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Dr Lyabwene M. Mtahabwa
Commissioner for Education

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Tanzania Institute of Education
132 Ali Hassan Mwinyi Road
Mikocheni Area
P.O. Box 35094
14112 Dar es Salaam

Mobile numbers: +255 735 041 170 / +255 735 041 168
Email: director.general@tie.go.tz
Website: www.tie.go.tz

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Editors:	Prof Ralph P. Masenge, Dr Makungu S. Mwanzalima, Dr Sylvester E. Rugeihyamu Dr Augustino I. Msigwa, Dr Andongwisye J. Mwakisisile, Dr James P. Mpele, Dr Thadei D. Sagamiko, and Dr Grayson K. Kakiko
Designers:	Mr Amani J. Kweka, Mr Harry A. Robert and Ms Eva G. Kwai
Illustrator:	Mr Godlove S. Kyando
Coordinator:	Mr Luckford D. Hamsini

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Dr Aneth A. Komba
Director General
Tanzania Institute of Education

Preface

This textbook, *Mathematics for Secondary Schools Form two*, is written specifically for Form Two students in the United Republic of Tanzania. The book is prepared in accordance with the 2023 Mathematics Syllabus for Ordinary Secondary Education, Form I–IV, issued by the Ministry of Education Science and Technology. It is a revised edition of Basic Mathematics for Secondary Schools Student's Book Form Two that was published in 2021 in accordance with the 2005 syllabus issued by the then, Ministry of Education and Vocational Training (MoEVT).

The textbook consists of eight chapters, namely: Rates and variations, Congruence, Similarity, Algebra, Exponents and radicals, Logarithms, Sets, and Trigonometry. In addition to the contents, each chapter contains activities, illustrations, projects, and exercises. You are encouraged to do all the activities, projects, and questions in the exercises. This will enhance your understanding and development of the intended competencies for this level.

Additional learning resources are available at TIE e-library https://ol.tie.go.tz

Tanzania Institute of Education

Chapter One

Rates and variations

Introduction

Understanding rates and variations is essential for grasping how quantities change in relation to each other. Some variables are interdependent, meaning a change in one affects the others. Many real-life situations involve such relationships. In this chapter, you will describe the concepts of rates and variations, explain the types of variations, and solve problems on rates and variations. The competence developed will enable you to solve real-life challenges such as finding shopping deals, assessing fuel efficiency, or predicting data trends in fields like economics, science, engineering, and many other applicattions.

Think

Comparing quantities and establishing relationships without the concept of rates an variations.

Rates

Rates show how one quantity is related to another quantity either increasing or decreasing its quantity. For example, speed of moving object is the ratio between the distance covered and time taken.

Engage in Activity 1.1 to explore the concept of rates in real life.

Example 1.1

If a car travels 150 kilometres in 3 hours. What is its speed?

Solution

Given distance $= 150\,\text{km}$ and time 3 hrs.

But, speed is the rate of change of distance with respect to time. That is,

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Thus, speed $= \frac{150 \text{ km}}{3 \text{hrs}}$

$= 50$ km per hour

Therefore, the rate of travel of the car is 50 km in every hour.

Example 1.2

If a water tank fills up with 200 litres of water in 5 minutes, what is the rate of flow of water?

Solution

Given the volume of water is 200 L and the time the tank takes to be filled is 5minutes.

$$\text{Rate of flow} = \frac{\text{Volume}}{\text{Time}}$$

$$= \frac{200 \text{ L}}{5 \text{min}}$$

$$= 40 \text{L/min}$$

Therefore, water enters the tank at the rate of 40 litres per minute.

Example 1.3

A student had two plant seedlings. She measured the rate at which the seedlings were growing. Seedling A grew 5 cm in 10 days and seedling B grew 8 cm in 12 days. Which seedling was growing more quickly?

Solution

The rates of growth of the two seedlings is computed as follows:

Rate of growth of seedling A

$$= \frac{5 \text{ cm}}{10 \text{ days}} = 0.5 \text{ cm per day}$$

Rate of growth of seedling B

$$= \frac{8 \text{ cm}}{12 \text{ days}} = 0.67 \text{ cm per day}$$

The growth rate of seedling B is higher than that of seedling A.

Therefore, seedling B was growing more quickly than seedling A.

Example 1.4

Two pipes, A and B are used to fill a water tank. Pipe A can fill the tank in 6 hours, while pipe B can fill the same tank in 4 hours. If both pipes are opened at the same time, how long will it take to fill the tank?

Solution

Pipe A fills the tank in 6 hours, so it fills $\frac{1}{6}$ of the tank in 1 hour.

Pipe B fills the tank in 4 hours, so it fills $\frac{1}{4}$ of the tank in 1 hour.

Combined rate = Rate of A + Rate of B

$$\text{Combined rate} = \frac{1}{6} + \frac{1}{4}$$

$$\text{Combined rate} = \frac{2+3}{12} = \frac{5}{12}$$

Therefore, both pipes together fill $\frac{5}{12}$ of the tank in 1 hour.

If $\frac{5}{12}$ of the tank is filled in 1 hour, then

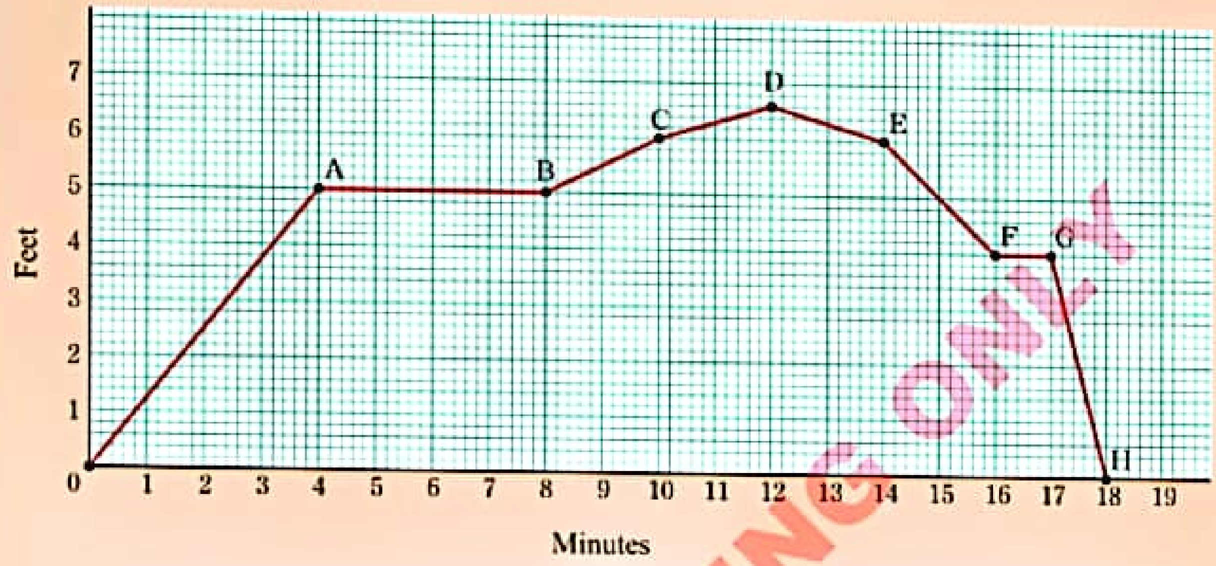
the time taken to whole tank $\frac{12}{12}$ will be;

$$\text{Time} = \frac{\text{Whole tank}}{\text{Combined rate}}$$

Exercise 1.1

1. What rate in metres per second is equivalent to a speed of 45 kilometres per hour?
2. A car covers a distance of 200 kilometres in 60 minutes. What is the speed of the car in metres per second?
3. A water tap takes 10 minutes to fill a 500 litres tank. Find the rate of flow of water in litres per second.
4. If premium petrol costs Tsh 2,500 per litre, how many litres can be purchased for Tshs 10,000?
5. Find the rate in kilometres per hour if a racing car travels $115\frac{1}{2}$ kilometres in 35 minutes.
6. Mwajuma walks 6 kilometres in 1 hour and John walks the same distance in 2 hours. What are their walking speeds? Who walks faster, and by how much?
7. A garden hose fills a 20 litre bucket in 4 minutes. What is the rate of flow in litres per minute?
8. A car travels 400 km using 25 litres of fuel. What is the car's fuel efficiency in kilometres per litre?
9. Mr. Magoda's salary increased from Tsh 800,000 to Tsh 1,000,000 in a year. What is the rate of change of his salary?
10. The temperature in Mrs. Kidunula's room increased from 15°C at 8 am to 25°C at 2 pm. What was the average rate of temperature change per hour?
11. A car accelerates from 0 to 60 km/h in 10 seconds. What is the rate of change of its speed in km/h per second?
12. Mr. Kilenzi's heartbeat increased from 60 beats per minute to 120 beats per minute during exercise over 3 minutes. What was the average rate of change in his heartbeats?
13. A family uses 80 GB of internet data in 30 days. What is the average daily rate of data usage in GB per day?
14. Pipe X can fill a tank in 8 hours, while Pipe Y can fill it in 5 hours. How long will it take to fill the tank if both pipes are opened simultaneously?
15. Pump A can fill a pool in 5 hours, Pump B in 4 hours, and Pump C in 10 hours. How long will it take to fill the pool if all three pumps are operated at the same time?

16. Study the following figure which shows different altitudes (in thousands feet) attained by a plane and then answer the questions that follow.



(a) During the first 5 minutes of flight, the plane took off to the altitude shown by point A. Estimate the rate of change in altitude.

(b) Between points E and F, the plane is descending. Estimate the rate of change of altitude during this time.

(c) Is the rate of change in (b) positive or negative? Give reasons for your answer.

(d) Find the rate of change in altitude between points C and D. Is this rate greater or less than the rate between points D and E? How does the graph visually show that the two rates are different?

(e) Can the rate of change between any two points be 0? Briefly explain and give an example from the figure.

17. Ana and Lina can weed a garden in 2 hours. Working alone, Ana can do the same work in 3 hours. How long would Lina weed the garden alone?

Exchange rates

In any country, peopleexpect to do transactions in in the currency of of their own country. When money from country A is to be used in country , it is necessary to exchange the currency of country A to the currency of country B. Various currencies in the world are linked together by exchange rates. This enable smooth transfer of money and payments to take place between countries.

Engage in Activity 1.2 to explore more about the concept of currency exchange in real-life situations.

Activity 1.2: Performing currency exchange rates

1. Choose a suitable computer Currency Conversion Application.
2. Analyse current exchange rates between the Tanzanian currency and other currencies of your choice using the computer application.
3. Compare the app's rates with market rates and evaluate its accuracy for financial decisions, especially for travel.

Exercise 1.2

Use any computer Currency conversion application to answer questions 1 - 10.

1. How much is Tsh 20,600 worth in Indian Rupees?
2. Convert 1 New Zealand Dollar into Saudi Arabia Riyal.
3. How much is Tsh 500,000 worth in Euros?
4. How many Yen are equivalent to Tsh 1?
5. How many Tanzanian shillings can a visitor from Kenya get for exchanging 930 Kenyan shillings?
6. Find the amount in Tanzanian shillings of each of the following:
   (a) 30,000 Euros
   (b) 4,200 Pula
   (c) 640 Rands
   (d) 12,000 Riyal
7. Exchange Tsh 300,000 into the following currencies:
   (a) Mozambican meticals
   (b) Malawian kwachas
   (c) Swiss francs
   (d) Indian rupees
8. How much is Tsh 6,000,000 worth in Pounds Sterling?
9. Mwanaisha bought story books for 200 AUD (Australian Dollars). How much did she spend in Tanzanian shillings?
10. Mr. Utaligolo wants to exchange 1,000 USD for EUR. If the exchange fee is 2%, how much will he receive after charging the fee?
11. Mrs. Uwemba is shopping in a country where 1 USD = 205 Local Currency (LC). If she spent 205,000 LC, how much did she spend in USD?
12. Ayota Stationery wants to buy items from an online store, which cost 2500 NOK (Norwegian Kroner). If the exchange rate is 1 NOK = 9 Local Currency, how much does the stationery pay in Local Currency?

Variations

Variation is a relationship where a change in one quantity leads to a proportional change in the other. It allows for the exploration of connections between two or more quantities. The four basic types of variations are direct, inverse, joint, and combined.

Direct variations

Direct variation is a relationship between two variables where one variable is a constant multiple of the other. In other words, if one variable increases or decreases, the other variable changes proportionally in the same direction.

This type of variation is useful for understanding how changes in one variable affect another in a directly proportional manner.

Engage in Activity 1.3 to explore the application of direct variation in real life.

> **Activity 1.3: Exploring direct variation in real life**
>
> 1. Learn about direct variation from books or from the internet.
> 2. Find real-life examples that illustrate direct variations.
> 3. Demonstrate mathematically how variables in the real-life scenarios relate and use the relationship to solve related problems.

If y is directly proportional to x, it can be written as $y \propto x$, where $\propto$ is a symbol of proportionality. The corresponding mathematical equation connecting x and y is formed by introducing a proportionality constant k, and replaces $\propto$ with an equal sign to get, $y = kx$.

For instance, if y varies directly as the square of x, then $y \propto x^2$ and the corresponding equation is $y = kx^2$, where k is the constant of proportionality.

For any two pairs of quantities x and y, say (x_1, y_1) and (x_2, y_2), two equations $y_1 = kx_1^2$ and $y_2 = kx_2^2$ are obtained. This implies that $k = \frac{y_1}{x_1^2} = \frac{y_2}{x_2^2}$. So, it is said that x and y vary directly if the ratios of the values of y to the values of x are proportional.

If x and y are any two quantities that are in direct variation, then $y = kx$. The nature of the equation $y = kx$ is a straight line passing through the origin, where k represents the gradient (slope) of the line. Figure 1.1 shows the relation $y \propto x$ for $k = 1$.

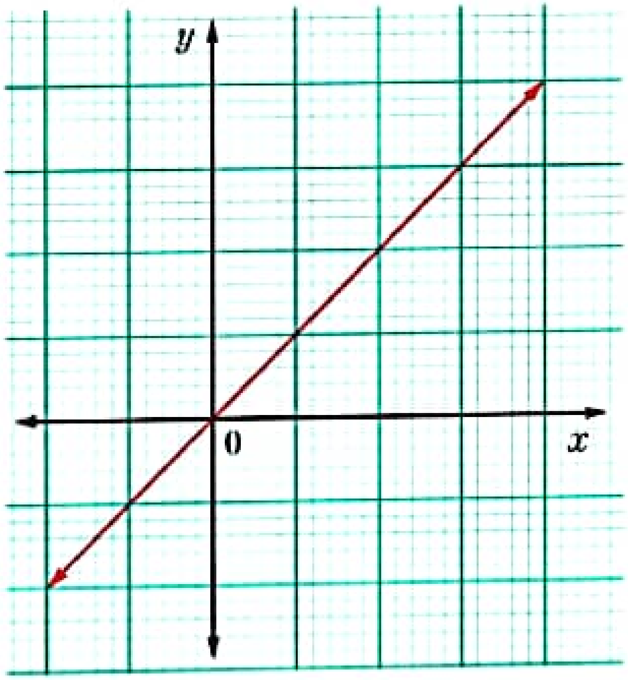


Figure 1.1: *Graph of the relation $y \propto x$*

From Figure 1.1, it can be observed that an increase (or decrease) in the quantity x results in a proportional increase (or decrease) in the quantity y, and vice versa.

Example 1.5

If y varies directly as x and $x = 15$ when $y = 4$, find the value of y when $x = 12$.

Solution

Given that $y \propto x$. This implies that $y = kx$, where k is the constant of proportionality.

Making k the subject of the equation gives

$$k = \frac{y}{x}$$

But, for any two pairs of quantities, $k = \frac{y_1}{x_1}$ and $k = \frac{y_2}{x_2}$.

Thus, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Given $x_1 = 15$, $x_2 = 12$, and $y_1 = 4$, the value of y_2 is obtained as follows:

$$y_2 = \frac{y_1 x_2}{x_1}$$

Substituting the given values gives

$$y_2 = \frac{4 \times 12}{15} = \frac{16}{5}$$

Therefore, the value of y is $\frac{16}{5}$ when $x = 12$.

Example 1.6

If x varies directly as the square of y and $x = 4$ when $y = 2$, find the value of x when $y = 8$.

Solution

Let $x_1 = 4$, $y_1 = 2$ and $y_2 = 8$. The variation equation is given by;

$$\frac{x_1}{x_2} = \frac{y_1^{\,2}}{y_2^{\,2}}$$

Substituting the values of x_1, y_1, and y_2 gives

$$\frac{4}{x_2} = \frac{2^2}{8^2}$$

$$\frac{4}{x_2} = \frac{4}{64}$$

Cross multiplication gives

$$4 \times x_2 = 4 \times 64$$

$$x_2 = \frac{4 \times 64}{4}$$

$$= 64$$

Therefore, the value of x is 64 when y is 8.

Example 1.7

A car travels 60 kilometres using 5 litres of diesel. How many litres of diesel are needed to travel 150 kilometres?

Solution

Let x denote the number of litres of diesel and y denote the number of kilometres. The equation for the variation becomes

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

Given $x_1 = 5$ litres, $y_1 = 60$ km, and $y_2 = 150$ km, the value of x_2 is given by:

$$x_2 = \frac{x_1 y_2}{y_1}$$

$$= \frac{5 \text{ litres} \times 150 \text{ km}}{60 \text{ km}}$$

$$= 12.5 \text{ litres}$$

Therefore, 12.5 litres of diesel are needed to travel 150 kilometres.

Exercise 1.3

1. If x varies directly as y and $x = 16$ when $y = 10$, find the value of y when $x = 20$.
2. The surface area of a circular object varies directly as the square of its radius. If its surface area is 78.5 cm^2, when the radius is 5 cm, find the surface area of the circular object when the radius is 7 cm.

3. If x varies directly as y and $x = 30$ when $y = 40$, find the value of x when $y = 16$.

4. A mason can build 100 metres of fence in 20 hours. How long will it take 5 masons with the same ability to build 875 metres of fence?

5. If x varies directly as $2y + 7$ and $x = 5$, when $y = 4$, find the value of y when $x = 6$.

6. If 8 men can assemble 16 machines in 12 days, how long will it take 15 men of the same ability to assemble 100 machines?

7. If y varies directly as the square root of x, and $y = 12$ when $x = 4$, find the value of y when $x = 9$.

8. If y varies directly as x and $y = 8$ when $x = 3$, find the value of y when $x = 18$.

9. Two variables x and y that vary directly have corresponding values as shown in the following table.

x	3	5	6	8	
y	17		34		58

(a) Find the rule connecting x and y.

(b) Fill in the missing values.

(c) Draw a graph which shows that $y \propto x$ for $k = 1$.

10. Study the following table and answer the questions that follow.

Hours worked	2	3	4	5
Earning (Tsh)	1,150	1,725	2,300	2,875

(a) Do earnings vary directly as the number of hours worked?

(b) If yes, calculate the constant of proportionality and find the equation that describes the relationship.

11. The volume of a sphere varies as the cube of its radius. Three solid spheres of diameters $\frac{3}{2}$ m, 2 m, and $\frac{5}{2}$ m are melted and combined to form a new solid sphere. Find the diameter of the new sphere.

12. When observing two buildings simultaneously, the length of each building's shadow varies directly with its height. If a 5 floor building has a shadow of length 20 m, how many floors of a building would form a shadow of length 32 m?

13. The resistance of a wire varies as the square of the diameter of its cross-section. Find the percentage change in the resistance when the diameter is (a) doubled (b) reduced by 20%.

14. Two variables A and x are related by the formula $A = ax^n$. The following set of data was generated based on this formula.

x	1	2	3	4
A	0.5	2	4.5	8

(a) Find the values of a and n.

(b) Find the value of A when $x = 5$.

15. A precious stone worth Tshs 15,600,000 is accidentally dropped and broken into three pieces. The weights of the pieces are in the proportions of 2:3:5, respectively. If the value of the precious stone varies directly as the cube of its weight, calculate the value of the remaining stone in percentage.

Inverse variations

A relationship between two or more variables is said to be an inverse variation if the value of one variable increases while the other value decreases, or vice versa. Engage in Activity 1.4 to explore further inverse variations.

Activity 1.4: Identifying inverse variations in daily life

1. Identify three real-life activities where increasing one variable decreases the other variable.
2. Perform one of the activities to experience how changes of the variables are related in the activity.
3. Find out through reading books and browsing the internet how to express these inverse relationships mathematically.
4. Use the mathematical expression to explain how changes in one variable affect the other, and present your findings.

Quantities with an inverse variation relationship are said to be inversely proportional to each other. In this case, the quantities vary inversely or in inverse proportion. Inverse proportion is sometimes referred to as indirect proportion.

For example, the number of men employed to cultivate a farm and time taken to complete the work are inversely related. Likewise, the time to travel to a certain place and the speed are inversely related.

Generally, if y has an inverse relationship with x, then y is proportional to the reciprocal of x. This relationship is denoted by;

$$y \propto \frac{1}{x}$$

The equation relating y and x is formed by introducing a constant of proportionality, k and replacing the symbol of proportionality with an equal sign (=). That is, the inverse variation between y and x becomes;

$$y = k\frac{1}{x}$$

If y varies inversely as the square of x, the equation connecting x and y is $y = \frac{k}{x^2}$ or $k = yx^2$.

A sketch of a relation $y \propto \frac{1}{x^2}$ for $k = 1$ is shown in Figure 1.3. Note that, the curve representing $y = \frac{k}{x^2}$ approaches both axes but does not touch them. This is because at $x = 0$ the curve is undefined.

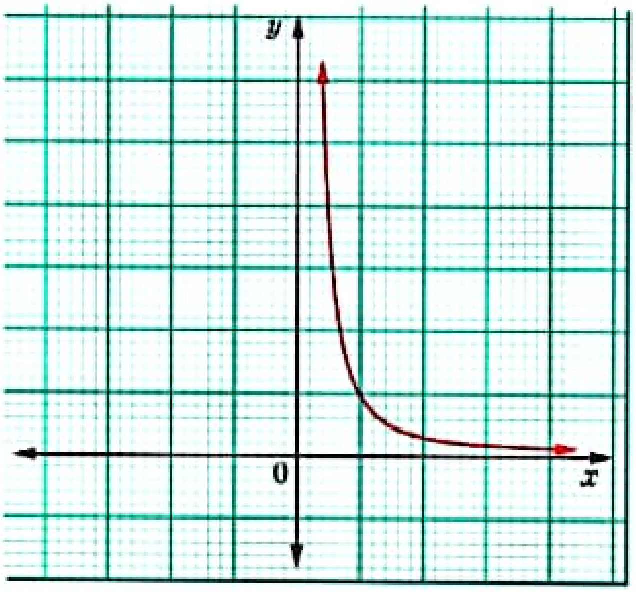


Figure 1.2: *Graph of the relation $y \propto \frac{1}{x}$ for $k = 1$ and $x > 0$*

From Figure 1.2, it can be observed that as x increases, it results in a proportional decrease in y or vice versa.

