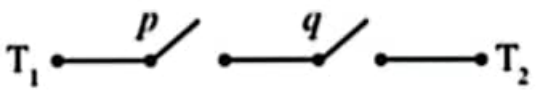
1. (a) Tautology (b) Tautology
   (c) Tautology (d) Not a tautology
   (e) Tautology
2. $\sim(pvq)$
4. (a) $pv \sim q$ (c) p
   (b) $\sim p$ (d) T
5. T
6. (a) F (b) ϕ (c) $qv \sim pv \sim r$

Exercise 2.6

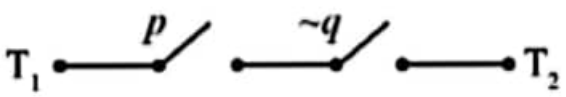
1. Not valid
2. Not valid
3. Valid
4. Not valid
5. Not valid
6. Not valid
7. Valid
8. Valid
9. Valid
10. Valid

Exercise 2.7

1. (a) $p \wedge q$



(b) $p \wedge \sim q$



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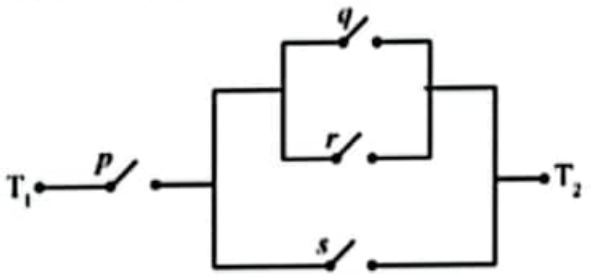
(c) $p \wedge (q \vee r) \vee s$

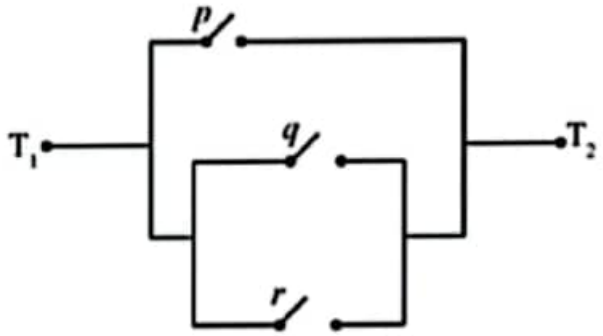


3 (a) $(p \wedge q) \vee \sim q \vee \sim r$

(b) $[\sim p \vee ((p \wedge q) \wedge \sim q)] \wedge [r \vee \sim r]$

(c) $(p \vee q \vee s) \wedge r$

5.



Exercise 2.8

1. $p \wedge \sim q$
2. $(p \wedge q) \vee (\sim p \wedge \sim q)$
3. $\sim p \wedge \sim q$
4. $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$
5. $p \wedge q$
6. $s_1 : (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$
   $\equiv \sim q \wedge r$
   $s_2 : (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$
   $\equiv q \wedge r$

Revision exercise 2

1. Not a mathematical statement. Because it is not possible to asign its truth value T or F.

2. (a) Converse: If a person is an adult, then a person is 20 years old.

Contrapositive: If a person is not an adult, then a person is not 20 years old.

Inverse: If a person is not 20 years old, then a person is not an adult.

(b) *Converse:* If I have a test today, then today is Friday.

Contrapositive: If I have no test today, then today is not Friday.

Inverse: If today is not Friday, then I have no test.

(c) *Converse:* If you are attractive, then you bought our clothes.

Contrapositive: If you are not attractive, then you did not buy our clothes.

Inverse: If you did not buy our clothes, then you are not attractive.

(d) *Converse:* If today is a holiday, then it is Saturday.

Contrapositive: If today is not a holiday, then it is not Saturday.

Inverse: If today is not Saturday, then it is not a holiday.

(e) *Converse:* If Nuru pass the examination, then she is intelligent.

Contrapositive: If Nuru do not pass the examination, then she is not intelligent.

Inverse: If Nuru is not intelligent, then she will not pass the examination.

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4. (a) Not a tautology (b) Tautology (c) Tautology (d) Tautology

5. They have the same truth values (They are logically equivalent).

6. (a) Tautology (b) Tautology (c) Not a tautology (d) Tautology
   (e) Not a tautology (f) Not a tautology (g) Tautology (h) Tautology
   (i) Not a tautology (j) Tautology

7. (a), (b), (c), and (e)

9. (a) Logically equivalent (b) Logically equivalent (c) Logically equivalent
   (d) Logically equivalent

10. (d) and (f)

11. (a) T (b) T (c) $p \vee \sim q \vee r$ (d) T

12. (a) Not a tautology (b) Not a tautology

13. (a) (i) $(p \wedge q) \vee r \equiv$ Lightness is clever and polite or she is humble.
    (ii) $\sim p \vee q \equiv$ Lightness is not clever or she is humble.
    (b) (i) $(p \vee q) \wedge p \rightarrow \sim q$ (ii) $(q \vee r) \wedge \sim p$ (iii) $(\sim q \wedge \sim r) \rightarrow \sim p$

14. (a) Valid (b) Valid (c) Not valid (d) Valid (e) Valid
    (f) Not valid (g) Valid (h) Not valid (i) Not valid (j) Not valid

16. (a) $p \wedge (q \vee r) \wedge s$ (b) $(p \vee \sim r) \vee [(\sim p \vee q) \wedge (q \vee r)]$ (c) $p \vee q \vee r$

17. (a) $(p \wedge \sim q) \vee (\sim p \wedge q)$ (b) $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

18. (a) Tautology (b) Not a tautology

19. $* = \wedge$

20. (a) $\sim q \wedge r$: T_1 —$\sim q$— —r— T_2
    (b) q: T_1 —q— T_2

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Answers

Chapter Three

Exercise 3.1

1. 109.4°
2. 78.7°
3. (a) 36.9°, 40.6°, 102.5°
   (b) 40.6°, 63.4°, 76°
   (c) 17.7°, 60.3°, 102°
4. $x-3y+7=0$
5. $m=\frac{18}{5}$, $n=\frac{22}{15}$, 53.6° or 126.4°
6. 47.7°, 132.3°
7. $17\sqrt{3}x+13y=126-68\sqrt{3}$
9. $5x+y=13$ and $x-5y=13$

Exercise 3.2

1. (a) 1 unit (b) 2.55 units
   (c) $\frac{3}{5}a$ units (d) $\frac{2h+4k}{5}$ units
   (e) 0 units, the point lies on the line
2. $p=\frac{-9\pm\sqrt{2}}{3}$
4. $\overline{PR}=5.06$ units, $\overline{QS}=0.95$ units, and $\overline{RS}=0.32$ units
6. $k=-5$ 7. $n=-1$
8. $a=-2$ or $a=34$

Exercise 3.3

1. (a) $x=4$ (b) $y=5$
   (c) $x^2+y^2=25$
   (d) $x^2+y^2+4x+6y-51=0$
   (e) $x^2+y^2-8x+12=0$
   (f) $x^2-3y^2+32y-64=0$
   (g) $3x^2+4y^2+6x-9=0$
   (h) $x=\frac{3}{8}$
2. $8x-6y-7=0$
3. $y^2=4ax$
4. $y^2-3x^2+34x=91$
5. $3x-11y=0$, $11x+3y=0$
6. $73x^2-156x+9y^2-18y-63=0$
7. $3x^2+3y^2+2x+16y+25=0$
8. $8x^2+8y^2-228x-224y+2540=0$
9. $2484x^2+2500y^2-19800x-25000y+400=0$
10. $2x+3y+10=0$

Exercise 3.4

1. (a) $\left(-\frac{7}{5}, \frac{14}{5}\right)$, $(25, 22)$
   (b) $(-1, 3)$, no external division
   (c) $\left(-5, \frac{32}{5}\right)$, $(7, 40)$
   (d) $\left(\frac{19}{3}, 6\right)$, $(8, 6)$
2. C(20, 22)
3. 0.41 units
4. P(−1, −3), Q(2, 3)
5. $m=3$, $n=2$
6. 2:9
8. $m=2$, $n=3$
9. 1:3

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10. $\left(\frac{73}{13}, \frac{64}{13}\right)$

11. A$\left(-4, \frac{-7}{3}\right)$ B$\left(\frac{-20}{7}, \frac{-9}{7}\right)$

Exercise 3.5

1. (a) $x^2+y^2-4y-12=0$
   (b) $x^2+y^2+4x-4y-17=0$
   (c) $x^2+y^2-4=0$
   (d) $4x^2+4y^2+4x-40y-299=0$
   (e) $8x^2+8y^2-32x+48y+103=0$

2. (a) Centre $(1, -2)$, radius $=\sqrt{45}$ units
   (b) Centre $(0, 0)$, radius $=5$ units
   (c) Centre $(0, 2)$, radius $=2$ units
   (d) Centre $(-1, 0)$, radius $=2$ units
   (e) Centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$, radius $=1$ unit

3. $x^2+y^2-8x-8y+16=0$

4. $(x+41)^2+(y+161)^2=144$

5. $4x^2+4y^2+40x-16y+67=0$

6. Centre $\left(2, \frac{-5}{2}\right)$, radius $=\frac{3\sqrt{3}}{2}$ units

8. Centre $(5, -6)$, radius $=2\sqrt{3}$ units

9. (b) Centre $(4, 1)$, radius $\frac{3\sqrt{6}}{2}$ units,
   diameter $=3\sqrt{6}$ units

10. $x^2+y^2+12x+8y-92=0$

Exercise 3.6

1. (a) $x^2+y^2+7x-5y+16=0$
   (b) $x^2+y^2-4x+4y-17=0$
   (c) $x^2+y^2+x+2y-3=0$
   (d) $x^2+y^2-x+4y-53=0$
   (e) $x^2+y^2+4x-3y=0$

2. (a) $x^2+y^2-6x+4y-12=0$
   (b) $x^2+y^2-x+4y-12=0$
   (c) $x^2+y^2-14x+10y-95=0$
   (d) $x^2+y^2-10x-8y+16=0$
   (e) $x^2+y^2-13x+3y+2=0$

3. $x^2+y^2-10x-10y+24=0$

4. C$\left(\frac{1}{2}, \frac{3}{2}\right)$, $r=\frac{9\sqrt{2}}{2}$ units

5. $x^2+y^2-2x+2y-23=0$

7. (a) C$(-4, 8)$ (b) $r=4\sqrt{10}$ units
   (c) $x^2+y^2-8x-16y-80=0$

9. $8x^2+8y^2-46x-35y-202=0$

Exercise 3.7

1. (a) Tangent, $6x-2y-5=0$;
   Normal, $x+3y=0$
   (b) Tangent, $3x-4y+25=0$;
   Normal, $4x+3y=0$.
   (c) Tangent, $2x-y-9=0$;
   Normal, $x+2y+3=0$

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8. (a) $\left(\frac{18}{5}, \frac{14}{5}\right)$ (b) $(6, -2)$

9. $4\sqrt{2}$ units

10. $-x+y+8=0$, $(4, -4)$

11. L(4,4), M(7, 6)

12. $\frac{6\sqrt{5}}{5}$ units

13. $x^2+y^2-12x-12y+36=0$,
$x^2+y^2-2x-2y+1=0$

14. $x^2=4by$

15. 5.9 units

16. 53.13°

17. $2x-y-4=0$, $x+2y-7=0$

18. $x-3y+3=0$, $3x+y-11=0$

19. 47.7°

20. $m=-2$, $\left(\frac{1}{3}, -\frac{1}{3}\right)$

21. $x^2+y^2-2x+2y-23=0$

22. $x^2+y^2+\frac{8}{5}x-\frac{16}{5}y-4=0$

centre $\left(-\frac{4}{5}, \frac{8}{5}\right)$,

radius $= \frac{6\sqrt{5}}{5}$ units

23. $x^2+y^2-4x+6y+8=0$

24. C(5, 6)

25. B(6, 4)

26. $x^2+y^2-2x-2y+1=0$

27. 30°

28. $6x+4y+23=0$

29. $16x^2+9y^2+24xy-156x-42y+249=0$

30. $x^2-4x+2y-3=0$

31. $x^2+y^2-10x-8y+16=0$

33. $x^2+y^2+6x-6y+8=0$

34. $x^2+y^2-4x-8y+15=0$

35. $a=2$, $b=0$

36. 36. (1, 2)

37. $x^2+y^2-12x-10y+36=0$

38. centre $(-2,4)$, radius $=4$ units,
length of tangent $=5$ units

39. $2x^2+2y^2-8x-12y+25=0$

40. $y=2x$

42. $c=-32$

43. $x-y-2=0$, $x+7y\pm10=0$

44. $12x+5y+119=0$; $12y-5x+49=0$

45. $y=2x$; $22x+19y=0$

46. $\sqrt{37}$ units

47. $\sqrt{12}$ units

49. $4x-3y-18=0$; 13.5 square units

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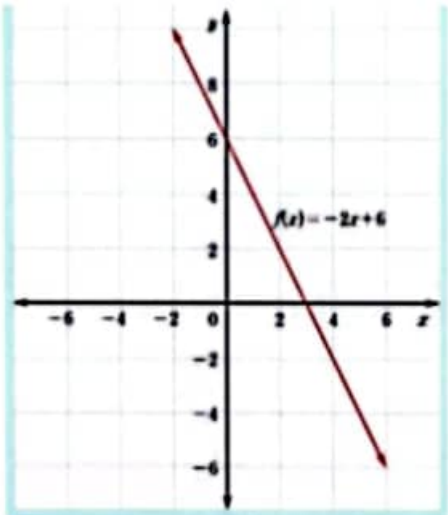
Answers

Chapter Four

Exercise 4.1

1.



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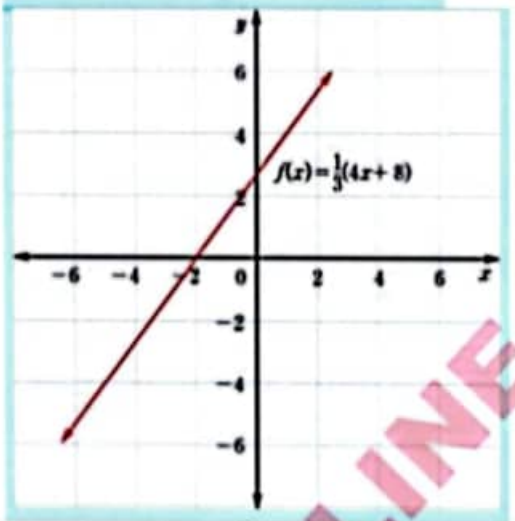

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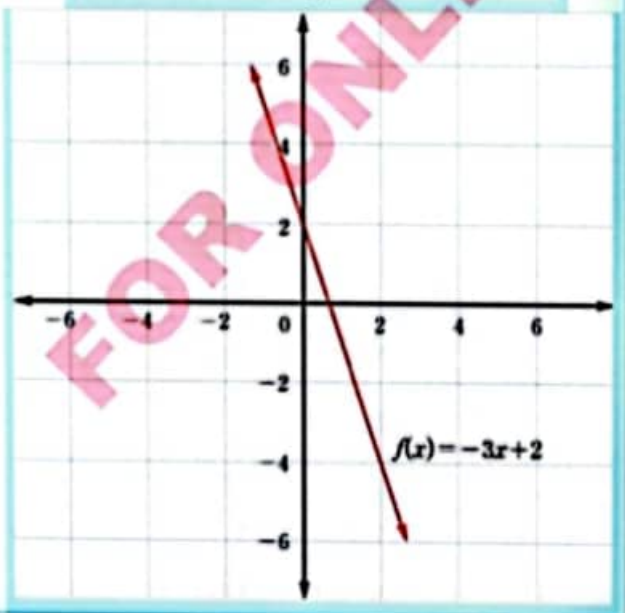

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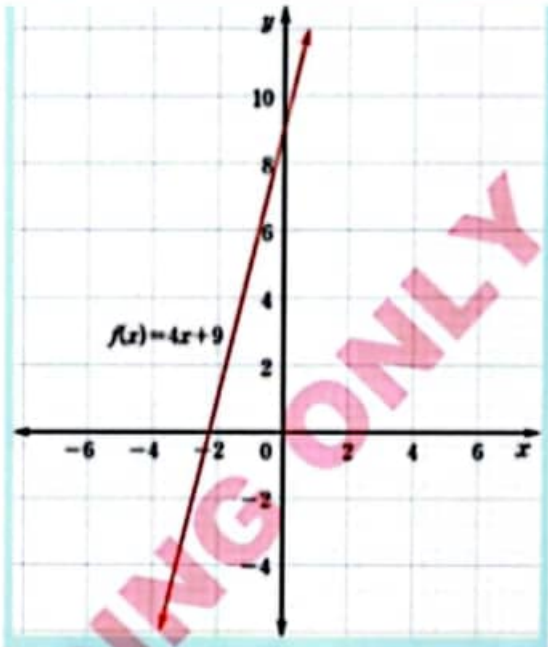

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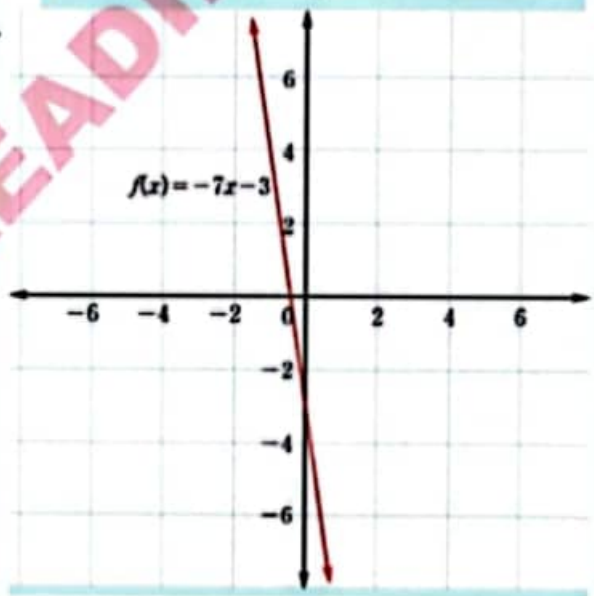

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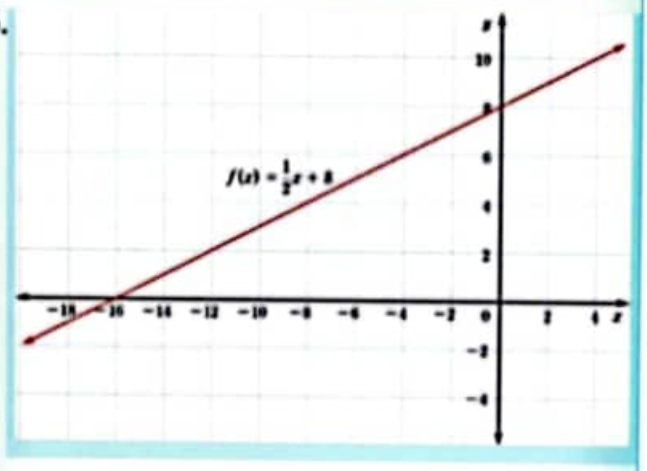

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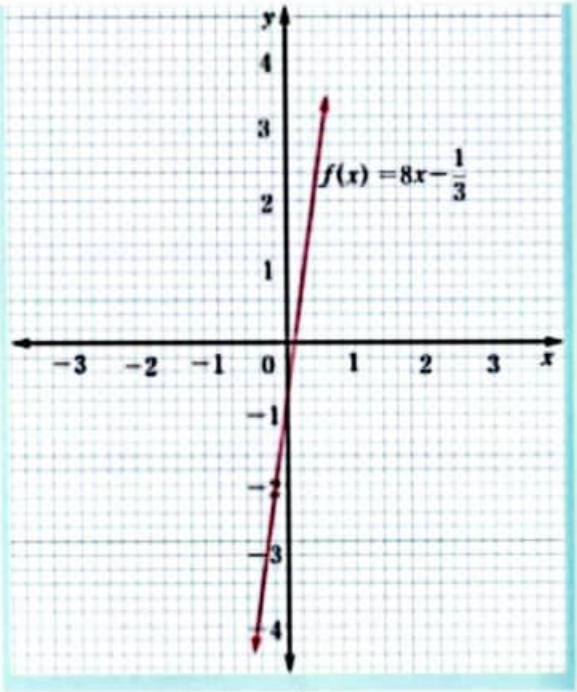

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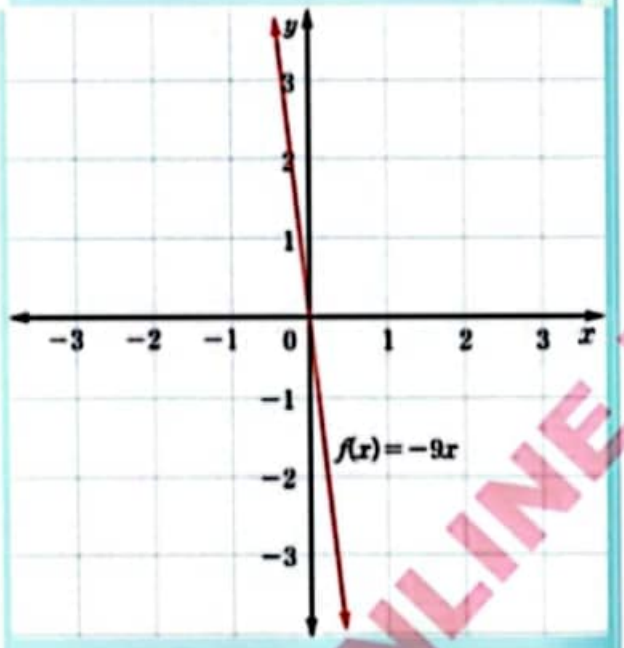

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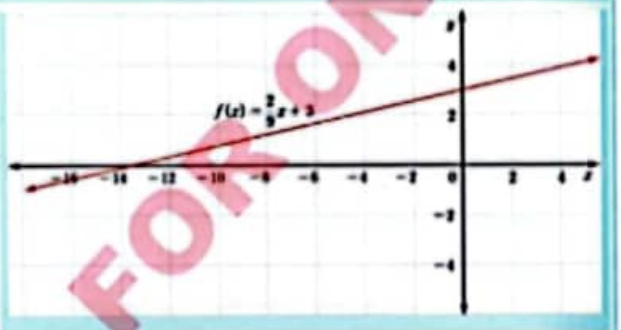

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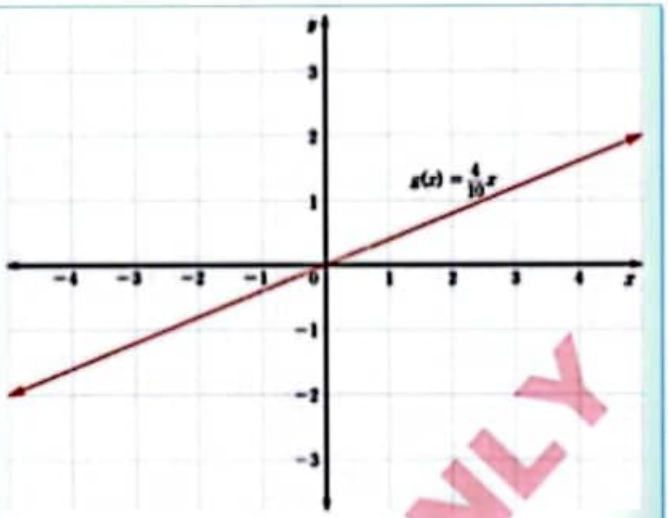
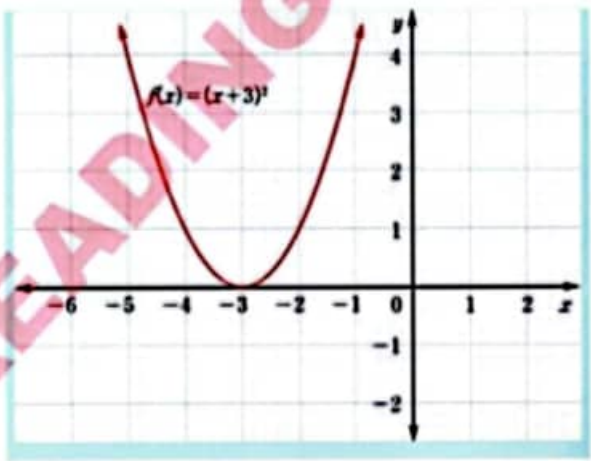


Exercise 4.2

1.



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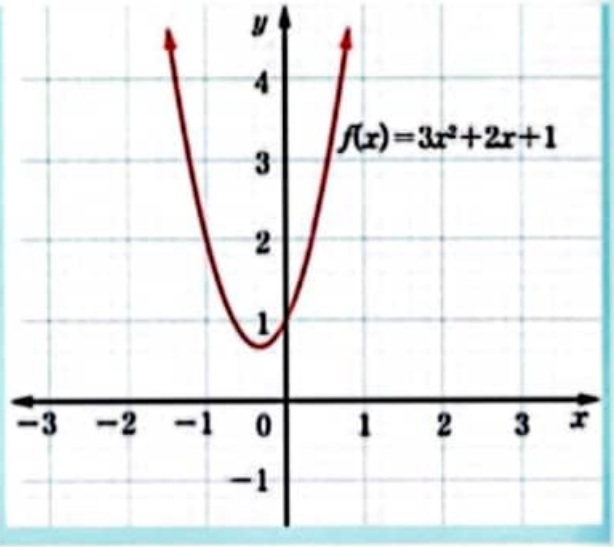

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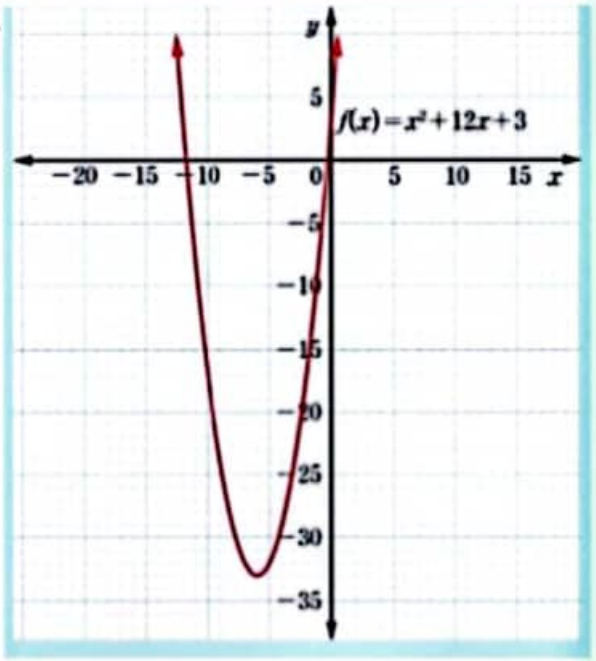
3.