The equation $y = \frac{k}{x}$ can be re-arranged to get $xy = k$. If x_1 and y_1 are the first set of inversely related variables, then the second set of values x_2 and y_2 can be computed using the relationship,

$$x_1y_1 = x_2y_2$$

Thus, $x_1y_1 = x_2y_2$ or $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

Example 1.8

If x varies inversely as y and $x = 2$ when $y = 3$, find the value of y when $x = 18$.

Solution

The statement $x \propto \frac{1}{y}$ implies that $x = \frac{k}{y}$, where k is the constant of proportionality.

Making k the subject of the equation gives,

$xy = k$, which implies that $x_1y_1 = x_2y_2$.

When $x_1 = 2$, $y_1 = 3$, and $x_2 = 18$, the value of y_2 is

$$y_2 = \frac{x_1y_1}{x_2} = \frac{2 \times 3}{18} = \frac{1}{3}$$

Therefore, the value of y is $\frac{1}{3}$ when the value of x is 18.

Example 1.9

If it takes 12 days for 10 men to assemble a machine, how long does it take 15 men with the same ability to assemble the same machine?

Solution

Let m be the number of men and d be the number of days. It is obvious that 15 men will take less time to assemble the machine than 10 men.

Thus, m and d vary inversely, that is, $m \propto \frac{1}{d}$, which implies that $m = \frac{k}{d}$.

Thus, $k = md$.

For the two pairs of quantities (m_1, d_1) and (m_2, d_2), it follows that,

$m_1d_1 = m_2d_2$.

Thus, $d_2 = \frac{m_1d_1}{m_2}$.

Given $d_1 = 12$ days, $m_1 = 10$ men and $m_2 = 15$ men, the value of d_2 is:

$$d_2 = \frac{10 \text{ men} \times 12 \text{ days}}{15 \text{ men}}$$
$$= 8 \text{ days}$$

Therefore, it takes 8 days for 15 men to assemble the same machine.

Example 1.10

The intensity of light is inversely proportional to the square of the distance D from the light source. Calculate the percentage change in intensity under the following conditions:

(a) The distance is halved.

(b) The distance is increased by 30%.

Solution

$$I \propto \frac{1}{D^2}$$

$$I = \frac{k}{D^2} \qquad \text{(i)}$$

(a) If the distance D is halved, the new distance is $D_1 = \frac{D}{2}$.

The new intensity $I_1 = \frac{k}{\left(\frac{D}{2}\right)^2}$

$$= \frac{4k}{D^2} \qquad \text{(ii)}$$

10. The time t in seconds that Mary takes to return home from school varies inversely with her average speed v in metres per second. If Mary gets back home in half an hour at an average speed of 10 m/s;
    (a) write and sketch the graph of the equation.
    (b) if Mary wants to get back home in 15 minutes, what must be her average speed?
    (c) why the graph in (a) does not cross either axes?
    (d) what happens to t as v increases? As v decreases? How does t and v vary?
11. The intensity of light varies inversely as the square of the distance from the light source. If the intensity from a light source 90 cm away is 12 lumen, how far should the light source be so that the intensity is 4 lumen?

Joint and combined variations

In some activities, one variable can depend on several other variables to operate effectively. Such relationships are described by joint and combined variations. Engage in Activity 4.5 to explore more about joint and combined variations in real-life activities.

Activity 1.5: Discovering joint and combined variations in daily life

1. Explore the concepts of joint and combined variations using books and online resources.
2. Find and describe real-life examples of such variations using daily practices such as formulas.
3. Share your final observations and discuss the examples you discovered.

Joint variation

Consider a formula for finding the area of a triangle. It is given by $\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$. In this equation, the area of a triangle varies directly as the product of its base and height. Variable relationships of this nature are known as joint variations. This specific variation can be generally expressed as

$$\text{Area} \propto \text{base} \times \text{height}.$$

A joint variation occurs when a variable is directly or inversely proportional to the product of two or more variables. Mathematically, if a variable z varies jointly as x and y, the general formula for joint variation can be written as:

$$z \propto xy \Rightarrow z = kxy$$

Example 1.11

Suppose y varies directly as x and z. Given $x = 4$, $z = 2$, and $y = 24$, find:

(a) The variation equation connecting x, y, and z.

(b) The value of y when $x = 5$ and $z = 6$.

Solution

(a) Since $y \propto x$ and $y \propto z$, it follows that

$y \propto xz$.

$y = kxz$.

The variation equation is $k = \frac{y}{xz}$, where k is a constant of proportionality.

Given $x = 4$, $z = 2$, and $y = 24$ it follows that

$$k = \frac{24}{4 \times 2}$$
$$= 3$$

Therefore, the variation equation is

$$y = 3xz.$$

(b) With $y = 3xz$, if $x = 5$ and $z = 6$, then

$$y = 3 \times 5 \times 6$$
$$= 90$$

Therefore, the value of y is 90.

Example 1.12

A bakery has a project to bake 240 cakes. With 4 bakers working for 5 days, they can complete the task. If the number of cakes is increased to 300 and the bakery decides to use 6 bakers, how many days will it take to bake all 300 cakes?

Solution

Let c be the number of cakes, b be the number of bakers, and d be the number of days.

More cakes can be produced if there are more bakers and more working days, assuming other factors remain constant. It implies that these variables vary jointly and are directly related.

Thus, $c \propto bd$.

$$c = kbd.$$

Given $c = 240$, $b = 4$, and $d = 5$, it implies that

$$240 = 4 \times 5 \times k$$
$$20k = 240$$
$$k = 12$$

From $c = kbd$.

$$c = 12bd$$

Given, $c = 300$, $b = 6$,

$$300 = 12 \times 6 \times d.$$
$$72d = 300$$
$$d = 4.2$$

Therefore, 4.2 days will be needed to bake 300 cakes or it requires 4 days, 4 hours, and 48 minutes to bake 300 cakes.

Example 1.13

Nine employees work 8 hours a day to complete a piece of work in 52 days. How long will it take 13 employees to complete the same work by working 6 hours a day?

Solution

Let w, h, and d represent the number of employees, hours, and days, respectively.

Thus, it follows that the number of employees varies inversely as the number of hours and days, that is,

$$w \propto \frac{1}{h} \text{ and } w \propto \frac{1}{d}.$$

The joint variation is given by the relation

$$w \propto \frac{1}{hd}.$$

The variation equation is

$w = \frac{k}{hd}$, where k is a constant of proportionality.

Thus,

$$w_1 = \frac{k}{h_1 d_1}$$
$$w_2 = \frac{k}{h_2 d_2}$$

Hence, $\frac{w_1}{w_2} = \frac{h_2}{h_1} \times \frac{d_2}{d_1}$.

Using $t = 5$ and $c = 20$, then

$$5 = \frac{20}{d}$$

$$5d = 20$$

$$d = 4$$

Therefore, it will take 4 days for 5 tailors to sew 20 clothes.

Example 1.16

The cost of a certain material varies directly with the quantity purchased and inversely with the number of suppliers. If the cost is Tshs 120,000 when the quantity is 100 units and there are 4 suppliers, find the constant of variation and the cost when the quantity becomes 150 units and there are only 3 suppliers.

Solution

Let C be the cost of materials q be the quantity purchased, and S be the number of suppliers.

Thus, $C \propto q$ and $C \propto \frac{1}{S}$. The combined variation is:

$$C \propto \frac{q}{S} \Rightarrow C = k\frac{q}{S}$$

Given $C =$ Tshs 120,000, $q = 100$, $S = 4$, then

$$k = \frac{CS}{q} = \frac{120000 \times 4}{100} = 4,800$$

Now, if $S = 3$ and $q = 150$, it follows that

$$C = \frac{4800 \times 150}{3} = 240,000$$

Therefore, the cost of materials will be Tshs 240,000.

Example 1.17

The monthly rent for an apartment varies directly with its area and inversely with the number of roommates. If Tsh 400,000 is the rent for an apartment of 900 square units with 3 roommates, find the rent for an apartment of 1,200 square units with 4 roommates.

Solution

Given $R \propto a$ and $R \propto \frac{1}{n}$, then

$$R \propto \frac{a}{n} \Rightarrow R = k\frac{a}{n}$$

$$k = \frac{R \times n}{a}$$

When $R =$ Tshs 400,000 $n = 3$, and $a = 900$, it follows that

$$k = \frac{400000 \times 3}{900}$$

$$= 1333.33$$

Now, for $a = 1200$ and $n = 4$,

$$R = \frac{4000 \times 1200}{3 \times 4} = 400,000$$

Therefore, the rent for the apartment will be Tshs 400,000.

Exercise 1.5

1. If y varies directly as the square of x and inversely as z, find the percentage change in y when x is increased by 10% and z is decreased by 20%.

2. Suppose P varies directly as V and inversely as the square root of R. If $P = 180$ when $R = 25$ and $V = 9$, find the value of P when $V = 6$ and $R = 36$.

3. The height h of a cone varies directly as its volume V and inversely as the square of its radius r. Write a formula for the height of the cone.
4. If y^2 varies directly as $x-1$ and inversely as $x+d$ and $x = 2$, $d = 4$ when $y = 1$, find the value of x when $y = 2$ and $d = 1$.
5. If two typists in a typing pool can type 210 pages in 3 days, how many typists working at the same speed will be needed to type 700 pages in 2 days?
6. Suppose x varies directly as y^2 and inversely as p. If $x = 2$, when $y = 3$ and $p = 1$, find the value of y when $x = 4$ and $p = 5$.
7. If V varies directly as the square of x and inversely as y, and if $V = 18$ when $x = 3$ and $y = 4$, find the value of V when $x = 5$ and $y = 2$.
8. Use a mathematical software to draw the following curves. Assume that the constant of proportionality is 1.
   (a) $y \propto \frac{1}{x}$
   (b) $y \propto \frac{1}{x^2}$
   (c) $y \propto \frac{1}{\sqrt{x}}$
9. The following table shows the values of y for some selected values of x. The variables x and y are connected by the relation, 'y varies inversely as x'. Calculate the missing values of y.

x	5	10	15	20
y	a	3	b	1.5

10. Express each of the following relations as an equation using k as a constant of proportionality.
    (a) c varies directly as p and q, and inversely as s.
    (b) d varies jointly as t and r^2.
    (c) d varies directly as y and the square root of z.
11. The heating cost H for a house varies directly with its size S in square metres and inversely with the efficiency rating E of the heating system. If H = 50,000 shillings for a house of 200 square metres with an efficiency rating of 4, find k and the heating cost for a house of 250 square metres with an efficiency rating of 5.
12. The shipping cost C varies directly with the mass W of the package and inversely with the number of packages n. If the cost is TShs 7,000,000 for a package weighing 20 kg, and 6 packages, find k and the cost of a package weighing 30 kg with 5 packages.

Chapter summary

1. A rate gives the change of one quantity with respect to another quantity.
2. Exchange rate is the conversion rate between different currencies.
3. Variation is the relationship in which the change in one quantity results in a proportional change in the other.
4. If $y = kx$, then y varies directly with x, or y is directly proportional to x. The constant k is called a constant of proportionality.
5. If y varies as $\frac{1}{x}$, then y is inversely proportional to x.
6. If a quantity varies as the product of two or more quantities, then it varies jointly with other quantities.
7. If both direct variation and inverse variation occur at the same time, then it is called a combined variation.

Revision exercise 1

1. If $y = kx$ and $y = 8$ when $x = 7$, find the value of k and the value of y when $x = 40$.

2. If y is directly proportional to x and $y = 10$ when $x = 4$, find the value of y when $x = 15$ and the value of x when $y = 8.4$.

3. If $y \propto x$ and $y = 16.5$ when $x = 3.5$, find the equation connecting x and y. Hence, find the value of x when $y = 21$.

4. If y is proportional to x^2 and if $x = 15$ when $y = 200$, find the equation connecting x and y. Find the value of y when $x = 8.5$.

5. If $y \propto \sqrt{x}$ and $y = 3.5$ when $x = 4$, express y in terms of x. What is the value of y when $x = 25$?

6. If $y \propto \frac{1}{x}$, find the values of a, b, and c in the following table.

x	a	1.2	8	c
y	6	b	1.5	0.8

7. Given that y varies directly as x and inversely as z. If $y = 10$ when $x = 8$ and $z = 5$, find the equation connecting x, y, and z. Find the value of y when $x = 6$ and $z = 2.5$.

8. If y varies jointly as x and z^2, and if $y = 13\frac{1}{3}$ when $x = 2.5$ and $z = \frac{4}{3}$, find the equation connecting the three variables. Find the value of x when $z = \frac{3}{2}$ and $y = 54$.

9. Suppose y varies directly as x^2 and inversely as $\sqrt{z}$. If $x = 8$, $y = 16$, and $z = 25$, find the value of y when $x = 5$ and $z = 9$.

10. Determine whether the data in the following tables have an inverse variation relationship. If yes, find the missing values.

(a)

x	y
7	10
9	12
12	15
	6

(b)

x	y
12	4
6	2
21	7
	3

(c)

x	y
−15	−8
−8	−15
	10

11. A dairy farm dispenses milk into a container at a rate of 45 litres per minute.
    (a) How much milk is dispensed in 2 hours?
    (b) How long will it take to empty a 2000 litre tank?

12. A water pump operates at a flow rate of 300 litres per minute.
    (a) Calculate the total volume of water pumped in 2 hours.
    (b) Calculate the time required to fill a 30,000-litre tank.

13. A truck travels 800 km and uses 90 litres of diesel.
    (a) Calculate the fuel consumption rate in litres per 100 km.
    (b) How much diesel will the truck use to travel 600 km at the same rate?
    (c) If the truck travels 1,300 km, how much diesel will it consume?
14. State whether distance and speed vary directly or inversely. Give reason.
15. If x and y vary inversely, use the given pair of values to find an equation which in each case relate the variables:
    (a) $x = 6, y = 4$
    (b) $x = 8, y = 12$
    (c) $x = 10, y = \frac{1}{3}$
16. A group of 10 volunteers can pack 400 boxes of food in 5 hours. If the organization decides to pack 600 boxes and increases the number of volunteers to 12, how many hours will it take to pack all 600 boxes?
17. A publishing company needs to print 2,000 copies of a book. With 4 printers, the task is completed in 10 days. If the company decides to print 3,000 copies and uses 6 printers, how many days will it take to complete the printing?
18. A logistics team of 8 members can package and ship 400 orders in 5 days. If the number of team members is increased to 12 and the number of orders to be shipped is raised to 600, calculate the number of days needed to complete the work.
19. The intensity I of light, in lux, on a surface varies directly with the power P in watts of light and inversely with the square of the distance d from the light source. A photographer is setting up lighting for a photoshoot. If the light intensity is 400 lux when a light source of 66 watts is placed 2 m away, what will be the intensity when a light source of 100 watts is placed 3 m away?
20. Machine A can produce 800 parts per hour, while Machine B can produce 1,000 parts per hour. There is an order of 24,000 parts.
    (a) How long will the machines take working together to process the order?
    (b) Machine B was on preventive maintenance, so Machine A had to work alone for the first 5 hours before Machine B joins in. How many more hours will they have to work together to finish the job?
21. Pipe A can fill a pond in 8 hours, while Pipe B can fill it in 3 hours. However, Pipe B has a defect that causes it to lose water at the rate of 1 litre every 20 minutes while filling. If the pond has a capacity of 200 litres and both pipes are opened simultaneously, how long will it take to fill the pond?

Chapter Two

Congruence

Introduction

In fields like architecture, engineering and design, precise measurements and patterns are crucial. Maintaining such patterns and measurements across objects and designs requires an understanding of the relationship between shapes and their properties. In this chapter, you will learn about postulates, proofs and theorems on congruence as well as the congruence of triangles. The competencies developed will enable you to solve problems such as determining the dimensions of sides of figures without taking actual measurements, designing and making objects of the same shapes and sizes such as worn-out parts.

Think

Without knowledge of congruence industries would take longer time to produce objects of the same shape and size.

The concept of congruence

The term congruence is derived from the Latin word *congruentia* which means agree or fit together exactly. Engage in Activity 2.1 to explore the characteristics of objects from real life situations which fit together.

Activity 2.1: Exploring congruence of objects

1. Create a shape of your choice in Ms-Word, copy and paste it, then drag it on top of the original and observe.
2. Enlarge the original shape, overlap it with the copied shape, and note any changes.
3. Use real objects (e.g., coins or books) to repeat task 1 (or 2 where necessary) and compare your observations.

In Activity 2.1, one may have noticed that same objects matched exactly on top of each other. This means that the objects have the same shape and size. Two or more objects with such characteristics are called congruent figures.

Postulates, theorems and proofs

Understanding postulates, theorems, and proofs is important for developing a solid foundation in mathematics. These concepts form the basis for logical reasoning and they are used to validate mathematical ideas.

Postulate

A postulate is a statement that is accepted as true without any proof. These statements are universally accepted, self-evident and can form the basis for further reasoning and making arguments. The following are examples of postulates.

(i) A circle can be drawn with any centre and radius.

(ii) A straight line can be drawn from any point to any other point.

(iii) All right angles are equal to each other.

Theorem

A theorem is an argument that has been proved to be true based on past established results, definitions or postulates. The following are examples of theorems.

(i) The sum of interior angles of a triangle is 180 degrees.

(ii) In a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

(iii) The sum of interior angles of a quadrilateral is 360 degrees.

Proof

A proof is a series of logical statements that are based on definitions, previously established facts, and postulates that may be used to conclude the truth of a mathematical argument. The following are common procedures when undertaking a proof.

1. Draw a clearly labelled diagram to represent a problem. Indicate all the information such as equal angles, parallel lines, and congruent segments.
2. Write down the given information based on the labels of the supporting figures.
3. State the argument which needs to be proved.
4. Where necessary, make additional constructions with dotted lines to make the proof clear.
5. In writing the proof;
   (i) Refer the figures you have planned to use in the proof.
   (ii) Provide arguments with reasons based on the given information or established facts.
   (iii) Start with statements whose validity are given or are obvious.
   (iv) The final statement is the conclusion about what was supposed to be proved.

Engage in Activity 2.2 to explore more about postulates, proofs, and theorems.

Activity 2.2: Exploring postulates, proofs, and theorems

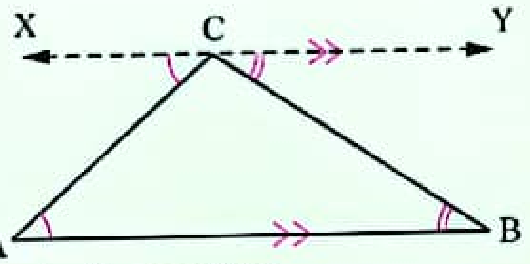
Identify a list of postulates, proofs, and theorems from reliable sources and briefly explain why each is accepted as a postulate, proof or theorem.

Example 2.1

Prove that the sum of interior angles of a triangle is 180°.

Solution

Consider the triangle ABC as shown in the following figure.



Required to prove that,

$\hat{CAB} + \hat{BCA} + \hat{ABC} = 180°$.

Proof

Construction: Draw line $\overline{XY}$ through C parallel to $\overline{AB}$. Thus,

$A\hat{C}X = C\hat{A}B$ (i)

(alternate interior angles as $\overline{XY} // \overline{AB}$)

$Y\hat{C}B = A\hat{B}C$ (ii)

(alternate interior angles as $\overline{XY} // \overline{AB}$)

But $A\hat{C}X + B\hat{C}A + Y\hat{C}B = 180°$ (iii)

(degree measure of a straight angle)

Substitute (i) and (ii) into (iii) to get

$C\hat{A}B + B\hat{C}A + A\hat{B}C = 180°$.

Therefore, the sum of interior angles of a triangle is $180°$.

Example 2.2

Prove that the sum of two interior angles of a triangle is equal to the exterior angle of the third interior angle.

Solution

Consider ΔABC with $\overline{AB}$ extended to D as shown in the following figure.

Required to prove that,

$C\hat{A}B + B\hat{C}A = C\hat{B}D$.

Proof

From the figure, it implies that

$C\hat{A}B + B\hat{C}A + A\hat{B}C = 180°$

(sum of interior angles of a triangle), and

$A\hat{B}C + C\hat{B}D = 180°$

(degree measure of a straight angle).

It follows that,

$C\hat{A}B + B\hat{C}A + A\hat{B}C = A\hat{B}C + C\hat{B}D$

But $A\hat{B}C$ is common to both sides of the equation.

Thus, $C\hat{A}B + B\hat{C}A = C\hat{B}D$.

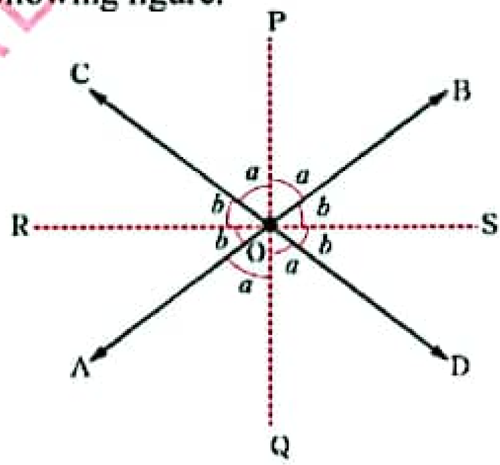
Therefore, the sum of two interior angles of a triangle is equal to the exterior angle of the third interior angle.

Example 2.3

Prove that the bisectors of the angles formed by two intersecting straight lines are at right angles to each other.

Solution

Consider the two straight lines $\overline{AB}$ and $\overline{CD}$ intersecting at O as shown in the following figure.



Required to prove that, $a + b = 90°$.

Construction: Draw the bisectors of the angles formed by intersecting lines.

Proof

From the figure, it implies that

$a + b + b + a = 180°$

(degree measure of a straight angle)

Thus, $2a + 2b = 180°$ or

$2(a + b) = 180°$

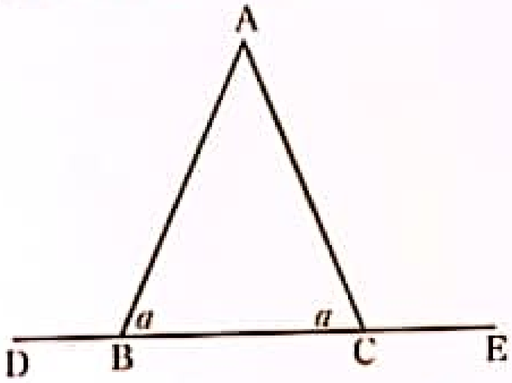
(distributive property)

$a + b = 90°$

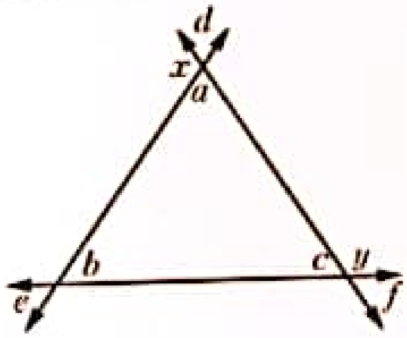
Therefore, the bisectors are at right angles to each other.