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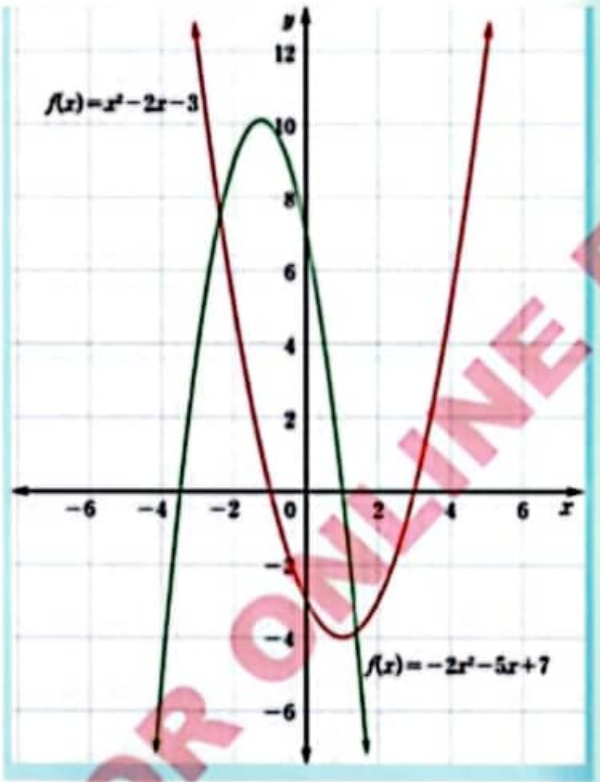

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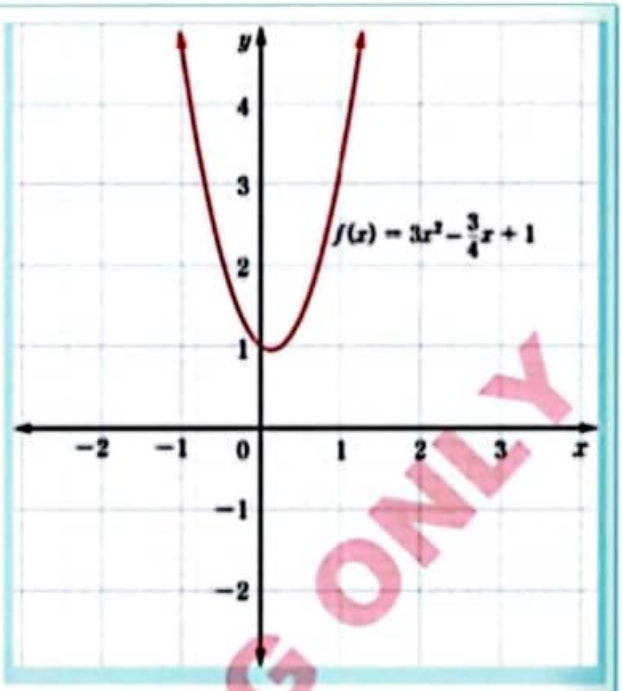

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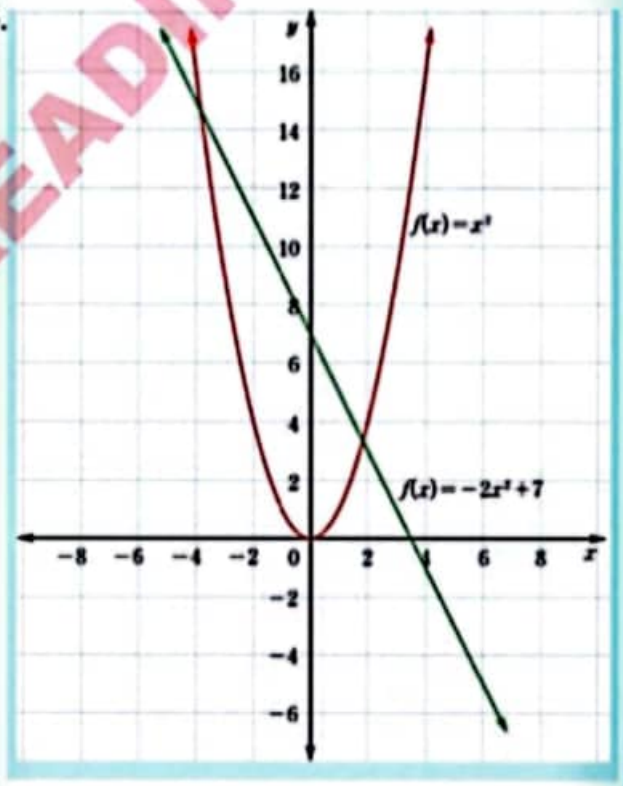

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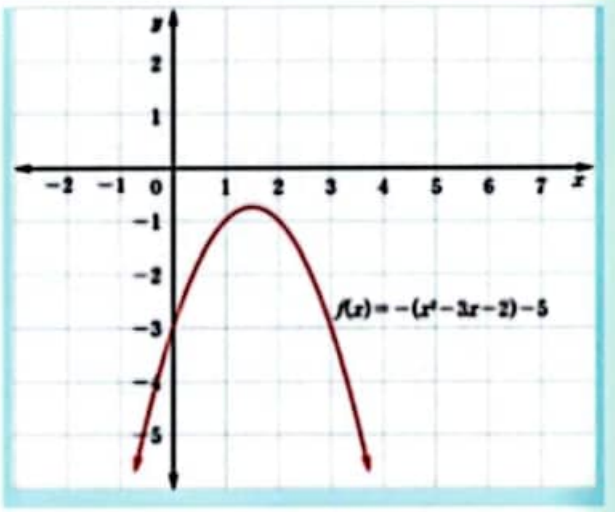
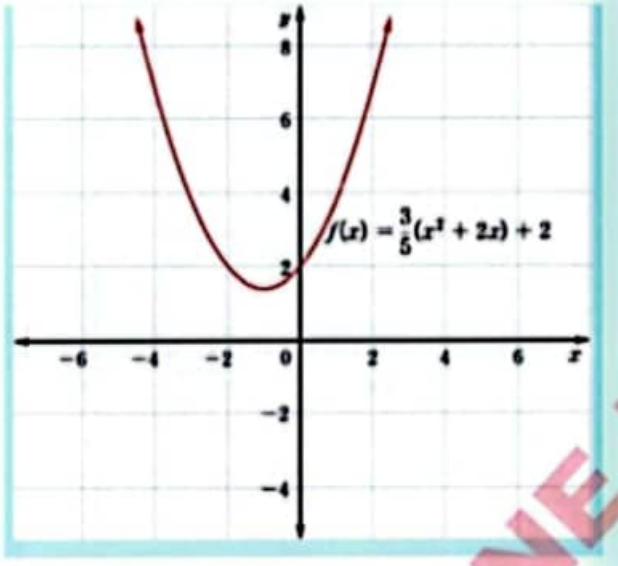
7.



8.



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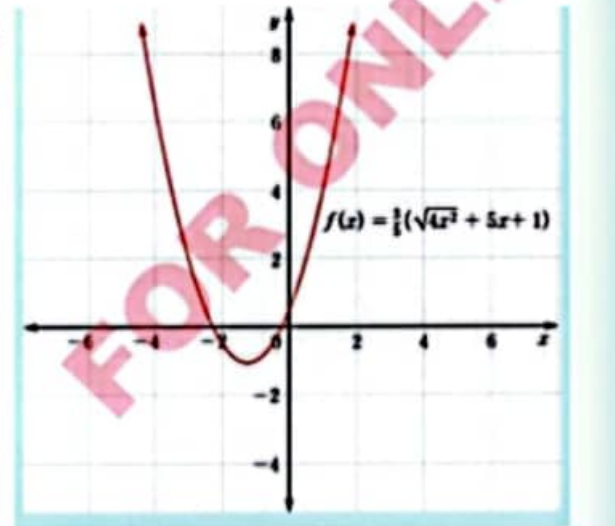
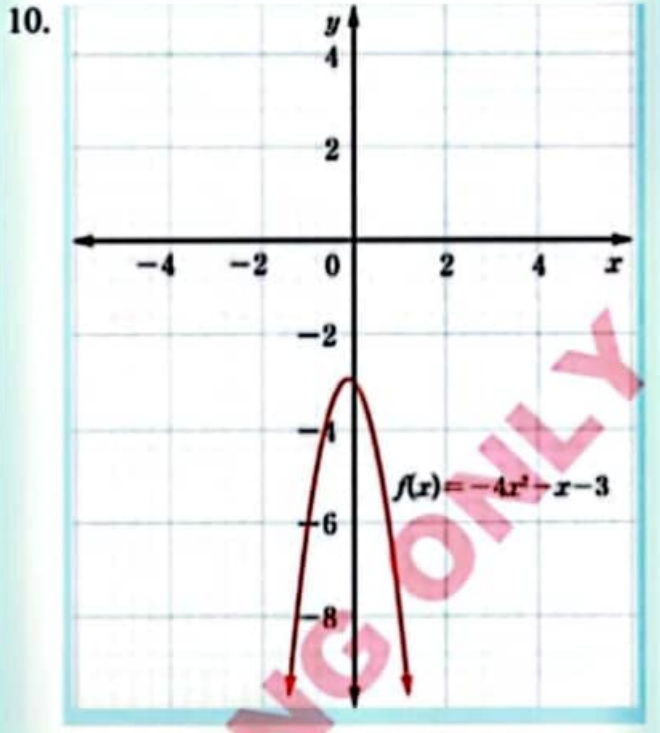
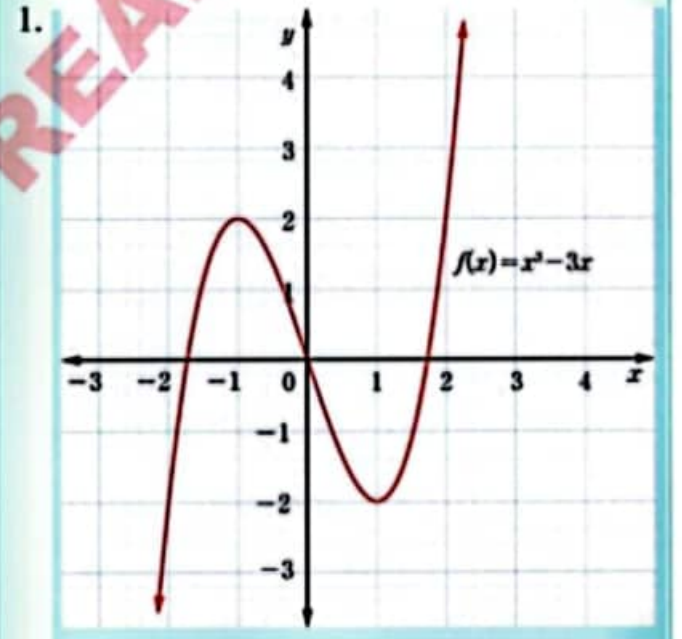


10.



Exercise 4.3

1.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

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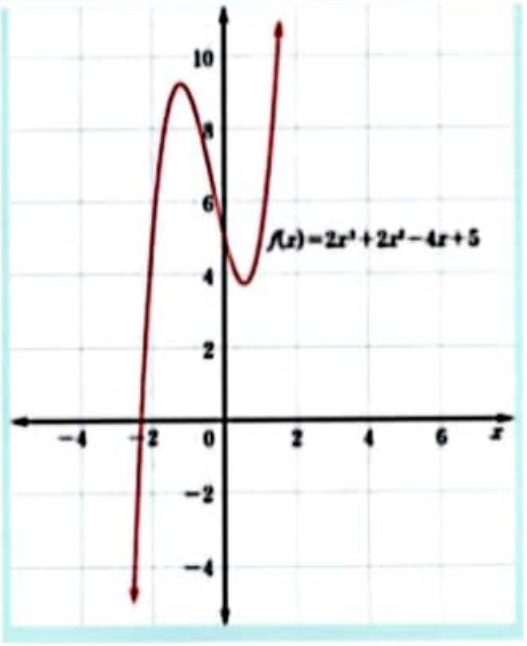
2.



Domain = $\{x : x \in \mathbb{R}\}$

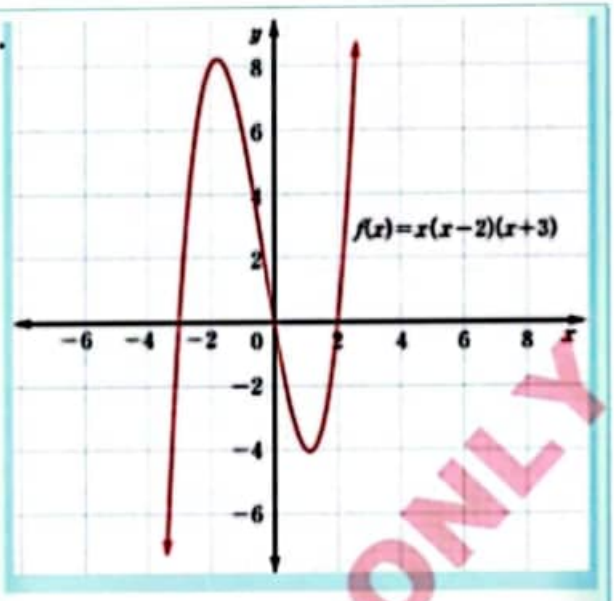
Range = $\{y : y \in \mathbb{R}\}$

3.

Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

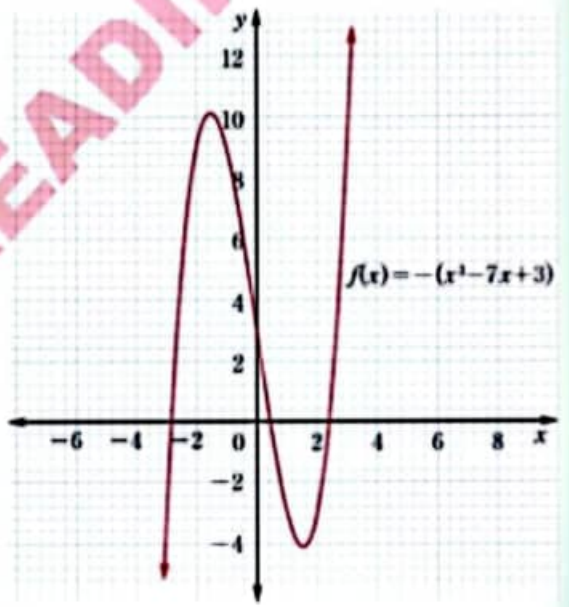
4.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

5.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

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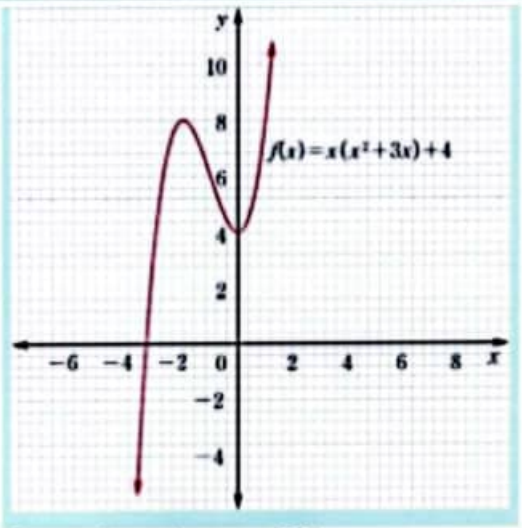
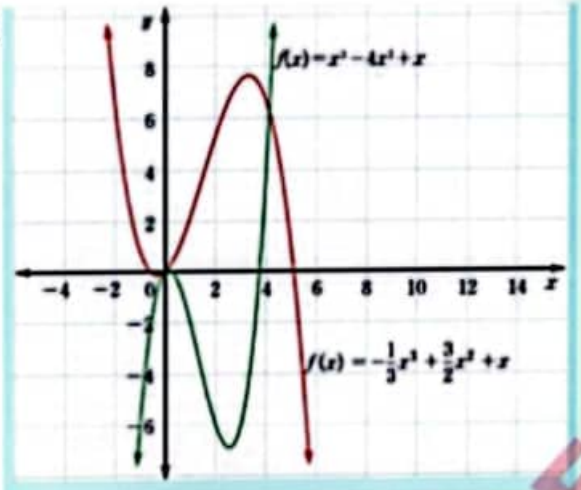
6.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

7.



Domain = $\{x : x \in \mathbb{R}\}$

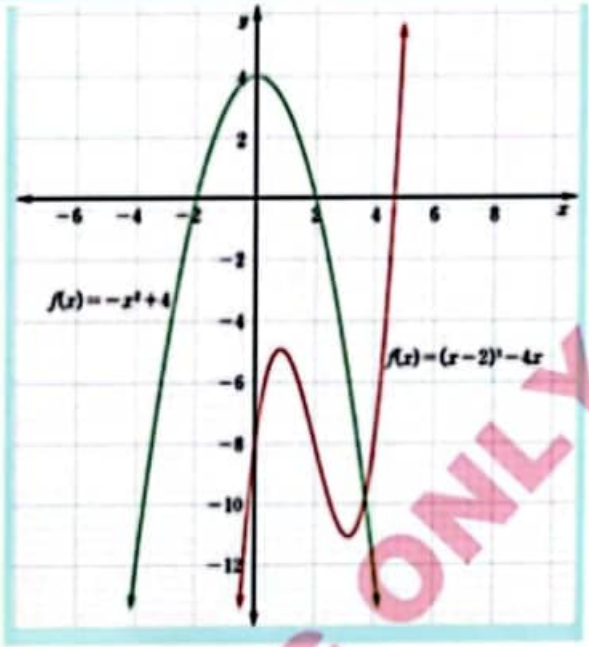
Range = $\{y : y \in \mathbb{R}\}$

8.

Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

9.



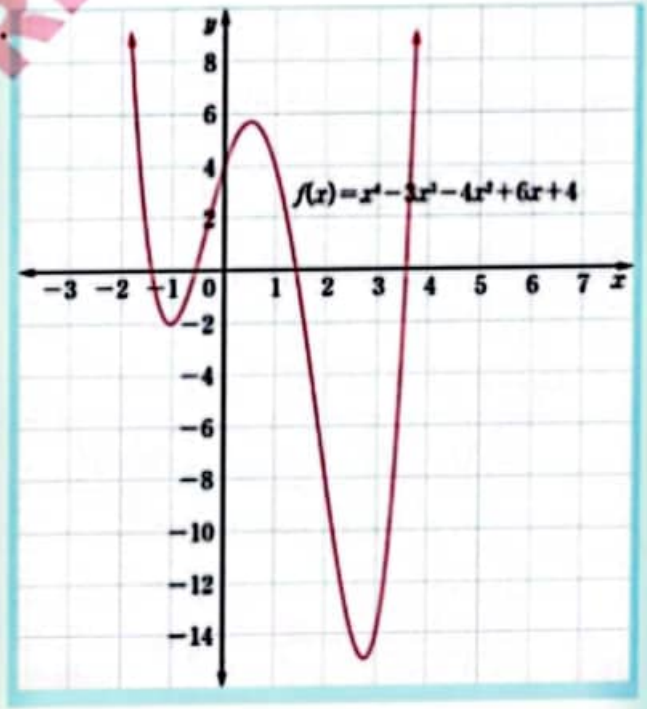
Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

Exercise 4.4

1. $x = -1, 0, 1, 2$

2.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \geq -15\}$

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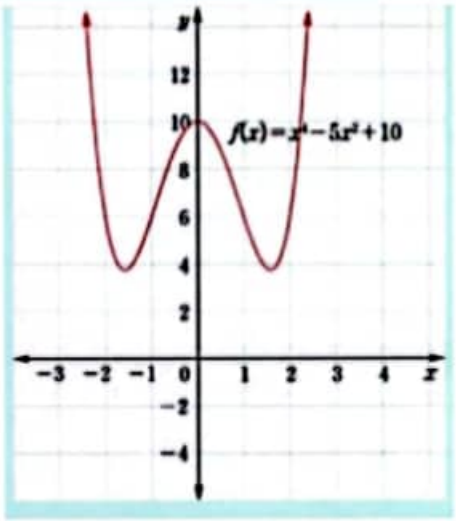
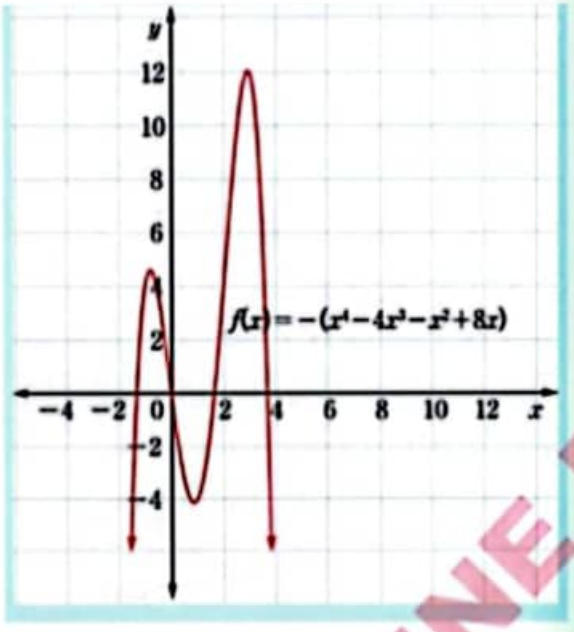
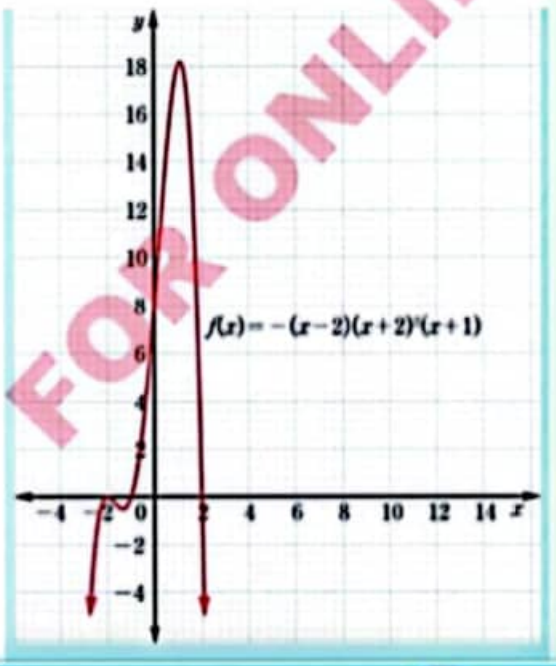
3. (a)



(b)



(c)



4.

5. Range: $\left\{y : y \in \mathbb{R}, y \geq -\frac{1}{4}\right\}$

6. $f(x)$ goes to large positive ∞

7. Domain $= \{x : x \in \mathbb{R}\}$

   Range $= \{y : y \in \mathbb{R}, y \geq 0\}$

8. Domain $= \{x : x \in \mathbb{R}\}$

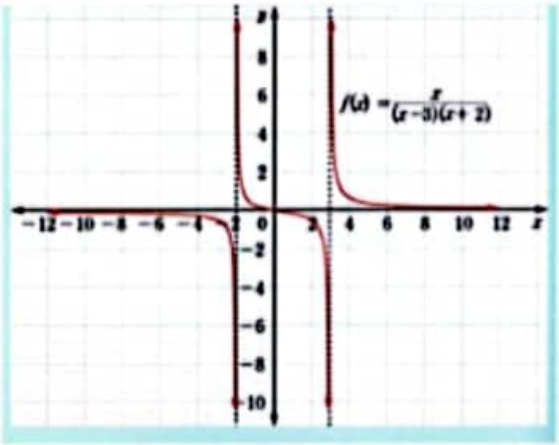
   Range $= \{y : y \in \mathbb{R}, y \leq 1\}$

9. $f(x)$ increases to $+\infty$

10. Range $= \{y : y \in \mathbb{R}, -20.9 \leq y \leq 126\}$

Exercise 4.5

1.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

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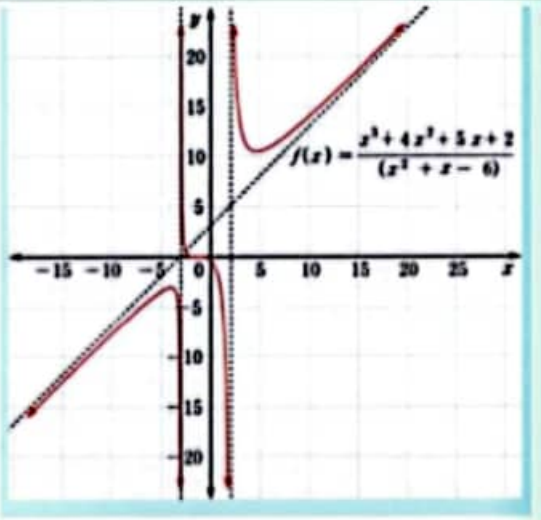
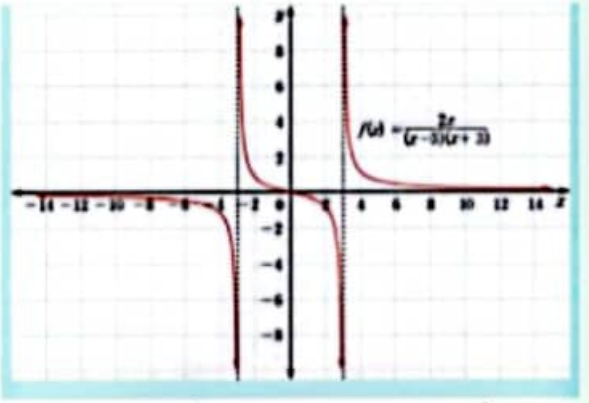
2.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

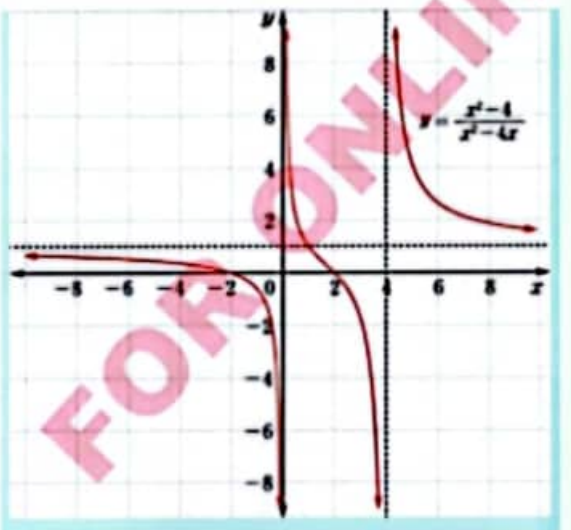
3.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $= \{y : y \in \mathbb{R}\}$

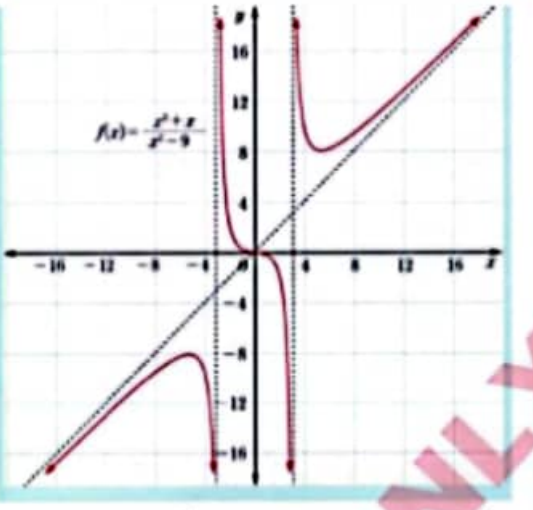
4.



Domain $= \{x : x \in \mathbb{R}, x \neq 0, x \neq 4\}$

Range $= \{y : y \in \mathbb{R}\}$

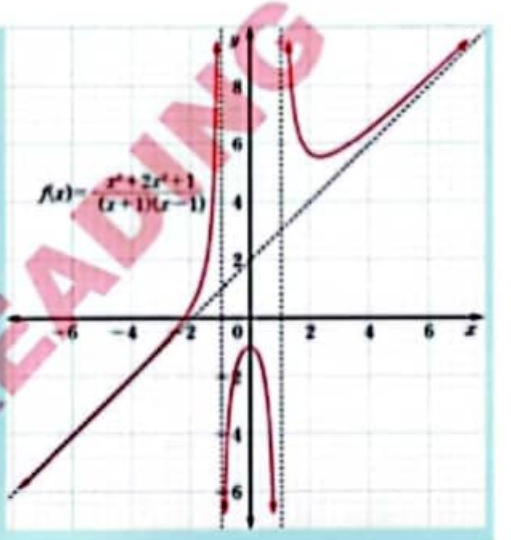
5.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $= \{y : y \in \mathbb{R}\}$

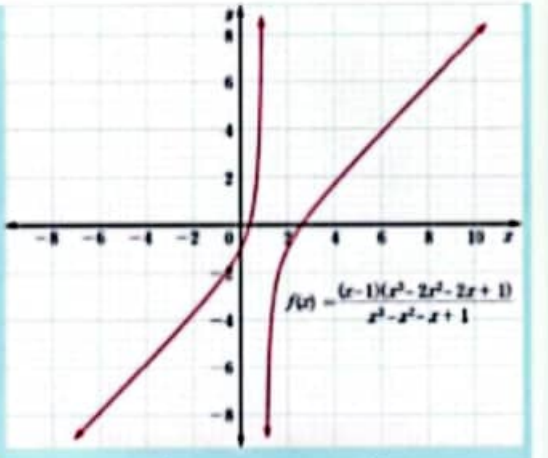
6.



Domain $= \{x : x \in \mathbb{R}, x \neq -1, x \neq 1\}$

Range $= \{y : y \in \mathbb{R}\}$

7.



Domain $= \{x : x \in \mathbb{R}, x \neq 1\}$

Range $= \{y : y \in \mathbb{R}\}$

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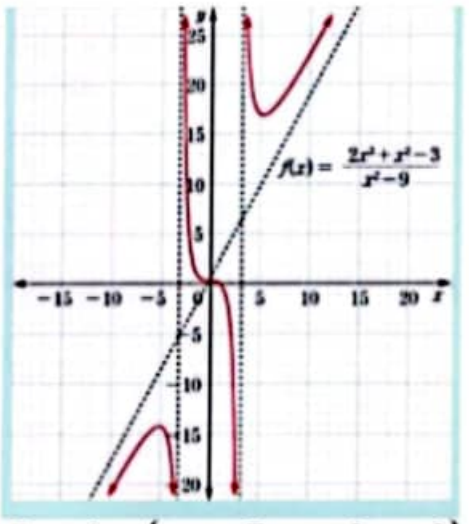
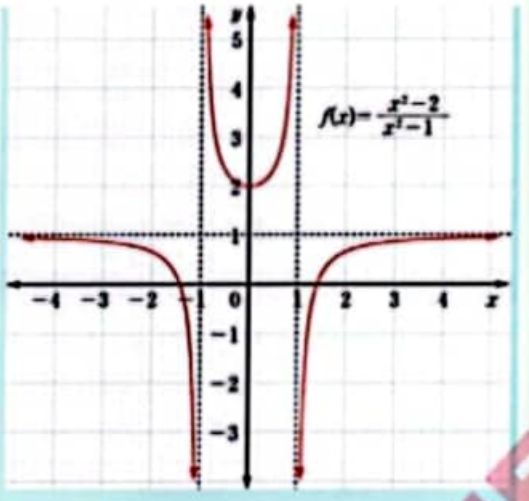
8.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $= \{y : y \in \mathbb{R}\}$

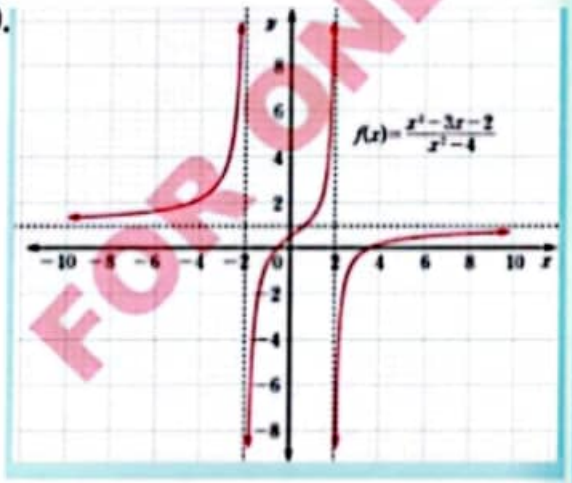
9.



Domain $= \{y : y \in \mathbb{R}, x \neq -1, x \neq 1\}$

Range $= \{y : y \in \mathbb{R}, y < 1, y \geq 2\}$

10.