Exercise 2.1

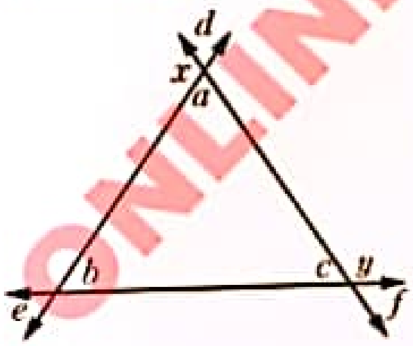
1. Use the following figure to prove that $A\hat{B}D = A\hat{C}E$.



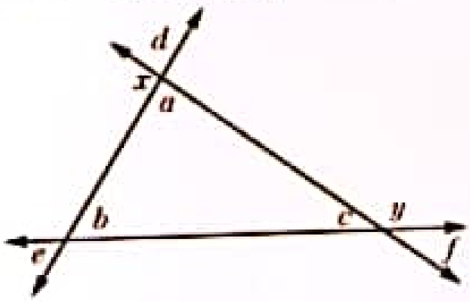
2. If $e = d$, use the following figure to prove that $a = b$.



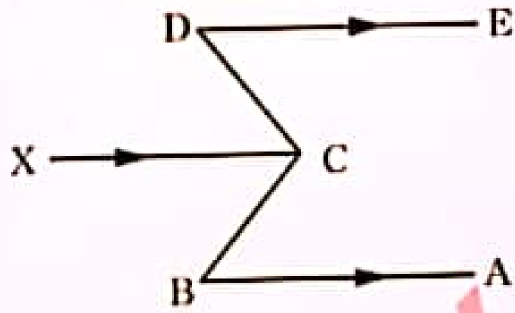
3. If $a = c$, use the following figure to prove that $x = y$.



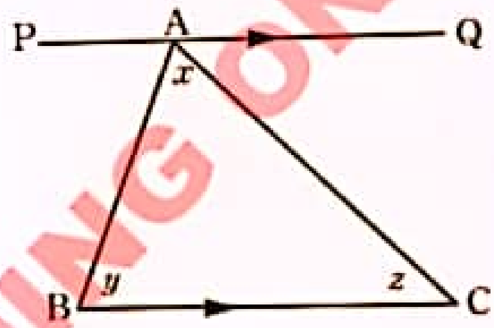
4. Use the following figure to prove that $a = c$, if $d + y = 180°$.



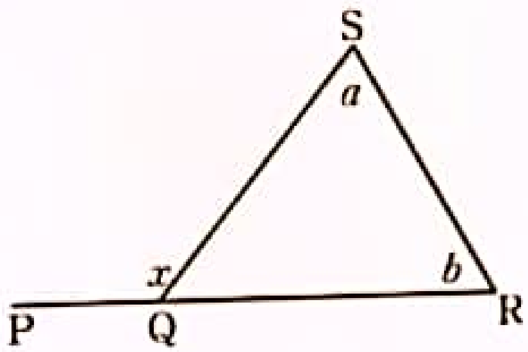
5. Use the following figure to prove that $B\hat{C}D = E\hat{D}C + C\hat{B}A$.



6. In the following figure, if $\overline{AC}$ bisects $B\hat{A}Q$, prove that $x = z$.

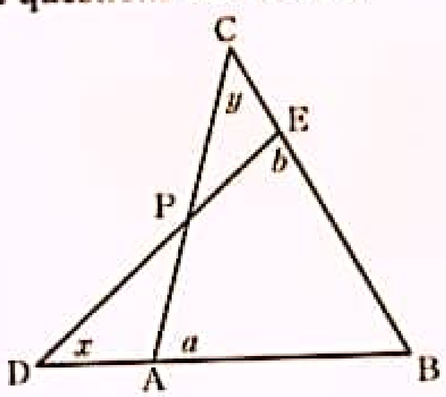


7. In the following figure, if $x = 2a$, prove that $a = b$



8. Prove that the sum of interior angles of a quadrilateral is equal to four right angles.

9. Use the following figure to answer the questions that follow.



(a) Prove that if $x = y$, then $a = b$.

(b) If $C\hat{A}B = B\hat{E}P$, prove that $B\hat{D}E = A\hat{C}B$.

Congruence of triangles

Two triangles ABC and PQR are said to be congruent if pairs of corresponding sides are equal and pairs of corresponding angles are equal. This fact is mathematically represented as $\Delta ABC \cong \Delta PQR$. The symbol $\cong$ means "congruent to".

Consider the triangle in Figure 2.1.

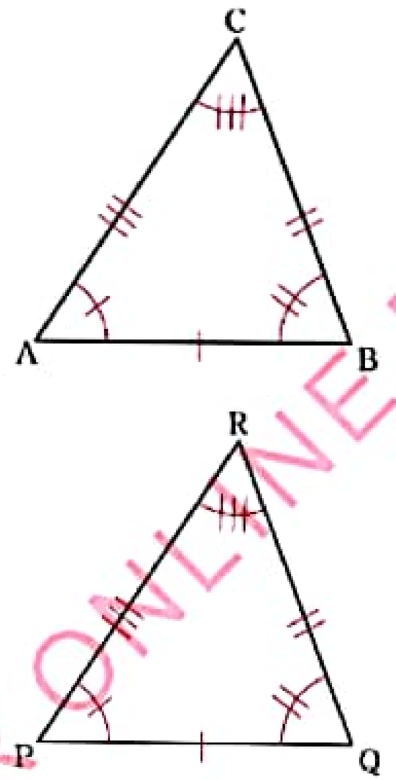


Figure 2.1: *Conguerence of triangles*

From Figure 2.1, if $\Delta ABC \cong \Delta PQR$, the pairs of corresponding sides are $\overline{AB}$ and $\overline{PQ}$, $\overline{BC}$ and $\overline{QR}$, and $\overline{AC}$ and $\overline{PR}$. Therefore, $\overline{AB} = \overline{PQ}$, $\overline{BC} = \overline{QR}$, and $\overline{AC} = \overline{PR}$.

Pairs of corresponding angles are $A\hat{B}C$ and $P\hat{Q}R$, $B\hat{C}A$ and $Q\hat{R}P$, and $B\hat{A}C$ and $Q\hat{P}R$. Therefore, $A\hat{B}C = P\hat{Q}R$, $B\hat{C}A = Q\hat{R}P$, and $B\hat{A}C = Q\hat{P}R$.

Postulates for congruence of triangles

Three conditions for congruence are sufficient to prove that two triangles are congruent. The postulates that are commonly used include the Side-Side-Side (SSS), Side-Angle-Side (SAS), Angle-Angle-Side (AAS), and Right angle-Hypotenuse-Side (RHS).

Side-Side-Side (SSS) postulate

The SSS postulate states that two triangles are congruent if the pairs of their corresponding sides are equal. The postulate is illustrated in Figure 2.2.

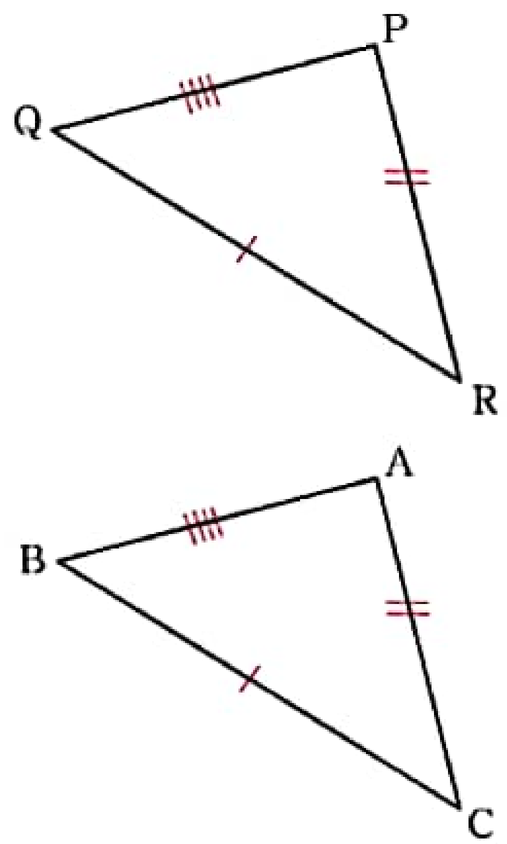


Figure 2.2: *Pair of congruent triangles satifying SSS postulate.*

From Figure 2.2, it follows that

$\overline{AB} = \overline{PQ}$ (given)

$\overline{BC} = \overline{QR}$ (given)

$\overline{AC} = \overline{PR}$ (given)

Since the pairs of the corresponding sides of triangles ABC and PQR are equal, then the two triangles are exactly the same. Therefore, it follows that,

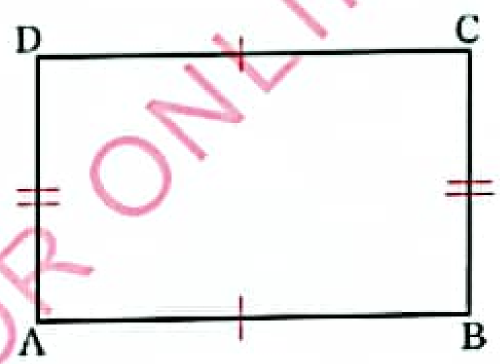
$\Delta ABC \cong \Delta PQR$ (by SSS).

Since the two triangles are congruent, it implies that, their corresponding angles are also equal, that is,

$C\hat{A}B = R\hat{P}Q$, $A\hat{B}C = P\hat{Q}R$ and $B\hat{C}A = Q\hat{R}P$.

Example 2.4

Use the following figure to prove that $\Delta ABC \cong \Delta CDA$, hence deduce that $D\hat{C}A = B\hat{A}C$.



Solution

Given, a rectangle ABCD in which $\overline{AB} = \overline{DC}$, and $\overline{AD} = \overline{BC}$.

Construct a line joining A and C.

Proof: From ΔABC and ΔCDA

$\overline{AB} = \overline{DC}$ (given)

$\overline{BC} = \overline{AD}$ (given)

$\overline{AC}$ is a common side to both triangles.

Therefore, $\Delta ABC \cong \Delta CDA$ (By SSS).

Hence $B\hat{A}C = D\hat{C}A$

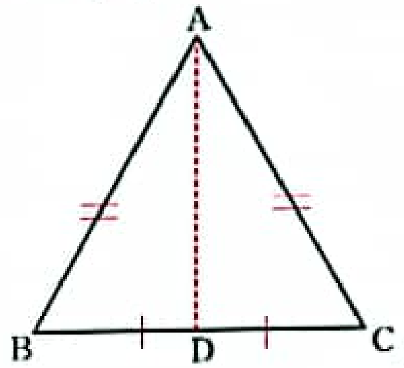
(definition of congruent triangles).

Example 2.5

Triangle ABC is an isosceles triangle in which $\overline{AB}$ and $\overline{AC}$ are equal. If D is the midpoint of $\overline{BC}$, prove that $\Delta ABD \cong \Delta ACD$.

Solution

Consider the ΔABC such that $\overline{AB} = \overline{AC}$ and D is a mid-point of $\overline{BC}$ as shown in the following figure.



Required to prove that $\Delta ABD \cong \Delta ACD$.

Construction: Use a dotted line to join the points A and D.

Proof: In ΔABD and ΔACD

$\overline{AB} = \overline{AC}$ (given)

$\overline{BD} = \overline{DC}$ (given)

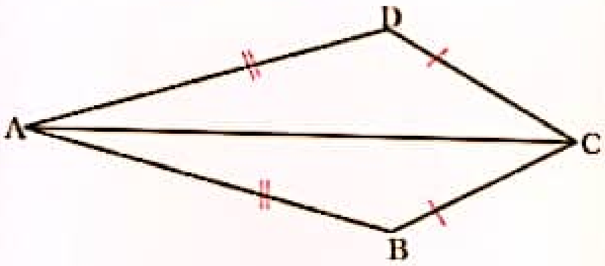
$\overline{AD}$ is a common side to both triangles.

Therefore,

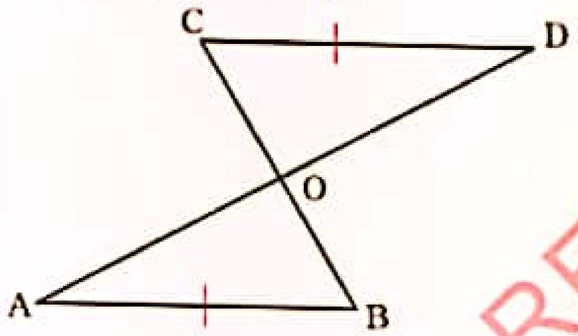
$\Delta ABD \cong \Delta ACD$ (by SSS).

Exercise 2.2

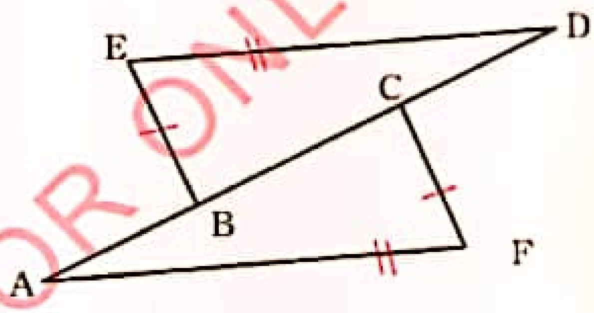
1. Use the following figure to prove that $A\hat{B}C = A\hat{D}C$.



2. In the following figure, $\overline{AD}$ and $\overline{BC}$ bisect each other at O. Prove that $\overline{AB}$ is parallel to $\overline{CD}$.



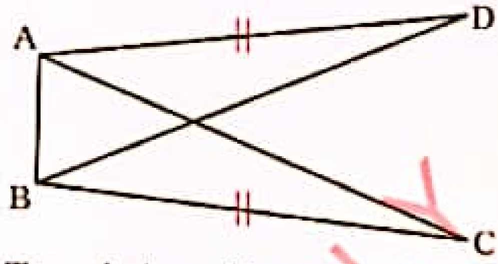
3. In the following figure, $\overline{AB} = \overline{CD}$ and ABCD is a straight line. Prove that $B\hat{A}F = C\hat{D}E$.



4. If OAB is a triangle in which $\overline{OA} = \overline{OB}$, and N is the mid-point of $\overline{AB}$. Prove that $O\hat{N}A = O\hat{N}B$.

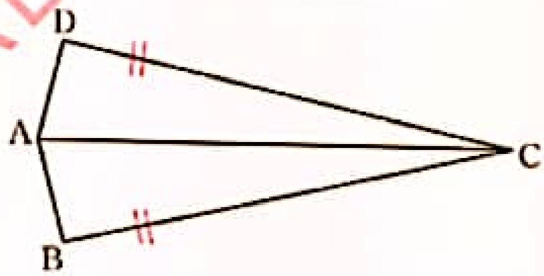
5. If $\overline{AB}$ and $\overline{CD}$ are two equal parallel chords of a circle with centre O, prove that $A\hat{O}B = C\hat{O}D$.

6. Use the following figure to prove that if $\overline{AC} = \overline{BD}$, then $A\hat{C}B = B\hat{D}A$.

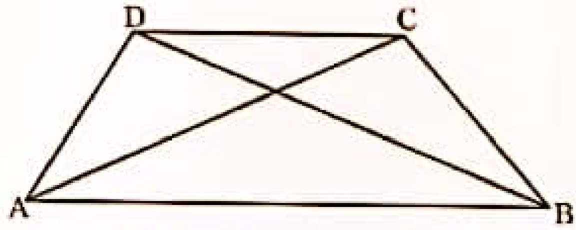


7. Two circles with centres at A and B, respectively intersect at points P and Q. Prove that $\overline{AB}$ bisects $P\hat{A}Q$ (Hint: use triangles APB and AQB).

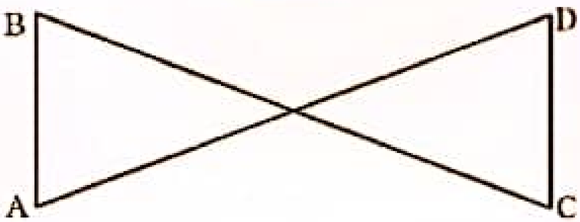
8. In the following figure, $\overline{AD} = \overline{AB}$ and $\overline{CD} = \overline{CB}$, prove that $\overline{AC}$ bisects $D\hat{A}B$ and $D\hat{C}B$.

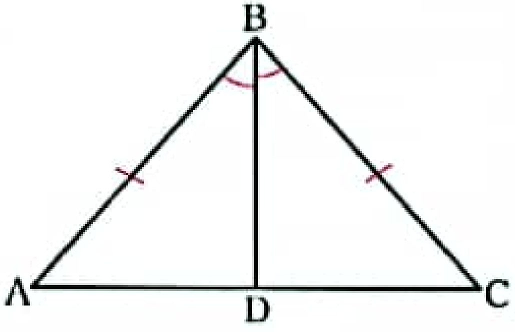


9. In the following figure, $\overline{AD} = \overline{BC}$ and $\overline{AC} = \overline{BD}$. Prove that $\Delta ABD \cong \Delta BAC$.



10. In the following figure $\overline{AD} = \overline{BC}$ and $\overline{AB} = \overline{CD}$, prove that $B\hat{A}D = D\hat{C}B$. (Hint: Join B and D)





Solution

Given ΔABC such that $\overline{BA} = \overline{BC}$ and $A\hat{B}D = D\hat{B}C$.

Required to prove that $\overline{AD} = \overline{DC}$.

Proof: In ΔABD and ΔCBD, it implies that

$\overline{BA} = \overline{BC}$ (given)

$A\hat{B}D = C\hat{B}D$ (given)

$\overline{BD}$ is a common side.

Thus, $\Delta ABD \cong \Delta CBD$ (by SAS)
Since the two triangles are congruent, it follows that all sides and angles are equal.

Therefore, $\overline{AD} = \overline{CD}$.

Note that, two triangles may have two equal sides and angles but not qualify to be congruent. This is described in ΔAB_1C and ΔAB_2C in Figure 2.4.

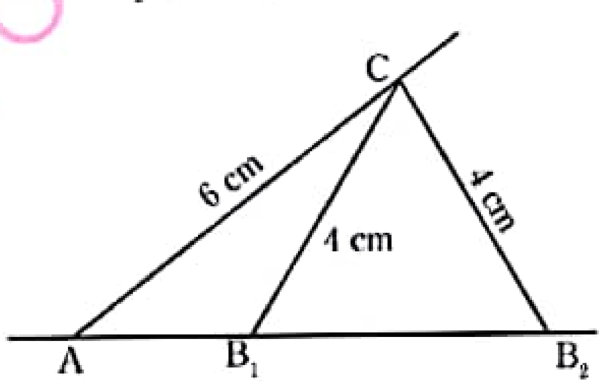
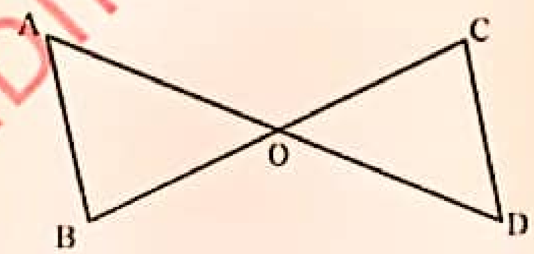


Figure 2.4: *Demonstrating non-congruence of triangles*

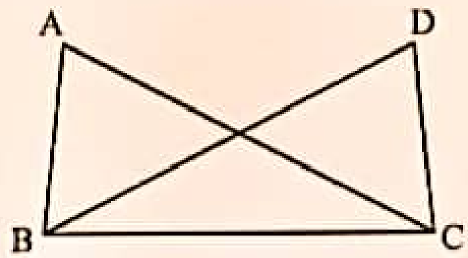
In Figure 2.4 ΔAB_1C and ΔAB_2C, have two pairs of equal sides ($\overline{CB_1}$,and $\overline{CB_2}$) and one pair of equal angles ($C\hat{A}B_1$). However, SAS postulate cannot be used because angle CAB_1 is not included between the two equal sides $\overline{CB_1}$, and $\overline{CB_2}$.

Exercise 2.3

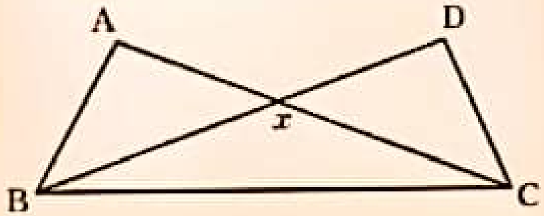
1. In the following figure, $\overline{AO} = \overline{OD}$ and $\overline{OB} = \overline{OC}$.
   (a) Prove that $\overline{AB} = \overline{CD}$.
   (b) Write the angle which is equal to $O\hat{A}B$.



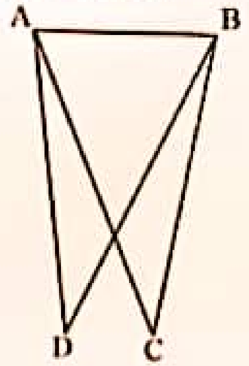
2. In the following figure, if $\overline{AB} = \overline{DC}$ and $A\hat{B}C = D\hat{C}B$, prove that $\overline{AC} = \overline{DB}$.



3. In the following figure, if $\overline{AX} = \overline{DX}$ and $\overline{BX} = \overline{CX}$, prove that $B\hat{A}C = C\hat{D}B$.

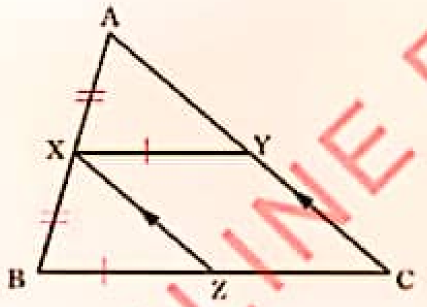


4. Use the following figure to prove that $A\hat{D}B = B\hat{C}A$, given $\overline{AC} = \overline{BD}$ and $B\hat{A}C = A\hat{B}D$.



5. Consider a quadrilateral ABCD such that $\overline{AB} = \overline{DC}$ and $\overline{AC}$ bisects $D\hat{A}B$. Prove that $\overline{AD} = \overline{BC}$.