Domain $= \{x : x \in \mathbb{R}, x \neq -2, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

Exercise 4.6

1. $f(g(x)) = -3x + 17, g(f(x)) = -3x - 1.$
   Not commutative.
2. (a) $f(x) = \dfrac{x^2 - 3x + 2}{2 - x}, f\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

   (b) $f(x) = 3x + 2, f\left(\dfrac{1}{2}\right) = \dfrac{7}{2}$
3. $(g \circ f)(x) = \{(-1, 1), (0, 5)\}$
   Domain $= \{x : -1, 0\}$, Range $= \{y : 1, 5\}$
4. (a) $(f \circ g)(x) = x^2 + 6x + 7$

   (b) $(f \circ g)(x) = x^2 - 7$

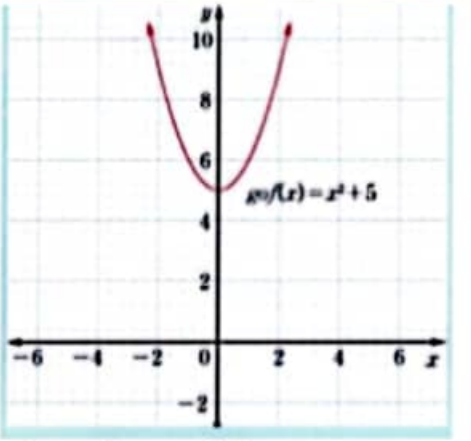
   (c) $(f \circ g)(x) = e^{x^2} - 2$ (d) $(f \circ g)(x) = x$
6. -7 7. $x = \pm 5$
8. (a) 34 (b) 112 (c) 584

   (d) 130 (e) 2706 (f) $x^2 - 2x + 3$
9. (a) $32x^2 - 312x + 685$

   (b) $512x^3 - 4576x + 10149$
10. (a) $f \circ g = \{(7, 5), (-5, 8)\}$ and
    $g \circ f = \{(3, 10), (2, 7), (4, 3)\}$
11. $g \circ f = \{(-2, 1), (0, 3)\}$
12. $f \circ g = \{(2, 6), (4, 7)\}$

Exercise 4.7

1.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}, y \geq 5\}$

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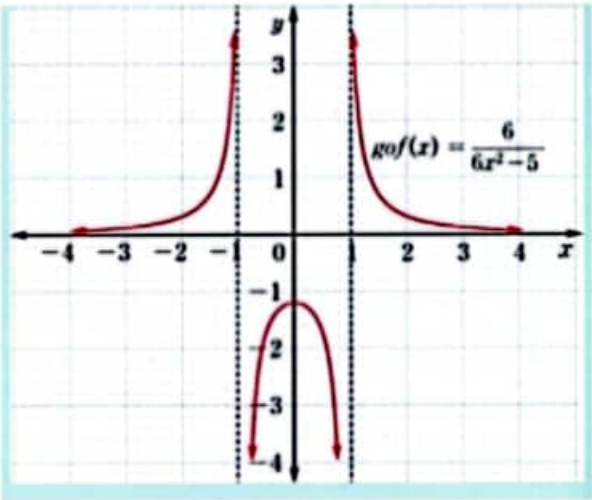
2. $g(x)=x-3$ and $g(x)=-x+1$

3. $(f\circ g)(x)=\{(3,6)\}$

4. 

5. $\sqrt{3}x+\dfrac{10-2\sqrt{3}}{2\sqrt{3}}$ or $-\sqrt{3}x-\left(\dfrac{10+2\sqrt{3}}{2\sqrt{3}}\right)$

6. $f(x)=\dfrac{x+1}{x^2-2x+5}$

7. $k=\dfrac{5}{3}$

8. $x=\pm4$ or $x=\pm2$

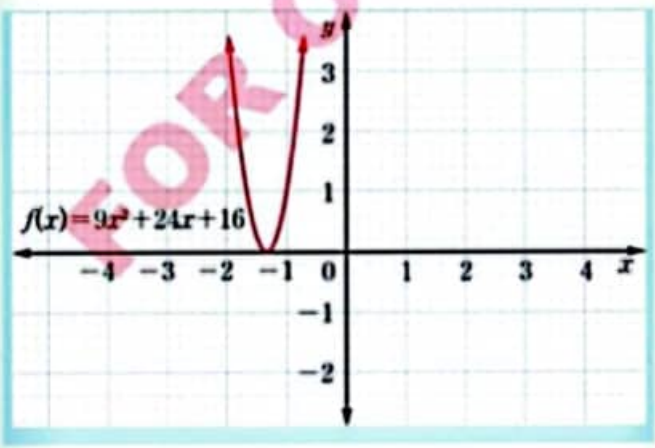
9. x-intercept $(x,y)=(1,0)$

   y-intercept $(x,y)=\left(0,-\dfrac{1}{2}\right)$

   Vertical asymptote $x=-2$

   Horizontal asymptote $y=1$

10. (a) $3x^2+4$ (b) $9x^2+24x+16$



11. (a) $\dfrac{1}{4}x^2-x+1$ (b) 7.5

12. (a) $(f\circ g)(x)=x+9$
    and $(g\circ f)(x)=x+9$

    (b) $(f\circ g)(x)=x^2+2$
    and $(g\circ f)(x)=x^2+2$

    (c) $(f\circ g)(x)=-9x+19$
    and $(g\circ f)(x)=3-9x$

Exercise 4.8

1. Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>0\}$

2. (a) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>-4\}$

   (b) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y<0\}$

   (c) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>-3\}$

   (d) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y<1\}$

   (e) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>3\}$

6. (a) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>7\}$

   (b) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>-1\}$

   (c) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>0\}$

   (d) Domain $=\{x:x\in\mathbb{R}\}$
   Range $=\{y:y\in\mathbb{R},\ y>1\}$

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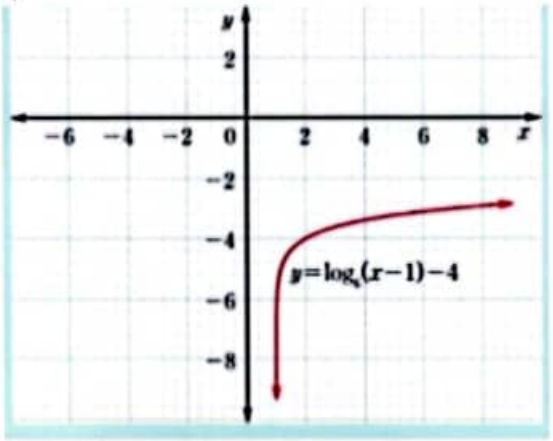
Exercise 4.9

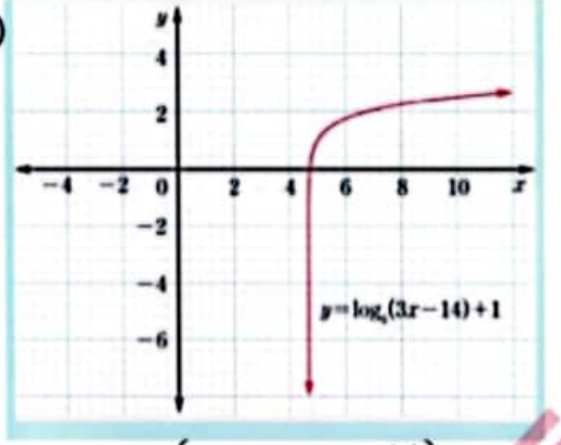
1. (a)



Domain $= \{x : x \in \mathbb{R}, x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

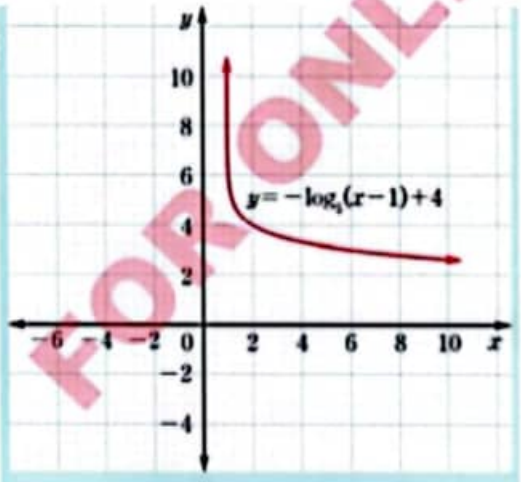
(b)



Domain $= \left\{x : x \in \mathbb{R}, x > \frac{14}{3}\right\}$

Range $= \{y : y \in \mathbb{R}\}$

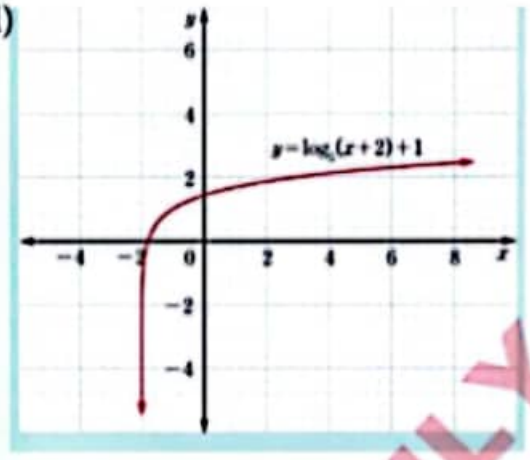
(c)



Domain $= \{x : x \in \mathbb{R}, x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

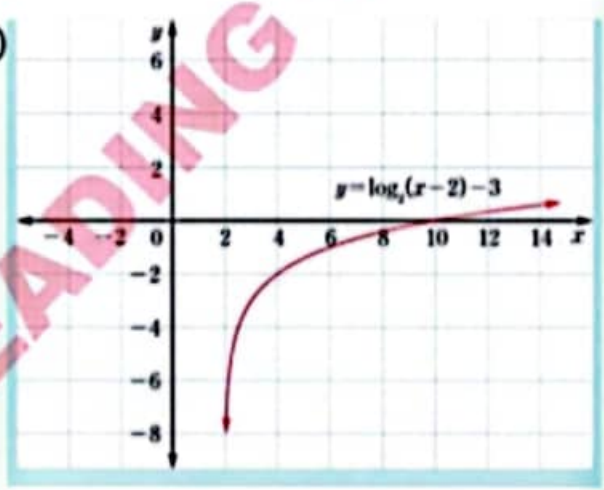
(d)



Domain $= \{x : x \in \mathbb{R}, x > -2\}$

Range $= \{y : y \in \mathbb{R}\}$

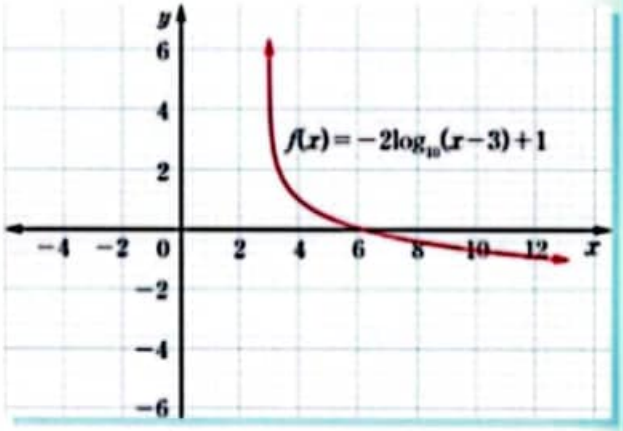
(e)



Domain $= \{x : x \in \mathbb{R}, x > 2\}$

Range $= \{y : y \in \mathbb{R}\}$

2.



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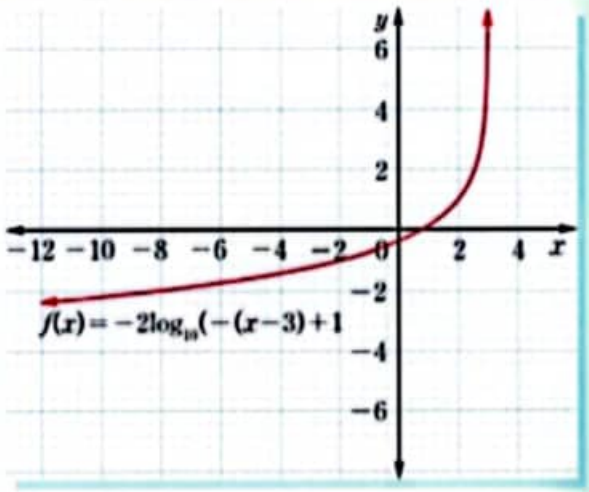
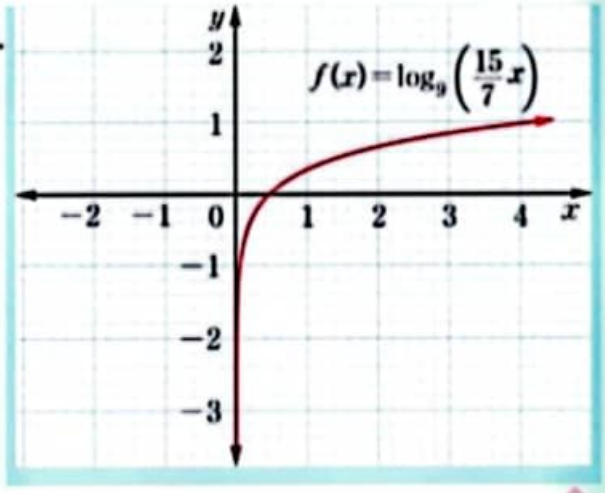
3.

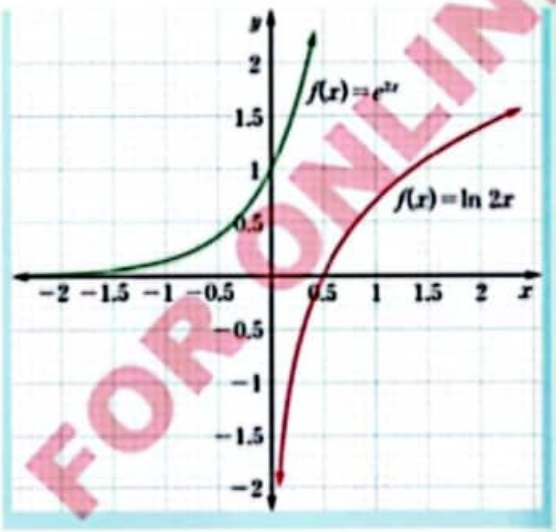


4.

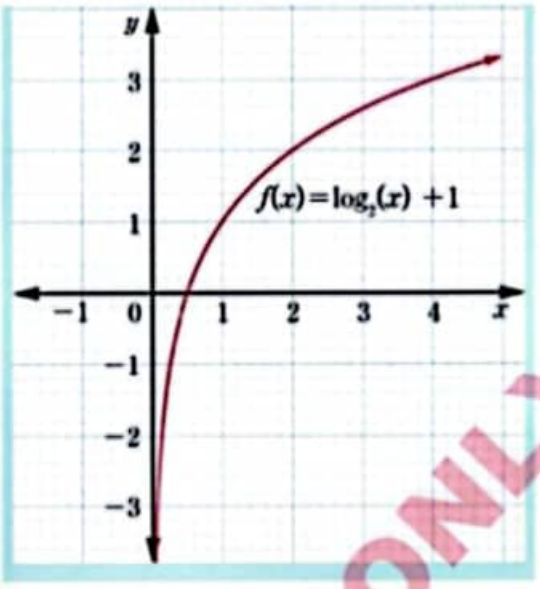


Domain$=\{x : x \geq 0\}$, Range $=\{y : y \in \mathbb{R}\}$

5.



6.



7.

(a) Domain$=\{x : x \in \mathbb{R}\}$,

Range $=\{y : y > -6\}$

(b) x-intercept $=(-0.7, 0)$

y-intercept $=(0, 10)$

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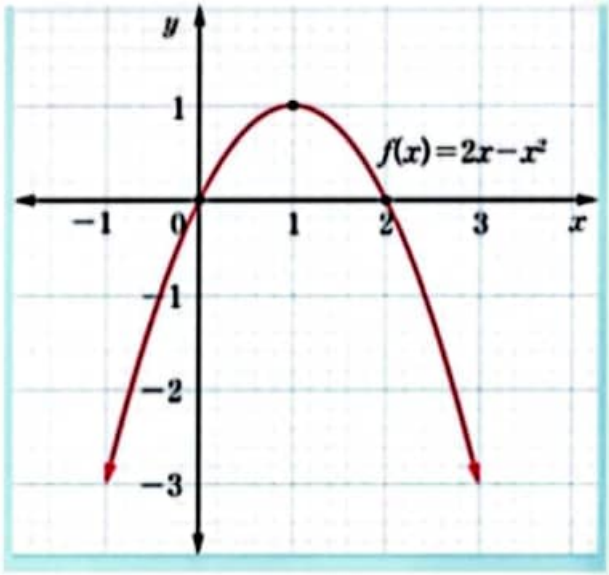
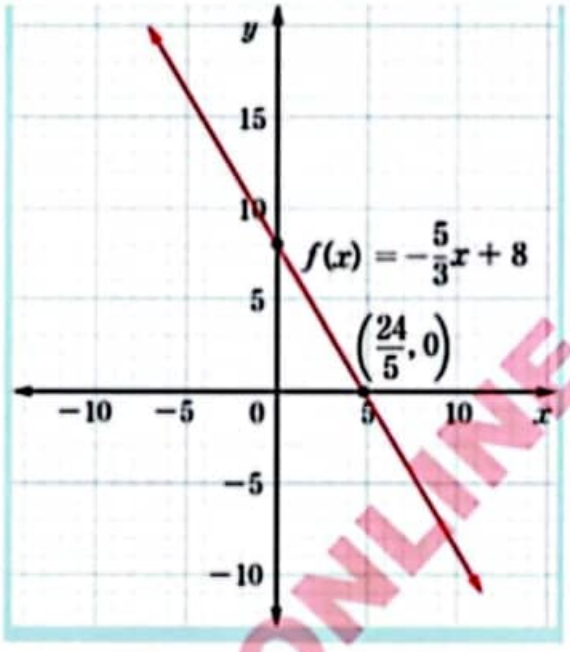
Revision exercise 4

1.

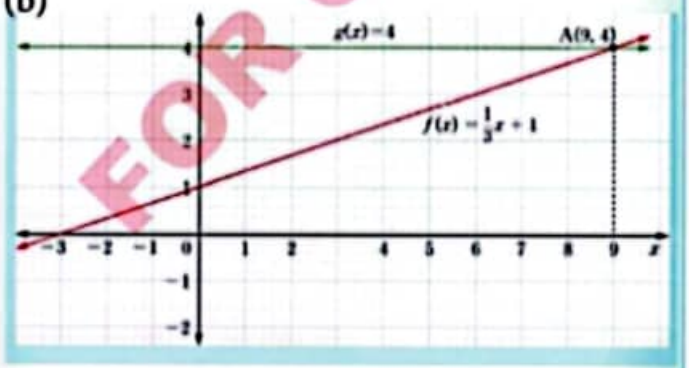


Domain$=\{x: x\in \mathbb{R}\}$, Range $=\{y: y\le 1\}$.

2. (a)

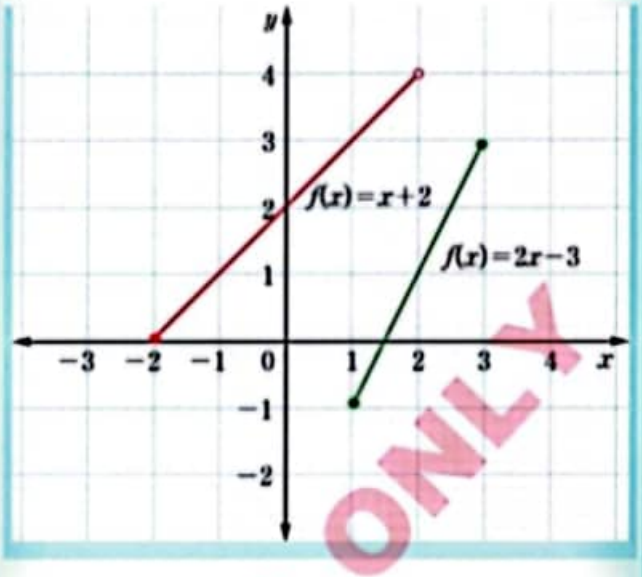


(b)

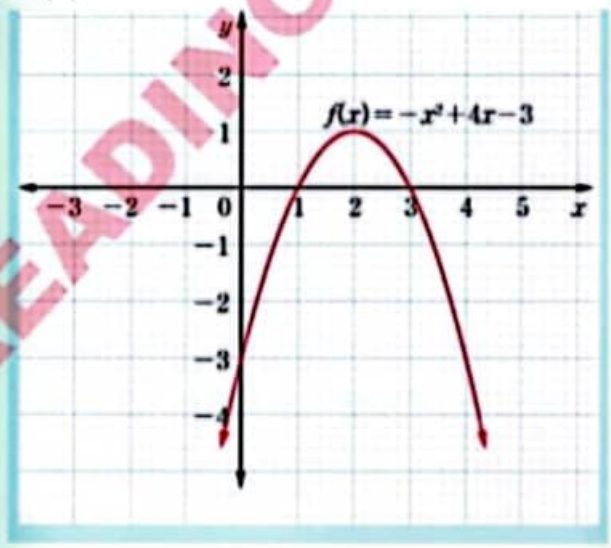


The point is (9, 4).

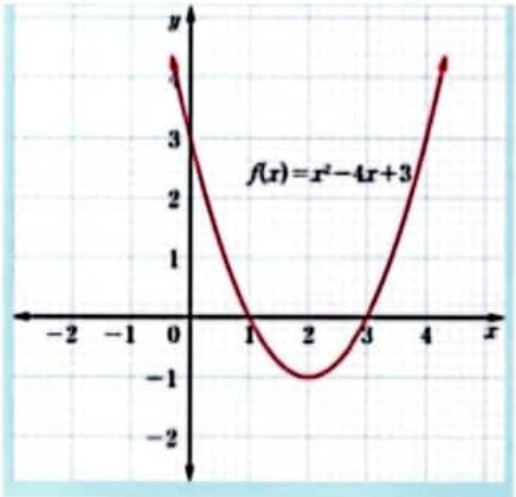
(c)



3. (a)



(b)



Domain $=\{x: x\in \mathbb{R}\}$, range $=\{y: y\ge -1\}$

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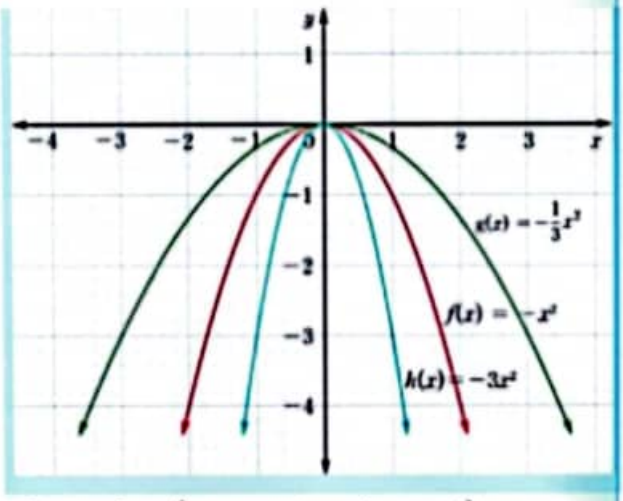
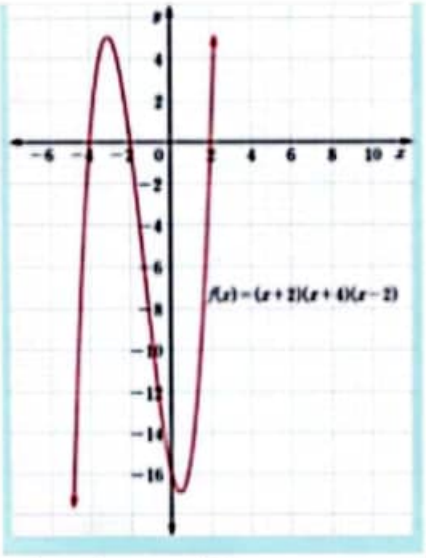
(c)



4. Domain $= \{x : x = 1 \text{ and } x = 2\}$

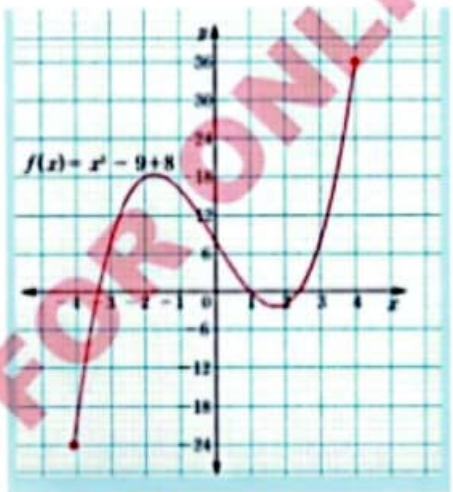
5. (a)



Domain $= \{x : x \in \mathbb{R}\}$

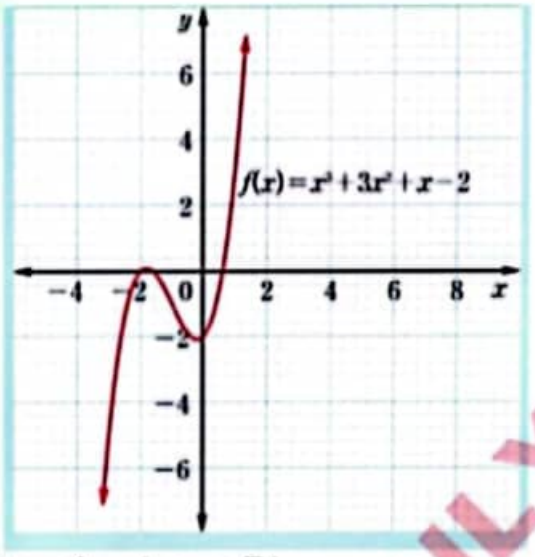
Range $= \{y : y \in \mathbb{R}\}$

(b)



Domain $= \{x : x \in \mathbb{R}, -4 \le x \le 4\}$

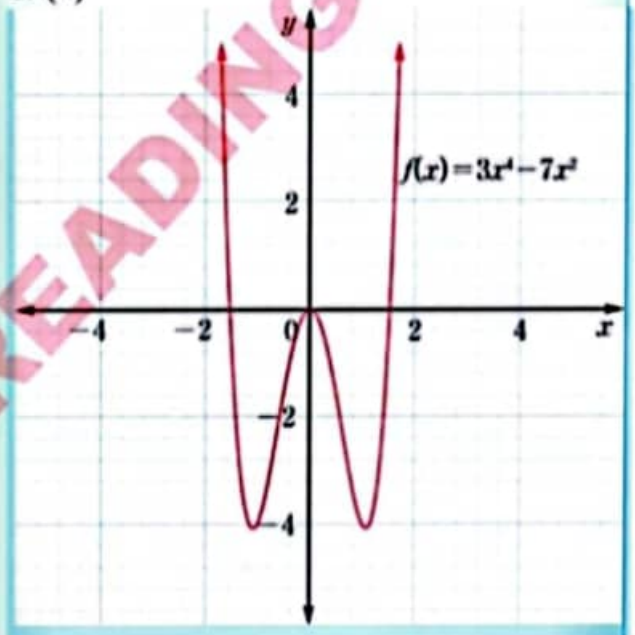
Range $= \{y : y \in \mathbb{R}, -20 \le y \le 36\}$



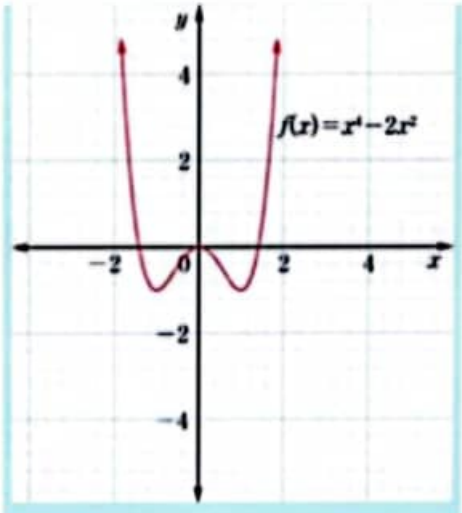
Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

6. (a)



(b)



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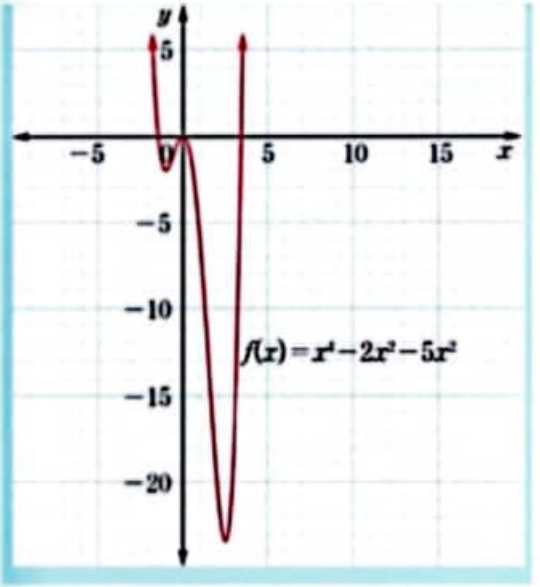
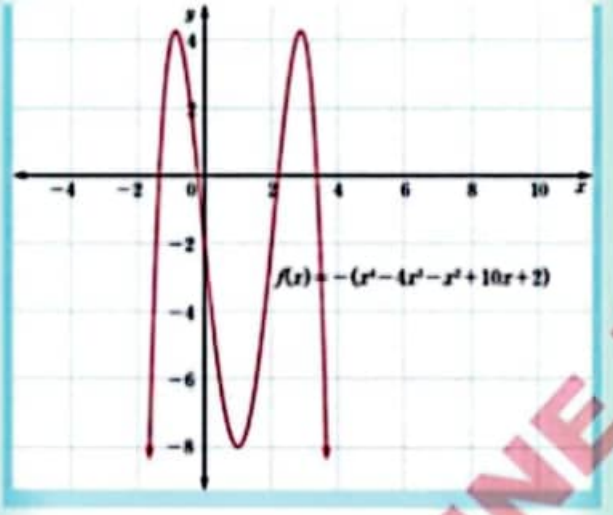
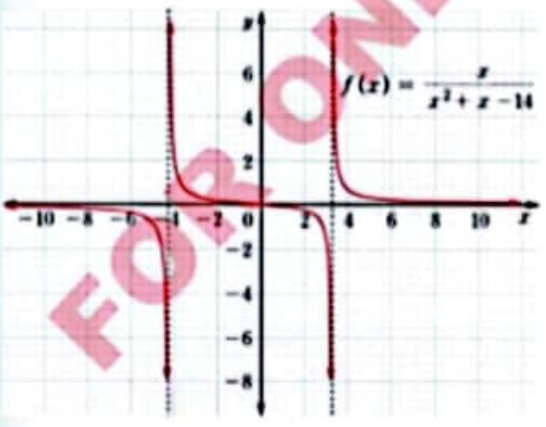
(c)