6. Use the following figure to prove that $\overline{XZ} = \overline{AY}$.



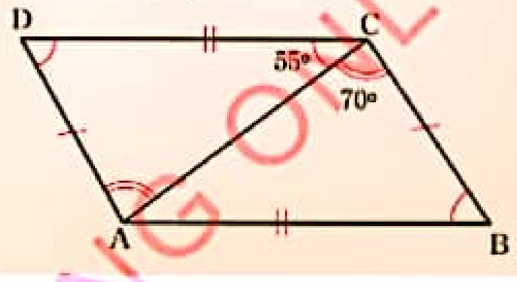
7. Consider a quadrilateral ABCD with two line segments $\overline{DC}$ and $\overline{AB}$ drawn apart such that $\overline{AB} = \overline{DC}$ and $A\hat{B}D = B\hat{D}C$. Prove that $D\hat{A}B = B\hat{C}D$.

8. A triangle ABC is such that $\overline{AC} = \overline{BC}$ and $\overline{DC}$ is the bisector of $B\hat{C}A$. Prove that $A\hat{D}C = B\hat{D}C$.

9. If O is the centre of a circle ACB and $A\hat{O}C = C\hat{O}B$, prove that $\overline{AC} = \overline{CB}$.

10. A circle ABCD is centered at O. If $\overline{AC}$ and $\overline{BD}$ are diameters of the circle and line segments $\overline{AD}$, $\overline{AB}$, and $\overline{CB}$ are drawn, prove that $\overline{AD} = \overline{BC}$.

11. Find the measure of $A\hat{D}C$ in the following figure.



Angle-Angle-Side (AAS) postulate

Two triangles are congruent if the angles in any two pairs of corresponding angles are equal and the lengths of a pair of corresponding sides are equal.

Figure 2.5 illustrates the postulate.

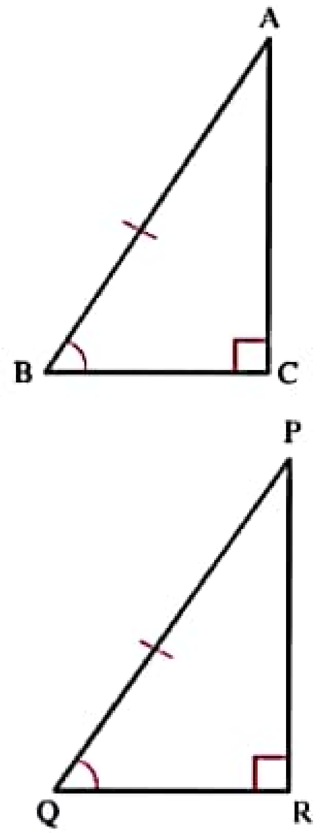


Figure 2.5: *Two congruent figures by AAS postulate*

Figure 2.5 shows that,

$\overline{BA} = \overline{QP}$ (given)

$A\hat{B}C = P\hat{Q}R$ (given)

$B\hat{C}A = Q\hat{R}P$ (given)

Thus, the two triangles satisfy the AAS postulate.

Therefore, $\Delta ABC \cong \Delta PQR$ (by AAS).

Since the two triangles are congruent, it follows that, all corresponding sides and angles are equal. That is,

$\overline{BA} = \overline{QP}$, $\overline{AC} = \overline{PR}$, and $B\hat{A}C = Q\hat{P}R$.

Similarly, if two angles and one included side of the first triangle are equal to two angles and an included side of the second triangle, then the triangles are congruent by Angle-Side-Angle (ASA) postulate as described in Figure 2.6.

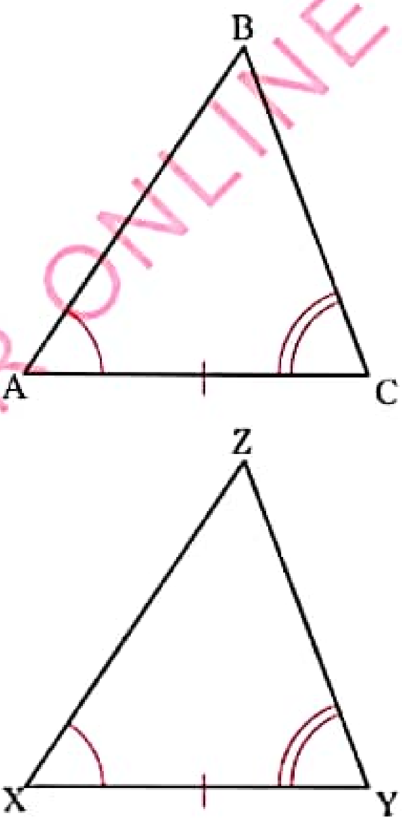


Figure 2.6: *Triangles satsfying ASA postulate*

Consider triangles ABC and XZY in Figure 2.6,

$B\hat{A}C = Z\hat{X}Y$ (given)

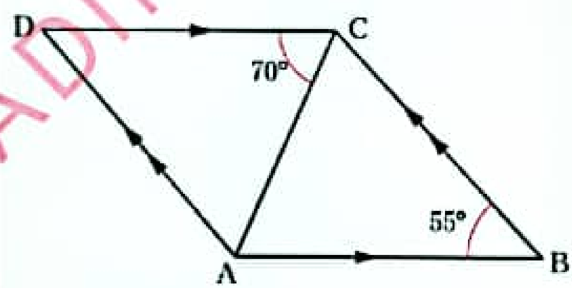
$\overline{AC} = \overline{XY}$ (given)

$A\hat{C}B = X\hat{Y}Z$ (given)

Therefore $\Delta ABC \cong \Delta XYZ$ (by ASA postulate).

Example 2.8

In the following figure, prove that $\Delta ABC \cong \Delta CDA$.



Solution

Given a parallelogram ABCD where $\overline{AC}$, is its diagonal.

Required to prove that,

$\Delta ABC \cong \Delta CDA$.

Proof: In ΔABC and ΔCDA it follows that,

$C\hat{A}B = A\hat{C}D$.

(alternate interior angles as $\overline{AB} // \overline{DC}$).

Similarly,

$A\hat{C}B = C\hat{A}D$

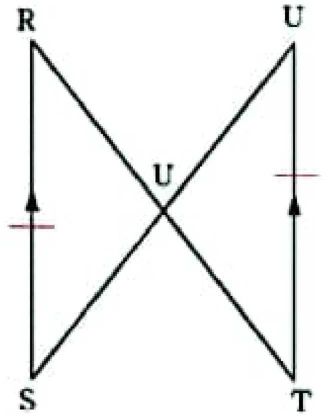
(alternate interior angles as $\overline{AD} // \overline{BC}$).

$\overline{AC}$ is a common side.

Therefore, $\Delta ABC \cong \Delta CDA$ (by AAS).

Example 2.9

In the following figure, prove that $\Delta RVS \cong \Delta TVU$.



Solution

Given that $\overline{RS} \parallel \overline{UT}$ and $\overline{RS} = \overline{UT}$.

Required to prove that $\Delta RVS \cong \Delta TVU$.

Proof: In ΔRVS and ΔTVU, it implies that

$S\hat{R}V = U\hat{T}V$ (alternate angles, $\overline{RS} \parallel \overline{UT}$).

$\overline{RS} = \overline{UT}$ (given)

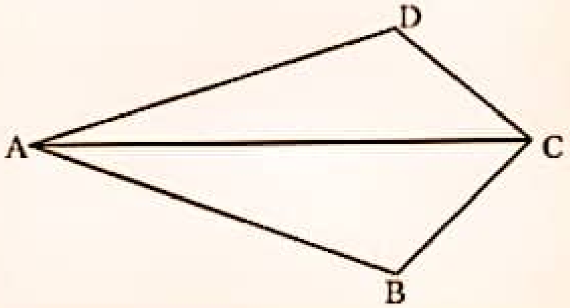
$R\hat{S}V = T\hat{U}V$ (alternate angles, $\overline{RS} \parallel \overline{UT}$)

Therefore,

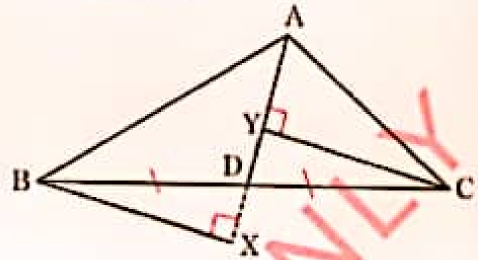
$\Delta RVS \cong \Delta TVU$ (by ASA postulate).

Exercise 2.4

1. In the following figure, $\overline{AC}$ bisects $B\hat{A}D$ and $B\hat{C}D$, prove that $\overline{AB} = \overline{AD}$.

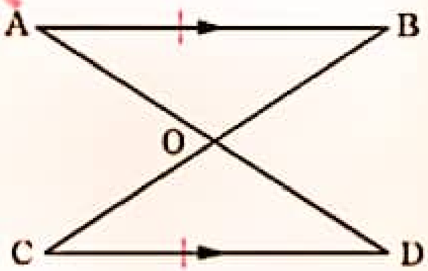


2. Given that X and Y are the feet of the perpendiculars from B and C to $\overline{AD}$ as shown in the following figure. Prove that $\overline{BX} = \overline{CY}$.



3. Line segments $\overline{CB}$ and $\overline{AD}$ intersect at O such that $\overline{AO} = \overline{OD}$. If $\overline{AB}$ is parallel to $\overline{CD}$, prove that $\overline{AB} = \overline{CD}$.

4. Consider the following figure.

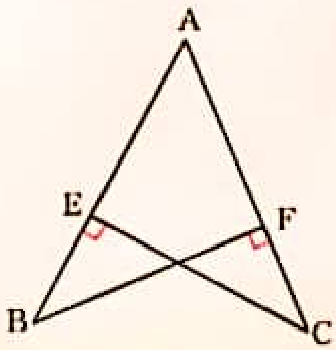


Prove that,

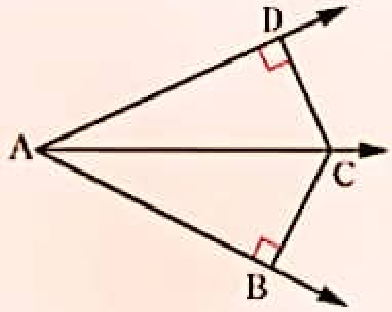
(a) $\overline{AO} = \overline{OD}$ (b) $\overline{CO} = \overline{OB}$

5. In ΔPQR, X is the midpoint of $\overline{PQ}$, Y and Z are the midpoints of $\overline{PR}$ and $\overline{QR}$, respectively. If XYRZ is a parallelogram, prove that $\overline{XY} = \overline{QZ}$.

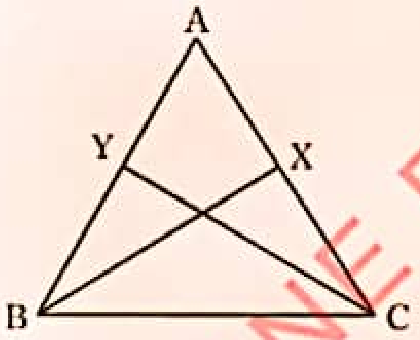
6. In the following figure, $\overline{AB} = \overline{AC}$. Prove that $\overline{BF} = \overline{CE}$.



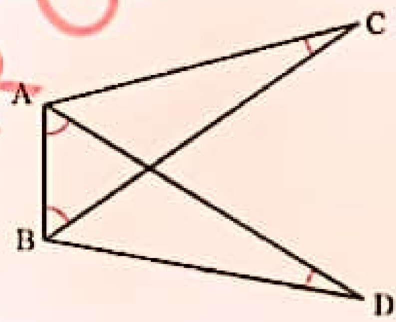
7. In the following figure, $\overline{AC}$ is the bisector of $B\hat{A}D$ and B and D are the feet of the perpendiculars from C. Prove that $\overline{AB} = \overline{AD}$.



8. In the following figure, $A\hat{B}C = A\hat{C}B$, $\overline{BX}$ bisects $A\hat{B}C$ and $\overline{CY}$ bisects $A\hat{C}B$. Prove that $\overline{BX} = \overline{CY}$.

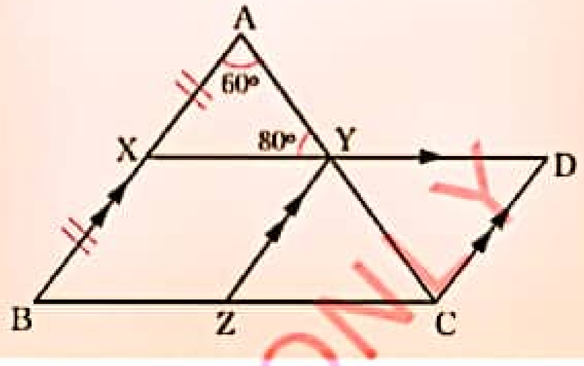


9. Use the following figure to prove that $\overline{BC} = \overline{AD}$, if $A\hat{B}C = B\hat{A}D$ and $B\hat{C}A = A\hat{D}B$.



10. Consider a quadrilateral ABCD such that $\overline{AD}$ is parallel to $\overline{BC}$ and O is the point of intersection of the diagonals. If $\overline{AD} = \overline{BC}$, prove that $\Delta AOD \cong \Delta COB$.

11. Find the value of $Y\hat{Z}C$ in the following figure.



Right angle-Hypotenuse-Side (RHS) postulate

Two right-angled triangles are congruent if their hypotenuses have equal length and a pair of the corresponding sides have equal length. Figure 2.7 illustrates the postulate.

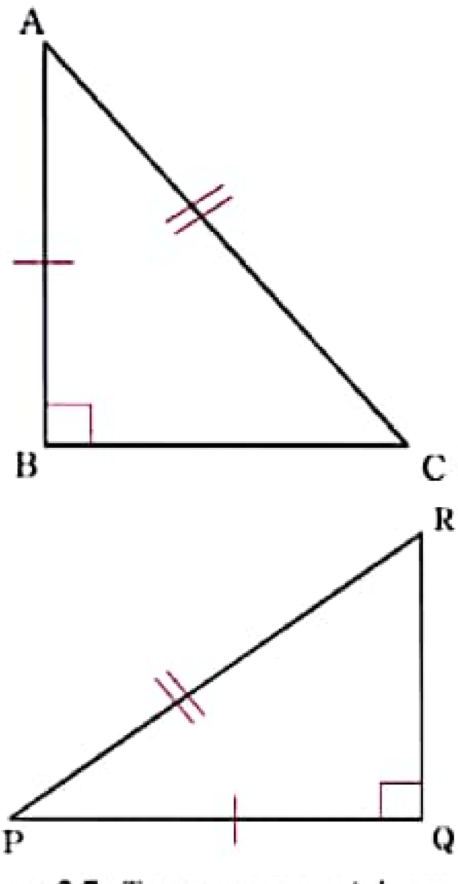


Figure 2.7: *Two congruent right-angled triangles with RHS postulate*

Figure 2.7 shows that,

$\overline{AB} = \overline{PQ}$ (given)

$\overline{AC} = \overline{PR}$ (given)

$A\hat{B}C = P\hat{Q}R = 90°$ (given).

If two triangles satisfy the conditions in the RHS postulate, then the triangles must fit exactly. Thus, Figure 2.7 shows that $\Delta ABC \cong \Delta PQR$ (by RHS).

Since the two triangles in Figure 2.7 are congruent, then, all the corresponding sides and angles are equal. That is,

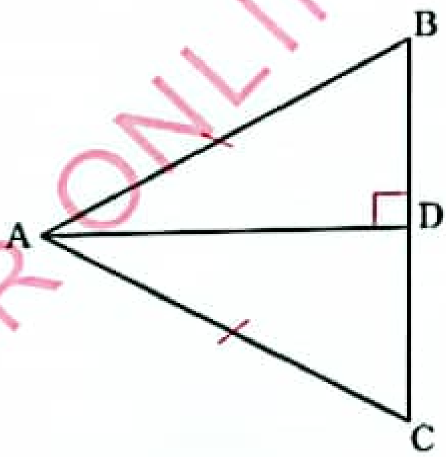
$\overline{BC} = \overline{QR}$

$B\hat{A}C = Q\hat{P}R$

$B\hat{C}A = Q\hat{R}P$.

Example 2.10

Use the following figure to prove that $\Delta ADB \cong \Delta ADC$ and $\overline{DB} = \overline{DC}$.



Solution

Given ΔABC such that $\overline{AD}$ is perpendicular to $\overline{BC}$.

Required to prove that

(a) $\Delta ADB \cong \Delta ADC$

(b) $DB = BC$

Proof: From ΔADB and ΔADC, it implies that

$\overline{AB} = \overline{AC}$ (given)

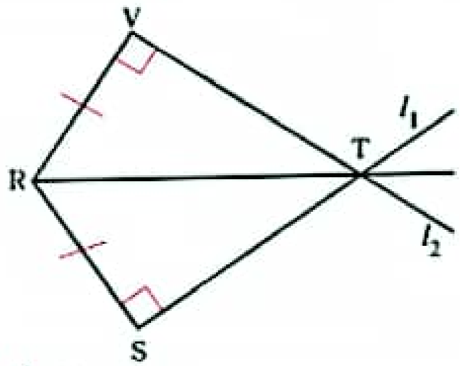
$A\hat{D}B = A\hat{D}C = 90°$ (given)

$\overline{AD}$ is a common side.

Therefore, $\Delta ADB \cong \Delta ADC$ (by RHS), hence $\overline{DB} = \overline{DC}$ (definition of congruence of triangles).

Example 2.11

In the following figure, point R is equidistant from two lines l_1 and l_2, which intersect at T. Prove that $R\hat{T}V = R\hat{T}S$.



Solution:

Required to prove that $R\hat{T}V = R\hat{T}S$.

Proof: From ΔRVT and ΔSRT, it follows that

$\overline{RV} = \overline{RS}$ (given)

$R\hat{V}T = R\hat{S}T$ (given)

$\overline{RT}$ is a common side.

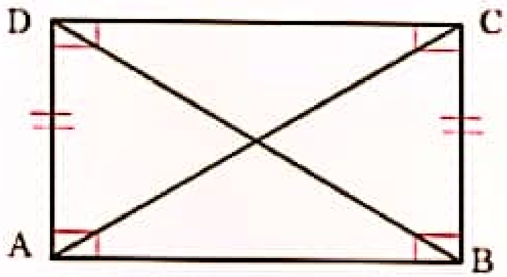
Thus, $\Delta RVT \cong \Delta SRT$ (by RHS).

Since $\Delta RVT \cong \Delta SRT$, it follows that all corresponding angles and sides are equal.

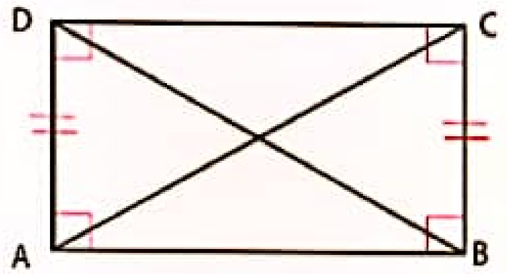
Therefore, $R\hat{T}V = R\hat{T}S$.

Exercise 2.5

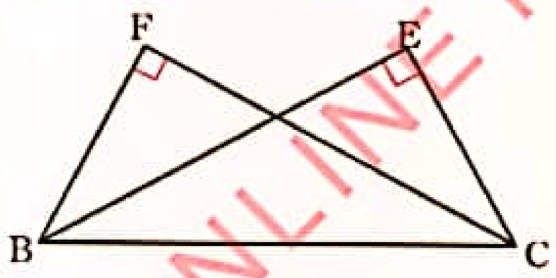
1. In the following figure, $\overline{AC} = \overline{BD}$, prove that $\Delta ABD \cong \Delta BAC$.



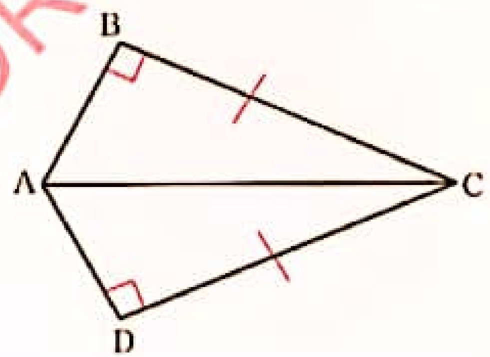
2. Use the following figure to prove that $\overline{AB}$ is parallel to $\overline{DC}$.



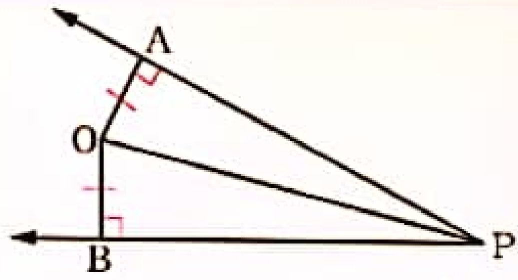
3. In the following figure $\overline{BF} = \overline{CE}$, prove that $\overline{CF} = \overline{BE}$.



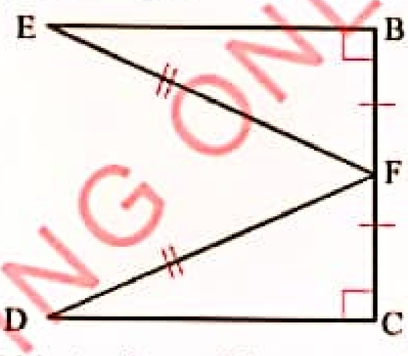
4. In the following figure, prove that $\overline{AC}$ bisects $B\hat{A}D$ and $B\hat{C}D$.



5. Use the following figure to prove that $\overline{PA} = \overline{PB}$.

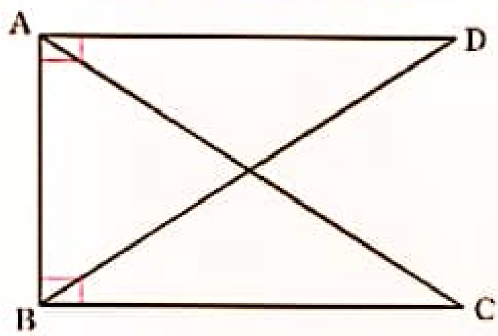


6. Use the following figure to prove that $\overline{BE} = \overline{CD}$.

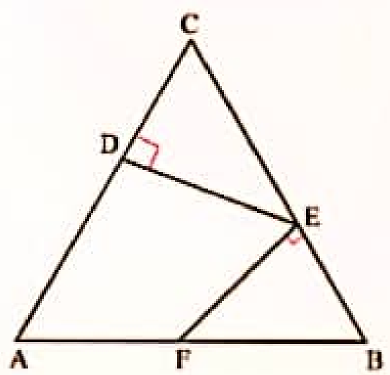


7. If N is the foot of the perpendicular from the centre O of a circle to a chord AB, prove that $\overline{AN} = \overline{NB}$.