(d)



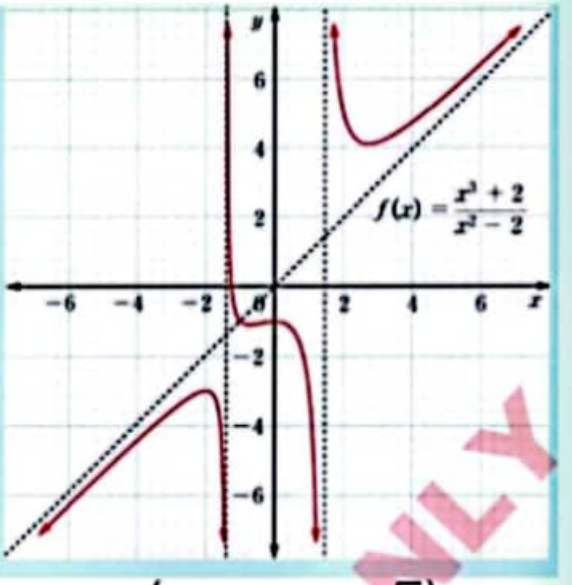
7. (a)



Domain = $\{x : x \in \mathbb{R},\ x \neq -4.27,\ x \neq 3.27\}$

Range = $\{x : x \in \mathbb{R}\}$

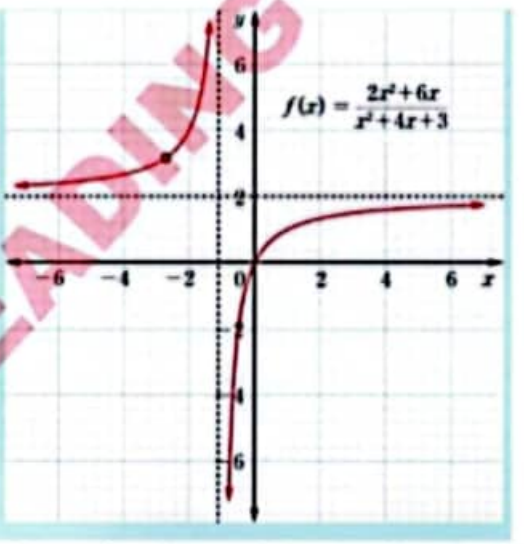
(b)



Domain = $\{x : x \in \mathbb{R}, x \neq \pm\sqrt{2}\}$

Range = $\{y : y \in \mathbb{R}\}$

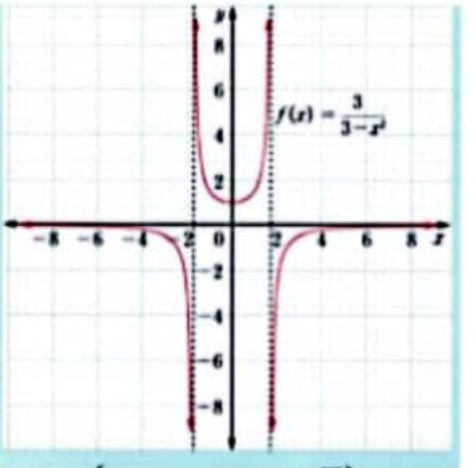
(c)



Domain = $\{x : x \in \mathbb{R}, x \neq -3,\ x \neq -1\}$

Range = $\{y : y \in \mathbb{R},\ y \neq 2\}$

(d)



Domain = $\{x : x \in \mathbb{R}, x \neq \pm\sqrt{3}\}$

Range = $\{y : y \in \mathbb{R},\ y < 0,\ y \geq 1\}$

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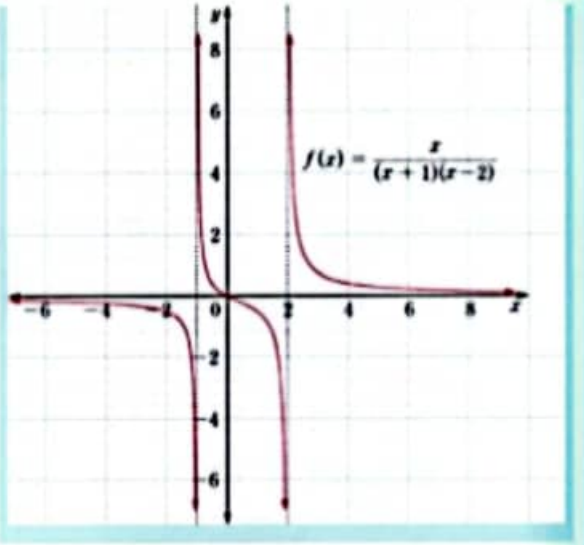
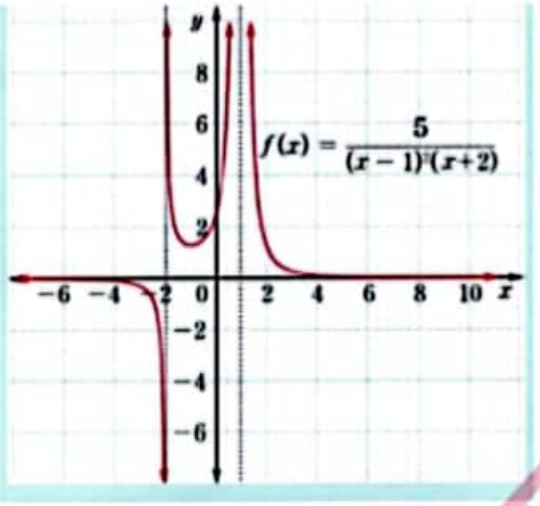
(e)



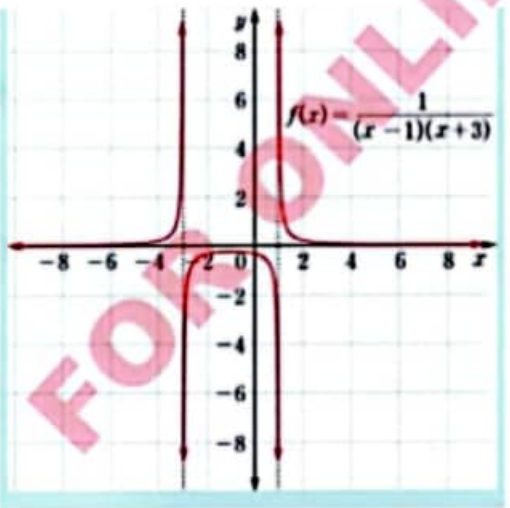
Domain = $\{x : x \in \mathbb{R}, x \neq -1, x \neq 2\}$
Range = $\{y : y \in \mathbb{R}\}$

(f)



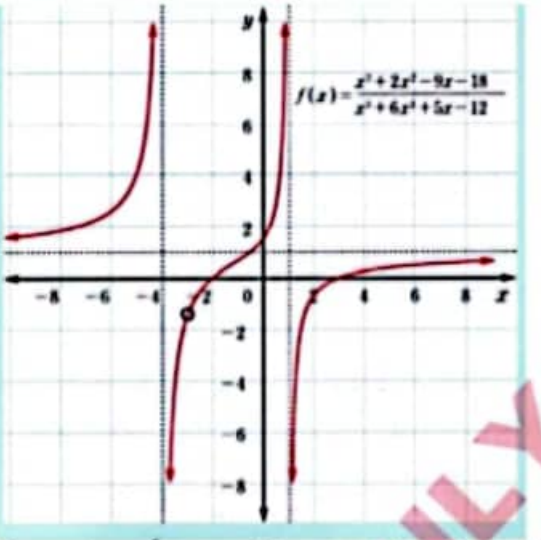
Domain = $\{x : x \in \mathbb{R}, x \neq -2, x \neq 1\}$
Range = $\{y : y \in \mathbb{R}, y \neq 0\}$

(g)



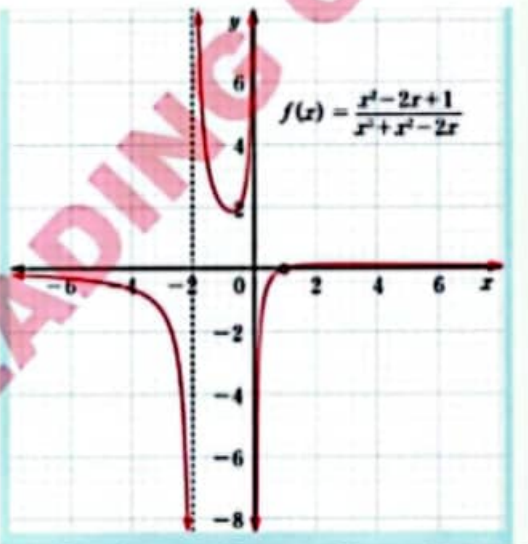
Domain = $\{x : x \in \mathbb{R}, x \neq -3, x \neq 1\}$
Range = $\left\{y : y \in \mathbb{R}, y < -\frac{1}{4}, y > 0\right\}$

(h)



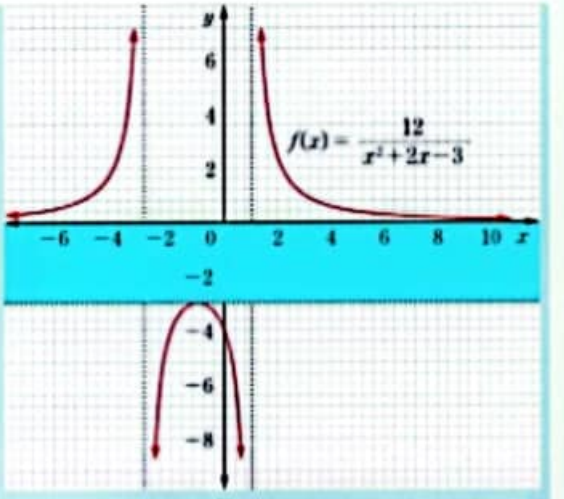
Domain = $\{x : x \in \mathbb{R}, x \neq -4, -3, x \neq 1\}$
Range = $\{y : y \in \mathbb{R}\}$

(i)



Domain = $\{x : x \in \mathbb{R}, x \neq -2, x \neq 0, x \neq 1\}$
Range = $\{y : y \in \mathbb{R}, y \geq 1.8 \text{ and } y \leq 0.3\}$

(j)



Domain = $\{x : x \in \mathbb{R}, x \neq -3, x \neq 1\}$
Range = $\{y : y \in \mathbb{R}, y \leq -3 \text{ or } y > 0\}$

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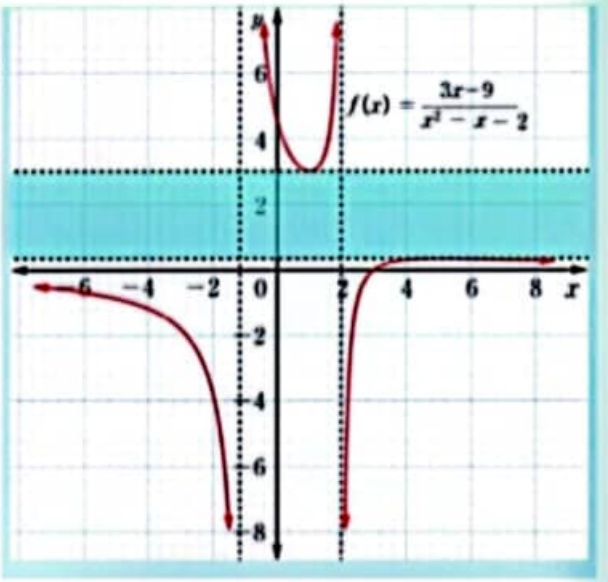
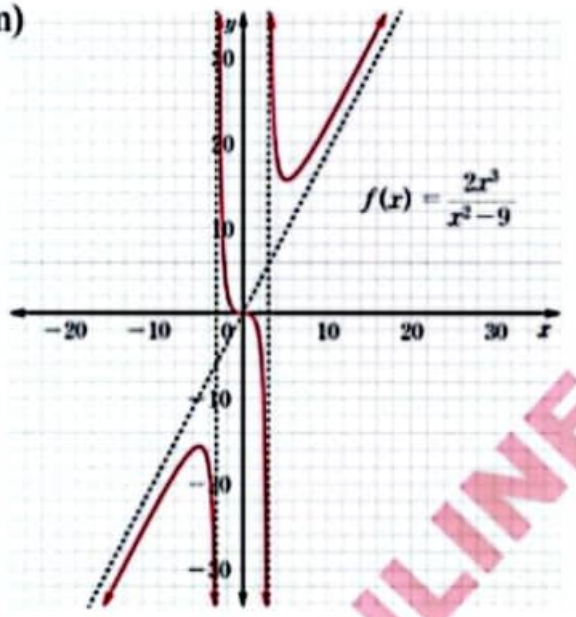
(k)



Domain=$\{x : x \in \mathbb{R}, x \neq -1, x \neq 2\}$

Range=$\left\{y : y \in \mathbb{R}, y \geq 3, y \leq \frac{-}{3}\right\}$

(m)



Domain=$\{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range=$\{y : y \in \mathbb{R}\}$

8. (a) $f \circ g(x) = \{(2,-2), (3, 2)\}$

   (b) $g \circ f(x) = \{(3, 1), (4, 6)\}$

9. (a) $f \circ g(x) = \{(7,12),(-1,19),(9,15)\}$

   (b) $g \circ f(x) = \{(1,3),(2,4)\}$

10. (a) $f \circ g(x) = x^2 - 6x + 10$

    (b) $g \circ f(x) = x^2 + 2x - 2$

(c) $g \circ f \circ h(x) = \cos^2 x + 2\cos x - 2$

11. $f(x) = \pm 3x - \frac{5}{4}$

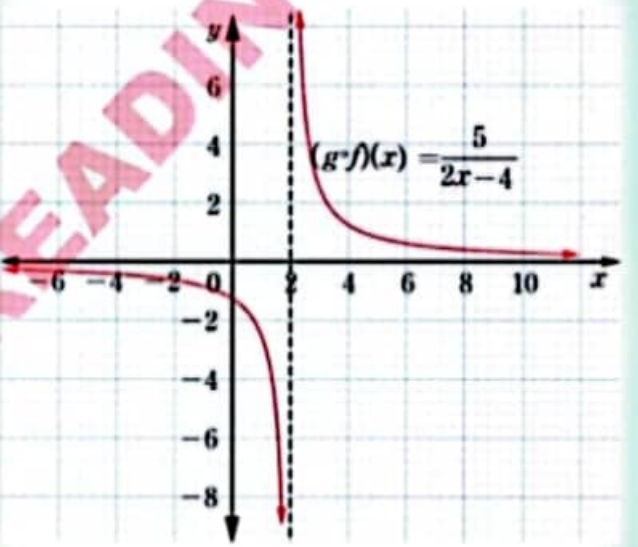
13. (a) $(f \circ g)(x) = \frac{-2x+14}{x-2}$

    (b) $(g \circ f)(x) = \frac{5}{2x-4}$

    (c) $(f \circ g)(3) = 8$

    (d) $(g \circ f)(x) = \frac{5}{4}$

(e)



Domain $= \{x : x \in \mathbb{R}, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}, y \neq 0\}$

14. $x = \pm 2.91$

15. (a) $\frac{3-3x}{x^2+4x+4}$ (b) $\frac{1}{4}$

16. (a) 2 (b) 4

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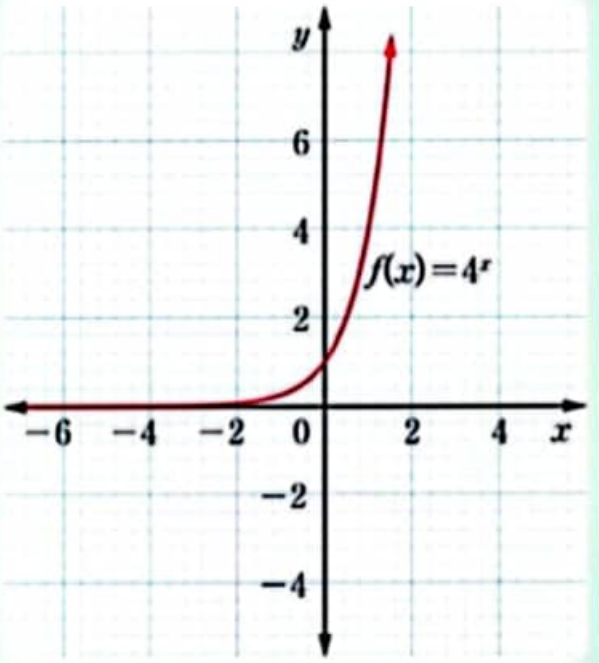
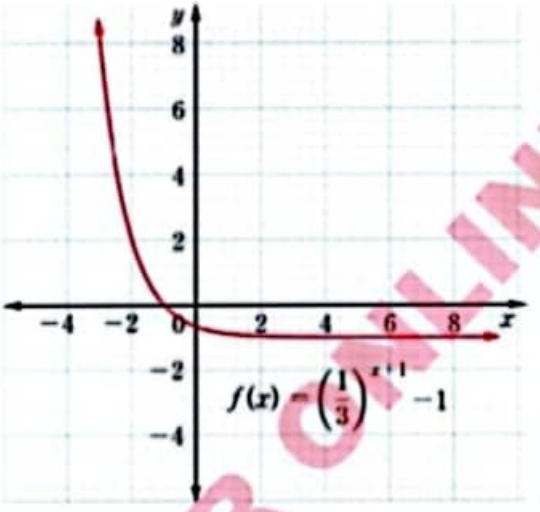
17. (b)



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y > 0\}$

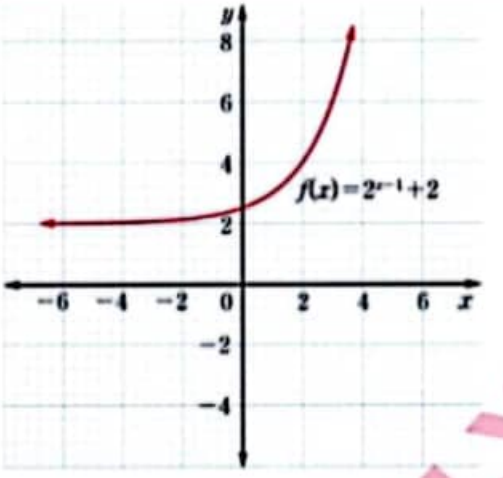
(c)



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y > -1\}$

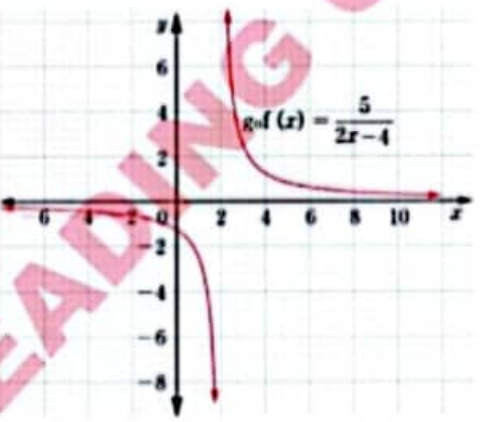
(d)



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y > 2\}$

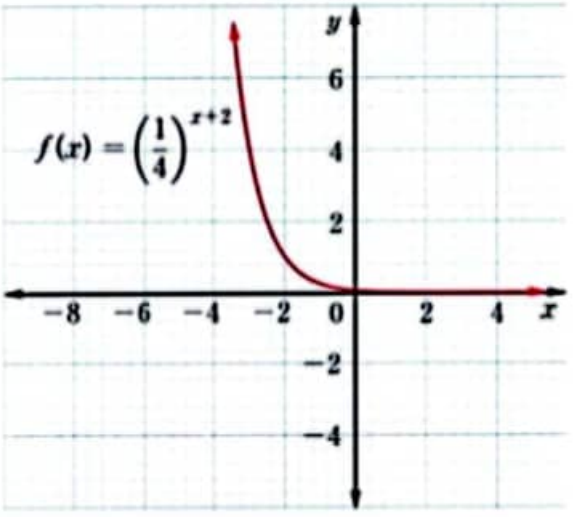
(e)



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y > 0\}$

(f)



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \geq 0\}$

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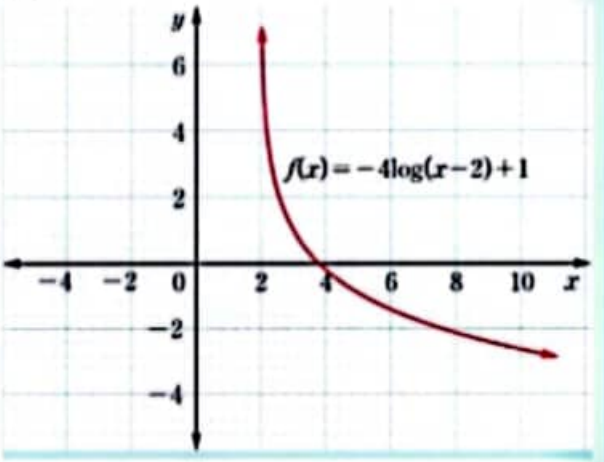
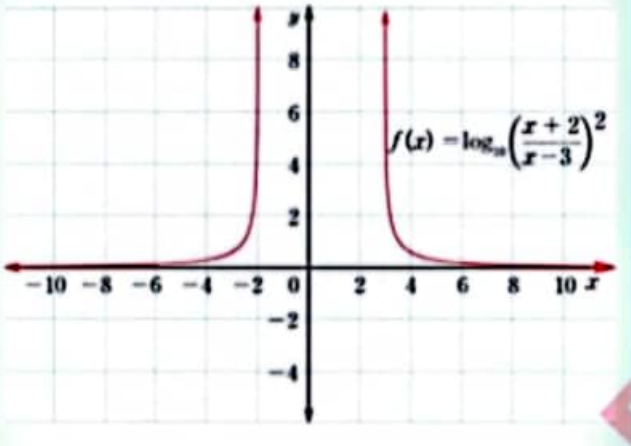
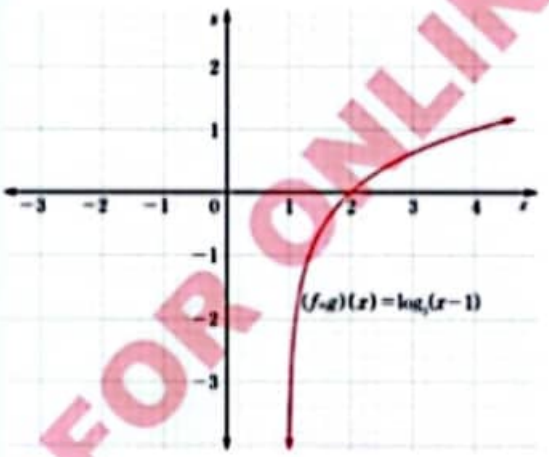
18. (a)



(b)



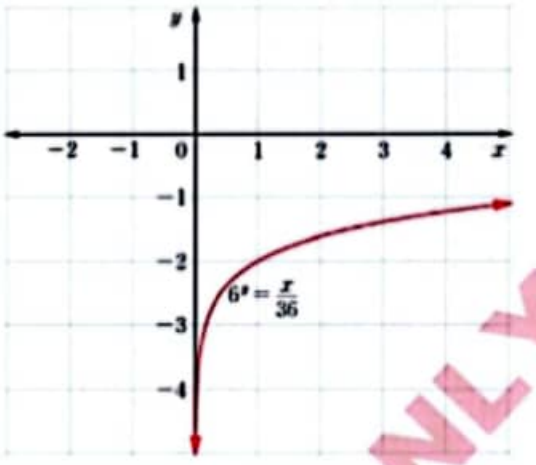
19. (a)



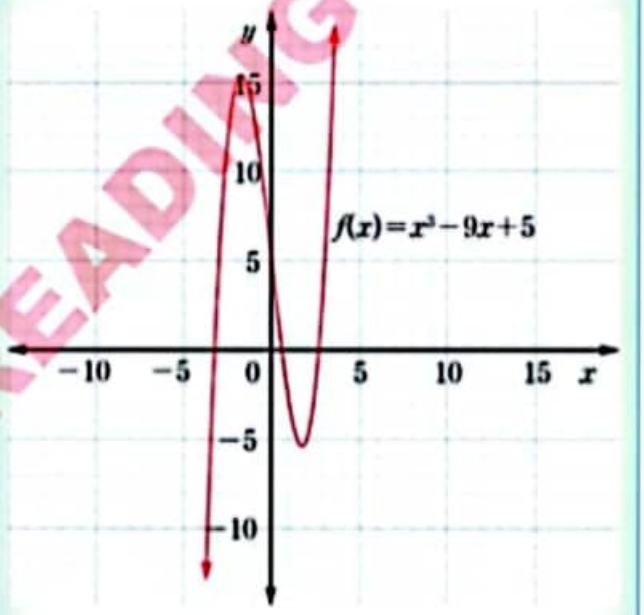
Domain $= \{x : x \in \mathbb{R}, x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

(b)



20.



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Answers

Exercise 5.1

1. (a) $\sum_{k=1}^{5}(3k+5)$ (b) $\sum_{k=1}^{14}2k$ (c) $\sum_{k=1}^{7}32\left(\frac{1}{2}\right)^{k-1}$ (d) $\sum_{k=1}^{11}k^2$

   (e) $\sum_{k=1}^{6}\frac{1}{2^k}$ (f) $\sum_{k=1}^{n}(-1)^k k^2$

2. (a) $4+5+6+7+8+9$ (b) $29+70+145$

   (c) $-2-10-24-44-70-102-140$ (d) $-3-7-15-31-63-127-255$

   (e) $\frac{1}{4}+\frac{1}{2}+1+2+4+8+16+32+64+128$

3. (a) 156 (b) $\frac{63}{2}$ (c) $\frac{42}{5}$ (d) 380 (e) 1

4. (a) $2n(n+1)$ (b) $\frac{1}{3}n(n+1)(n+5)$ (c) $\frac{1}{12}n(n+1)(3n^2+23n+46)$

5. (a) $\sum_{k=1}^{4}k(2k-1)$ (b) $\sum_{k=1}^{5}(-x)^{k-1}$ (c) $\sum_{k=1}^{\infty}ky^k$ (d) $\sum_{k=1}^{8}(3k-7)$

   (e) $\sum_{k=1}^{4}\frac{1}{2k+1}$ (f) $\sum_{k=1}^{4}(3k-2)(3k+1)$

6. (a) 4, 8, 16 (b) 2, 6, 12 (c) $2, \frac{1}{2}, \frac{4}{15}$ (d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}$

   (e) 1, −4, 9 (f) 18, 300, 1,134

7. $\frac{1}{6}n(n+1)(3n^2+11n-140)$, $\frac{1}{6}(7,997,000)$

8. 48,124,960 9. 1,871,040

Exercise 5.3

1. $\frac{2}{1-q}, \frac{-1}{1-q}$ 2. (a) 6 (b) $-\frac{117}{8}$ 3. (a) $x^2-5x-3=0$

   (b) $3x^2-10x-4=0$

5. $\sqrt{\frac{140}{17}}$ 6. $9x^2+55x+6=0$

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7. (a) $4x^2+40x+51=0$ (b) $4x^2-37x+9=0$ (c) $x^2-5x-6=0$
   (d) $3x^2+37x+12=0$ (e) $2x^2-7x=0$ (f) $2x^2+25x+72=0$

10. $4x^3-13x^2+48x-64=0$

11. (a) $x^3-3x^2-2x-32=0$ (b) $x^3+4x^2-x-11=0$

12. (a) $-\frac{1}{12}$ (b) 23 (c) 64 (d) $-\frac{1}{12}$ (e) $-\frac{5}{12}$ (f) 18

Exercise 5.4

1. (a) $-43x^5+86x^4-56x^3+12x^2-24x+20$
   (b) $7x^5-58x^4+8x^3-12x^2-24x+120$

2. $a=\frac{41}{7}, b=-\frac{114}{7}$, and $c=-\frac{648}{7}$

3. 890

4. $e=9, f=-2, g=-11$

5. (a) $x^4-4x^3-2x^2+12x+9$ (b) $-3x^4+7x^3-6x^2+3x-1$

6. (a) $m=10, n=-7, p=4$

7. (a) quotient x^2+2x+1, remainder 3 (b) quotient $x^3-8x+10$, remainder -11
   (c) quotient x^5-2x^4+2, remainder -5 (d) quotient $x-13$, remainder 64
   (e) quotient $x^3+4x^2+12x+50$, remainder 206.
   (f) quotient $2x^3-x^2+x-1$, remainder 2.

8. (a) quotient $2x^3+2x+8$ remainder $-x+6$.
   (b) quotient x^2-x-1, remainder $2x+2$.
   (c) quotient $2x^3-x^2+x-1$, remainder 2.
   (d) quotient $x^3+ax^2+a^2x+a^3$, remainder 0.