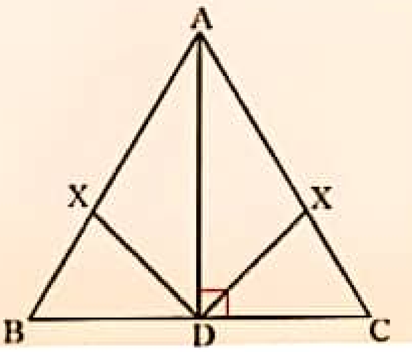
8. Use the following figure to prove that $A\hat{C}B = B\hat{D}A$ if $\overline{AC} = \overline{BD}$



9. In the following figure $\overline{AC} = \overline{BC}$ and $\overline{DE} = \overline{FE}$, prove that $\overline{CD} = \overline{EB}$.



10. In the following figure, D is the midpoint of $\overline{BC}$, X and Y are points on $\overline{AB}$ and $\overline{AC}$, respectively. If $\overline{DX} = \overline{DY}$ and $D\hat{X}B = D\hat{Y}C = 90°$. Prove that $A\hat{B}D = A\hat{C}D$.

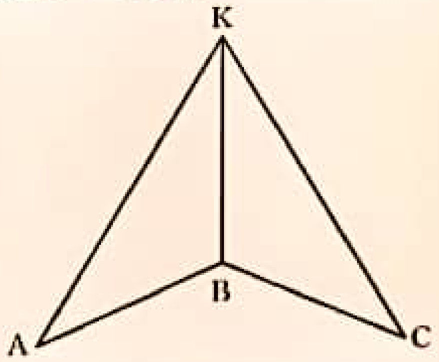


Chapter summary

1. Two figures are said to be congruent if they have exactly the same size and shape.
2. A postulate a statement that is accepted without any proof.
3. A theorem is an argument that has been proven to be true based on past established results or definitions or postulates.
4. A proof is a series of logical statements that are based on definitions, previously established facts, and postulates that may be used to conclude the truth of mathematical arguments.
5. Two triangles are congruent if:
   (a) The sides of one triangle have equal lengths to the corresponding sides of the other triangle (SSS).
   (b) The lengths of two sides and the included angle of one triangle are respectively equal to the lengths of two corresponding sides and the included angle of other triangle (SAS).
   (c) (i) Two angles and the included side of one triangle are respectively equal to the corresponding two angles and the included side of the other triangle (ASA).
   (ii) Two angles and non-included side of one triangle are respectively equal to the corresponding two angles and a non-included side of the other triangle (AAS).
6. Two right-angled triangles are congruent if their hypotenuses and a pair of sides have equal length.

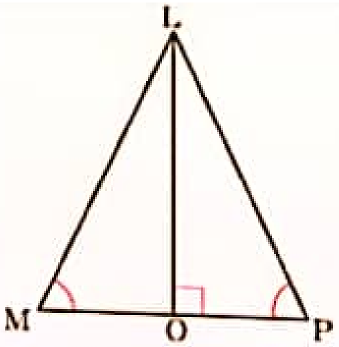
Revision exercise 2

1. In the following figure, $\overline{BA} = \overline{BC}$ and $\overline{KA} = \overline{KC}$. Prove that $B\hat{A}K = B\hat{C}K$.

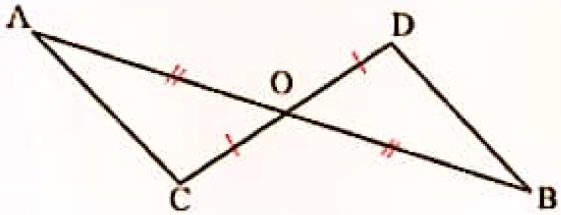


2. A quadrilateral MNOP has the property that $\overline{MN} = \overline{OP}$ and $\overline{MP} = \overline{NO}$. Prove that $P\hat{M}N = N\hat{O}P$.

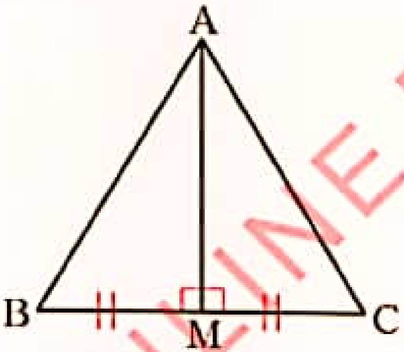
3. Use the following figure to prove that $\overline{ML} = \overline{PL}$.



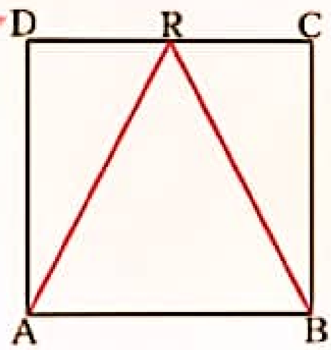
4. In the following figure, prove that $A\hat{C}D = B\hat{D}C$.



5. Use the following figure to prove that $B\hat{M}A = C\hat{M}A = 90°$, given that $\overline{AB} = \overline{AC}$.

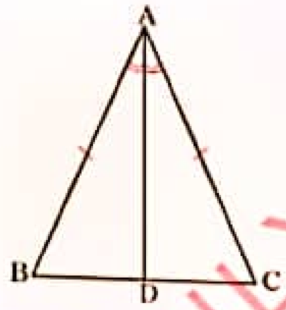


6. If ABCD is a square and $\overline{AR} = \overline{BR}$, prove that R is the midpoint of $\overline{DC}$.



7. Prove that the line segment from the vertical angle of an isosceles triangle to the mid-point of its base is perpendicular to the base.

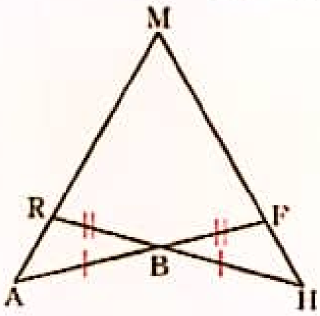
8. In the following figure, prove that $\overline{BD} = \overline{DC}$.



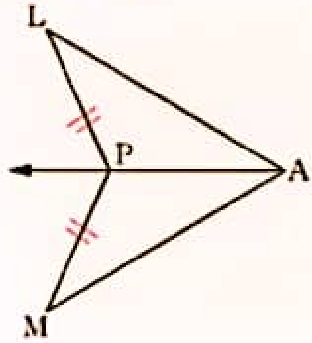
9. Prove that the bisector of the vertical angle of an isosceles triangle is perpendicular to the base at its mid-point.

10. In the following figure $\overline{AB} = \overline{HB}$ and $\overline{RB} = \overline{BF}$. Prove that:

(a) $R\hat{A}B = F\hat{H}B$ (b) $\overline{AM} = \overline{HM}$

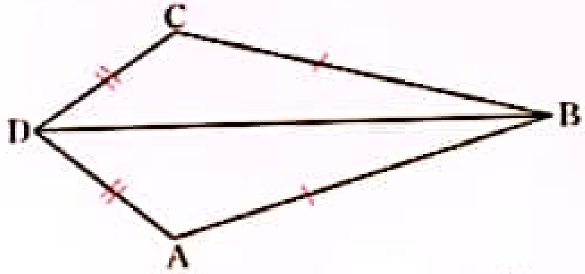


11. Use the following figure to prove that $\overline{AP}$ bisects $L\hat{A}M$, given that $\Delta ALP \cong \Delta AMP$.

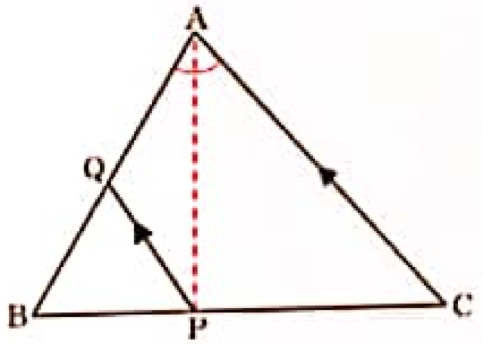


12. Prove that the perpendicular from the vertex to the base of an isosceles triangle bisects the base and the vertical angle.

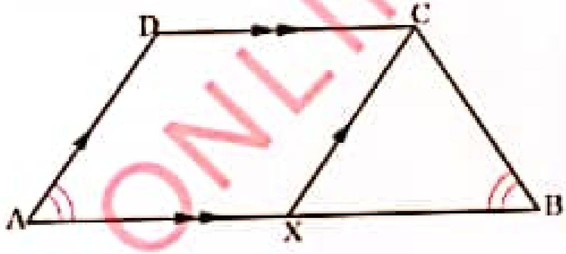
13. Use the following figure to prove that $B\hat{A}D = B\hat{C}D$.



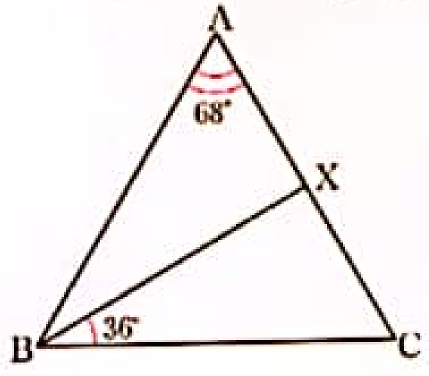
14. If $\overline{PA}$ bisects the angle $B\hat{A}C$ and Q is the point on $\overline{AB}$ such that $\overline{QP} // \overline{AC}$. Prove that $\overline{AQ} = \overline{QP}$.



15. Given the trapezium ABCD such that $D\hat{A}X = C\hat{B}X$ and $\overline{CX}$ is drawn parallel to $\overline{AD}$ as shown in the following figure. Prove that ΔBXC is an isosceles triangle.

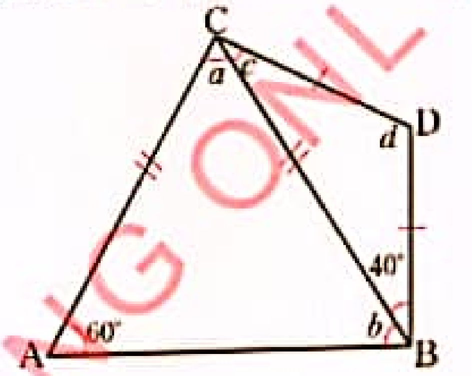


16. If $\overline{AB} = \overline{AC}$, use appropriate congruence postulate to find the value of $A\hat{X}B$ in the following figure.

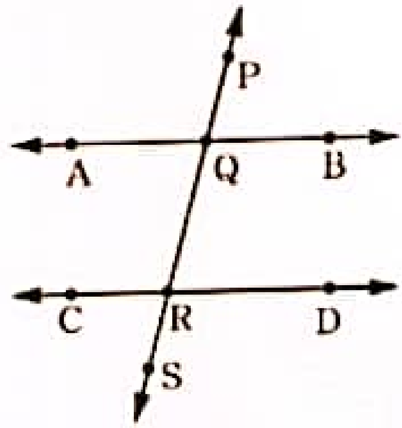


17. If ΔPQR is an equilateral triangle such that $\overline{PQ}$ is extended to S so that $\overline{QS} = \overline{QR}$. Calculate the measure of $Q\hat{R}S$.

18. Use the following figure to find the values of *a*, *b*, *c*, and *d*.



19. If $\overline{AB} // \overline{CD}$, prove that $P\hat{Q}B = C\hat{R}S$.



20. Prove that a quadrilateral in which its diagonals bisect each other is a parallelogram.

21. Prove that a quadrilateral in which both pairs of opposite angles are equal is a parallelogram.

22. Prove that the base angles of an isosceles triangle are congruent.

Chapter Three

Similarity

Introduction

Understanding similar triangles is crucial in scaling, designing structures, even in photographing, where proportions must be maintained. All of these skills are possible for someone with knowledge of similar figures. In this chapter, you will learn the concept of similar figures, recognise the properties of similar triangles and explain postulates, proofs, and theorems of congruent triangles. The competencies developed will be applied in constructions and in architectural matters such as finding heights of buildings, bridges and trees, where tape measures cannot be used conveniently and many other applications.

Think

A world without knowledge of scales, proportions, sizes and comparisons.

Similar figures

The previous chapter discussed congruent figures which are figures with identical shape and size. This chapter introduces the concept of similar figures which advances the concept of congruent figures. Engage in Activity 3.1 to explore the concept of similar figures.

Activity 3.1: Recognizing similarities between figures

1. Cut a rectangle from the edge of a plain paper or a manila sheet as shown in the following figure, then compare the rectangle with the original paper in terms of shape and size.

2. Measure the lengths of sides and angles of both shapes, then find the ratio of the corresponding side lengths.
3. Use your observations to make a general conclusion about the shapes, and explore additional sources such as the internet and books to enrich your competence.
4. Reflect on real objects around you with similar features and characteristics and share your insights.

Similar figures are geometric shapes that have the same shape but differ in size. They maintain the same ratio of corresponding sides and angles. Study Figure 3.1 which illustrates the concept of similar objects.

Figure 3.1: *Pictures of similar blackboards*

In Figure 3.1, the objects are similar because they have the same shape but differ in size. Based on this experience, a simple method of obtaining similar figures is by uniformly scaling (enlarging or shrinking) the original figure. Engage in Activity 3.2 to explore more about similarity of figures.

Activity 3.2: Using computer applications to demonstrate similarity of figures

1. Use a software of your choice (MS Word, Desmos, GeoGebra) to create various shapes.
2. Copy and paste the shapes into a new workspace, then resize by enlarging or reducing the figures.
3. Observe the changes in size in terms of sides and angles and discuss your findings based on the similarity of figures.

Similar triangles

Triangles are similar when their corresponding angles are equal and their corresponding sides are proportional. Consider the pair of triangles shown in Figure 3.2.

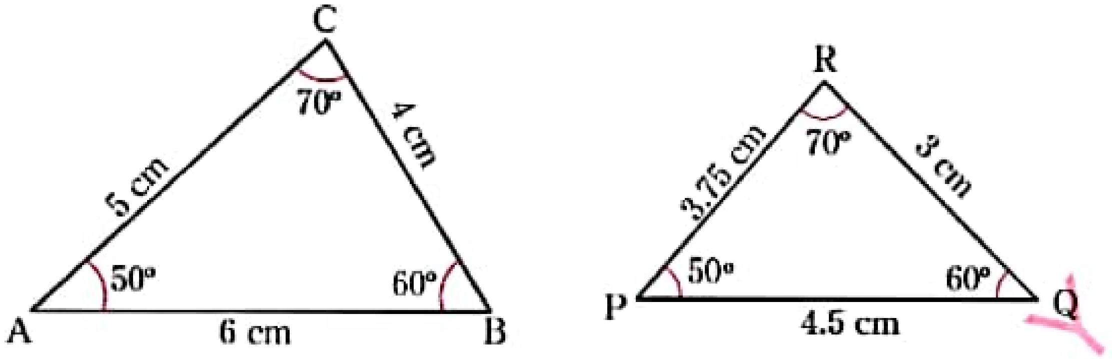


Figure 3.2: *Similar triangles*

From Figure 3.2, $C\hat{A}B$ of ΔABC corresponds to $R\hat{P}Q$ of ΔPQR and each measures 50°, $A\hat{B}C$ corresponds to $P\hat{Q}R$ and each measures 60°, and $B\hat{C}A$ corresponds to $Q\hat{R}P$ and each measures 70°.

Since the corresponding angles are equal, then the two triangles are similar. Also, $\overline{AB}$ corresponds to $\overline{PQ}$, $\overline{BC}$ corresponds to $\overline{QR}$, and $\overline{CA}$ corresponds to $\overline{RP}$. The ratios of the corresponding sides are given by:

$$\frac{\overline{AB}}{\overline{PQ}}=\frac{6\text{ cm}}{4.5\text{ cm}}=\frac{4}{3},\quad \frac{\overline{BC}}{\overline{QR}}=\frac{4\text{ cm}}{3\text{ cm}}=\frac{4}{3}\text{ and }\frac{\overline{CA}}{\overline{RP}}=\frac{5\text{ cm}}{3.75\text{ cm}}=\frac{4}{3}.$$

Therefore, $\frac{\overline{AB}}{\overline{PQ}}=\frac{\overline{BC}}{\overline{QR}}=\frac{\overline{CA}}{\overline{RP}}=\frac{4}{3}.$

Since the corresponding sides are proportional, the two triangles are similar. Therefore, ΔABC is similar to ΔPQR and is denoted by $\Delta ABC \sim \Delta PQR$. The symbol ~ means "similar to."

Similar triangles (polygons) are named according to the order of their vertices. For instance, in ΔABC and ΔPQR in Figure 3.2, it can be deduced from the order of the vertices that AB of the first triangle corresponds to PQ of the second triangle. Consequently, BC corresponds to QR, and AC corresponds to PR.

Example 3.1

Given that $\Delta SLK \sim \Delta NFR$, identify all the corresponding angles and the corresponding sides.

Solution

Using the order of vertices of the two similar triangles, $S\hat{L}K$ corresponds to $N\hat{F}R$, $K\hat{S}L$ corresponds to $R\hat{N}F$, and $L\hat{K}S$ corresponds to $F\hat{R}N$. Also, $\overline{LS}$ of ΔSLK corresponds to $\overline{FN}$ of ΔNFR, $\overline{SK}$ of ΔSLK corresponds to $\overline{NR}$ of ΔNFR, and $\overline{KL}$ of ΔSLK corresponds to $\overline{RF}$ of ΔNFR.

Example 3.2

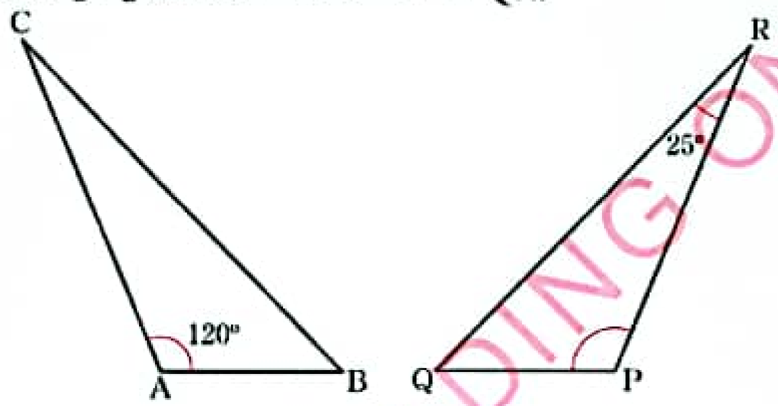
Given that $\Delta ABC \sim \Delta PQR$, find the value of $A\hat{B}C$ if:

(a) $B\hat{A}C = 120°$ and $P\hat{R}Q = 25°$

(b) $Q\hat{P}R + B\hat{C}A = 145°$

Solution

Consider the following figures of ΔABC and ΔPQR.



(a) Since $Q\hat{R}P$ corresponds to $B\hat{C}A$, then $Q\hat{R}P = B\hat{C}A = 25°$.

In ΔABC, $A\hat{B}C + B\hat{C}A + B\hat{A}C = 180°$ (sum of interior angles in a triangle)

$A\hat{B}C + 25° + 120° = 180°$

Thus, $A\hat{B}C = 180° - 145° = 35°$.

Therefore, $A\hat{B}C = 35°$.

(b) Since $Q\hat{P}R$ corresponds to $B\hat{A}C$, then $Q\hat{P}R = B\hat{A}C$.

Thus, $Q\hat{P}R + B\hat{C}A = B\hat{A}C + B\hat{C}A = 120° + 25° = 145°$.

But $A\hat{B}C + B\hat{A}C + A\hat{C}B = 180°$ (sum of interior angles in a triangle). It follows that

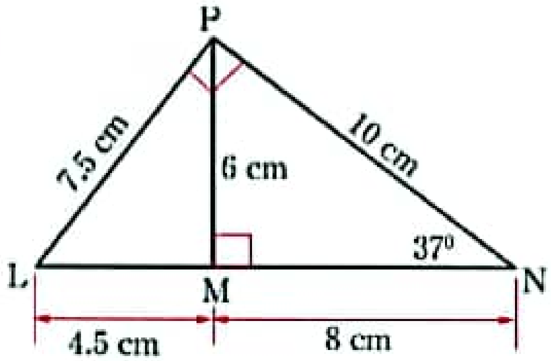
$145° + A\hat{B}C = 180°$

$A\hat{B}C = 180° - 145° = 35°$

Therefore, $A\hat{B}C = 35°$.

Example 3.3

In the following figure, name the triangles which are similar and determine the constant of proportionality needed to show their similarity.



Solution

Consider ΔPMN and ΔLPN

$M\hat{N}P = L\hat{N}P$ (common)

$N\hat{M}P = L\hat{P}N = 90°$(given).

$M\hat{P}N = P\hat{L}N$ (third angles of the triangles)

Therefore, $\Delta PMN \sim \Delta LPN$ (1)

Consider ΔLMP and ΔLPN

$M\hat{L}P = P\hat{L}N$ (common)

$L\hat{M}P = N\hat{P}L = 90°$ (given)

$L\hat{P}M = L\hat{N}P$ (third angles of the triangles)

$\Delta LMP \sim \Delta LNP$ (2)

Relating (1) and (2), it can be observed that

$\Delta LPN \sim \Delta PMN \sim \Delta LMP$

Since $\Delta LPN \sim \Delta PMN$, then the constant of proportionality is given by

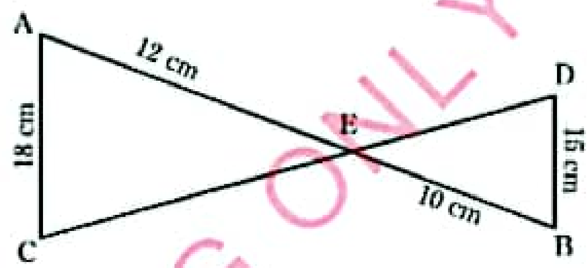
$$\frac{\overline{LP}}{\overline{PM}} = \frac{\overline{PN}}{\overline{MN}} = \frac{\overline{NL}}{\overline{NP}}.$$

Thus, $\dfrac{7.5 \text{ cm}}{6 \text{ cm}} = \dfrac{10 \text{ cm}}{8 \text{ cm}} = \dfrac{12.5 \text{ cm}}{10 \text{ cm}} = \dfrac{5}{4}$.

Therefore, the constant of proportionality needed to show their similarity is $5:4$ or $\dfrac{5}{4}$.

Example 3.4

In the following figure, find the constant of proportionality needed to obtain a pair of similar triangles, if $\overline{CE}:\overline{DE} = 1.2:1$. Name this pair of similar triangles.