9. (a) 2 (b) 7 (c) 319 (d) 2549 (e) 131

10. $t=30$, no factors

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Exercise 5.5

1. $\left\{x \in \mathbb{R} : -\frac{1}{3} < x < 7\right\}$
2. $\left\{x \in \mathbb{R} : \frac{5}{3} < x < 11, x \neq 4\right\}$
3. $\left\{x \in \mathbb{R} : x > \frac{3}{2} \text{ or } x < -\frac{7}{2}\right\}$
4. $\left\{x \in \mathbb{R} : x < -\frac{1}{2} \text{ or } x > \frac{3}{2}\right\}$
5. $\{x \in \mathbb{R} : -4 < x < 5\}$
6. $\{x \in \mathbb{R} : x > 3 \text{ or } x < -1\}$
7. $\{x \in \mathbb{R} : -1 < x < 1\}$
8. $\{x \in \mathbb{R} : -1 < x < 0 \text{ and } x > 1\}$
9. $\{x \in \mathbb{R} : x \leq -1 \text{ or } 0 < x \leq 1\}$
10. $\{x \in \mathbb{R} : x \leq 3 \text{ or } x \geq 5\}$
11. $\{x \in \mathbb{R} : 0 \leq x \leq 5\}$
12. $\left\{x \in \mathbb{R} : x \geq -1 \text{ or } x \leq -5 \text{ or } -2 < x < \frac{-3}{2}\right\}$
13. $\left\{x \in \mathbb{R} : \frac{-3}{2} < x < 1\right\}$
14. $\{x \in \mathbb{R} : 1 < x < 3 \text{ or } x < -1\}$
15. $\{x \in \mathbb{R} : -1 < x < 0 \text{ or } 0 < x < 1\}$
16. $\left\{x \in \mathbb{R} : \frac{2}{3} < x < 1 \text{ or } \frac{3}{2} < x < 4\right\}$
17. $\left\{x \in \mathbb{R} : -1 < x < \frac{}{3} \text{ or } 1 < x < 2\right\}$
18. $\left\{x \in \mathbb{R} : x < \frac{-1}{2} \text{ or } x > \frac{-3}{7}\right\}$
19. $\{x : x \in \mathbb{R}\}$

Exercise 5.6

2. $\begin{pmatrix} 25 & 37 & 53 \\ 119 & 31 & 55 \\ 68 & -9 & 0 \end{pmatrix}$
4. (a) $\begin{pmatrix} -5890 \\ -190 \\ 9310 \end{pmatrix}$ (b) Not possible
   (c) Not possible
5. $c = -50$, $d = 4$, $e = 9$.
6. $\begin{pmatrix} 1220 \\ 620 \\ 10 \end{pmatrix}$
8. (a) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$
   (b) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$
9. $\begin{pmatrix} 2580 \\ 2170 \\ 2292 \end{pmatrix}$ where, 2580, 2170, and 2292 are the total points from the departments of Geography, Chemistry, and Biology, respectively.

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10. (a)
$$\begin{array}{l} \\ 1^{st}\text{ student} \\ 2^{nd}\text{ student} \\ 3^{rd}\text{ student} \end{array}\begin{array}{c} \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ \begin{pmatrix} 8 & 12 & 16 \\ 5 & 9 & 10 \\ 15 & 18 & 12 \end{pmatrix} \end{array}$$

(b) The total sum of money spent by the 1st, 2nd and 3rd students are Tshs 18,000, Tshs 12,000 and Tshs 22,500, respectively.

Exercise 5.7

1. (a) -532 (b) 380

2. (a) $\begin{pmatrix} -16 & 8 & -34 \\ -14 & -46 & 63 \\ -24 & 12 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & -9 & -10 \\ 1 & -12 & -5 \\ -10 & -5 & 0 \end{pmatrix}$

3. (a) $\begin{pmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 15 & -3 & -21 \\ -6 & -18 & 2 \\ -30 & 6 & 10 \end{pmatrix}$

4. (a) -212 (b) 25

9. (a) $m = 4$ (b) $r = -10$

10. (a) $\begin{pmatrix} 17 & 2 & -6 \\ -4 & 4 & 6 \\ -2 & 6 & 9 \end{pmatrix}$ (b) 144

Exercise 5.8

1 (a) $R^{-1} = \frac{1}{100}\begin{pmatrix} 2 & 10 & 26 \\ 16 & -20 & 8 \\ 22 & 10 & -14 \end{pmatrix}$

(b) $x = 1,\ y = -2,\ z = 3$

2. (a) $(x, y, z) = (3, 1, 4)$

(b) $(x, y, z) = (3, 2, 1)$

(c) $(x, y, z) = -4.714, -2.071, -5.86$

(d) $(x, y, z) = (-1, 2, 3)$

(e) $(x, y, z) = (8, -11, 3)$.

3. (a) $\begin{pmatrix} 8 & 8 & 8 \\ -7 & 11 & -1 \\ -5 & 1 & 13 \end{pmatrix}$

(b) $x = 1,\ y = 3,\ z = -2$

4. $\frac{1}{15}\begin{pmatrix} -4 & 11 & 0 \\ -7 & 8 & 0 \\ 27 & -3 & 15 \end{pmatrix}$

5. $(a, b, c) = (2, 5, 1)$

6. $(I_1, I_2, I_3) = (1.3, -4.5, -3.7)$

7. $(k, m, n) = (-5, 10, 2)$

8. $q = 40{,}000,\ r = 75,\ s = 30$

9. 5 units of simple, 8 units of medium, 8 units of complex

10. $(x, y, z) = (2, 3, 1)$ units

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Exercise 5.9

1. (a) $32x^5+240x^4z+720x^3z^2+1080x^2z^3+810xz^4+243z^5$

   (b) $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$

   (c) $32x^5+160x^3+320x+\frac{320}{x}+\frac{160}{x^3}+\frac{32}{x^5}$

2. (a) $140\sqrt{2}$ (b) 24476 (c) $40\sqrt{2}$ (d) 98 .

3. $1+10x+55x^2+210x^3+\cdots$; 1.106

4. (a) 1.0743 (b) 1279.2 (c) 7272.2

5. $m=-9$ and $n=46$

7. $32+80y+80y^2+40y^3+10y^4+y^5$; 32.08008

8. $1024+1280y+720y^2+240y^3$; 1159

9. $4-28c+85c^2-146c^3+155c^4+\cdots$

10. $81a^4-54a^3b+\frac{27}{2}a^2b^2-\frac{3}{2}ab^3+\frac{1}{16}b^4$; 757335.0625

Exercise 5.10

3. $1+2x+\frac{5}{2}x^2+\cdots; -\frac{1}{3}<x<\frac{1}{3}$

4. $-\frac{2}{3}<y<\frac{2}{3}$

5. $1-a-\frac{7}{2}a^2+\cdots; -\frac{1}{3}<a<\frac{1}{3}$

6. $1-4y-24y^2-224y^3$; 2.499

8. $1+\frac{1}{3}t-\frac{1}{9}t^2+\cdots$; 2.080

10. $-\frac{1}{4}-\frac{1}{2}n-\frac{3}{16}n^2-\cdots; -1<n<1$

Exercise 5.11

2. $n=55$ 4. $r=7$ 5. (a) ${}^9C_r(-1)^r\frac{2^{18-3r}}{5^{9-2r}}y^{9-2r}$ (b) ${}^5C_r(2y)^{5-r}\left(\frac{-1}{y}\right)^r$

3. 27 (c) $(-1)^r\,{}^{12}C_r y^{24-3r}$ (d) $(-1)^r\,{}^6C_r t^{12-2r}s^r$

8. $p=3, q=5, n=6$ 9. (a) ${}^{15}C_{10}\left(\frac{1}{6}\right)^5$ (b) 45 (c) $\frac{17}{54}$ (d) −3432

10. (a) 14 (b) −16

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Exercise 5.12

1. $\frac{4}{x+2}-\frac{2}{x+1}$
2. $\frac{2}{x-3}-\frac{4}{x-1}$
3. $\frac{5}{12(x+1)}+\frac{34}{3(x-2)}-\frac{43}{4(x-3)}$
4. $\frac{3}{13(x+1)}+\frac{5x-17}{13(x^2+3)}$
5. $1-\frac{14}{x}+\frac{12x-6}{x^2-x+1}$
6. $\frac{8x+13}{3(x^2+2)}+\frac{1}{3(x+1)}$
7. $y-5+\frac{2}{y+1}+\frac{1}{y+2}-\frac{3}{y+3}$
8. $-\frac{3}{2(x-1)}+\frac{1}{(x-1)^2}+\frac{2}{x-2}-\frac{1}{2(x-3)}$
9. $\frac{16}{3(x+1)}+\frac{-13+11}{3(x^2-x+1)}$
10. $\frac{36}{5(t^2+2)}+\frac{8}{5\sqrt{3}(t+\sqrt{3})}-\frac{8}{5\sqrt{3}(t-\sqrt{3})}$

Exercise 5.13

1. $\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}$; $s_\infty=\frac{1}{4}$
2. $\frac{1}{2n-1}-\frac{1}{2n+1}$; $\frac{1}{2}-\frac{1}{2n+1}$
3. $2-\frac{1}{n+1}-\frac{2}{n+2}$; $s_\infty=2$
5. $\frac{1}{2}+\frac{n+3}{(2n+1)(2n+3)}$
6. $\frac{1}{n}-\frac{1}{n+2}$
7. $\frac{1}{4(2k-1)}-\frac{1}{4(2k+3)}$
9. $\frac{3}{8}-\frac{2n+1}{(2n+2)(2n+6)}$
10. (a) $\frac{1}{6}-\frac{n+2}{(n+3)(n+4)}$ (b) $\frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$ (c) $\frac{1}{9}-\frac{1}{9(n+1)}$

Revision exercise 5

3. (a) $x^3+9x^2+45x+7=0$ (b) $x^3+x^2-17x-49=0$
   (c) $x^3-6x^2+14x-8=0$
4. Factors: $x+1, x-1$, and x^2+4, $\frac{1}{x+1}-\frac{2}{x-1}+\frac{3x+2}{x^2+4}$
5. $k=2\sqrt{5}-2$
6. $\frac{3}{1+x}+\frac{4-3x}{1+x^2}$; $7-6x-x^2+7x^4-6x^5$, $a=7, b=-6, c=-1, d=0, e=7, f=-6$

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7. $n=15$ 8. $\frac{2}{(x-2)^6}+\frac{5}{(x-2)^7}+\frac{4}{(x-2)^8}$

11. $\left\{x: x<-2, -1\le x\le\frac{2}{3} \text{ or } x>4\right\}$

13. (a) $\frac{\sqrt{3}}{36\left(x-\sqrt{3}\right)}-\frac{\sqrt{3}}{36\left(x+\sqrt{3}\right)}-\frac{1}{6\left(x^2+3\right)}$. (b) $\frac{2}{x}-\frac{2x}{x^2+3}+\frac{2-5x}{\left(x^2+3\right)}$.

14. $\begin{pmatrix}11 & -4 & 1\\ -25 & 9 & -2\\ 15 & -5 & 1\end{pmatrix}$ 15. $x=30,\ y=20,\ z=-60$ 16. $x=\frac{1}{2},\ y=1,$ and $z=\frac{3}{2}$

17. (a) $1-3x+6x^2-10x^3+\cdots$ (b) $1-\frac{2}{3}x^2-\frac{1}{9}x^4-\frac{4}{81}x^6+\cdots$

18. $\frac{21875}{128}x^4$ 20. $\frac{3}{3x-1}-\frac{1}{x+1}+\frac{1}{(x+1)^2}$, Coefficient $=n(-1)^n-3^{n+1}$

21. (a) 1.00499 (b) 0.9933

22. $n=11$; 55, 165, and 330.

24. (a) 0 (b) $(y-x)(z-x)(z-y)$

25. $2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\ldots,\ -4<x<4.$ 29. $1+2x+3x^2+4x^3+\cdots$

30. $x=1,\ y=-4,\ z=-4$ 31. (a) $\begin{pmatrix}19 & 31\end{pmatrix}$ (b) $\begin{pmatrix}-6 & -4 & 13\\ -1 & -2 & 3\\ 3 & -6 & -10\end{pmatrix}$

33. $\frac{2}{9}+\frac{11}{27}x+\frac{20}{27}x^2;\ -\frac{3}{2}<x<\frac{3}{2}$

36. (a) $k=-6,\ k=6$ (b) $k<-6$ or $k>6$

37. (a) $x<-5$ or $x>\frac{1}{5}$ (b) $2<x<3$ or $x>8$

38. $n\left(n^2+6n+11\right)$; 296,616

39. (b) 3 carton of concentrated product, 1 carton of diluted product and 2 cartones of dry product.

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41. (a) (i) $-28x^4+51x^3-71x^2+62x-35$ (ii) $8x^4-7x^3-7x^2+29x+48$

(b) $m=17,\ n=15,\ p=19,\ q=-16,\ r=6,\ t=-4$

42. (a) $10x^5-4x^4-80x^3+52x^2-8x$ (c) x^3-10x^2-4x+2

(b) $\frac{1}{10}x+4+\frac{8x+2}{10x^2-4x}$ (d) $2x^6+68x^4-72x^3+48x^2-64x+8$

43. (a) $x^5+4x^4+12x^3+48x^2+194x+668$; Remainder $=2,675$

(b) $x^5+4x^4+12x^3+48x^2+194x+668$; Remainder $=2,675$

44. (a) $\left\{x\in\mathbb{R}:-2\le x\le\frac{-3}{2}\right\}$ (b) $\{y\in\mathbb{R}:1<y<3\}$ (c) No solution

45. (a) $p=12,\ q=9,\ r=5,\ s=\frac{50}{3}$ (b) $x-1$ is a factor of $p(x)$

46. (a) $\begin{bmatrix} 99 & 11 & -121 \\ -108 & 3 & 156 \\ -54 & -15 & 111 \end{bmatrix}$ (d) $|N|=297$

(b) $\begin{bmatrix} 99 & -11 & -121 \\ 108 & 3 & -156 \\ -54 & 15 & 111 \end{bmatrix}$ (e) $\frac{1}{297}\begin{bmatrix} 99 & 108 & -54 \\ -11 & 3 & 15 \\ -121 & -156 & 111 \end{bmatrix}$

(c) $\begin{bmatrix} 99 & 108 & -54 \\ -11 & 3 & 15 \\ -121 & -156 & 111 \end{bmatrix}$

47. $(I_1, I_2, I_3)=(5, 4, 6)$

48. $(m, n, p)=(-4, 8, 12)$

49. (b) $\frac{224}{27}x^{20}$ and $\frac{448}{9}x^{16}$

50. (a) $\frac{1}{2(2n+1)}-\frac{1}{2(2n+3)}$ (c) $\frac{90}{4669}$

(b) $\frac{1}{6}+\frac{1}{2(2n+3)}$ (d) $\frac{1}{6}$

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Answers

Exercise 6.1

1. $x = 8.1\text{cm}$, $h = 12.8676\text{cm}$
2. $\overline{VW} = 20$ units, $\overline{UV} = 40$ units, $\overline{UW} = 20\sqrt{5}$ units,
3. $\overline{BD} = 13.4\text{cm}$
4. $\overline{JK} = 10$ cm, $\overline{KL} = 10$ cm, $\overline{JL} = 14.14$ cm
5. $\text{cosec}\,D = \frac{169}{119}$, $\sec D = \frac{169}{120}$
6. (a) $1\frac{1}{12}$ (b) $2\frac{2}{5}$
7. $\hat{M} = 45°$, $\hat{N} = 45°$
8. $5\sqrt{5}$ cm, 53.13° and 126.87°
9. $\overline{PR} = 13.97$ cm
10. 407 metres
11. $h = 46.86$ metres

Exercise 6.2

1. (a) $\cos y$ (b) $\cos\theta$
   (c) 1 (d) $\sec\theta$
3. $\theta = -172.8°, -7.2°, 14.5°, 165.5°, 187.2°$
4. (a) $x^2 + y^2 = 1$
   (b) $(x-2)^2 + (y+1)^2 = 1$
   (c) $xy = 3$
5. (a) $\theta = 60°, 120°, 240°, 300°$
   (b) $\theta = 70.5°, 289.5°$
   (c) $\theta = 26.6°, 45°, 225°, 206.6°$

Exercise 6.3

1. (a) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$
   (b) $-\frac{36}{325}, \frac{323}{325}, -\frac{36}{323}$
2. (a) $\frac{204}{325}, -\frac{253}{325}, -\frac{204}{253}$
   (b) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$
6. (a) $157\frac{1}{2}°, 337\frac{1}{2}°$
   (b) 49.1°, 229.1°
   (c) 56.5°, 236.5°
8. (a) $\frac{\sqrt{3}}{2}$ (c) $\frac{\sqrt{6}+\sqrt{2}+6}{8}$
   (b) $\sqrt{3}$

Exercise 6.4

3. (a) (c) $1 - \sin x$
   (b) $\cot\theta$ (d) $\cot x$
5. (a) $x + 2y^2 = 0$
   (b) $x = 2y^2 + 4y + 1$
   (c) $y = \frac{4x(1-x^2)}{1-6x^2+x^4}$
   (d) $\frac{(x-4)^2}{9} + \frac{(y-7)^2}{81} = 1$
6. (a) 22.5°, 112.5°, 202.5°, 292.5°
   (b) 45°, 121°, 225°, 301°
   (c) 14.47°, 165.53°
   (d) 0°, 48.18°, 180°, 311.82°, 360°

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8. (a) $\frac{336}{527}, -\frac{336}{625}, -\frac{625}{527}$ (b) $-\frac{28560}{239}, -\frac{28560}{28561}, \frac{28561}{239}$

10. (a) $-\frac{\sqrt{3}}{3}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\frac{1}{2}$ (d) $\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}}$

11. (a) 1 (a) $\frac{\sqrt{2}}{2}$ (b) $2\sqrt{2}$

Exercise 6.5

1. (a) 0°, 112.62°, 360°
   (b) 53.1°, 323.1°
   (c) 48.4°, 205.34°
   (d) 119.56°, 346.7°
   (e) 114.2°, 335.7°
   (f) 188.4°, 319.72°
   (g) 101.34°, 355.4°
   (h) 36.9°, 241.92°
   (i) 80.72°, 234.04°
   (j) 36.87°, 126.87°

2. (a) −30°, 90°
   (b) 76.71°, −150.45°
   (c) −4.9°, −129.88°
   (d) −20.52°, 28.11°, 159.48°
   (e) −24.3°, 114.29°
   (f) 0°, 45°, 180°
   (g) −8.3°, −155.43°
   (h) −180°, −106.85°, −90°, −16.85°, 0°, 73.16°, 90°, 163.16°, 180°
   (i) −40.72°, 72.6°
   (j) 51.33°, 128.67°

3. (a) Maximum $\sqrt{2}$, minimum $-\sqrt{2}$
   Maximum occurs at 45°
   Minimum occurs at 225°

   (b) Maximum $\sqrt{2-\sqrt{3}}$, minimum $-\sqrt{2-\sqrt{3}}$ Maximum occurs at 255°
   Minimum occurs at 75°

   (c) Maximum 5, minimum − Maximum occurs at 143.13° Minimum occurs at 306.87°

   (d) Max. $\sqrt{\frac{201}{4}+7\sqrt{3}}$, min. $-\sqrt{\frac{201}{4}+7\sqrt{3}}$,
   Maximum occurs at 83.42°,
   Minimum occurs at 263.42°

   (e) Maximum $\sqrt{10}$, minimum $-\sqrt{10}$,
   Maximum occurs at 71.57°,
   Minimum occurs at 251.57°

   (f) Maximum 5, minimum −5,
   Maximum occurs at 36.87°,
   Minimum occurs at 216.87°

   (g) Maximum 17, minimum −17,
   Maximum occurs at 118.07°,
   Minimum occurs at 293.96°

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(h) Maximum $\sqrt{34}$, minimum $-\sqrt{34}$,
Maximum occurs at $30.96°$,
Minimum occurs at $210.96°$

(i) Maximum $\sqrt{37}$, minimum $-\sqrt{37}$,
Maximum occurs at $99.46°$,
Minimum occurs at $279.46°$

(j) Maximum $\frac{1}{2}$, minimum $-\frac{1}{2}$,
Maximum occurs at $45°$,
Minimum occurs at $135°$.

4. Maximum $\sqrt{13}$, minimum $-\sqrt{13}$
Maximum occurs at $33.69°$
Minimum occurs at $-146.31°$

5. $\cos\theta + 2\sin\theta \equiv \sqrt{5}\sin(\theta + \alpha)$
Maximum $\sqrt{5}$, minimum $-\sqrt{5}$,
Maximum occurs at $63.43°$,
Minimum occurs at $-116.565°$

Exercise 6.6

1. $\frac{\pi}{2}n + (-1)^n \frac{\pi}{12}$
2. $\frac{\pi}{5}n + \frac{\pi}{20}$
3. $2\pi n \pm \frac{\pi}{4}$
4. $2.251 + 2\pi n$; $6.0928 + 2\pi n$
5. $\pi n + 1.6512$; $1.769 + \pi n$
6. $1.0616 + n\pi + 1.9148 + n\pi$
7. $\pi n + (-1)^n \frac{\pi}{2}$ or $\pi n - (-1)^n \frac{\pi}{6}$
8. $1.7 + 2\pi n$; $5.563 + 2\pi n$
9. $0.864 + n\pi$; $2.972 + n\pi$
10. $2\pi n + \frac{11}{6}\pi$.

Exercise 6.7

1. (a) $2\sin 45° \cos 5°$
(b) $2\cos 45° \sin 25°$
(c) $2\cos 40° \cos 15°$
(d) $2\sin 55° \sin 20°$

2. (a) $-\frac{1}{2}(\cos 95° - \cos 15°)$
(b) $\frac{1}{2}(\sin 165° - \sin 55°)$
(c) $\frac{1}{2}(\sin 70° + \sin 10°)$
(d) $\frac{1}{2}(\cos 85° + \cos 15°)$.

3. (a) $\theta = \frac{\pi n}{4}$ or $\theta = \pi n \pm \frac{\pi}{3}$
(b) $x = \pi n \pm \frac{\pi}{4}$ or $x = 2\pi n \pm \frac{2\pi}{3}$
(c) $x = \frac{1}{4}\pi n$, *or* $\frac{2}{3}\pi n \pm \frac{2}{9}\pi$
(d) $\theta = \frac{n\pi}{3}$ or $\frac{\pi n}{4} - (-1)^n \frac{\pi}{8}$
(e) $x = \frac{n\pi}{4}$ or $2\pi n \pm \frac{\pi}{3}$
(f) $x = \pi n + (-1)^n \frac{\pi}{4}$ or
$x = \frac{\pi}{3}n \pm \frac{\pi}{12}$

6. $\frac{a}{b}$, $\frac{2a - b + a^2 b}{1 - a^2 + 2ab}$

7. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and
$\cos 2\theta = 2\cos^2\theta - 1$

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8. (a) 22.5°, 60°, 67.5°, 112.5°, 120°, 157.5°
   (b) 50°, 140°
   (c) 20°, 90°, 100°, 140°
   (d) 0°, 45°, 90°, 120°, 135°, 180°

10. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, 2\pi$

11. (a) $\tan\frac{5\lambda}{2}$ (b) $-126°, -54°, 18°, 90°, 162°$

12. (a) $\tan\frac{17A}{2}$ (b) $\cot\frac{2}{3}\pi$

Exercise 6.8

1. (a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{3}$ (c) $\frac{\pi}{2}$
   (d) $-\frac{3\pi}{4}$ (e) $\frac{2\pi}{9}$ (f) $\frac{3\pi}{2}$
   (g) $\frac{7\pi}{4}$ (h) $\frac{9\pi}{4}$

2. (a) 60° (b) 300°
   (c) 210° (d) −120°
   (e) 540° (f) 252°
   (g) 382.5° (h) 360°

3. (a) $\frac{\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{4}, \frac{5\pi}{4}$
   (c) $\frac{\pi}{4}, \frac{3\pi}{4}$ (d) $\frac{7\pi}{6}, \frac{11\pi}{6}$
   (e) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (f) $\frac{\pi}{3}, \frac{4\pi}{3}$

4. (a) 28.07° (b) 98.55°
   (c) −135.22° (d) −48.70°
   (e) 251.53° (f) −291.06°

5. (a) 1.14 (b) 0.559 (c) 1.48

6. $\frac{7\pi}{12}, \frac{\pi}{4}$

7. (a) $\frac{7\pi}{8}$ (b) $\frac{11\pi}{36}$ (c) $\frac{5\pi}{9}$

8. (a) 0.71π (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$

9. 360°, 135°

10. (a) −64.3° (b) 270°
    (c) −114.6° (d) 439.3°

Exercise 6.9

1. (a) $1+5\theta$ (b) 4
   (c) $\frac{1}{1+2\theta}$ (d) $\frac{4\theta}{1-2\theta^2}$
   (e) $\frac{1}{4}\sqrt{2}\left(\theta^2+2\theta-2\right)$ (f) $-\frac{1}{2}\theta$

2. 5 3. 0.294 6. $8\theta^2$

7. $\frac{2}{3}\left[\left(1-2\theta^2\right)\sin 3\alpha + 2\theta\cos 3\alpha\right]$

9. $\frac{18\theta^2}{1+2\theta+6\theta^2}$

10. (a) $1+2\theta-\theta^2$ (b) $\tan\alpha+\theta\sec^2\alpha$

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Exercise 6.10

1. Domain $= \left\{x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$
   Range $= \{y \in \mathbb{R} : \text{except } -1 < y < 1\}$
2. Domain $= \left\{x \in \mathbb{R} : x \neq \frac{\pi}{4} + \frac{\pi n}{2}, n \in \mathbb{Z}\right\}$, Range $= \{y \in \mathbb{R}\}$
3. Domain $= \{x \in \mathbb{R} : x \neq \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$, Range $= \{y : y \in \mathbb{R}\}$
4. Domain $= \{x \in \mathbb{R} : x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$
   Range $= \{y \in \mathbb{R} : y \in (-\infty, -1] \cup [1, \infty)\}$
5. Domain $= \left\{x \in \mathbb{R} : x \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \ldots\right\}$, Range $= \{y \in \mathbb{R} : \text{except } -1 < y < 1\}$
6. Domain $= \{x \in \mathbb{R} : -\pi \leq x \leq \pi\}$, Range $= \{y \in \mathbb{R} : -2 \leq y \leq 2\}$
7. Domain $= \left\{x \in \mathbb{R} : x \neq \frac{\pi}{4} + \frac{\pi n}{2}, n \in \mathbb{Z}\right\}$, Range $= \{y : y \in \mathbb{R}\}$
8. Domain $= \left\{x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$, Range $= \left\{y \in \mathbb{R} : \text{except } -\frac{1}{2} < y < \frac{1}{2}\right\}$
9. Domain $= \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$, Range $= \{y \in \mathbb{R} : -6 \leq y \leq 6\}$
10. Domain $= \left\{-2\pi \leq x \leq 2\pi, x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\right\}$, Range $= \{y : y \in \mathbb{R}\}$

Exercise 6.11

1. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$ (b) $\frac{}{4}$ (c) $\sqrt{3}$ (d) 1
2. (a) $\frac{4}{3}$ (b) $-\frac{13}{85}$ (c) $\frac{87}{425}$ (d) $\frac{56}{65}$
4. (a) $\sqrt{1-x^2}$ (b) $\frac{x}{\sqrt{1+x^2}}$
5. (a) $x = 0.7862$ (b) $x = \pm 1, x = \frac{1}{2}$ (c) $x = 0.2808$ (d) $x = \sqrt{6-4\sqrt{2}}$
   (e) $\frac{1}{3}$ (f) $\frac{1}{6}$ (g) $\pm\sqrt{2}$ (h) 0 (i) $\pm\sqrt{2}$ (j) $\pm\frac{3}{4}$

10. 0

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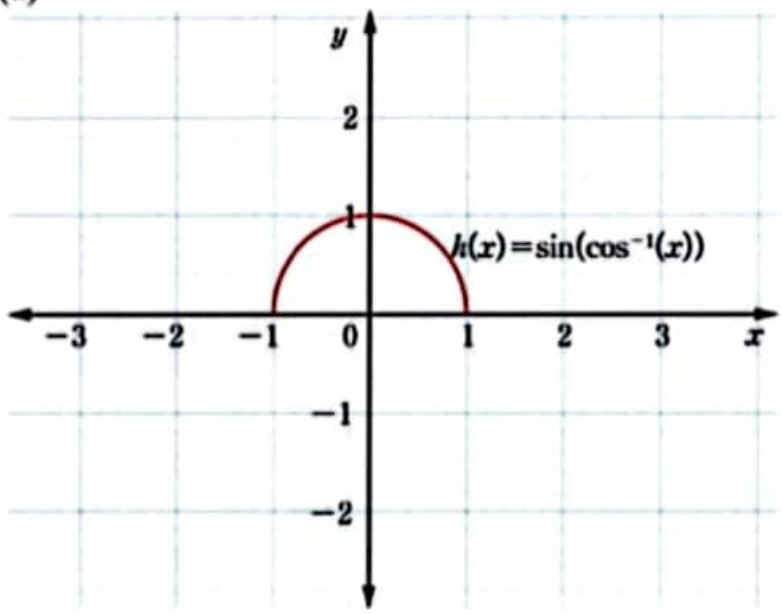
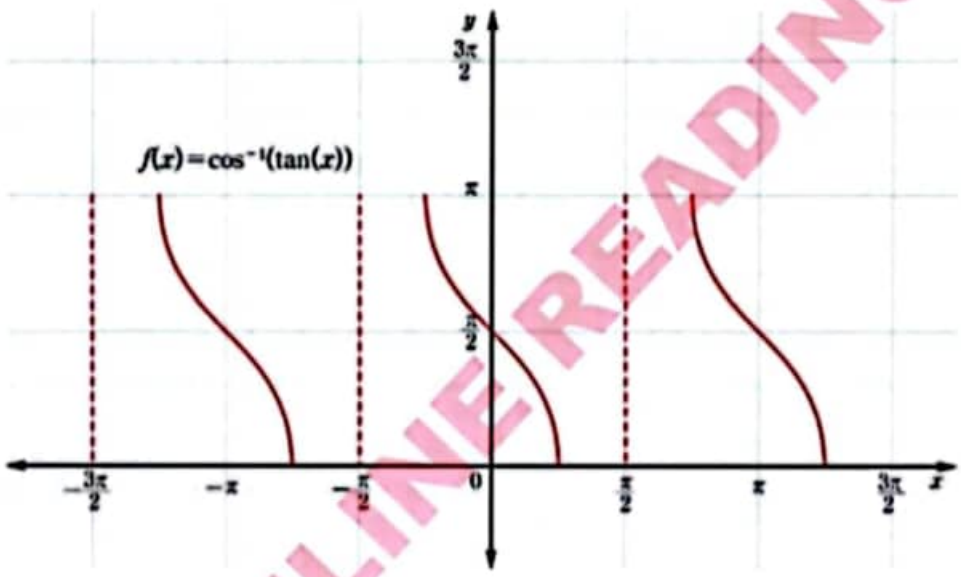
Exercise 6.12