Solution

The ratio of lengths of corresponding sides are given by;

$$\overline{CE}:\overline{DE} = \frac{\overline{CE}}{\overline{DE}} = \frac{1.2}{1} = \frac{12 \text{ cm}}{10 \text{ cm}} = \frac{6}{5}$$

$$\overline{AE}:\overline{BE} = \frac{\overline{AE}}{\overline{BE}} = \frac{12 \text{ cm}}{10 \text{ cm}} = \frac{6}{5}$$

$$\overline{AC}:\overline{BD} = \frac{\overline{AC}}{\overline{BD}} = \frac{18 \text{ cm}}{15 \text{ cm}} = \frac{6}{5}$$

Thus, $\dfrac{\overline{CE}}{\overline{DE}} = \dfrac{\overline{AE}}{\overline{BE}} = \dfrac{\overline{AC}}{\overline{BD}} = \dfrac{6}{5}$.

Therefore, $\Delta ACE \sim \Delta BDE$ (corresponding sides are proportional) and $\dfrac{6}{5}$ is the constant of proportionality.

Exercise 3.1

1. (a) Given that $\Delta PQR \sim \Delta TSM$, identify the corresponding angles and the corresponding sides.

(b) Given that $\Delta PQR \sim \Delta LMN$ and $\Delta PQR \sim \Delta ABC$, identify the corresponding angles and corresponding sides between ΔABC and ΔLMN.

2. (a) One rectangle is 10 cm long and 5 cm wide, while another measures 10 cm in length and 4 cm in width. Determine whether the two rectangles are similar or not, and give reasons for your answer.

(b) A rectangle has a length of 23 cm and width of 16 cm. A second rectangle has length of 12 cm and width of 9 cm. Are the two rectangles similar? Explain your answer.

3. Given that ΔABC and ΔLMN are similar, find the value of $A\hat{C}B$, if:

(a) $A\hat{B}C = 70°$ and $M\hat{N}L = 40°$.

(b) $A\hat{B}C + M\hat{L}N = 130°$

(c) $L\hat{N}M$ and $B\hat{A}C$ are complementary and $N\hat{L}M = 56°$

4. Given that

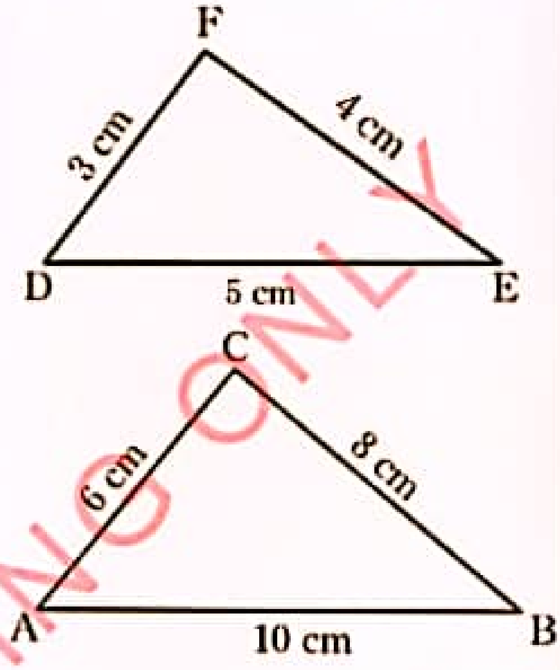
$$\frac{\overline{AB}}{\overline{KL}} = 2, \frac{\overline{BT}}{\overline{LS}} = 2 \text{ and } \frac{\overline{TA}}{\overline{SK}} = 2;$$

(a) name the triangles which are similar.

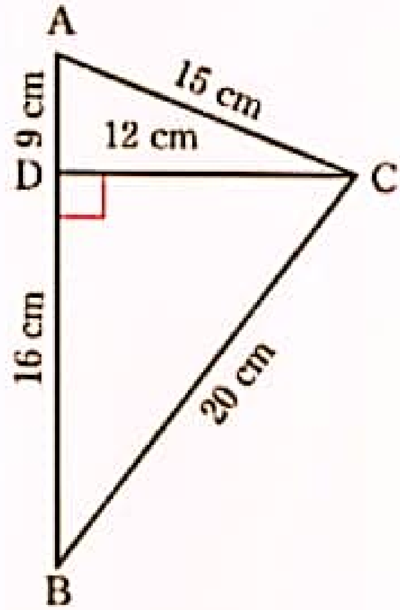
(b) identify the corresponding angles.

5. In each of the following figures, determine the constant of proportionality so that the pair of the triangles are similar. In each case, state the pair of similar triangles.

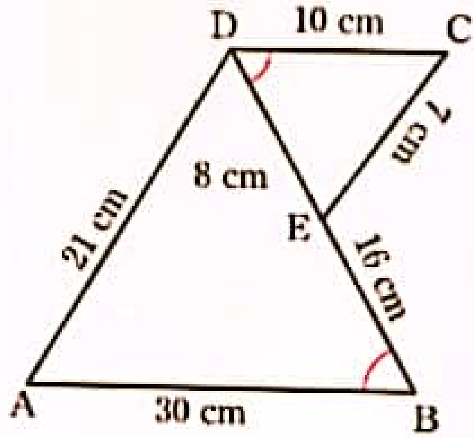
(a)



(b)

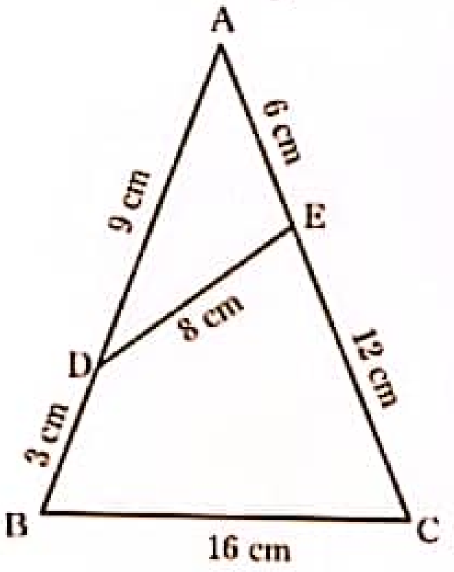


(c)

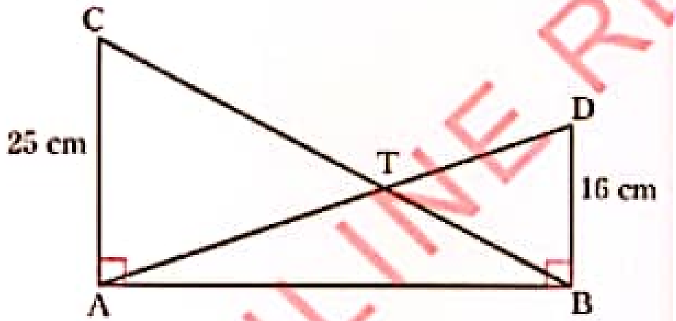


6. Use the vertices A, D, B, C, and E of the following figure to identify

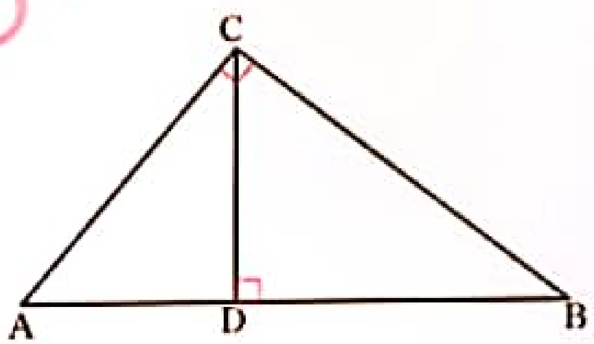
all similar triangles, and hence state their corresponding angles.



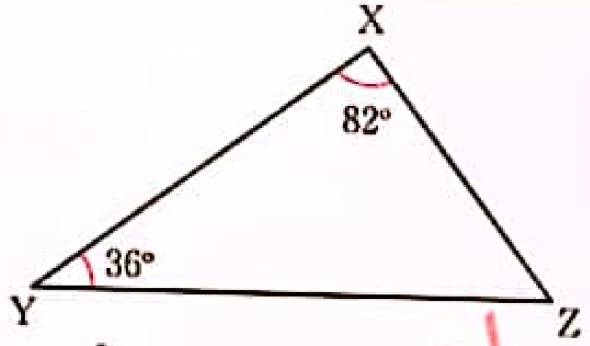
7. Is there enough information in the following figure to determine if ΔACT and ΔDBT are similar? Explain your answer.

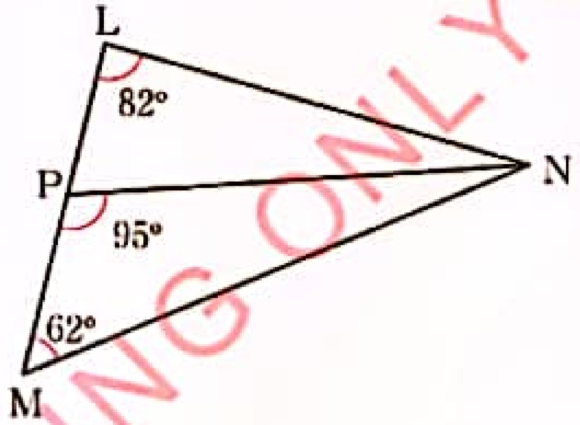


8. In the following figure, name the triangles which are similar to ΔADC.



9. In figure LMN, name the triangle which is similar to ΔXYZ.





10. Which of the following geometric figures are always similar?

(a) Circles

(b) Hexagons

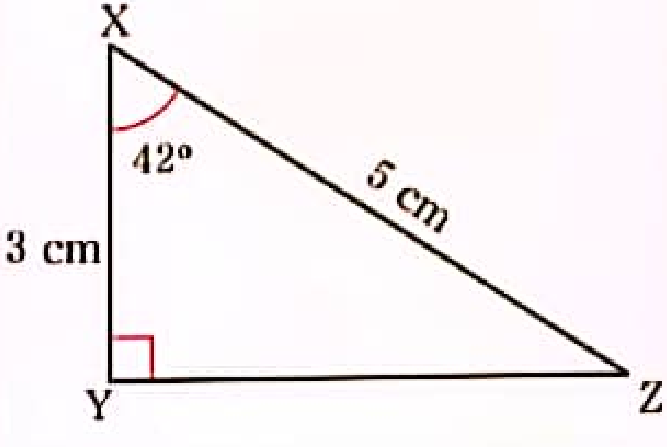
(c) Rhombuses

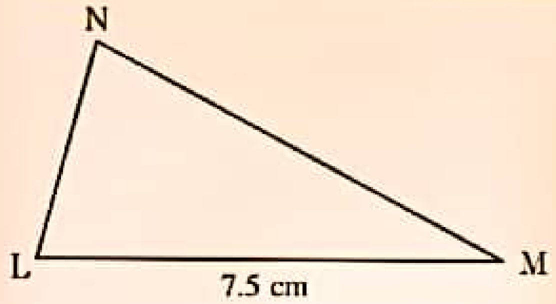
(d) Rectangles

(e) Squares

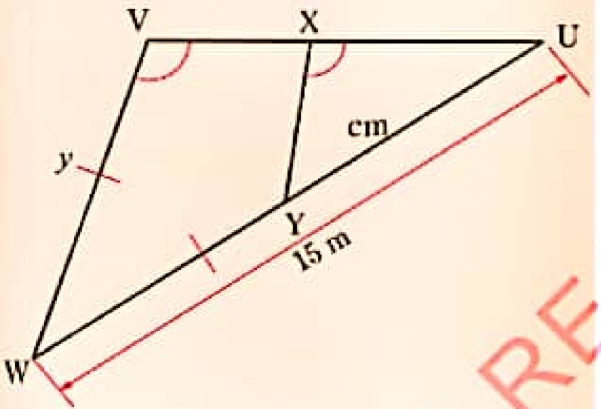
(f) Congruent polygons

11. If $\Delta YZX \sim \Delta LMN$, use the figures to find the measures of the remaining angles and lengths of sides of each triangle.

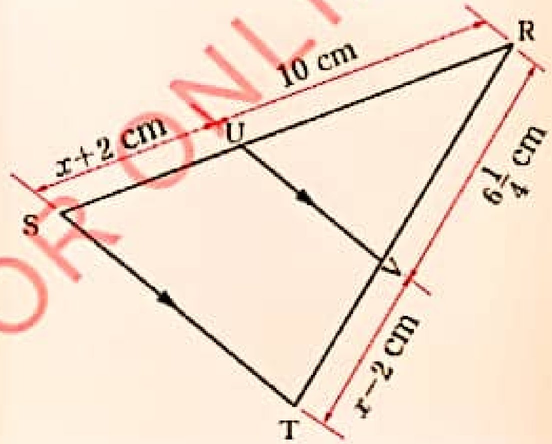




12. It is given that ΔUVW and ΔUXY are similar with $U\hat{X}Y = W\hat{V}X$ and WY = VW. Find the value of *y*.



13. In the following figure, find the value of x.



14. Show that the perimeters of similar triangles have the same ratio as their corresponding sides.

15. If *k* units are added to the length of each side of two similar triangles, are the new triangles still similar? Justify your answer.

16. A teacher at Magoda Secondary School has tasked Adolfina with finding the height of a tall tree in front of the office. Using the knowledge of similarity, how would you advise Adolfina to accomplish this task?

Similarity theorems

Similarly theorems are used to solve problems based on similar triangles.

If ΔABC ~ ΔXYZ, it means;

(a) Corresponding angles are equal. That is $A\hat{B}C = X\hat{Y}Z$, $C\hat{A}B = Z\hat{X}Y$, and $B\hat{C}A = Y\hat{Z}X$.

(b) The ratio of corresponding sides is equal. That is, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$

> **Note**
>
> The value of the constant ratio obtained from the ratio of corresponding sides of similar figures is called the constant of proportionality or scale factor.

To prove similarity of two triangles, only one of the following conditions is sufficient.

1. Two pairs of corresponding angles are equal. This introduces an Angle-Angle (AA) similarity theorem.
2. Three corresponding sides are proportional. This is described by

a Side-Side-Side (SSS) similarity theorem.

3. Two corresponding sides are proportional and one pair of corresponding angles (formed by these sides) are equal is described by the Side-Angle-Side (SAS) similarity theorem.

Angle-Angle (AA) similarity theorem

The AA similarity theorem states that, two triangles are similar if two pairs of corresponding angles are equal. This implies that, if two pairs of angles are equal, the third pair of angles will also be equal as described in Figure 3.3.

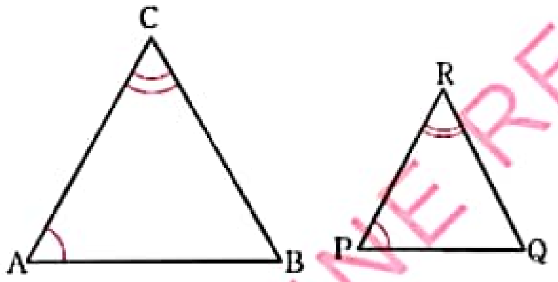


Figure 3.3: *Similar triangles by the Angle-Angle theorem*

In Figure 3.3, it can be observed that,

$B\hat{A}C = Q\hat{P}R$, $A\hat{C}B = P\hat{R}Q$

Thus, $A\hat{B}C = P\hat{Q}R$ (third pairs of angles of triangles)

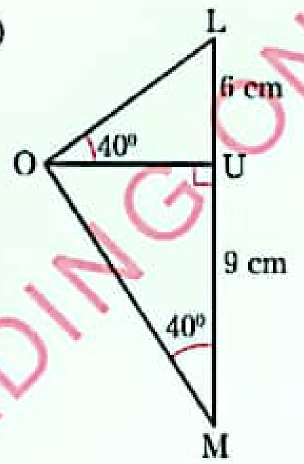
Therefore, $\Delta ABC \sim \Delta PQR$ and

$$\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{BC}}{\overline{QR}} = \frac{\overline{AC}}{\overline{PR}}.$$

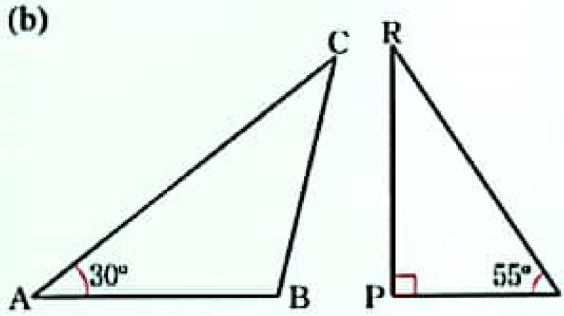
Example 3.5

For each pair of triangles in the following figures, determine whether they are similar or not. Indicate the similarity theorem used to support your argument.

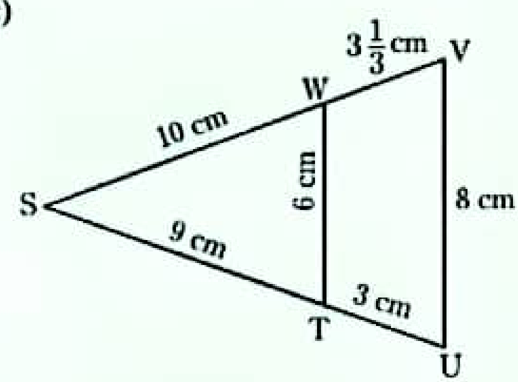
(a)



(b)



(c)



Solution

(a) Required to prove that: $\Delta OUL \sim \Delta MUO$.

Proof

$L\hat{O}U = O\hat{M}U = 40°$ (given)

$O\hat{U}L = M\hat{U}O = 90°$ (given)

$U\hat{L}O = U\hat{O}M = 50°$

(third angles in triangles)

Therefore, $\Delta OUL \sim \Delta MUO$

(by AA-similarity theorem).

(b) ΔCBA and ΔRPQ are not similar because the corresponding angles are not equal.

(c) Required to prove that ΔSVU ΔSWT

Proof

The ratio of the corresponding sides

$\frac{\overline{SW}}{\overline{SV}} = \frac{10 \text{ cm}}{13\frac{1}{3} \text{ cm}} = \frac{3}{4}$,

$\frac{\overline{ST}}{\overline{SU}} = \frac{9 \text{ cm}}{12 \text{ cm}} = \frac{3}{4}$, and

$\frac{\overline{TW}}{\overline{UV}} = \frac{6 \text{ cm}}{8 \text{ cm}} = \frac{3}{4}$.

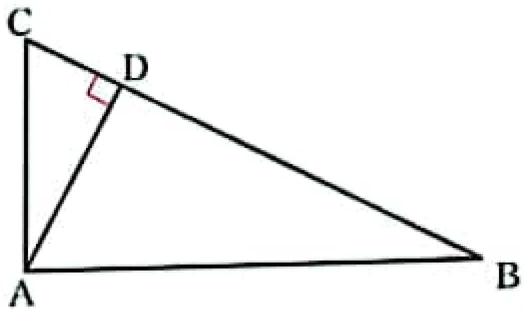
Therefore, $\Delta SVU \sim \Delta SWT$

(by SSS-Similarity theorem).

Example 3.6

Use the following figure to prove that,

(a) $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{CD}}{\overline{AC}}$ (b) $\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}}$



Solution

(a) The numerators contain the vertices A, D, and C while the denominators contain A, B, and C. Thus, the two triangles are ADC and ABC.

From ΔADC and ΔABC. It follows that,

$A\hat{D}C = C\hat{A}B$ (each measures 90°)

$A\hat{C}D = A\hat{C}B$ (common)

Thus, $\Delta DCA \sim \Delta ACB$ (by AA - similarity theorem).

Since $\Delta DCA \sim \Delta ACB$, then the ratio of the corresponding sides are equal, that is:

$$\frac{\overline{CD}}{\overline{AC}} = \frac{\overline{CA}}{\overline{CB}} = \frac{\overline{DA}}{\overline{AB}}$$

Therefore, $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{DC}}{\overline{AC}}$ or $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{CD}}{\overline{AC}}$.

(b) Vertices appearing in the numerators are A, B and D, while in the denominators are A, D and C. Thus, the triangles ADC and BDA are obtained.

From ΔADC and ΔBDA, it follows that

$A\hat{D}B = A\hat{D}C$ (each measures 90°)

$A\hat{B}D = C\hat{A}D = A\hat{B}C$ (using the proof in (a) above)

Thus, $\Delta BDA \sim \Delta ADC$

(by AA-similarity theorem).

Since $\Delta BDA \sim \Delta ADC$, then the ratio of the corresponding sides are equal, that is:

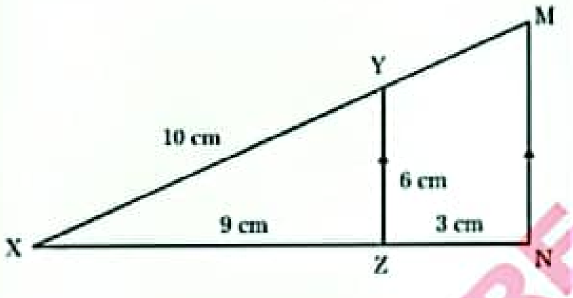
$$\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}} = \frac{\overline{AD}}{\overline{CD}}$$

Therefore, $\frac{\overline{DB}}{\overline{DA}} = \frac{\overline{BA}}{\overline{AC}}$.

Example 3.7

In the following figure, calculate:

(a) $\overline{MY}$ (b) $\overline{MN}$



Solution

(a) Since $\overline{ZY} // \overline{MN}$, it follows that,

$X\hat{Y}Z = X\hat{M}N$ (corresponding angles)

$X\hat{Z}Y = X\hat{N}M$ (corresponding angles)

Hence, $\Delta XYZ \sim \Delta XMN$ (by AA – similarity theorem)

Thus, $\frac{\overline{XY}}{\overline{XM}} = \frac{\overline{XZ}}{\overline{XN}} = \frac{\overline{YZ}}{\overline{MN}}$, this means that,

$$\frac{10\text{ cm}}{\overline{XM}} = \frac{9\text{ cm}}{12\text{cm}} = \frac{6\text{ cm}}{\overline{MN}}$$

From $\frac{10\text{ cm}}{\overline{XM}} = \frac{9\text{ cm}}{12\text{cm}}$, it implies that

$$\overline{XM} = \frac{10\text{ cm}\times 12\text{ cm}}{9\text{ cm}} = 13\tfrac{1}{3}\text{ cm}.$$

But,
$\overline{MY} = \overline{XM} - \overline{XY} = 13\tfrac{1}{3}\text{ cm} - 10\text{ cm} = 3\tfrac{1}{3}\text{ cm}$

Therefore, $\overline{MY} = 3\tfrac{1}{3}$ cm.