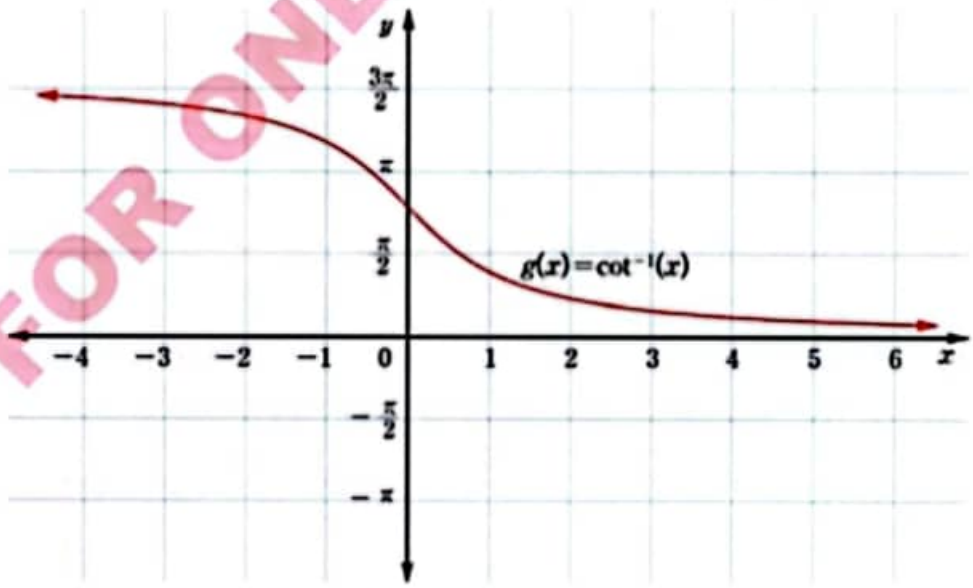
1. (a)



(b)



(c)



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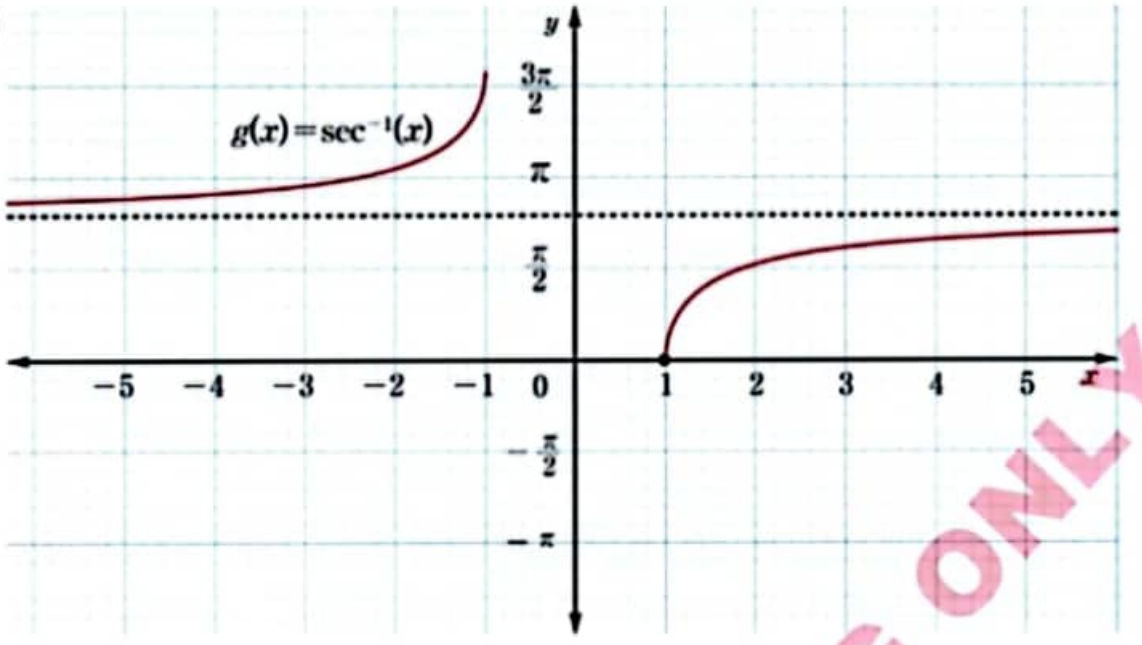
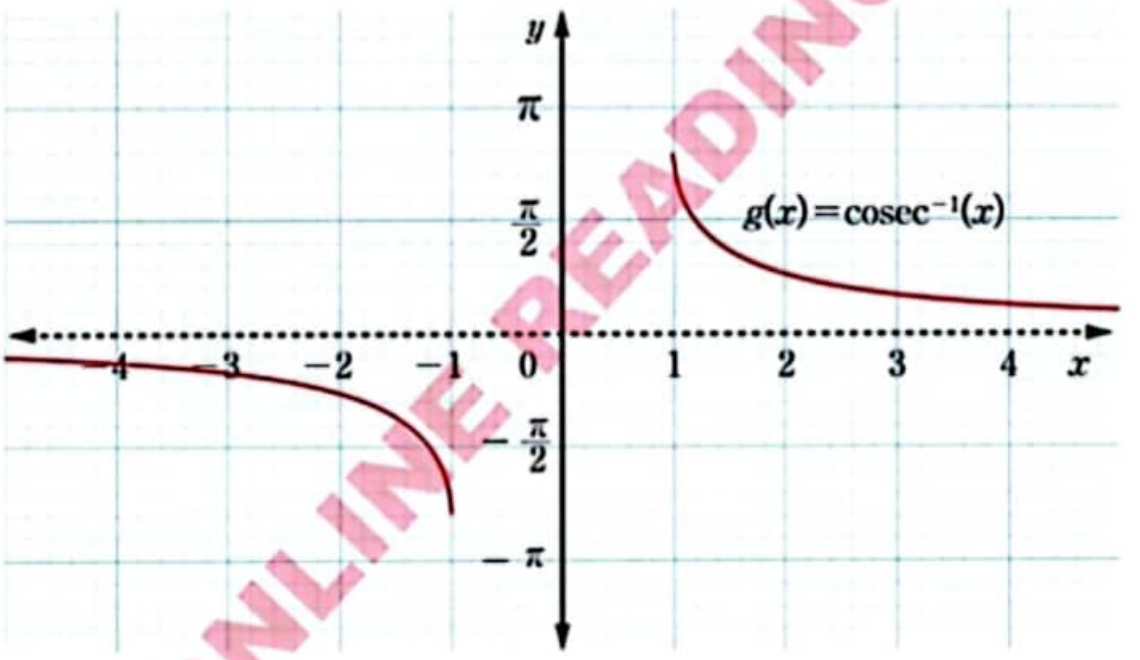
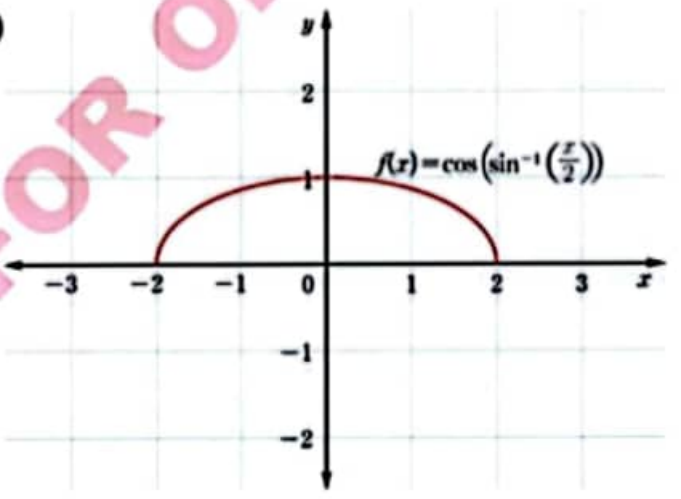
(d)



(e)



(f)



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2. (a) Domain $= \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ (b) Domain $= \{x : x \in \mathbb{R}\}$

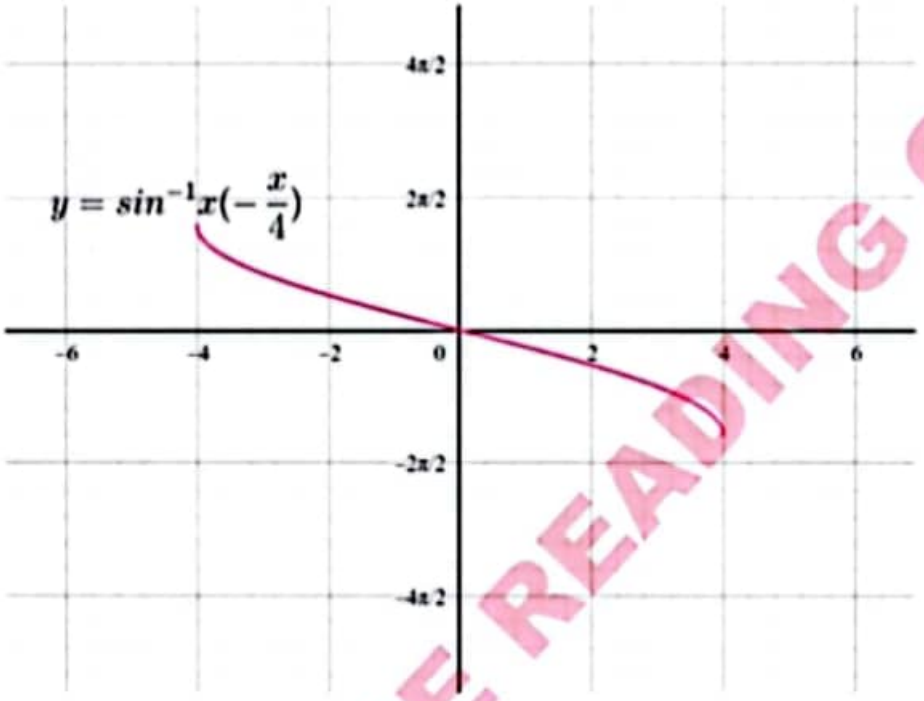
Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$ Range $= \{y \in \mathbb{R} : 0 < y < \pi\}$

(c) Domain $= \{x : x \in \mathbb{R}\}$

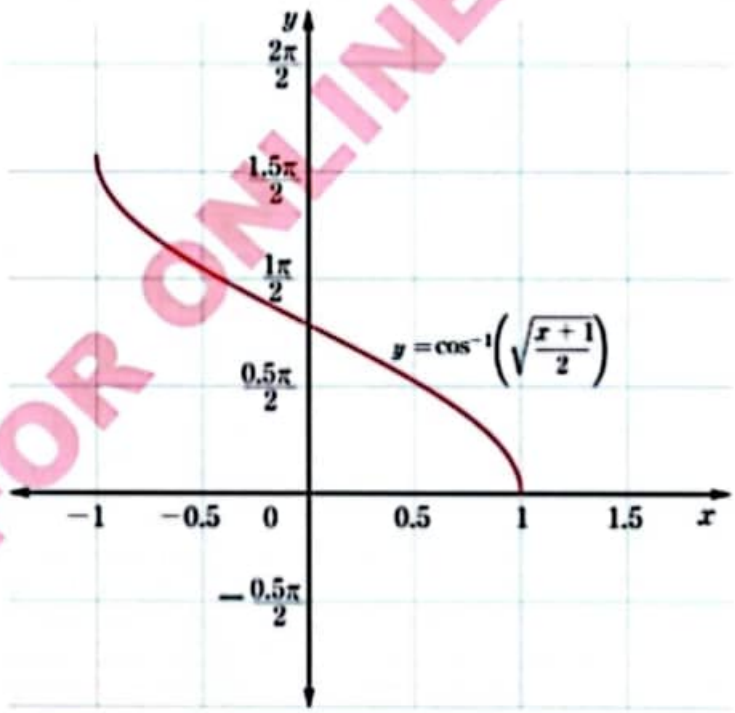
Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

3. (a) $0 < \cot^{-1} x < \pi$

4.



5.



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Revision exercise 6

1. (a) $-\left(\frac{15\sqrt{135}+168}{315-8\sqrt{135}}\right)$ (b) $-\left(\frac{315-8\sqrt{135}}{408}\right)$ (c) $\frac{15\sqrt{135}-168}{408}$

2. (a) $\left(\frac{x-9}{4}\right)^2+\left(\frac{y-7}{15}\right)^2=1$ (b) $y=\frac{4x^2-46x+94}{x^2-10x+16}$

   (c) $x=2\left(1-y^2\right)$ (d) $yx^3+12x^2-256=0$

3. (a) $2\tan 2\theta$ (b) $-\frac{3\sin 3\theta}{4}$ (c) $-\tan x$ (d) $\frac{\cos 3y}{\cos 8y}$

5. (a) $460°$ (b) $316.55°$ (c) $429.7°$ (d) $-150°$

6. (a) $\frac{73}{72}\pi$ (b) $\frac{19}{30}\pi$

7. (a) -3π (b) $-\frac{35}{6}\pi$ (c) $\frac{9641}{14400}\pi$

8. (a) $9\theta+2$ (b) $\frac{4}{3}$ (c) $-\frac{9}{8}$ (d) $6\theta^2$

10. (a) $-144.1°, -58.5°, 35.9°, 121.5°$ (b) $139.26°, -107.38°$

11. (a) $2\pi n,\ 2\pi n+\frac{\pi}{2},\ \frac{\pi(1-2n)}{5}$ (b) $\theta=2\pi n$, or $2\pi n+\frac{1}{2}\pi$

    (c) $\theta=\frac{2}{5}\pi n+\frac{1}{10}\pi$, or $\theta=-\frac{2}{3}\pi n+\frac{1}{6}\pi$

    (d) $\theta=360°n+69.2°$ or $\theta=360°n-32.3°$ (e) $\frac{4\pi n\pm\pi}{a+b},\ \frac{4\pi n\pm\pi}{a-b}$

    (f) $\theta=360°n-19°47'$ or $\theta=360°n-47°35'$

    (g) $\theta=\frac{\pi n}{3}$ or $\theta=\pi n\pm\frac{\pi}{3}$ (h) $\theta=180°n\pm60°$ or $\theta=180°n\pm24.09°$

12. (a) $(x, y)=(15°, 75°);(75°, 15°);(195°, 255)$
    $(15°, 255);(195°, 75°)$.

    (b) $(x,y)=(45°,165°)$ or $(x,y)=(165°,45°)$

13. $\frac{52}{39}$ 14. $27.2°, 152.8°, 207.2°, 332.8°$ 15. $26.6°, 206.6°$

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16. (a) 103.1°, 330° (b) 36.9°, 270° (c) 0°, 112.6°
(d) 28.1°, 208.1°, 159.5°, 339.5°
(e) 76.7°, 209.6° (f) 90°, 330°

17. (a) 13, 67.4° (b) 5, 53.13° (c) 10, 53.13° (d) $\sqrt{2}$, 45°

18. (a) $\sqrt{2}$, 45°; $-\sqrt{2}$, 225° (b) 5, 53.1°; −5, 233.1°
(c) 17, 298.1°; −17, 118.1° (d) $\sqrt{37}$, 170.5°; $-\sqrt{37}$, 350.5°

19. Maximum = 5, $x = 53.13°$
Minimum = −5, $x = -126.9°$

20. (a) $-\frac{1}{2}\sin\alpha$ (b) $\frac{2}{3}(1+\theta)$ (c) $\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$

21. $3-\frac{5}{2}\cos\left(2x-\tan^{-1}\frac{3}{4}\right)$,
Maximum = 5.5
Minimum = 0.5

22. 22.5°, 112.5°

24. $t=\frac{1}{2}$ or $t=\frac{1}{p}$, 53.12° and 60°

25. (a) $2\cos\frac{23}{2}\theta\sin\frac{9}{2}\theta$ (b) $2\sin 11\theta\cos 2\theta$ (c) $-2\sin\frac{11}{10}\theta\sin\frac{43}{10}\theta$
(d) $2\cos\left(\frac{7}{2}\theta\right)\cos\theta$

28. $x=\frac{1}{2}$, $x=\frac{1}{3}$ 35. −360°, −90°, 0°, 360°

36. 0°, 10.32°, 60°, 70.32°, 120°, 130.32°, 180°, 190.32°, 240°, 250.32°, 300°, 310.32, 360°

37. $\tan 2x=\frac{2t}{1-t^2}$, $x=\pi n, \frac{2\pi}{3}+\pi n, \frac{\pi}{3}+\pi n$

39. (a) $\sqrt{\frac{4+\sqrt{15}}{4}}$ (b) $\pm\sqrt{\frac{4-\sqrt{15}}{8}}$ (c) $\pm\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}$

FOR ONLINE READING ONLY

Answers — Chapter Seven

Exercise 7.1

1. Min $f = 6000x + 5000y + 25200000$
   Subject to: $x \leq 1100$
   $y \leq 1200$
   $x + y \geq 500$
   $x + y \leq 1500$
   $x \geq 0,\ y \geq 0$

2. Min $w = 1200x + 800y + 1682000$
   Subject to: $x \leq 60$
   $y \leq 60$
   $x + y \leq 160$
   $x + y \geq 112$
   $x \geq 0,\ y \geq 0$

3. Min $w = -(2500x + 2000y) + 575000$
   Subject to: $x \leq 30$
   $y \leq 25$
   $x + y \leq 40$
   $x + y \geq 20$
   $x \geq 0,\ y \geq 0$

4. Min $w = 25000x + 35000y + 2100000$
   Subject to: $x \leq 30$
   $y \leq 25$
   $x + y \geq 10$
   $x + y \leq 50$
   $x \geq 0,\ y \geq 0$

5. Min $f = 500x + 1407500$
   Subject to: $x \leq 40$
   $y \leq 55$
   $x + y \leq 80$
   $x + y \geq 30$
   $x \geq 0,\ y \geq 0$

6. Min $f = -(30 + 10y) + 82000$
   Subject to: $x \leq 8$
   $y \leq 5$
   $x + y \geq 4$
   $x + y \leq 6$
   $x \geq 0,\ y \geq 0$

7. Min $w = 4x + 2y + 11000$
   Subject to: $x \leq 2000$
   $y \leq 1500$
   $x + y \geq 2000$
   $x + y \leq 3000$
   $x \geq 0,\ y \geq 0$

8. Min $w = 1000x + 7000y + 190000$
   Subject to: $x \leq 5$
   $y \leq 5$
   $x + y \geq 4$
   $x + y \leq 8$
   $x \geq 0,\ y \geq 0$

9. Min $w = 400x + 300y + 380000$
   Subject to: $x \leq 500$
   $y \leq 600$
   $x + y \geq 500$
   $x + y \leq 900$
   $x \geq 0,\ y \geq 0$

FOR ONLINE READING ONLY

Exercise 7.2

1.

To \ From	D_1	D_2
C_1	80	0
C_2	40	10

Minimum cost is 2,500

2.

To \ From	Misungwi	Ngudu
Mabuki	20	0
Misasi	5	10

Minimum cost is Tshs 5,450

3.

To \ From	H_1	H_2
P	0	500
Q	500	100
R	400	0

Minimum cost is Tshs 53,000

4.

To \ From	M_1	M_2
A	0	5
B	5	0
C	3	1

Minimum cost is Tshs 1,550

5.

To \ From	A	B
Malaga	0	15,000
Falulu	20,000	0
Tina	10,000	5,000

Minimum cost is Tshs 3,200,000

6.

To \ From	S_1	S_2
P_1	5,000	30,000
P_2	20,000	0
P_3	25,000	0

Minimum cost is Tshs 3,200,000

7.

To \ From	W_1	W_2
C_1	10	50
C_2	50	0
C_3	40	0

Minimum cost is Tshs 5,100,000

8.

To \ From	B_1	B_2
C_1	5	15
C_2	0	15
C_3	20	0

Minimum cost is Tshs 8,750

9.

To \ From	L_1	L_2
Msigwa	35,000	0
Asha	45,000	15,000

Minimum cost is Tshs 1,015,000,000

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Revision exercise 7

1. (a) Min $w = 1000x - 4000y + 3718000$

   (b) $x \leq 100$

   $y \leq 100$

   $x + y \geq 80$

   $x + y \leq 160$

   $x \geq 0,\ y \geq 0$

2. Min $w = 21000x - 2200y + 1456700$

   Subject to: $x \leq 12000$

   $y \leq 15000$

   $x + y \geq 2000$

   $x + y \leq 20000$

   $x \geq 0,\ y \geq 0$

3. Min $f = 150x + 170y + 2872000$

   Subject to: $x \leq 3000$

   $y \leq 2400$

   $x + y \geq 1600$

   $x + y \leq 4200$

   $x \geq 0,\ y \geq 0$

4. Min $f = 15x + 5y + 3200000$

   Subject to: $x \leq 25000$

   $y \leq 20000$

   $x + y \geq 15000$

   $x + y \leq 40000$

   $x \geq 0,\ y \geq 0$

5. (a) 0 bags of groundnuts from T_1 to M_1
   600 bags of groundnuts from T_1 to M_2
   700 bags of groundnuts from T_2 to M_1
   0 bags of groundnuts from T_2 to M_2

   (b) Minimum cost is 51,500 Tanzanian shillings

6. (a) Min $f = 25000x + 16000y + 101650000$

   Subject to: $x \leq 180$

   $y \leq 140$

   $x + y \geq 150$

   $x + y \leq 200$

   $x \geq 0,\ y \geq 0$

   (b) 10 tonnes from Dar es Salaam to Kibaha
   140 tonnes from Dar es Salaam to Chalinze
   170 tonnes from Morogoro to Kibaha
   0 tonnes from Morogoro to Chalinze

7. The minimum cost is Tshs 44,000.
   The transportation is as follows;
   500 litres from N to D
   3,000 litres from N to E
   3,500 litres from N to F
   4,000 litres from Q to D
   0 litres from Q to E
   0 litres from Q to F

8. The minimum cost is Tshs 64,000.
   The crates should be supplied as follows;
   0 crates from W_1 to R_1
   200 crates from W_1 to R_2
   700 crates from W_1 to R_3
   600 crates from W_2 to R_1
   300 crates from W_2 to R_2
   0 crates from W_2 to R_3

9.

From / To	A	B
P_1	70	0
P_2	90	30

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Minimum Transportation cost is Tshs 40,600

10.

	A	B	C
Warehouse 1	50	150	0
Warehouse 2	100	0	200

The minimum cost is Tshs 236,000

11. (a) The objective function is
Min $z = (40x + 20y) + 45200$

(b) Inequalities associated to the transportation problems are
$x + y \leq 160$
$x + y \geq 50$
$x \leq 70$
$y \leq 120$
$x \geq 0,\ y \geq 0$

(c) The problem is not balanced since total supply from sources is not equal to the total demands.

12. (a) Min $f = 400x + 240y + 5120000$
Subject to: $x \leq 4000$
$y \leq 3000$
$x + y \geq 6000$
$x + y \leq 4000$
$x \geq 0,\ y \geq 0$

(b) The problem is balanced since total demands is equal to total supply. That is;
Total supply is
$6000 + 3000 = 9000$ bricks.
Total demand is
$4000 + 3000 + 2000 = 9000$ bricks.

FOR ONLINE READING ONLY

Answers — Chapter Eight

Exercise 8.1

1. (a) $f'(x)=5$

   (b) $f'(t)=3t^2-2t$

   (c) $f'(x)=-3$

   (d) $f'(x)=4x-3$

   (e) $f'(v)=2$

   (f) $f'(x)=(3-x)^{-2}$

   (g) $f'(t)=5kt^4$

   (h) $f'(x)=(2x)^{-\frac{1}{2}}$

2. (a) $f'(x)=5;\ f'(1)=5$

   (b) $f'(x)=3x^2-2x;\ f'(2)=8$

   (c) $f'(x)=1;\ f'(2)=1$

   (d) $f'(x)=2x;\ f'(3)=6$

   (e) $f'(x)=-2x^{-3};\ f'(4)=-\frac{1}{32}$

3. $f'(x)=5-4x;\ f'(3)=-7;\ f'(-1)=9$

4. (a) $y=1080;\ \frac{dy}{dx}=540$

   (b) $y=2;\ \frac{dy}{dx}=9$

   (c) $y=1;\ \frac{dy}{dx}=-1$

Exercise 8.2

1. (a) $f'(1)=-2$

   (b) $f'(0)=0$

   (c) $f'(-1)=20$

   (d) $f'(-2)=-37$

2. (a) $f'(x)=\frac{1}{2}ax-3b$

   (b) $f'(x)=\frac{9}{25}x^{\frac{4}{5}}-\frac{4}{25}x^{-\frac{6}{5}}$

   (c) $f'(x)=\frac{3}{2}\sqrt{x}-\frac{2}{\sqrt{x}}$

3. (a) $f'(x)=4x-12$

   (b) $g'(x)=\frac{3}{4}(2x-3)$

   (c) $h'(x)=\frac{2}{3}(x-1)$

4. $a=2$ 5. $\left(-2,\frac{1}{2}\right)$

6. $x=\pm\frac{2\sqrt{3}}{3}$

7. $t=0, t=-1;\ x'(2)=36$

8. $h'(1)=9$

9. $4x-8x^{-3};\ 15\frac{7}{8}$

10. $\frac{dy}{dx}=-200(x+2);-200$

FOR ONLINE READING ONLY

Exercise 8.3

1. (a) $f'(x)=4x^3+6x^2-2$

   (b) $f'(s)=24s^3+60s^2+64s-32$

   (c) $f'(y)=-96y^3+36y^2+96y-27-3y^{-4}-2y^{-2}$

   (d) $f'(t)=70t^4-32t^3+12t^2+30t-22$

2. $x=\dfrac{-6\pm\sqrt{46}}{5}$

3. $\dfrac{dy}{dx}=\dfrac{17\sqrt{3}}{6}$

4. (a) $f'(x)=\dfrac{5x^2-23}{2\sqrt{x-3}}$

   (b) $g'(x)=2x^5\left[6(x+2)^5+5x(x+2)^4\right]$

   (c) $-3x^{-4}(1+x)^{\frac{1}{2}}+\dfrac{1}{2}x^{-3}(1+x)^{-\frac{1}{2}}$

   (d) $k'(x)=-9x^8+8x^7+6x^5-5x^4-6x^2$

3. $g'(x)=2(4x-1)(x+4)^{-3}$

7. (a) 1 (b) 7, −7

8. $f'(r)=-6r^5-4r^3+2r$

9. $\dfrac{dz}{dy}=6y^2-10y-4$

10. $f'(u)=2(u+1)(3u+1)(u-1)^3$; $f'(-5)=-24192$

Exercise 8.4

1. (a) $-\dfrac{16}{(3x-2)^2}$ (b) $-\dfrac{4(x+1)}{(x-1)^3}$ (c) $-\dfrac{2\left(x^2+x+2\right)}{\left(x^2-2\right)^2}$

2. (a) $(1+x)^{-\frac{1}{2}}(3+x)^{-\frac{3}{2}}$; $f'(2)=0.052$

   (b) $\dfrac{3}{2(x+3)^2}\sqrt{\dfrac{x+3}{x}}$; $f'(2)=0.095$

   (c) $\dfrac{1}{2}x\left(10x^2+33x-12\right)(x+3)^{-\frac{3}{2}}$; 8.408

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3. (a) $-\frac{1}{t^4}$ (b) $1-\frac{3}{q^2}$ (c) $\frac{2}{3}$

4. $\frac{9}{4}$ 5. $x=0$ and $x=-2$ 9. $\frac{-4s^2+20s-22}{(s-3)^2(s-4)^2}, -\frac{208}{9}$

Exercise 8.5

1. (a) $f'(x)=2x^3\left(x^4-2\right)^{-\frac{1}{2}}$ (b) $f'(x)=-3x^2\left(x^3+1\right)^{-2}$

2. (a) $\frac{dy}{dx}=32x\left(x^2+1\right)^{15}$

   (b) $\frac{dy}{dz}=-6\left(3z^2+8z-3\right)\left(z^3+4z^2-3z-3\right)^{-7}$

   (c) $\frac{dy}{dx}=18x\left(3x^2-4\right)^2$

   (d) $\frac{dy}{dx}=x\left(x^2+5\right)^{-\frac{1}{2}}$

3. (a) $f'(t)=(t-3)\left(t^2-6t+7\right)^{-\frac{1}{2}}$

   (b) $f'(t)=-21(3-t)^{20}$

   (c) $g'(z)=\frac{3}{2}z^2\left(3-z^3\right)^{-\frac{3}{2}}$

   (d) $z'=\frac{2}{5}\left(x+\frac{1}{x}\right)^{-\frac{3}{5}}\left(1-\frac{1}{x^2}\right)$