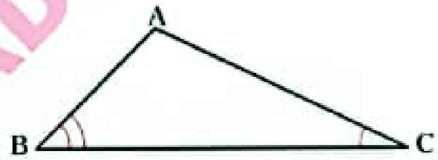
(b) From $\frac{6\text{ cm}}{\overline{MN}} = \frac{9\text{ cm}}{12\text{cm}}$,

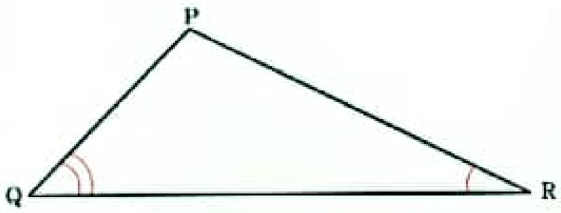
$$\overline{MN} = \frac{6\text{ cm}\times 12\text{ cm}}{9\text{ cm}} = 8\text{ cm}.$$

Therefore, $\overline{MN} = 8$ cm.

Example 3.8

In the following figure, $\Delta ABC \sim \Delta PQR$. If $\overline{AC} = 4.8$ cm, $\overline{AB} = 4$ cm, and $\overline{PQ} = 9$ cm, find the value of $\overline{PR}$.





Solution

Given $\Delta ABC \sim \Delta PQR$, it follows that,

$$\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{AC}}{\overline{PR}}$$

That is, $\frac{4\text{ cm}}{9\text{ cm}} = \frac{4.8\text{ cm}}{\overline{PR}} \Rightarrow$

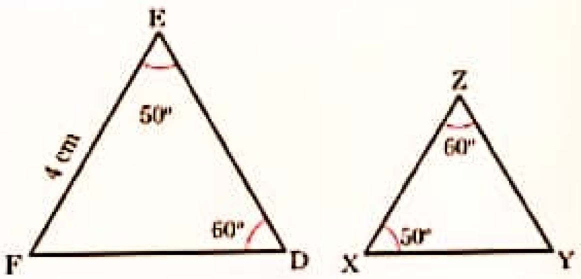
$$4\text{cm}\times \overline{PR} = 9\text{ cm}\times 4.8\text{ cm}$$

$$\overline{PR} = \frac{9\text{ cm}\times 4.8\text{ cm}}{4\text{cm}} = 10.8\text{ cm}.$$

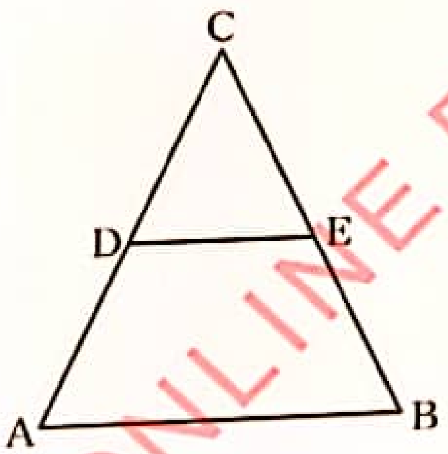
Therefore, $\overline{PR} = 10.8$ cm.

Exercise 3.2

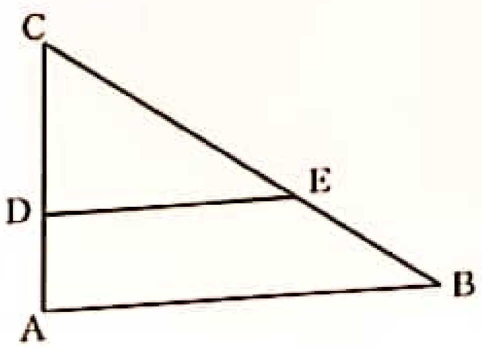
1. Determine whether the following triangles are similar or not. Indicate the similarity theorem used to support your answer.



2. In the following figure, if $\Delta CDE \sim \Delta CAB$, prove that $\overline{DE} \parallel \overline{AB}$.



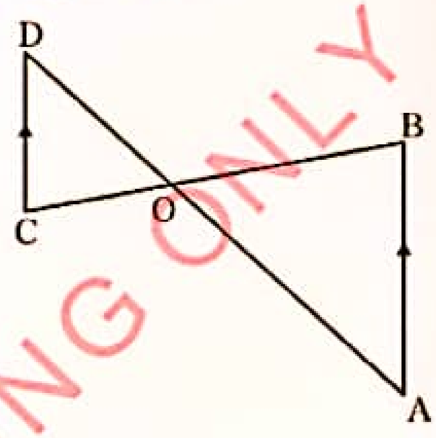
3. In the following figure, if $\overline{DE} \parallel \overline{AB}$, prove that $\Delta CDE \sim \Delta CBA$.



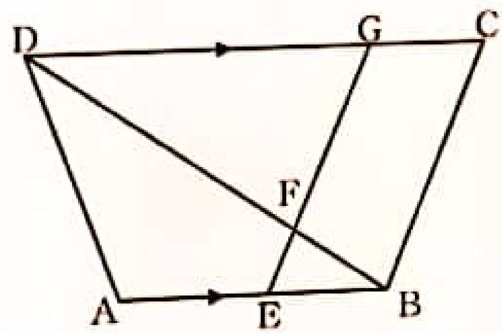
4. In the following figure, if $\overline{CD} \parallel \overline{AB}$, then prove that:

   (a) $\Delta ABO \sim \Delta DCO$

   (b) $\frac{\overline{AO}}{\overline{OD}} = \frac{\overline{BO}}{\overline{OC}}$



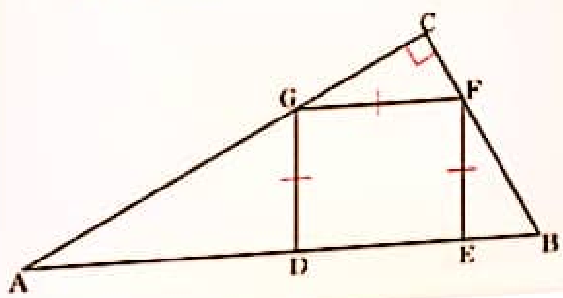
5. In the following figure, E is a point on $\overline{AB}$ and G is a point on $\overline{DC}$ and $\overline{AB} \parallel \overline{DC}$. Prove that $\frac{\overline{DG}}{\overline{BE}} = \frac{\overline{FD}}{\overline{FB}}$.



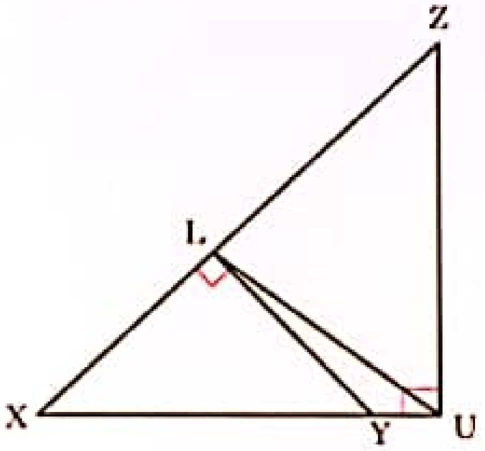
6. In the following figure, DEFG is a square and $A\hat{C}B$ is a right angle. Prove that:

   (a) $\Delta CFG \sim \Delta DGA$

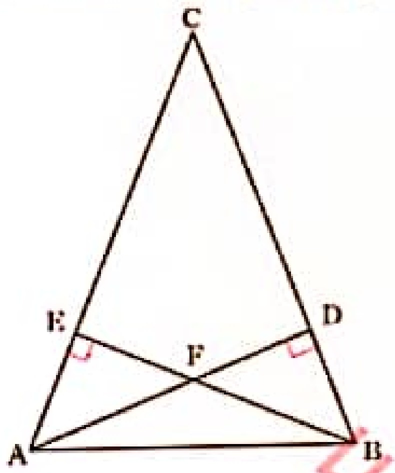
   (b) $\Delta EBF \sim \Delta CFG$



7. Name two similar triangles in the following figure.



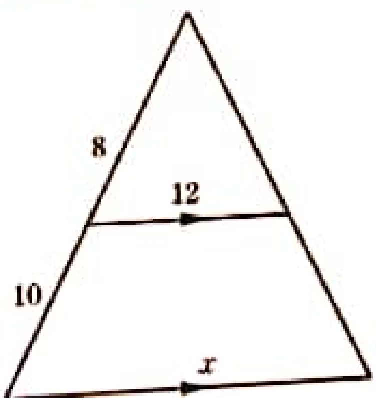
8. Use the following figure to prove that:



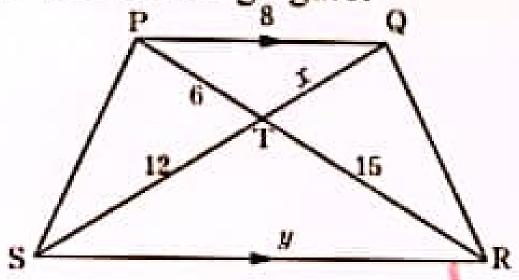
(a) $\Delta ADC \sim \Delta BEC$

(b) $\frac{\overline{AF}}{\overline{BF}} = \frac{\overline{EA}}{\overline{DB}}$

9. To determine the value of x in the figure below, Dora writes the equation $\frac{12}{8} = \frac{x}{10}$. Is this equation correct? Explain. If it is incorrect, write the correct equation and find the value of x.



10. In the following figure:



(a) List all pairs of congruent angles, giving reasons.

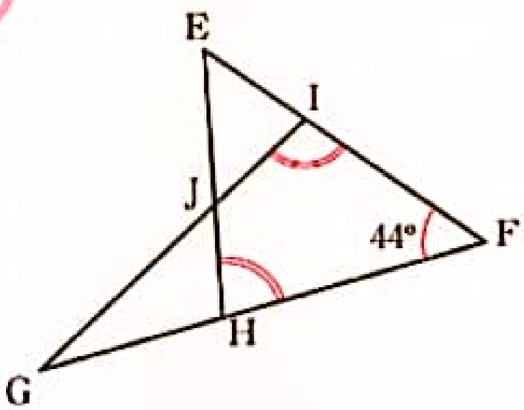
(b) Name pairs of similar triangles, giving a similarity theorem to justify your answer.

(c) Find the values of x and y.

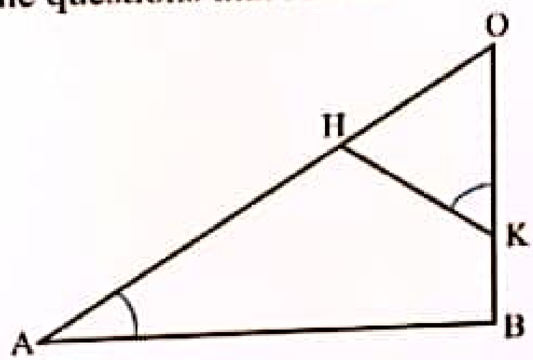
11. In the following figure,

(a) Show that $\Delta EFJ \sim \Delta GFI$

(b) If $H\hat{E}I : G\hat{I}F = 1:3$, find $<J\hat{E}F$



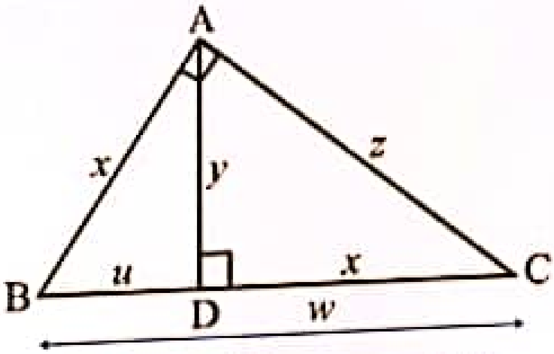
12. Study the following figure and answer the questions that follow.



(a) Name the triangle which is similar to ΔAOB and give reasons

(b) If $OA = 10\text{cm}$, $OB = 8\text{cm}$, $OK = 6\text{cm}$, $AB = 7\text{cm}$, calculate OH and HK.

13. In the following figure, ΔABC is a right-angled at A and AD is an altitude.



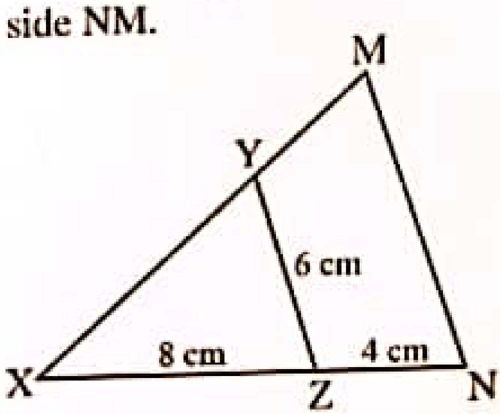
(a) Name with reasons, two triangles which are similar to ΔABD

(b) Show that $y^2 = uv$, $x^2 = uw$, and $xz = yw$.

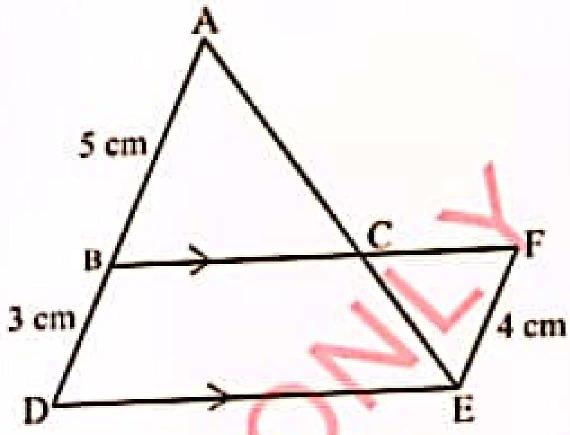
(c) If $\overline{AB} = 40$ cm, $\overline{AC} = 30$ cm, $\overline{BC} = 50$ cm, calculate $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$.

14. A man wishes to find the width of a river. There is a post at A on the far bank directly opposite post A and another at E so that A, C, and E align with each other, with ED being at the right angles with the bank. BC, DC, and DE are measured and found to be 117m, 26m, and 16 m, respectively. Find the width of the river.

15. In the following figure, find the proportionality constant between similar triangles and the length of side NM.



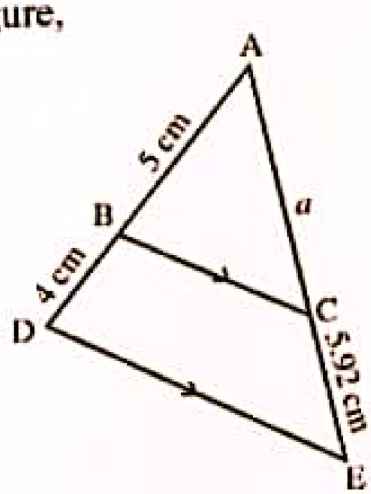
16. Study the following figure and answer the questions that follow.



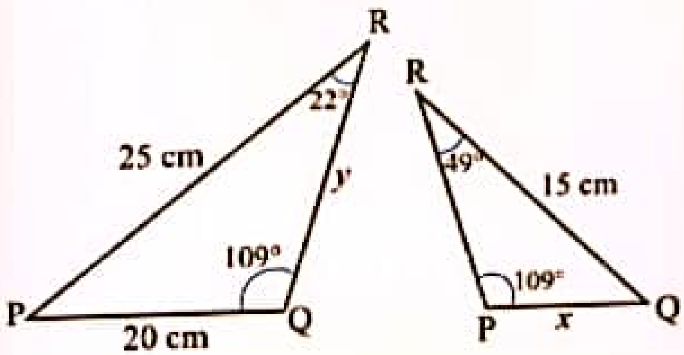
(a) State a triangle which is similar to triangle ABC.

(b) Calculate the constant of proportionality between triangles identified in part (a).

17. Find the value of *a* n the following figure,



18. Use the following figures to show that $\Delta RPQ \sim \Delta ABC$ and hence calculate the values of x and y.



Side-Side-Side (SSS) similarity theorem

The SSS similarity theorem states that, two triangles are similar if three pairs of corresponding sides are proportional. Figure 3.4 describes the SSS theorem.

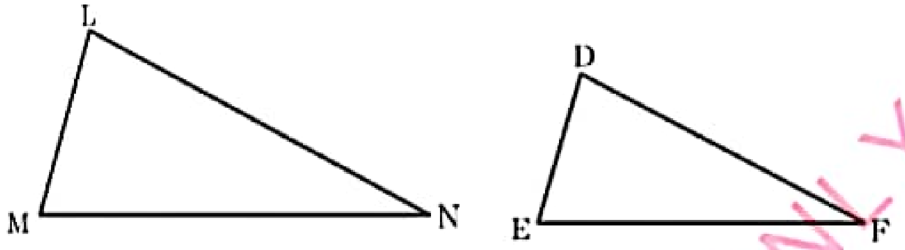


Figure 3.4: *Similar triangles by the Side-Side-Side theorem*

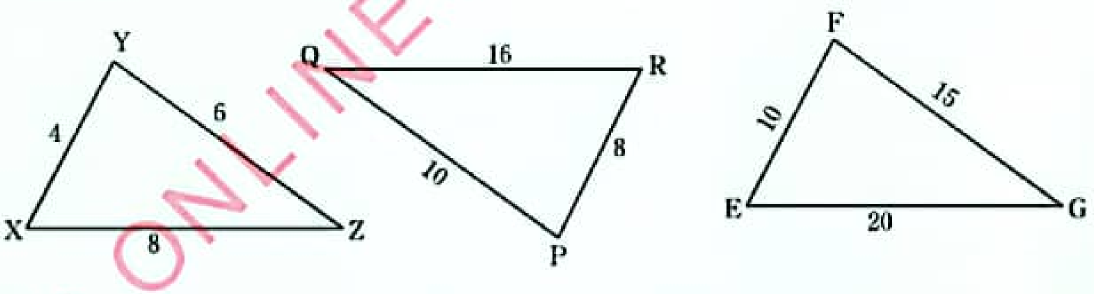
In Figure 3.4, it can be observed that, if the ratio of their sides are such that

$\frac{\overline{LM}}{\overline{DE}} = \frac{\overline{LN}}{\overline{DF}} = \frac{\overline{MN}}{\overline{EF}}$, then $\Delta LMN \sim \Delta DEF$

Therefore, $L\hat{M}N = D\hat{E}F$, $M\hat{N}L = E\hat{F}D$ and $M\hat{L}N = E\hat{D}F$.

Example 3.9

With reasons, identify a pair of triangles which are similar in each of the following triangles.



Solution

To show that the triangles are similar, find the ratio of the lengths of corresponding sides (shortest, longest and the remaining sides).

Comparing ΔXYZ and ΔPQR

Ratio of shortest sides: $\frac{XY}{PR} = \frac{4}{8} = \frac{1}{2}$

Ratio of longest sides: $\frac{XZ}{QR} = \frac{8}{16} = \frac{1}{2}$

Ratio of remaining sides: $\frac{YZ}{PQ} = \frac{6}{10} = \frac{3}{5}$

The ratios of the corresponding sides are not all equal. Therefore ΔXYZ and ΔPQR are not similar.

Comparing ΔXYZ and ΔEFG

Considering ΔXYZ and ΔPQR:

Ratio of shortest sides: $\frac{XY}{EF} = \frac{4}{10} = \frac{2}{5}$

Ratio of longest sides: $\frac{XZ}{EG} = \frac{8}{20} = \frac{2}{5}$

Ratio of remaining sides: $\frac{YZ}{FG} = \frac{6}{15} = \frac{2}{5}$

The ratios of the corresponding sides are equal. Therefore, $\Delta XYZ \sim \Delta PQR$.

Comparing ΔPQR and ΔEFG.

Ratio of shortest sides: $\frac{PR}{EF} = \frac{8}{10} = \frac{4}{5}$

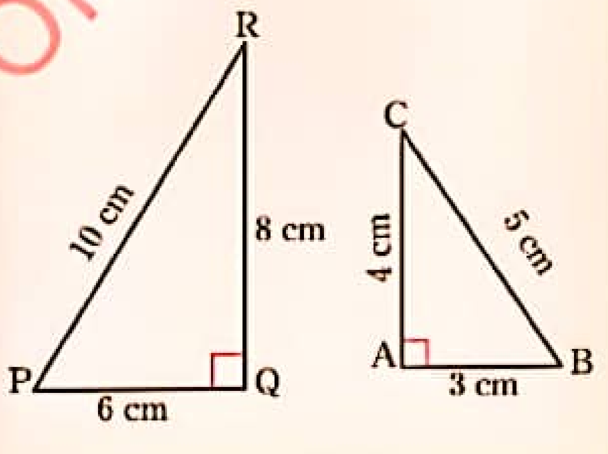
Ratio of longest sides: $\frac{QR}{EG} = \frac{16}{20} = \frac{4}{5}$

Ratio of remaining sides: $\frac{PQ}{FG} = \frac{10}{15} = \frac{2}{3}$

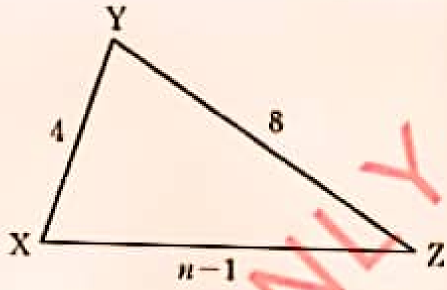
The ratios of the corresponding sides are not all equal. Therefore, ΔPQR and ΔEFG are not similar.

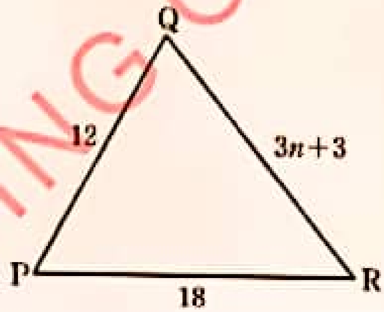
Exercise 3.3

1. Study the following pairs of figures determine whether they are similar or not. Indicate the similarity theorem used to support your answer.



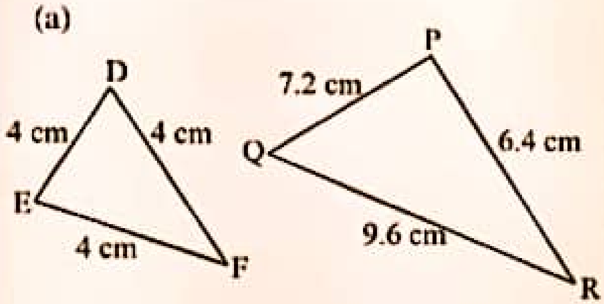
2. Find the value of n which makes $\Delta PQR \sim \Delta XYZ$.