   (e) $y'=2.9\left(t^3-\sqrt{t}\right)^{1.9}\left(3t^2-\frac{1}{2\sqrt{t}}\right)$

5. (a) $f'(t)=-\frac{8t^3+9t^2-5}{3\sqrt[3]{\left(2t^4+3t^3-5t+6\right)^4}}$ (b) $f'(-1)=0.06189$

6. $r'=\frac{r^4+1}{r^2\sqrt{r^4-1}}$ 8. $x'=\frac{1}{3u^{\frac{3}{2}}\left(1-\frac{2}{\sqrt{u}}\right)^{\frac{2}{3}}}$

7. $\theta'=-10(4-2\theta)^4$ 9. $\frac{15}{16}$

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Exercise 8.6

1. (a) $3y^2\frac{dy}{dx}$ (b) $1+\frac{dy}{dx}$

3. $\frac{5}{2(x-2)(x+3)}$

4. $\frac{108(x+2)^3}{\sqrt[3]{25(3x+6)^4}}$

7. $\frac{\sqrt{2}}{4}$

8. $\frac{2}{3}$

9. $y'=\frac{x}{\sqrt{3x^2-27}}$

10. $z'=\pm\frac{4}{3}$

Exercise 8.7

1. (a) $\frac{dy}{dx}=-\frac{4x}{y(1+x^2)^2}$

(b) $\frac{dy}{dx}=\frac{1-2y}{2x+2y-1}$

(c) $\frac{dy}{dx}=\frac{6x^2y^4-8x^3y^3-4x^2y^3-y}{12x^4y^4-12x^3y^3+4x^3y^2+x}$

(d) $\frac{dy}{dx}=-\frac{y^2+2x}{3y^2+2xy}$

2. $\frac{dy}{dx}=\frac{1-2xy^3}{3x^2y^2}$

3. $\frac{dy}{dx}=\frac{13}{9}$

5. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2+2y-2xy}$

6. $\frac{dz}{dx}=\frac{6z-3-2x}{2z-6x-2}$

9. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2-2xy+2y}$

10. $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

13. (a) $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$

(b) $\frac{dy}{dx}=\frac{4x\sqrt{xy}-y}{2\sqrt{xy}+x}$

(c) $\frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$

Exercise 8.8

1. (a) $2\cos 2x$

(b) $2x\sin x+x^2\cos x$

(c) $-\frac{2}{x^2}\cos\left(\frac{2}{x}\right)$

(d) $-12\sin 2x$

(e) $2x\sin\left(1-x^2\right)$

(f) $-\frac{1}{x^4}\left(x\sin x+3\cos x\right)$

(g) $\cos x+\cos 2x$

(h) $\frac{\sqrt{x+2}}{2(x+2)}\sec^2\sqrt{x+2}$

(i) $4\sec^2 4x$

(j) $12x\cos\left(3x^2-1\right)\sin\left(3x^2-1\right)$

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(k) $\frac{1}{1+\cos x}$

(l) $\frac{-\sin x+\cos x+1}{(1+\cos x)^2}\sin\left(\frac{1-\sin x}{1+\cos x}\right)$

2. (a) $2\pi-2$ (b) 4

5. $\frac{\cos x}{x}-\frac{\sin x}{x^2}$

7. $\frac{\theta\cos\sqrt{(\theta^2-1)}}{\sqrt{(\theta^2-1)}}$

10. (a) $-\frac{y^2}{2xy-2\sin 2y}$

(b) $\frac{\sin(x+y^2)-3x^2}{4y^3-2y\sin(x+y^2)}$

(c) $\sec^2 x$

(d) $\frac{2\sec^2 x}{(1-\tan x)^2}$

Exercise 8.9

2. $-\frac{1}{1+x^2}$

4. $\frac{x^2}{(a^2-x^2)\sqrt{a^2-x^2}}$

5. $\frac{1-(1+x^2)y^3\cos x}{(1+x^2)(1+3y^2\sin x)}$

6. $-\frac{1}{1+x^2}$

9. $2x\sin^{-1}x+\frac{x^2}{\sqrt{1-x^2}}$

10. (a) $\frac{5}{\sqrt{1-25x^2}}$

(b) $-\frac{\sqrt{2}}{\sqrt{1-2x^2}}$

Exercise 8.10

1. (a) $4x(5x-16)(x-8)^2$

(b) $2\sec^2 2x+10\tan 5x\sec^2 5x$

(c) $\frac{(x^2+6x+4)}{(x^2-4)^2}$

(d) $-(3x^2+2)\sin(x^3+2x+1)$

(e) $7\cos(7x+4)$

2. (a) $2\cos 2x\cos 4x-4\sin 2x\sin 4x$

(c) $\frac{2}{\sqrt{1-4x^2}}-\frac{2}{\sqrt{2-x^2}}$

(d) $3\sec(3x+6)\tan(3x+6)$

(e) $-\frac{(7x+29)}{15}(2x+1)^{-\frac{4}{5}}(x-8)^{-\frac{5}{3}}$

(f) $2\text{cosec}2x\cot 2x\text{cosec}^2(\text{cosec}2x)$

Exercise 8.11

1. (a) $y'=\frac{5}{5x-4}$ (b) $y'=\ln x$

(c) $y'=2\sin(\ln x)$

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2. $y' = \dfrac{2}{(x+3)(x+5)}$

3. $y' = \dfrac{2x}{x^2+3} + \dfrac{3x^2}{x^3+2}$

4. (a) $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$

   (b) $f'(x) = \dfrac{4}{x} - \dfrac{6}{3x-5}$

   (c) $f'(x) = (\sin x)^x [\ln(\sin x) + x\cot x]$

   (d) $f'(x) = \dfrac{6x}{2+x^2} - \dfrac{12x^2}{1-x^3}$

5. (a) $y' = (\sin x)^{\tan x}(1 + \sec^2 x \ln \sec x)$

   (b) $y' = \dfrac{2x}{x^2+3} - 7\operatorname{cosec} 2x$

   (c) $y' = \dfrac{2x^2-4}{x(x^2-4)}$

   (d) $y' = x^x(1+\ln x)$

6. $f'(x) = \dfrac{2\operatorname{cosec} 2x}{\ln \tan x}$

7. $2\sec 2\theta$

11. $y' = \dfrac{3}{4(\ln 2)^2}$

12. $\dfrac{dy}{dx} = \dfrac{1}{2}$

Exercise 8.12

1. (a) $f'(x) = 2xe^{x^2}$

   (b) $g'(x) = e^x(1 - 12x - 6x^2)$

   (c) $h'(x) = e^x\left(\dfrac{1}{x} + \ln x\right)$

   (d) $f'(x) = x^3 9^x \ln 9 + 9^x 3x^2$

2. $y' = (\cos x - \sin^2 x)e^{\cos x}$

5. $f'(x) = 2xe^x + x^2e^x$

6. (a) $y' = 4\sin(8x)e^{\sin^2 4x}$

   (b) $y' = (2\ln 3)3^{2x}$

   (c) $y' = (1 + x\ln 3)3^x$

   (d) $x' = \sec^2\theta e^{\tan\theta}$

   (e) $x' = 3t^2 + 6te^{-3t^2} + \sec t \tan t$

   (f) $v' = (2u+2)e^{u^2+2u-8}$

10. $y' = \dfrac{x^3 \ln 2x}{e^x \sin x}\left[\dfrac{3}{x} + \dfrac{1}{x\ln 2x} - 1 - \cot x\right]$

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Exercise 8.13

2. $\frac{dy}{dx}=\frac{x\cos x-2\sin x}{x^3}$;

$\frac{d^2y}{dx^2}=\frac{\left(6-x^2\right)\sin x-4x\cos x}{x^4}$

3. (a) $\frac{dy}{dx}=\frac{(2t+1)^2}{t^2}$

(b) $\frac{d^2y}{dx^2}=\frac{-2(2t+1)^3}{3t^3}$

(c) $\frac{d^2y}{dx^2}$ $\frac{9}{4}$

4. (a) $\frac{dy}{dx}=-\tan t$

(b) $\frac{d^2y}{dx^2}=-\frac{1}{2}e^{-t}\sec^3 t$

(c) $-\frac{1}{2}$

6. (a) -1 (b) 48

7. $\frac{dy}{dx}=\frac{1}{5}$

9. $\frac{2\ln(x)-3}{x^3}$; $\frac{2\ln(2)-3}{8}$

10. 1

Exercise 8.14

1. 0.18

2. (a) 3.001 (c) 2.005

(b) 127.25

3. $0.03x^3\text{cm}^3$ 7. 1.8π cm^2

4. 80π m^3; 16π m^2 8. 1.25%

5. 40π cm^3 10. $\frac{1}{2}y\%$

6. 0.015

Exercise 8.15

1. 32π cm^2/s

2. 157.5 m

3. $\frac{1}{8\pi}$cm/s

4. 384π cm^3/min

5. 192π cm^2/s

6. 1.5 cm/s

7. $0.04\text{cm}^2/\text{s}$

8. 4.096cm, 0.3576cm/min

9. $3\text{cm}^2/\text{s}$

11. $v=15$ m/s, $a=12$ m/s^2

12. (a) $v(t)=3t^2-24t+36$, $a(t)=6t-24$

(b) when $t=2$ and $t=6$

(c) $t=4$

(d) $2<t<6$

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Exercise 8.16

1. $(-1, 3)$ and $\left(\frac{1}{2}, -\frac{15}{4}\right)$ 2. $(0, 0)$ and $(4, -32)$ 3. $(5, -210)$ and $(-1, 6)$

4. (a) $(1, -9)$ (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ (c) $(1, 2)$ and $(-1, -2)$

5. (a) $\left(\frac{1}{3}, \frac{76}{27}\right)$ and $(-1, 4)$ (b) $(-2, 0)$ and $\left(\frac{2}{3}, \frac{256}{27}\right)$ (c) $(2, -8)$ (d) $(3, 4)$

6. $k = 2$; Turning point $(1, -3)$ and $\left(\frac{1}{3}, -\frac{77}{27}\right)$ 7. $(0, -9)$ 10. $y = x^3 - 3x + 5$

Exercise 8.17

1. $48\,m^2$ 2. $x = \frac{3}{2}; v = \frac{243}{4}\,cm^3$ 3. 20 4. Maximum point $\left(\frac{2}{3}, 5\sqrt{13}\right)$

5. Maximum point $\left(\frac{7}{3}, \frac{4}{27}\right)$, Minimum point $(3, 0)$

6. (a) Maximum point $\left(\frac{4}{3}, \frac{4}{27}\right)$ minimum point $(2, 0)$; x-intercept $= 1$ and 2, y-intercept $= -4$

   (b) Inflexion point $(0, 0)$ minimum point $(3, -27)$; x-intercept $= 0$ and 4, y-intercept $= 0$

   (c) Maximum point $(2, 44)$ minimum point $(-3, -81)$; x-intercept $= 3.6$ and -5.1, y-intercept $= 0$

   (d) Maximum point $\left(\frac{1}{3}, \frac{1}{81}\right)$, infexion point $(0, 0)$; x-intercept $= 0$ and $\frac{5}{12}$, y-intercept $= 0$

   (e) Maximum point $(0, -4)$; x-intercept $= 2$, and -2; y-intercept $= -4$

   (f) Maximum point $\left(-\frac{1}{6}, \frac{49}{54}\right)$, minimum point $\left(\frac{1}{2}, \frac{3}{2}\right)$; x-intercept $= 1$, y-intercept $= -1$

7. $\frac{2x^2 + 24}{x}$ cm, $8\sqrt{3}$ cm 8. $10.98\ m^2$ 9. $r = \sqrt{\frac{200}{3\pi}}$ cm, $= 20\sqrt{\frac{2}{3\pi}}$ cm 10. $1250\,m^2$

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Exercise 8.18

2. $\frac{4}{5}+\frac{3}{5}x-\frac{2}{5}x^2-\frac{1}{10}x^3+\cdots$

3. (a) $4x-8x^2+\frac{64}{3}x^3-64x^4+\cdots$

(b) $1+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{5}{24}\left(x-\frac{\pi}{2}\right)^4+\cdots$

(c) $\sqrt{2}-\sqrt{2}\left(x+\frac{\pi}{4}\right)+\frac{3}{\sqrt{2}}\left(x+\frac{\pi}{4}\right)^2-\frac{11}{3\sqrt{2}}\left(x+\frac{\pi}{4}\right)^3+\frac{19}{4\sqrt{2}}\left(x+\frac{\pi}{4}\right)^4+\cdots$

(d) $\frac{1}{2}-\frac{\sqrt{3}}{2}x-\frac{1}{4}x^2+\frac{1}{4\sqrt{3}}x^3+\frac{1}{48}x^4+\cdots$

4. $-\frac{1}{2}<x<\frac{1}{2}$

5. (a) $1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots$

(b) $x-\frac{1}{3!}x^3+\cdots$

(c) $1+(\ln a)x+\frac{(\ln a)^2}{2}x^2+\frac{(\ln a)^3}{6}x^3+\frac{(\ln a)^4}{24}x^4+\cdots$

(d) $1+2x-2x^2+4x^3-10x^4+\cdots$

6. $1-4\theta^2+\frac{16}{3}\theta^4-\frac{128}{45}\theta^6+\cdots$

7. $1+\frac{3}{2}x+\frac{9}{8}x^2+\frac{9}{16}x^3+\cdots$

9. $1-\frac{1}{2}h^2+\frac{1}{24}h^4+\cdots$

Exercise 8.19

1. $\frac{\partial p}{\partial V}=-k\frac{T}{V^2}$; $\frac{\partial p}{\partial T}=k\frac{1}{V}$

2. $\frac{\partial z}{\partial x}=2xy^3$, $\frac{\partial z}{\partial y}=3x^2y^2$

3. $\frac{\partial z}{\partial x}=\frac{1}{y^2}+\frac{2y}{x^3}$ and $\frac{\partial z}{\partial y}=-\frac{2x}{y^3}-\frac{1}{x^2}$

4. (a) $2x\sin\left(x^3y\right)+3x^2y\left(x^2+4y^2\right)\cos\left(x^3y\right)$ (b) $\frac{x^2+2xy-y^2}{(x+y)^2}$ (c) $6(3x+y)$

5. (a) $\frac{\partial f}{\partial x}=3yx^2+2y^2x$, $\frac{\partial f}{\partial y}=x^3+2x^2y$ (b) $\frac{\partial f}{\partial x}=2x+3y$, $\frac{\partial f}{\partial y}=3x$

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(c) $\frac{\partial f}{\partial x} = 4x^3y^3 + 16xy + 5, \frac{\partial f}{\partial y} = 3x^4y^2 + 8x^2 + 4y^3$

(d) $\frac{\partial f}{\partial x} = \frac{2x}{y} - \frac{y^2}{x^2}, \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + \frac{2y}{x}$

6. (a) $\frac{\partial^2 z}{\partial x^2} = -x^2\sin(x-2y) + 4x\cos(x-2y) + 2\sin(x-2y)$

(b) $\frac{\partial^2 z}{\partial y^2} = -4x^2\sin(x-2y)$

(c) $\frac{\partial^2 z}{\partial x \partial y} = -4x\cos(x-2y) + 2x^2\sin(x-2y)$

(d) $\frac{\partial^2 z}{\partial y \partial x} = -4x\cos(x-2y) + 2x^2\sin(x-2y)$

7. $\frac{\partial^2 z}{\partial x^2} = -\frac{x}{\left(y^2-x^2\right)^{\frac{3}{2}}}, \frac{\partial^2 z}{\partial y^2} = \frac{2x^3y - 4xy^3}{2\left(y^4 - y^2x^2\right)^{\frac{3}{2}}}$

8. $\frac{\partial^2 z}{\partial x^2} = -0.71$

10. (a) $\frac{\partial T}{\partial l} = \frac{\pi}{\sqrt{gl}}$ (b) $\frac{\partial T}{\partial g} = -\frac{\pi}{g}\sqrt{\frac{l}{g}}$

9. $\frac{\partial^2 z}{\partial y^2} = -6$

Revision exercise 8

1. (a) $\frac{\cos x}{x} - \frac{\sin x}{x^2}$ (b) $-2\sin 2x$ (c) $x\cos x + \sin x$ (d) $-5\sin(5x+2)$

3. (a) $5\cos x + 4\sin x$ (b) $\frac{xe^{2y} - 8x^3 \ln\sin 2y}{4x^4\cot 2y - e^{2y}\left(1+x^2\right)}$

(c) $\frac{\cos x - \sin x}{2(1+\sin x\cos x)}$ (d) $\frac{4-3x}{\sqrt{1+8x-3x^2}}$

4. (a) $\tan 2\theta$ (b) $-\sec^3 2\theta$ (c) 0.4142

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5. (a) $\dfrac{-2y^3-6xy}{6xy^2+3x^2}$

(b) $\dfrac{y\left(x^2+y^2+2\right)+2x^3\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}{x\left(x^2+y^2+2\right)-2x^2y\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}$

(c) $-\left(\dfrac{2x^2\ln x+x^2+y^2}{2xy\ln x}\right)$

(d) $\dfrac{40xy^2+2y^2\sec^2 2x-3x^2\sin y}{x^3\cos y-2y\tan 2x-40x^2y}$

6. (a) $4e^x\left(1+e^x\right)^3$ (c) $e^x\left(x^2-3\right)+2xe^x$ (e) $-5e^{1-x}$

(b) $\dfrac{1-x}{e^x}$ (d) $\dfrac{1-3\ln x}{x^4}$

7. (a) $2xe^{4x}+4x^2e^{4x}$ (b) $\dfrac{e^x-1}{2\sqrt{e^x-x}}$

(c) $6x^5-16x^3+2$ (d) $\dfrac{6x}{\left(1+x^2\right)^2}$

8. $40\sqrt{2}$ m

9. (a) $(2+x)^2e^{-x}=4-x^2+\dfrac{1}{3}x^3$

(b) $\log_e y=(y-1)-\dfrac{1}{2}(y-1)^2+\dfrac{1}{3}(y-1)^3$

(c) $\dfrac{\cos x}{\sqrt{1-x}}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3+\dfrac{49}{384}x^4$

11. (a) 0 12. $\dfrac{902}{1215}$

13. (a) $\sin(x+h)=\sin x+\dfrac{h\cos x}{1!}-\dfrac{h^2\sin x}{2!}-\dfrac{h^3\cos x}{3!}+\dfrac{h^4\sin x}{4!}+\cdots$

(b) $\sin\left(\dfrac{\pi}{6}+h\right)=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}h-\dfrac{1}{4}h^2-\dfrac{\sqrt{3}}{12}h^3+\dfrac{1}{48}h^4+\cdots$

(c) $\tan\left(\dfrac{\pi}{3}+h\right)=\sqrt{3}+4h+4\sqrt{3}h^2+\dfrac{40}{3}h^3+\dfrac{44}{3}h^4+\cdots$

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(d) $\tan\left(\frac{\pi}{6}+x\right)=\frac{\sqrt{3}}{3}+\frac{4}{3}x+\frac{4}{3\sqrt{3}}x^2+\frac{8}{9}x^3+\frac{4}{3\sqrt{3}}x^4+\cdots$

14. (a) $\frac{-2x\sin 2x-\cos 2x}{2x^2}$ (d) $\frac{-9}{2\sqrt{5-9x}}$

(b) $16x+4$ (e) $4\cos 4\theta+2\sin 2\theta$

(c) $6(1-6x)^{-2}$

16. (a) $\ln 3, \frac{1}{3}$ (b) $\ln 3+\frac{1}{3}x-\frac{1}{9}x^2$

17. Turning point $=(0,-1)$

18. (a) 0.9316 (b) 0.1745

19. $\frac{1}{14}$ cm/s 20. 16 cm 21. $\frac{1}{28\pi}$ cm/s

23. (a) $-706.8583\,\text{cm}^2$ (b) $-4417.8647\,\text{cm}^3$

24. (a) $\frac{-\sin x}{2+\cos x}+\frac{\cos x}{3-\sin x}$ (b) $-\frac{1}{x^2+1}$

(c) $2\sec 2x$ (d) $\frac{-4-x^3}{2x^3\sqrt{x^3+1}}$

25. (b) $x+-x\ +\cdots$

26. (a) $1-\frac{9}{2}x^2+\frac{27}{8}x^4$ (b) $2x-\frac{4}{3}x^3+\frac{4}{15}x^5$ (c) $81+216t+216t^2$

27. (a) $\frac{-3x^2-14x-27}{(x^2-9)^2}$ (b) $3e^{3x}(x-4)^6+6(1+e^{3x})(x-4)^5$

(c) $\frac{-\sec^2 x}{\sqrt{1-(1+\tan x)^2}}$ (d) $\frac{4\sec^2 x+e^x\sqrt{1+y}-4\ln\sin y}{4x\cot y-\frac{1}{2}e^x\frac{\sqrt{1+y}}{1+y}}$

28. (a) $v=-5$ m/s (b) -4.25 m/s, 5 m/s^2
$a=-4$ m/s^2

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29. (a) $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{25-x^2-y^2}}$; $\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{25-x^2-y^2}}$

(b) $\frac{\partial z}{\partial x} = (2x+y)e^{x^2+xy}$; $\frac{\partial z}{\partial y} = xe^{x^2+xy}$

(c) $\frac{\partial z}{\partial x} = 3x^2+2y$; $\frac{\partial z}{\partial y} = 2x+4y^3$

(d) $\frac{\partial z}{\partial x} = \frac{3x^2}{y^2}+\frac{y^3}{x^2}$; $\frac{\partial z}{\partial y} = -\frac{2x^3}{y^3}-\frac{3y^2}{x}$

(e) $\frac{\partial z}{\partial x} = 4x-6y$; $\frac{\partial z}{\partial y} = -6x+2y$

(f) $\frac{\partial z}{\partial x} = 2\cos(2x+3y)$; $\frac{\partial z}{\partial y} = 3\cos(2x+3y)$

30. 1.85 cm

31. Length $l_1 = \frac{100\pi}{4+\pi}$ cm and $l_2 = \frac{400}{4+\pi}$ cm

32. $\frac{-3}{\pi}$ m/min

33. −86 m/s

34. Velocity $= (-4t+10)$ m/sec, acceleration $= -4$ m/s^2

35. (a) $\frac{\partial z}{\partial x} = 16$ (b) $\frac{\partial^2 z}{\partial x \partial y} = -8$

37. $r = \sqrt[3]{\frac{63}{\pi}}$ cm; $h = \left(\sqrt[3]{\frac{63}{\pi}}\right)^2$ cm

38. $(-1,2)$ maximum; $(1,2)$ minimum and $(0,0)$ point of inflexion

39. 0.16%

40. $\frac{d^2y}{dx^2} = -\frac{2}{3}$

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Answers

Exercise 9.1

1. (a) x^6+c (b) $2x^{\frac{1}{2}}+c$
   (c) $3x^{-7}+c$ (d) $x^{\frac{1}{2}}+c$
   (e) $x^{\frac{2}{3}}+c$ (f) $\frac{3}{2}x^{-2}+c$
2. (a) $\frac{1}{3}x^3+c$ (b) $5e^x+c$
   (c) $\frac{1}{4}\ln|x|+c$ (d) $2\sin\theta+c$
   (e) $-\frac{1}{2}\cos\theta+c$
   (f) $4\ln|\sec\theta|+c$
3. (a) $-\frac{3}{x}+c$ (b) $2x^{\frac{1}{2}}+c$
   (c) $\frac{12}{5}t^{\frac{5}{4}}+c$ (d) $-2e^x+c$
   (e) $10\cos\theta+c$ (f) $\frac{5}{4}x^1+c$

Exercise 9.2

1. $\frac{5}{4}x^4+\frac{8}{3}x^3-\frac{3}{2}x^2+5x+c$
2. $-\frac{3}{2}x^4+3x^3+2x^2-3x+c$
3. $\frac{2}{5}x^{\frac{5}{2}}+x^2+3x+c$
4. $\frac{48}{7}t^{\frac{7}{4}}-\frac{18}{7}t^{\frac{7}{2}}+c$
5. $\frac{1}{12}x^4+\frac{5}{3}x^3-\frac{7}{2}x^2+2x+c$
6. $-\frac{1}{2}t^{-2}-\frac{2}{5}t^5+7\ln|t|+t+c$
7. $2x^{\frac{1}{2}}-2x^{\frac{3}{2}}+\frac{6}{5}x^{\frac{5}{2}}-\frac{2}{7}x^{\frac{7}{2}}+c$
8. $-\frac{3}{x}+\frac{\sin x}{2}+c$
9. $\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{3}x^{\frac{1}{2}}+c$
10. $8\ln|x|+\frac{5}{x}-\frac{3}{x^2}+c$
11. $\ln|t|+\frac{5}{4}t^4+\frac{2}{t^2}+c$
12. $\ln|x|-\frac{\cos x}{5}-\frac{\sin x}{3}+c$
13. $6x-x^2+c$
14. $\frac{1}{3}x^3+2x^2+x+c$
15. $\frac{1}{5}x^5-x^2-2x^{\frac{3}{2}}+c$
16. $t^4-t^3-t^2+t+c$
17. $11e^x+c$
18. $\frac{y^2}{2}-\frac{2}{y}+c$
19. $2\sin\theta-2\cos\theta+c$
20. $\frac{4}{3}\sin x+c$
21. $x^4+\frac{7}{3}x^3-\frac{3}{2}\sin x+c$
22. $\frac{1}{3}x^3-\frac{1}{2}x^2-\frac{3}{2}x^{-\frac{2}{3}}+\frac{1}{x}+c$

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23. $\ln|t|+\frac{5}{4}t^4+\frac{2}{t^2}+c$

24. $-\frac{1}{r}+\frac{1}{3r^3}+\frac{3r^2}{2}+c$

Exercise 9.3

1. $\frac{1}{54}(6x-9)^9+c$
2. $-\frac{1}{5}(1-2x)^{\frac{5}{2}}+c$
3. $-\frac{1}{6}(3t-1)^{-2}+c$
4. $-3\ln|1-x|+c$
5. $12e^{\frac{x}{3}}+c$
6. $-\frac{1}{3}\ln|4-3t|+c$
7. $-2\sin(1-3x)+c$
8. $-\frac{1}{7}\ln|5-7x|+c$
9. $\frac{1}{2}\ln|\sec(2\theta+1)|+c$
10. $\frac{\ln|2x-1|}{2}+c$
11. $-\frac{e^{3-4x}}{14}+c$
12. $-10\cos\left(\frac{x}{2}-1\right)+c$
13. $\frac{e^{2x+2}}{2}+c$
14. $\frac{1}{a}\sin(ax\pm b)+c$
15. $\frac{1}{a}\ln|\sec(ax\pm b)|+c$

Exercise 9.4

1. $\frac{1}{2}(x^3+x^2-3)^2+c$
2. $-\frac{1}{2}(1+e^{-t})^2+c$
3. $\frac{4}{9}(3t^3-1)^2+c$
4. $\ln|1+e^x|+c$
5. $\ln|\ln|x||+\ln|x|+c$
6. $-\frac{1}{2}\sin^2\theta+c$ or $\frac{1}{2}\cos^2\theta+c$
7. $-\ln|\cos x-\sin x|+c$
8. $\frac{x^3}{2}+\frac{6}{5}x^{\frac{5}{2}}+c$
9. $\ln|\cos\theta+\sin\theta|+c$
10. $\ln|x|+\frac{1}{2}[\ln|x|]^2+c$
11. $\ln|3+4x+9x^2-6x^3|+c$
12. $\frac{1}{4}[\ln(|x+2|)]^2+c$
13. $\frac{2}{3}\ln\left|1+x^{\frac{3}{2}}\right|+c$
14. $x-2\sqrt{x}+2\ln|\sqrt{x}+1|-3+c$
15. $\frac{1}{3}(x^2-1)^{\frac{3}{2}}+c$

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16. $\frac{1}{2}e^{x^2+4x}+c$

17. $\frac{1}{6}\sin(3e^{x^2})+c$

18. $t-2\ln|1+e^t|+c$

19. $-\frac{1}{x\ln x}+c$

20. $-\frac{1}{\ln x}+c$

21. $e^{\tan^{-1}x}+c$

22. $\ln|\tan(x+1)|+c$

Exercise 9.5

1. $\frac{1}{2}e^{2x}\left(x^2-x+\frac{1}{2}\right)+c$

2. $x\tan^{-1}(3x)-\frac{1}{6}\ln(1+9x^2)+c$

3. $x\cos^{-1}\left(\frac{1}{2}x\right)-\sqrt{4-x^2}+c$

4. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

5. $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+c$ or

   $\frac{1}{2}x(\sin x)^2+\frac{1}{4}\cos x\sin x-\frac{1}{4}x+c$

6. $x^2\sin x+2x\cos x-2\sin x+c$

7. $\frac{1}{4}x^4\ln|x|-\frac{1}{16}x^4+c$

8. $\frac{1}{2}(x^2-2x)\ln|2x|+x-\frac{x^2}{4}+c$

9. $x\ln|3x|-x+c$

10. $\frac{1}{5}e^{2x}(2\cos x+\sin x)+c$

11. $-\frac{1}{5}e^{-x}(2\cos(2x)+\sin(2x))+c$

12. $\frac{2}{5}e^{\frac{1}{2}x}(2\sin x+\cos x)+c$

13. $\frac{1}{10}e^{4x}(2\sin 2x-\cos 2x)+c$

14. $\frac{1}{13}e^{2x}(3\sin 3x+2\cos 3x)+c$

15. $\theta\tan\theta+\ln|\cos\theta|+c$

16. $-e^{-3x}\left(\frac{1}{3}x^2+\frac{2}{9}x+\frac{2}{27}\right)+c$

17. $x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+c$

18. $\frac{1}{3}x^3\ln|x|-\frac{1}{9}x^3+c$

19. $2\sqrt{x}\ln|x|-4\sqrt{x}+c$

20. $-\frac{1}{6x^6}\ln|x|-\frac{1}{36x^6}+c$

21. $\frac{1}{3}(1+x)^3\ln|3x|-\frac{1}{3}\ln|x|-\frac{1}{9}x^3-\frac{1}{2}x^2-x+c$

23. $I_n=x^ne^x-nI_{n-1}$,

    $I_4=x^4e^x-4x^3e^x+12x^2e^x-24xe^x+24e^x+c$

Exercise 9.6

1. $\ln\left|\frac{(x+5)^2}{x-2}\right|+c$

2. $\frac{1}{36}\ln|x-1|+\frac{1}{6(x+5)}-\frac{1}{36}\ln|x+5|+c$

3. $\ln\left|\frac{x-1}{(9+x^2)^{\frac{1}{2}}}\right|-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$

4. $\ln\left|\frac{s^2}{(s-1)^2}\right|-\frac{1}{s-1}-\frac{1}{s}+c$

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5. $\ln\left|\frac{(2x+1)^{\frac{3}{2}}}{x-2}\right|+\frac{2}{2-x}+c$

6. $\ln\left|x^2(x+2)^3\right|+\frac{1}{x}+c$

7. $\frac{2}{t}-\frac{2t}{t^2+3}+\frac{2-5t}{(t^2+3)^2}+c$

8. $\ln\left|(y-3)^{\frac{1}{5}}y^2(y+2)^{\frac{9}{5}}\right|+c$

9. $\ln\left|\frac{(x+2)^3}{(x+1)^2}\right|+c$

10. $\ln\left|\frac{x(x-1)}{x+1}\right|+c$

11. $\ln\left|\left(\frac{t-1}{t+4}\right)^{\frac{1}{5}}\right|+c$

12. $-\frac{1}{x-1}-\frac{2}{(x-1)^2}+c$

13. $-\frac{1}{4(x+3)^4}+\frac{9}{5(x+3)^5}-\frac{x-3}{(x+3)^6}+c$

14. $x-\frac{1}{2}\tan^{-1}(x)+\ln\left|\left(\frac{x-1}{x+1}\right)^{\frac{1}{4}}\right|+c$

15. $5\ln|x-3|+\frac{2}{x-3}+c$

16. $\ln\left|\frac{t^{\frac{2}{5}}}{(t+5)^{\frac{7}{5}}}\right|+c$

17. $\ln\left|\left(\frac{x-4}{x+4}\right)^{\frac{15}{8}}\right|+x+c$

18. $3x+\frac{1}{3}x^3+\frac{4}{3}\ln|x-1|-\frac{13}{3}\ln|x+2|+c$

19. $\ln\left|r\left(\frac{r-1}{r+1}\right)^{\frac{1}{2}}\right|+c$

20. $\ln\left|\frac{(x+2)^4}{(x+5)^3}\right|+c$

21. $\frac{x^2}{2}+\frac{1}{4x}-\frac{1}{4x^2}-2x+\frac{1}{8}\ln x+\frac{31}{8}\ln(x+2)+c$

Exercise 9.7

1. $\frac{1}{2}\left(\sin x-\frac{1}{7}\sin 7x\right)+c$

2. $\frac{1}{16}(-4\cos 2x-\cos 8x)+c$

3. $\frac{1}{20}(5\sin 2x+\sin 10x)+c$

4. $\frac{1}{12}(3\cos 2x-\cos 6x)+c$

5. $\frac{1}{24}\cos 6\theta-\frac{1}{16}\cos 4\theta-\frac{1}{8}\cos 2\theta+c$

6. $\frac{1}{12}(3\sin 2\theta-\sin 6\theta)+c$

7. $-\cos 2x+\frac{1}{3}\cos^3 2x-\frac{1}{10}\cos^5 2x+c$

8. $\frac{1}{8}[4x-2-\sin(4x-2)]+c$

9. $\frac{1}{96}\sin(12x+4)+\frac{1}{12}\sin(6x+2)+\frac{3}{8}x+c$

10. $\frac{1}{96}(\sin 12x-8\sin 6x+36x)+c$

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11. $\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+c$

12. $\frac{3}{4}(x-1)-\frac{1}{2}\sin(2x-2)+\frac{1}{16}\sin(4x-4)+c$

13. $\sin x-\frac{1}{3}\sin^3 x+c$

14. $\cos x\left(\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x-1\right)+c$

15. $\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta+c$

16. $\cos^3 x\left(\frac{1}{5}\cos^2 x-\frac{1}{3}\right)+c$

17. $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+c$

18. $\frac{1}{64}(8x-\sin 8x)+c$

19. $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c$

20. $-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+c$

21. $\frac{1}{12}\tan^4(3\theta)+\frac{1}{18}\tan^6(3\theta)+c$

22. $\frac{1}{4}\sec 2x\tan 2x+\frac{1}{4}\left|\sec 2x\tan 2x\right|+c$

Exercise 9.8

1. $\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+c$

2. $\frac{\sqrt{5}}{45}\tan^{-1}\left(\frac{\sqrt{5}x}{9}\right)+c$

3. $3\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)+c$

4. $\frac{1}{2}\tan^{-1}\left(\frac{z+1}{2}\right)+c$

5. $\frac{1}{3}\ln\left|\frac{\sqrt{9+4y^2}-3}{y}\right|+c$

6. $-\frac{\sqrt{4+z^2}}{4z}+c$

7. $\tan^{-1}x+c$

8. $\frac{1}{2}\ln\left(y^2+9\right)+\tan^{-1}\frac{y}{3}+c$

9. $\frac{1}{2}\tan^{-1}\left(\frac{4y}{7}\right)+c$

10. $\frac{\sqrt{2}}{2}\sin^{-1}\sqrt{2}x+c$

11. $6\sin^{-1}\left(\frac{2}{3}x\right)+c$

12. $\sin^{-1}\left(\frac{\sqrt{5}}{5}x\right)+c$

13. $x\sqrt{1-4x^2}+\frac{1}{2}\sin^{-1}(2x)+c$

14. $5\left(\sin^{-1}\left(\frac{10}{\sqrt{10}}(x+2)\right)\right)+$
$\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{\sqrt{10}}{10}(x+2)\right)\right)+c$

15. $-\frac{\sqrt{9-x^2}}{9x}+c$

16. $\frac{1}{2}\left(\sin^{-1}(x+1)+\frac{1}{2}\sin\left(2\sin^{-1}(x+1)\right)\right)+c$

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17. $\frac{9}{2}\left[\sin^{-1}\frac{1}{3}(x-2)+\frac{1}{9}(x-2)\sqrt{5+4x-x^2}\right]+c$

18. $\frac{1}{3}(t^2+9)^{\frac{3}{2}}-9\sqrt{t^2+9}+c$

19. $2\sin^{-1}(\frac{1}{2}x)-\sin(2\sin^{-1}(\frac{1}{2}x))+c$

20. $\frac{1}{8}(\sin^{-1}x-\frac{1}{4}\sin(4\sin^{-1}x))+c$

21. $\frac{1}{2}\tan^{-1}(\frac{x+2}{2})+c$

22. $\frac{2}{5}\sqrt{5}\tan^{-1}\left(\frac{\sqrt{5}}{5}(x+5)\right)+c$

23. $\sin^{-1}(\frac{1}{2}x+1)+c$

24. $-2\sqrt{9-x^2}-7\sin^{-1}(\frac{1}{3}x)+c$

25. $\frac{2}{3}\tan^{-1}(\frac{1}{3}(x-2))+c$

26. $\frac{2s^2+9}{8(4s^2+9)^{\frac{1}{2}}}+c$

27. $7\ln\left|\sqrt{x^2+1}+x\right|+\frac{\sqrt{2}}{16}\left[\sin^{-1}\left(\sqrt{2}x\right)\right]+\frac{1}{2}\sin\left[2\sin^{-1}\left(\sqrt{2}x\right)\right]+c$

Exercise 9.9

1. $\tan\frac{1}{2}x+c$

2. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan(\frac{1}{2}\theta)\right)+c$

3. $\frac{-1}{1+\tan x}+c$

4. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{2}\tan x\right)+c$

5. $\ln\left|\tan\left(\frac{x}{2}\right)\right|+c$

6. $\tan^{-1}(\sin x)+c$

7. $-\ln\left|1+\cos^2 x\right|+c$

8. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+c$

9. $\frac{1}{6}\ln\left|\frac{3\tan x+1}{3\tan x-1}\right|+c$

10. $\frac{1}{2}\tan\theta+c$

11. $\ln\left|\sec^2\left(\frac{x}{2}\right)\right|+c$

12. $\ln\left|\tan(\frac{x}{2})\right|+c$

13. $\frac{1}{4}\ln\left|\tan(4x)+\sec(4x)\right|+c$

14. $\sin^{-1}(\frac{1}{2}\sin x)+c$

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15. $-\ln\left|\tan x+\sec x\right|+2\sin x+c$

16. $\frac{1}{2}\ln\left|\frac{\tan\frac{x}{4}+2}{\tan\frac{x}{4}-2}\right|+c$

17. $\frac{1}{3}\left(\ln\left|\frac{\tan\frac{x}{2}}{3}+1\right|-\ln\left|\frac{\tan\frac{x}{2}}{3}-1\right|\right)+c$

18. $2\ln\left|\tan\left(\frac{x}{4}\right)\right|+c$

19. $\frac{1}{3}\tan\left(\frac{3}{2}y\right)+c$

20. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}\right)-1\right)+c$

21. $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}+1\right)\right]+c$

22. $\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{\theta}{4}\right)\right)+c$

23. $\frac{1}{3}\ln\left|\frac{\cos x}{\cos x-3}\right|+c$

24. $\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)+1}{\tan\left(\frac{\theta}{2}\right)+3}\right|+c$

25. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{3}\right)\right)+c$

26. $\ln\left|\frac{2\sin x-4}{2\sin x-2}\right|+c$

27. $\tan^{-1}(\tan x+1)+c$

28. $-\theta+\frac{10\sqrt{3}}{3}\tan^{-1}\left[\sqrt{3}\tan\left(\frac{\theta}{2}\right)\right]+c$

29. $\ln\left|\tan\left(\frac{x}{2}\right)+2\right|+c$

30. $\sin x\sqrt{1+\sin x}-\frac{1}{3}(1+\sin x)^{\frac{3}{2}}+\sqrt{1+\sin x}+c$

Exercise 9.10

1. $\frac{1}{2}\ln\left|x^2+5\right|+\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

2. $\frac{1}{4}\ln\left|4x^2+12x+10\right|+\frac{1}{2}\tan^{-1}(2x+3)+c$

3. $\frac{1}{2}0-\frac{1}{2}\ln\left|\cos 0+\sin 0\right|+c$

4. $-\frac{7}{10}\ln\left|\sin\theta+3\cos\theta\right|-\frac{1}{10}\theta+c$

5. $\ln\left|x^2+2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{1}{3}(x+1)\right)+c$

6. $\ln\left|y+3\right|-\frac{1}{y+3}+c$

7. $\frac{3}{2}\ln\left|u^2+2u+5\right|+\frac{5}{2}\tan^{-1}\left(\frac{1}{2}(u+1)\right)+c$

8. $\frac{11}{17}\ln\left|\cos\theta+4\sin\theta\right|-\frac{10}{17}\theta+c$

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9. $\frac{3}{5}\ln|3+\cos x+2\sin x|+\frac{4}{5}x-\frac{12}{5}\tan^{-1}\left[\tan\left(\frac{x}{2}\right)+1\right]+c$

10. $\frac{1}{2}x+\frac{5}{2}\ln|\cos x+\sin x|+c$

11. $\frac{3}{13}\ln|2\cos\theta+5\sin\theta|+\frac{2}{13}\theta+c$

12. $\frac{3}{106}\ln|7\cos 2x-2\sin 2x|-\frac{37}{53}x+c$

Exercise 9.11

1. $\frac{7^x}{\ln 7}+c$

2. $\frac{(\ln x)^2}{2\ln 10}+c$

3. $\frac{8^x}{\ln 2}+c$

4. $-\frac{x^2}{\ln 10}+c$

5. $\frac{3^{2x}}{2\ln 3}+c$

6. $-\frac{6^{1-2y}}{2\ln 6}+c$

7. $\frac{-1}{4^x\ln 4}+c$

8. $\frac{-9^{3-7x}}{7\ln 9}+c$

9. $\frac{10^{x-1}}{\ln 10}+c$

10. $\frac{x}{\ln 10}(\ln|3x|-1)+c$

11. $\frac{1}{2\ln 2}(\ln|x|)^2+c$

12. $\frac{2^x}{\ln 2}+\frac{x}{\ln 10}(\ln|5x|-1)+c$

13. $\frac{(2x+1)\ln|2x+1|}{2\ln 10}-\frac{x}{\ln 10}+c$

14. $\frac{\left(x^2+x-1\right)\left[\ln|x^2+x-1|-1\right]}{\ln 10}+c$

15. $\frac{x}{6\ln 10}(\ln|2x|-1)+c$

16. $\frac{x}{\ln 10}(\ln|\frac{1}{4}x|-1)+c$

17. $-\frac{(\frac{1}{3})^x}{\ln 3}+c$

18. $\frac{\tan x}{\ln 10}(\ln|\tan x|-1)+c$

19. $\frac{x}{\ln 2}(\ln|x|-1)+c$

20. $\frac{(\ln|3x|)^2}{4\ln 5}+c$

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Exercise 9.12

1. 16
2. $\frac{52}{3}$
3. 32
4. 180
5. $\frac{15}{2}$
6. 2
7. $\ln 2$
8. $\frac{2\sqrt{2}}{3}$
9. $-\frac{12}{\pi}$
10. $\frac{55}{4}$
11. $\ln 2+\frac{69}{4}$
12. $\frac{113}{24}$
13. 7174089
14. $\frac{9\sqrt{3}-1}{5}$
15. $\frac{5}{128}$
16. $3\ln 2$
17. $3(e-1)$
18. $\frac{1}{3}(2\ln 2-\ln 3)$
19. 2
20. $\frac{-3}{7}\ln 2$
21. 0
22. $\frac{1}{2}\ln 7$
23. $\frac{e^{12}-1}{4e}$
24. $10\left(\cos(1)-\cos\left(\frac{\pi-4}{4}\right)\right)$
25. 2
26. $\frac{1}{2}(1-e^{-2})$
27. $\frac{16\sqrt{2}-4\sqrt[4]{2}}{9}$
28. $\frac{2}{15}$
29. $\ln(\frac{21}{5})$
30. $\frac{2}{3}$
31. 1
32. 6
33. 1
34. $\frac{3}{2}$
35. $\frac{40}{3}$
36. $\frac{(\ln 2)^2}{4}$
37. $\frac{3\pi}{8}+1$
38. $3+4\ln 2$
39. 9
40. $\frac{1}{6}$
41. $\frac{e^5-1}{2}$
42. $2\ln 2-2\ln 3$
43. $\frac{3}{10}$
44. $12\ln 3-8$
45. $\frac{1}{36}-\frac{7}{36e^6}$
46. $\frac{26}{3}\ln 2-\frac{59}{18}$
47. $-\frac{\ln 2-\ln 3}{2}$
48. $3\ln 3-5\ln 2$
49. $\frac{3\sqrt{3}}{8}$
50. $\frac{203}{480}$
51. 0
52. $\frac{2}{35}$
53. $\frac{2}{15}$
54. π
55. $\frac{5}{2}$
56. $\frac{\pi}{4}$
57. $\frac{1}{2}(\tan^{-1}(\frac{1}{2})-\frac{\pi}{4})$
58. $\frac{4}{3}\tan^{-1}(\frac{1}{3})$
59. $\frac{\pi}{4}$
60. $6-4\sqrt{2}-7\sin^{-1}(\frac{1}{3})$

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Exercise 9.13

1. $\frac{1}{3}$ square units
2. $\frac{1}{2}ab\pi$ square units
3. $\frac{\pi}{2}$ square units
4. $\frac{9}{2}(1-e^{-2})$ square units
5. 18 square units
6. $\frac{80}{\pi}$ square units
7. 3π square units
8. $\left(\frac{1}{2}-\frac{4\pi-3\sqrt{3}}{24}\right)$ square units
9. $\frac{45}{4}$ square units
11. $\sqrt{257}$ units
12. 4.44 units
13. 12.0397 units
14. $\frac{\pi}{2}r$ units
15. 21.2563 units
16. 2π units
17. $78\frac{2}{3}$ square units
18. $\frac{3\pi a}{2}$ square units

Exercise 9.14

1. $\frac{96}{5}\pi$ cubic units
2. $\frac{93}{5}\pi$ cubic units
3. $\frac{25}{8}\pi^2$ cubic units
4. 16π cubic units
5. $\frac{64}{3}\pi$ cubic units
6. $\frac{80}{\pi}$ cubic units
7. (a) $\frac{4\pi ab^2}{3}$ cubic units (b) $\frac{4\pi a^2 b}{3}$ cubic units
8. $\frac{16\pi}{5}$ cubic unit
9. 2500π cubic units
10. $\frac{404\pi}{5}$ cubic unit
11. (a) $\frac{62\pi}{5}$ cubic units (b) 30π cubic units
    (c) $\frac{423\pi}{80}$ cubic units (d) $\frac{10\pi}{3}$ cubic units
12. $\frac{\pi}{2}(e^2-1)$ cubic units
13. $\frac{3\pi}{10}$ cubic units
14. 9π cubic units
15. $\left(2-\frac{\pi}{4}\right)$ square units
16. $\frac{\pi}{24}$ square units
17. $\frac{\pi}{8}$ square units
18. $\frac{4\pi-7\sqrt{3}}{16}$ square units

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Revision Exercise 9

1. (a) $\frac{2}{9}(3x-1)^{\frac{3}{2}}+c$ (b) $\frac{x^5}{5}+x^3+c$

   (c) $\frac{1}{2}\ln\left|e^{2x}+1\right|+c$ (d) $\frac{1}{5}x^5+3x^4+16x^3+32x^2+c$

   (e) $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$ (f) $\frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{2}\ln\left(1+x^2\right)+c$

2. $\frac{\pi}{12}$ 3. $\frac{\ln 3}{4}$

4. (a) $-\frac{116}{15}$ (b) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{\theta}{2}\right)\right)+c$

   (c) $\frac{1}{5}(0+1)(20-3)^{\frac{3}{2}}+c$ (d) $x^2e^x-2xe^x+2e^x+c$

5. 0.1794

7. (a) $\frac{1}{4}\ln 3$ (b) $-\frac{\sqrt{-x^2+4}}{4x}+c$

   (c) $\frac{1}{2}(\sin x-\frac{1}{7}\sin 7x)+c$ (d) $-\frac{e^x\cos x}{2}+\frac{e^x\sin x}{2}+c$

   (e) $\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4}{\sqrt{23}}\tan\left(\frac{\theta}{2}\right)+\frac{1}{\sqrt{23}}\right)+c$ (f) $\frac{1}{2}\tan^{-1}\left(\frac{\theta-3}{2}\right)+c$

   (g) 2.3752 (h) $\frac{1}{6\ln 24}24^{6x}+c$

9. (a) $\sin(x^2-5)+c$ (f) $\frac{1}{3}\cos(\cos 3x)+c$

   (b) $-\cos(x^3-2x^2+1)+c$ (g) $\frac{1}{3}\left(\frac{\tan^3 3y}{3}+\frac{2\tan^5 3y}{5}+\frac{\tan^7 3y}{7}\right)+c$

   (c) $-2\sqrt{2}\cos\left(\frac{x}{2}\right)+c$ (h) $\frac{1}{2}\left(-\frac{1}{11}\cos 11t-\frac{1}{5}\cos 5t\right)+c$

   (d) $\frac{1}{3}\left(\frac{\sin^9(3z)}{9}-\frac{2\sin^{11}(3z)}{11}+\frac{\sin^{13}(3z)}{13}\right)+c$

   (e) $\frac{1}{2}\left(\cos^3 2t\sin 2t+\frac{3}{8}\left(2t-\frac{1}{4}\sin 8t\right)\right)+c$

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(i) $\ln|\sec x| - \sec^2 x + \frac{1}{4}\sec^4 x + c$

(j) $\frac{3}{2}\left(-\frac{1}{5}\cos^5\left(\frac{2x}{3}\right) + \frac{1}{7}\cos^7\left(\frac{2x}{3}\right)\right) + c$

(k) $e^{4+\sin x} + c$

(l) $\frac{-32-5\sqrt{2}}{252\sqrt{2}}$

(m) 0

(n) $\frac{1}{6}(\sin x - \cos x)^6 + c$

10. $I_6 = -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{8}\left(x - \frac{1}{2}\sin 2x\right)\right] + c$ and

$I_7 = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + c$

11. (a) $\frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right) + c$

(b) $\frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c$

(c) $\frac{2}{3}(1+\ln x)^{\frac{3}{2}} + c$

(d) $2\left(\frac{1}{3}(x-1)^{\frac{3}{2}} + \sqrt{x-1}\right) + c$

(e) $\frac{1}{2}\ln|x^2+4x+8| - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+2)\right) + c$

(f) $\frac{\pi\sqrt{3}}{9}$

11. (a) $\frac{1}{2\ln 2} - \frac{1}{e}$ (b) $-\frac{1}{2}\left(\frac{1}{2}\ln 3 - 1\right)$ (c) $\frac{7}{6} + \ln 2 - \ln 3$ (d) $\frac{1}{2}$

14. (a) $-5\sec^{-1}\frac{1}{5}x + \sqrt{x^2-25} + c$ (b) $\frac{4\sqrt{3}\pi - \pi + 2\ln 2 - 4}{6}$

(c) $\frac{1}{2\ln 10}$ (d) $\frac{\pi}{4}$

15. (a) $\frac{1}{4}\left(\sin^{-1}\left(\sin^2 x\right)\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\sin^2 x\right)\right) + c$

(b) $2e^{\sqrt{x-2}} + c$

(c) $-\frac{}{3^{\sqrt{x}}\ln 3} + c$

(d) $\frac{1}{2}\ln|x^2+2x+5| + c$

(e) $\frac{1}{2}\ln|x^2+1| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

(f) $\frac{x^2}{2} + x + 2\ln|x+2| + \ln|x-3| + c$

(g) $\ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + c$

(h) $\tan^{-1}(x) - \frac{1}{2(x^2+1)} + c$

(k) $-\frac{2}{3(x^3+1)} + c$

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(i) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+3}{4}\right)+c$

(j) $1+\ln|x|-\ln|1+\ln|x||+c$

(l) $\frac{1}{3}(x+1)^{\frac{3}{2}}+\frac{1}{3}(x-1)^{\frac{3}{2}}+c$

(m) $\frac{1}{2\ln 7}7^{2x}+c$

16. (a) $\frac{13}{10}\ln 3+\frac{11}{20}\pi$ (b) $\frac{1}{2}\ln|\theta^2+16|+8\tan^{-1}\left(\frac{\theta}{4}\right)+c$ (c) $\frac{1}{4}\ln|\sec 4\theta+\tan 4\theta|+c$

(d) $\frac{3}{2}\ln|\cos x+\sin x|+\frac{1}{2}x+4\sqrt{2}\ln\left[\frac{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{x}{2}\right)-1+\sqrt{2}}\right]+c$

17. (a) $\frac{35}{256}\pi$ (b) $\frac{128}{315}$

19. (a) $\frac{1328\sqrt{3}}{2835}$ (b) $\frac{17}{36}+\frac{3}{32}\ln 3$

21. (b) $\frac{1}{3}$

22. $\ln(1+\sqrt{2})$ units

23. (a) 1 square unit
(b) $\pi(e-1)$ cubic units

24. $\left(-37e^{-5}+2\right)$ units

26. The total sales for 4 months is 3702.80.

27. $\frac{2\pi+\sqrt{3}+8}{16}$ square units

28. $\frac{17}{6}$ units

30. (a) P(4, 16), Q(−6.25, −25)
(b) $(68-45\ln 3)$ square units.

31. (a) $21\frac{1}{3}$ square units

32. (a) $\frac{8\pi-9\sqrt{3}}{64}$ square units

(b) $\frac{1}{48}\left(99\pi+32\sqrt{2}+66\right)$ square units

33. $\frac{56\pi}{3}$ cubic units

34. $\frac{32}{5}\pi$ cubic units

35. $\frac{19}{3}\pi$ cubic units

37. $\frac{a^2}{4k}\left[e^{2\pi k}-e^{-2\pi k}\right]$ square units

38. $\frac{32}{3}$ square units

40. (a) 768π cubic units
(b) 8 square units

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Glossary

Absolute value inequalities	expressions with absolute function as well as inequality signs.
Adjoint of a matrix	a transpose of a matrix containing co-factors of elements of the matrix.
Anti-derivative	opposite of the derivative of a function.
Argument	a set of connected propositions that form a definite statement.
Asymptote	a line that the distance between a curve and the line approaches zero as one or both of the x or y coordinates tends to infinity.
Biconditional statement	a statement formed by a combination of conditional statements.
Binomial theorem	an algebraic method of expanding a binomial expression to any power without lengthy multiplication.
Cartesian coordinate system	a system used to determine a point uniquely in xy-plane through two numbers.
Cofactor of an element	the matrix obtained when the column and row of a chosen element in a matrix are removed.
Column matrix	a matrix whose elements are arranged in an order of a single column.
Complement of a set	a set of elements in the universal that are not in the set under consideration.
Composite function	a function formed by substituting one function into another function.
Compound angle	an algebraic sum of two or more angles which are added or subtracted through trigonometric functions.
Compound statement	a declarative sentence formed by more than one declarative sentence.
Conditional statement	a statement formed by two statements in which the second statement is a logical consequence of the first statement.
Conjunction statement	a compound statement formed by joining two statements with the connector "and".
Constraints	inequalities or equations which connect the decision variables under certain restrictions or limitations.

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Contrapositive of a conditional statement	a statement formed by interchanging the premise and the conclusion of the inverse statement.
Converse of a conditional statement	a statement formed by exchanging the premises and conclusion of the original conditional statement.
Definite integral	the integral function which has limit of integration.
Derivative of a function	the rate of change of a function with respect to a given variable.
Determinant	a scalar value that is a function of the entries of a square matrix.
Disjunction statement	a compound statement formed by joining two statements with the connector "or".
Double angle formulae	are formulas in trigonometry that deals with the double angles of trigonometric functions.
Equivalent logical expressions	expressions which have the same truth values in the truth table.
Feasible region	the region of the graph which contains all the points which satisfy all the constraints of the system.
Feasible solution	a solution that satisfies all constraints.
Finite set	a set with a fixed number of elements.
Hole	a removable discontinuity that exist on the graph of rational function at any input value that causes both the numerator and denominator of the function to be equal to zero
Horizontal asymptote	a horizontal line parallel to the axis of the independent variable.
Identity matrix	a square matrix in which all elements of the principal diagonal are 1's and all other elements are zeros.
Indefinite integral	is the integral function which has no limit of integration.
Indices	is a number which shows how many times it has been multiplied by itself.
Infinite set	a set containing uncountable number of elements.
Integration by parts	a process that finds the integral of the products of functions in terms of the integral of the product of their derivative and anti-derivative.
Intersection of sets	a set which contains all the elements that are common to all involved sets.

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Inverse of a conditional statement	a conditional statement formed by negating both the premise and conclusion.
Inverse of a matrix	a matrix that, when multiplied by the given matrix gives the multiplicative identity.
Limits	limits describe how a function behaves near a point, instead of at that point.
Locus	a set of points whose location is determined by one or more specified conditions.
Mathematical induction	a mathematical technique which is used to prove that if the statement is true for the n^{th} iteration, then it is also true for $(n+1)^{th}$ iteration.
Matrix	a set of numbers arranged in a rectangular array, having m rows and n columns and enclosed by the square or ordinary brackets.
Minor of determinant	the determinant of a sub-matrix formed by deleting the rows and columns in a given matrix.
Natural logarithms	logarithms to the base e (where e = 2.718281828459).
Normal to a circle	a straight line drawn at 90° to the tangent line at the point where the tangent touches the circle.
Objective function	a linear function whose value is to be either minimized or maximized subject to the constraints defined over the set of feasible solutions.
Oblique asymptote	occurs when the degree of the numerator is greater than that of the denominator.
Optimal points	a point where the objective function attains its maximum or minimum value.
Optimal value	the value from the optimal solution that maximizes or minimizes the objective function of the linear programming problems.
Orthogonal circles	circles intersecting in such a way that the tangents at the points of intersection are perpendicular.
Proposition	a declarative sentence which can be either true or false but not both.
Radian	the measure of an angle subtended at the centre of a circle by an arc whose length is equal to the radius of that circle.
Remainder theorem	a theorem used to calculate the remainder of the division of any polynomial by another polynomial without applying the long division method.

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Row matrix	a matrix whose elements are arranged in a single row.
Scalar matrix	a diagonal matrix with equal-valued elements along the diagonal.
Sequence	a set of numbers, or algebraic expressions, which can be obtained from a proceeding one by a definite rule.
Series	an expression showing the sum of terms in a sequence by linking the terms of the sequence with sign of addition.
Simple statement	a declarative sentence that is either true or false.
Square matrix	a matrix having the same number of rows and columns.
Tangent to a circle	a straight line which touches the circle at only one point.
Trigonometric identities	equations that are true for every value of variables occurring on both sides of an equation.
Trigonometric ratios	are values of all trigonometric functions based on the value of the ratio of sides in a right-angled triangle.
Trigonometric substitution	the substitution of trigonometric functions for other expressions.
Truth table	a table that shows the validity of a compound statement depending on the truth values of its simple statements.
Union sets	a set containing all the elements available in the individual sets under consideration.
Universal set	a set containing all elements or members of all related sets without any repetition of elements.
Valid argument	an argument that has all true premises and a true conclusion.
Venn diagram	a diagram that uses circles or ovals to show relationship between sets and their elements.

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