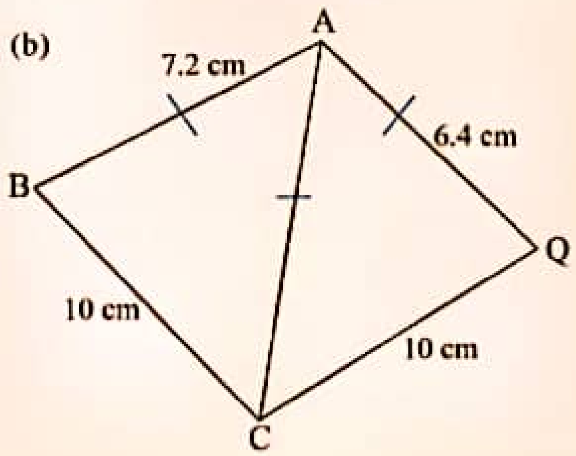


3. Confirm if each of the following pairs are similar triangles.

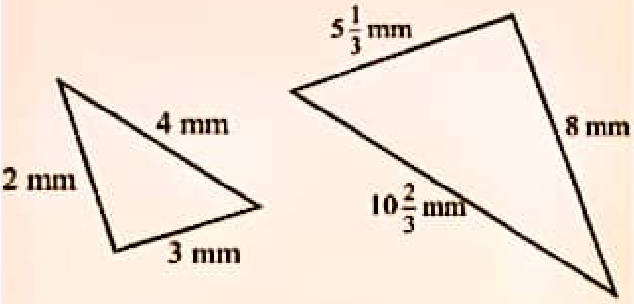
(a)



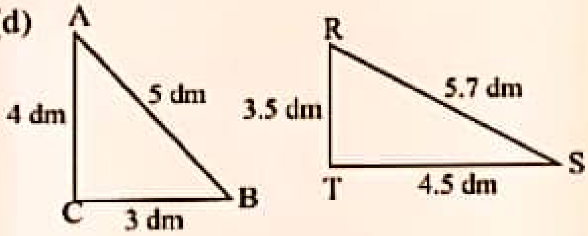
(b)



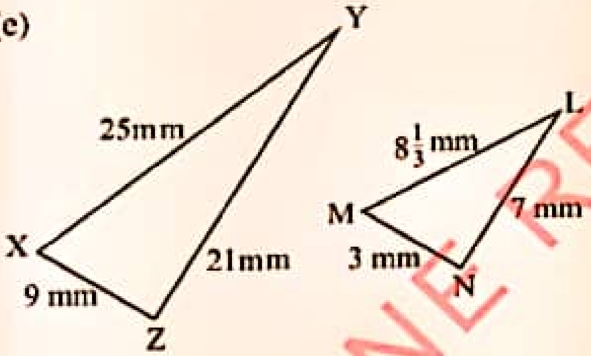
(c)



(d)



(e)



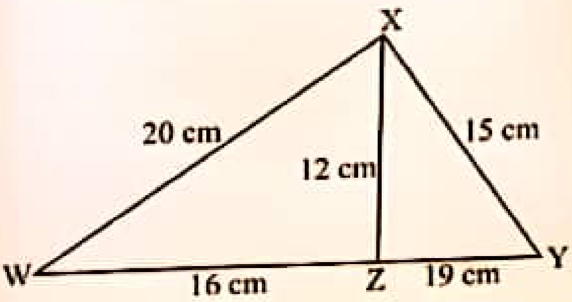
4. It is given that $AB = \frac{2}{3}XY$,

$BC = \frac{2}{3}YZ$, $BC = \frac{2}{3}YZ$ and

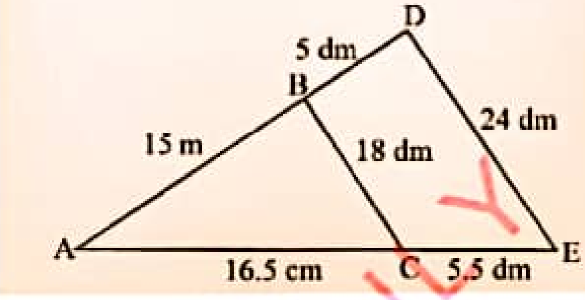
$AC = \frac{2}{3}XZ$, show that

$\Delta ABC \sim \Delta XYZ$.

5. Show that $\Delta XYZ \sim \Delta YZX$.



6. Identify similar triangles with justifications from the following figure.



Side-Angle-Side (SAS) similarity theorem

The SAS similarity theorem states that, two triangles are similar if two pairs of their corresponding sides are proportional and one pair of corresponding angles (formed by these sides) are equal. Figure 3.5 describes the SAS theorem.

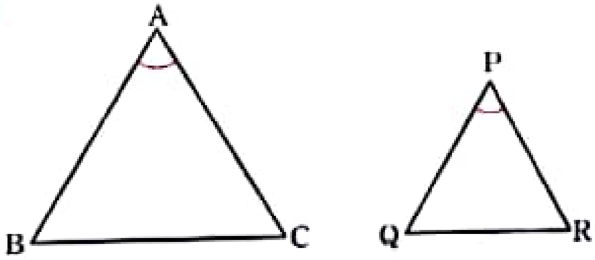


Figure 3.5: *Similar triangles by Side-Angle-Side theorem*

In Figure 3.5, if

$\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{AC}}{\overline{PR}}$ and $B\hat{A}C = Q\hat{P}R$, the

$\Delta ABC \sim \Delta PQR$.

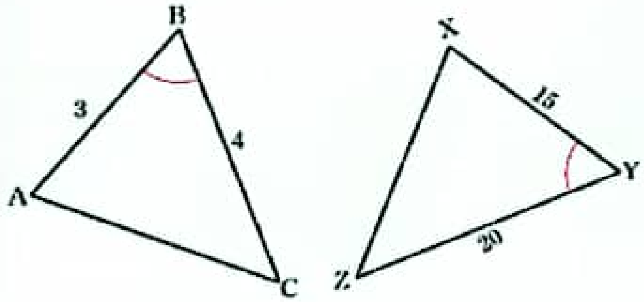
Therefore,

$A\hat{B}C = P\hat{Q}R$ and

$A\hat{C}B = P\hat{R}Q$.

Example 3.10

Given the following triangles, prove that ΔABC and ΔXYZ are similar.



Proof

Ratio of shortest sides: $\frac{AB}{XY} = \frac{3}{15} = \frac{1}{5}$

Ratio of longest sides: $\frac{BC}{YZ} = \frac{4}{20} = \frac{1}{5}$

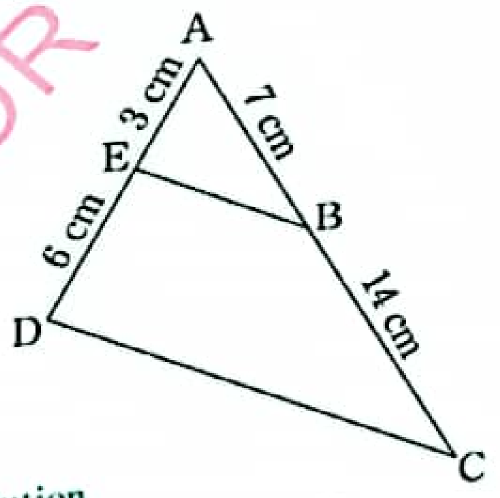
$A\hat{B}C = X\hat{Y}Z$ (given)

Thus, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{1}{5}$ and $A\hat{B}C = X\hat{Y}Z$

Therefore, $\Delta ABC \sim \Delta XYZ$ (SAS theorem)

Example 3.11

Given the following figure, show that $\Delta AED \sim \Delta ABE$.



Solution

From ΔACD and ΔABE

Ratio of the shortest sides:

$$\frac{\overline{AE}}{\overline{AD}} = \frac{3\text{cm}}{(3+6)\text{cm}} = \frac{3}{9} = \frac{1}{3}$$

Ratio of the longest sides:

$$\frac{\overline{AB}}{\overline{AC}} = \frac{7\text{cm}}{(7+14)\text{cm}} = \frac{7}{21} = \frac{1}{3}$$

$\angle BAE = \angle CAD$ (Common).

Thus $\frac{\overline{AE}}{\overline{AD}} = \frac{\overline{AB}}{\overline{AC}} = \frac{1}{3}$ and

$\angle BAE = \angle CAD$.

Therefore, $\Delta ACD \sim \Delta ABE$.

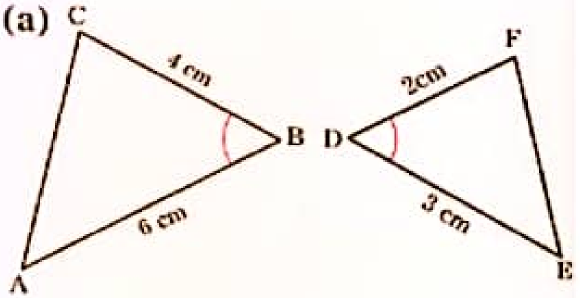
Activity 3.3: Exploring similarity through perpendicular bisectors

1. Use a geometrical software of your choice (or any other method) to draw a triangle ΔPQR.
2. Construct a line perpendicular to $\overline{QR}$ through point P, and label the intersection as S.
3. Measure segments $\overline{PQ}$, $\overline{PS}$, $\overline{PR}$, and $\overline{RS}$.
4. Calculate the ratios $\frac{\overline{PQ}}{\overline{PC}}$ and $\frac{\overline{PR}}{\overline{PS}}$.
5. Drag point C until $\frac{\overline{PQ}}{\overline{PS}} = \frac{\overline{PR}}{\overline{RS}}$.
6. For which value of $\angle PQR$ are ΔPQR and ΔPRS similar?
7. Share your findings about the theorem that supports your answer in task 6, and explain how it applies to the similarity observed in this activity.

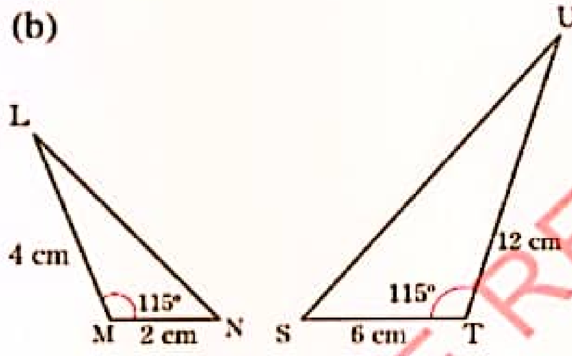
Exercise 3.4

1. In each of the following pairs of figures, determine whether they are similar or not. Indicate the similarity theorem used to support your answer.

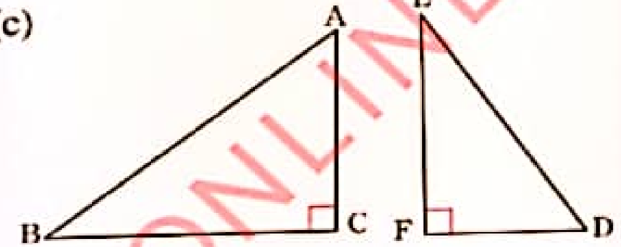
(a)



(b)

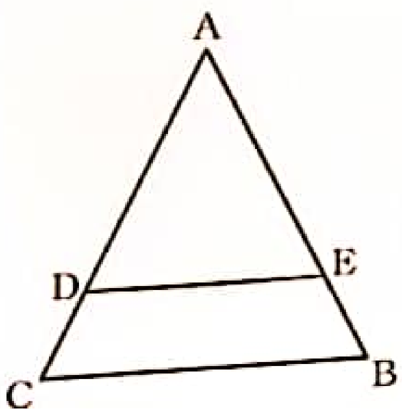


(c)



2. In the following figure, $\frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$.

Show that $\Delta ABC \sim \Delta AED$.

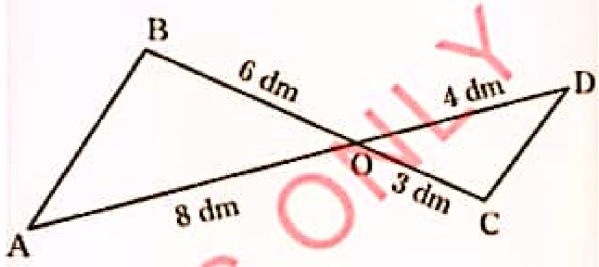


3. In the following figure, if

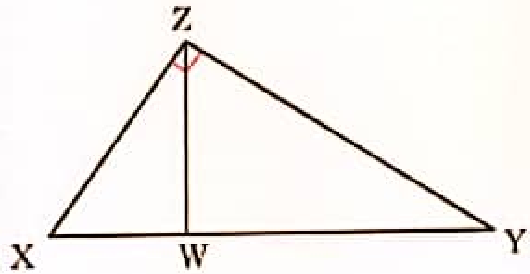
$\overline{OA} = 8$ dm, $\overline{BO} = 6$ dm,

$\overline{OD} = 4$ dm and $\overline{OC} = 3$ dm.

Verify that $\Delta AOB \sim \Delta DOC$.



4. In the following figure, ΔXYZ is a right-angled triangle and $\frac{\overline{XW}}{\overline{XZ}} = \frac{\overline{XZ}}{\overline{XY}}$. Prove that $\overline{ZW}$ is perpendicular to $\overline{XY}$.



5. To determine the height of a tree, Adelfina decided to stand in front of a tree so that the tip of her shadow coincides with the tip of the shadow of the tree. She stood 30 m from the base of the tree. If her height is 160 cm and her shadow is 220 cm:

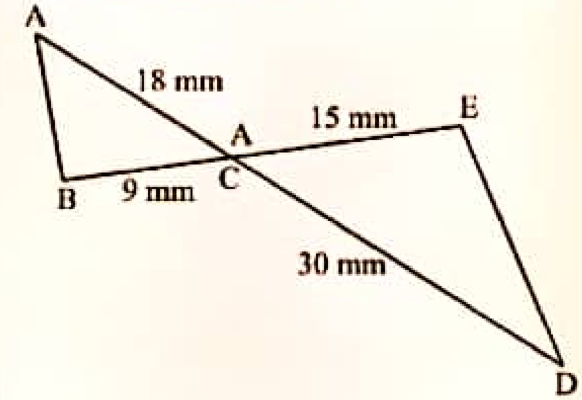
(a) Draw a diagram to show the given information.

(b) What theorem of similarity of triangles could she use to calculate the height of the tree?

(c) What is the height of the tree to the nearest metre?

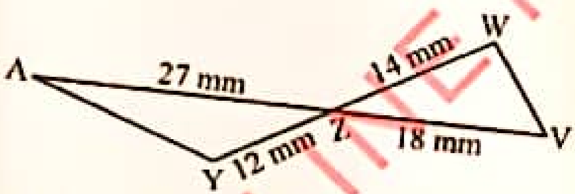
(d) Lemalai, who is 190 cm tall, did the same procedure described above. His shadow is 150 cm long. How far was he from the tree?

6 Study the following figure and show that ΔACB ~ ΔDCE.

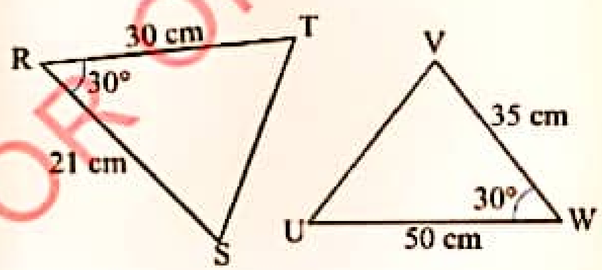


7. Show with reasons that the following triangles are similar.

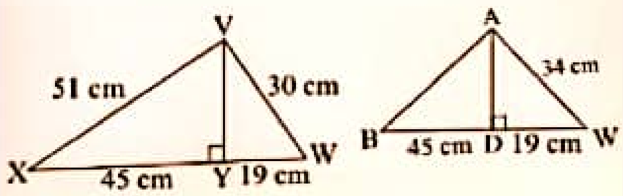
(a)



(b)



8. In the following triangles, ΔABC ~ ΔVWX.



(a) Find the scale factor of ΔVWX to ΔABC.

(b) Find the ratio of the area of ΔVWX to area of ΔABC.

(c) Explain how the results in (a) are related with the results in (b)

Chapter summary

1. **AA Similarity theorem**

   If the correspondence between triangles is such that two pairs of corresponding angles are equal, then the triangle are similar.

2. **SAS Similarity theorem**

   If the correspondence between two triangles is such that the lengths of two pairs of corresponding sides are proportional and the included angles are equal, then the triangles are similar.

3. **SSS Similarity theorem**

   If the correspondence between two triangles is such that the lengths of corresponding sides are proportional, then the triangles are similar.

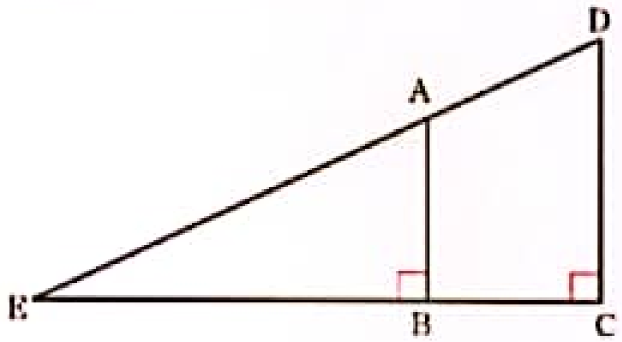
4. Similar figures have the same shape.

5. In similar figures, the ratios of the lengths of corresponding sides are equal. That is, corresponding sides are proportional. The value of the ratio is called the constant of proportionality or scale factor.

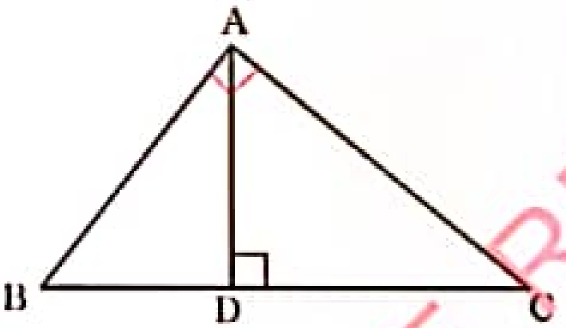
6. The symbol used to indicate similarity between figures is " ~ ".

Revision exercise 3

1. In the following figure determine the pairs of equal angles and the proportional sides which make $\Delta AEB \sim \Delta DEC$.



2. Name two triangles similar to ΔABC in the following figure.

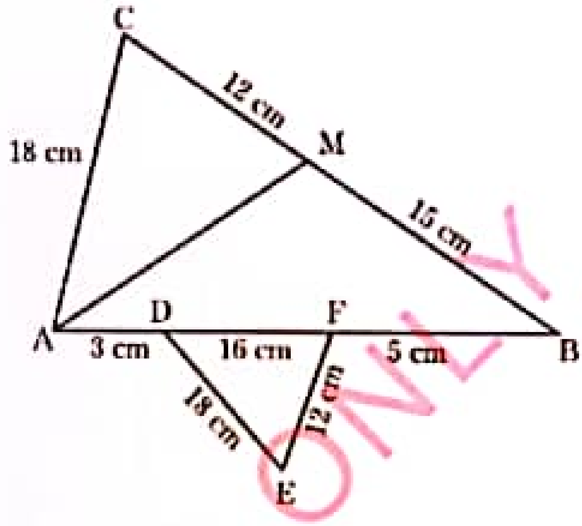


3. If $\Delta LMN \sim \Delta PQR$ and $\Delta PQR \sim \Delta ABC$, mention equal angles in ΔABC and ΔLMN, hence mention all proportional sides in ΔPQR and ΔABC. Find the ratio of proportionality for the similarity of ΔPQR and ΔABC.

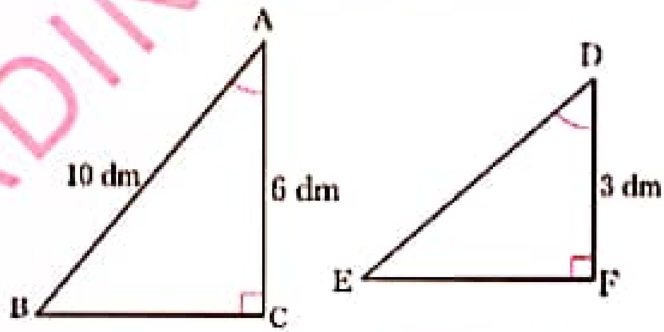
4. A triangle ABC is such that $\overline{CA}$ is extended to X and $\overline{BA}$ is extended to Y so that $\overline{XY} // \overline{BC}$. Prove that $\frac{\overline{AX}}{\overline{AC}} = \frac{\overline{XY}}{\overline{CB}}$.

5. A trapezium PQRS is such that $\overline{PQ} // \overline{RS}$. If the diagonals intersect at X, prove that $\frac{\overline{PX}}{\overline{RX}} = \frac{\overline{QX}}{\overline{SX}}$.

6. Name a triangle which is similar to ΔABC in the following figure.



7. Use similarity to find the value of $\overline{ED}$ in the following figures.

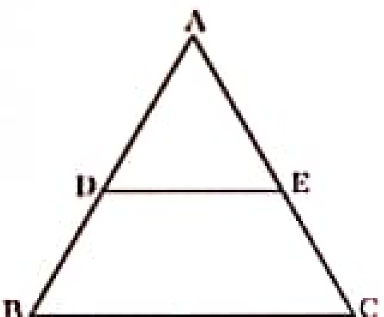


8. If $\Delta PQR \sim \Delta LMN$, $\overline{PR} = 20$ dm, $\overline{NL} = 10$ dm, $\overline{NM} = 12$ dm and $\overline{LM} = 9$ dm, find the lengths of the other sides of ΔPQR.

9. Prove that any two equilateral triangles are similar.

10. In ΔPMT and ΔQNS, $P\hat{M}T = Q\hat{S}N = M\hat{T}P = Q\hat{N}S$. Prove that $\overline{PM} \times \overline{NS} = \overline{QS} \times \overline{MT}$

11. Study the following figure, where $\overline{DE}$ is parallel to $\overline{BC}$. Answer the questions that follow.



(a) If $\overline{AD} = 4$ dm, $\overline{AB} = 8$ dm and $\overline{DE} = 10$ dm, find the value of $\overline{BC}$.

(b) If $\overline{AE} = 5$ cm, $\overline{AC} = 15$ cm, $\overline{BC} = 24$ cm, find the value of $\overline{DE}$.

(c) If $\overline{AD} = 7$ dm, $\overline{DE} = 11$ dm, $\overline{BC} = 22$ dm, find the value of $\overline{BD}$.

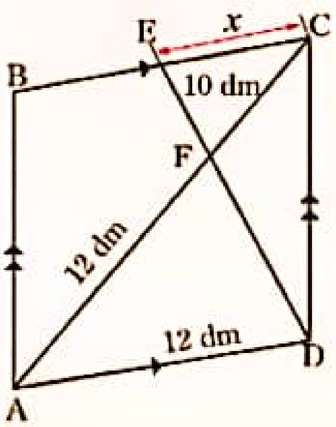
(d) If $\overline{BD} = 9$ dm, $\overline{DE} = 20$ dm, $\overline{BC} = 35$ dm, find the value of $\overline{AD}$.

(e) If $\overline{AD} = 10$ cm, $\overline{DE} = 24$ cm, $\overline{BC} = 84$ cm, find the value of $\overline{AB}$.

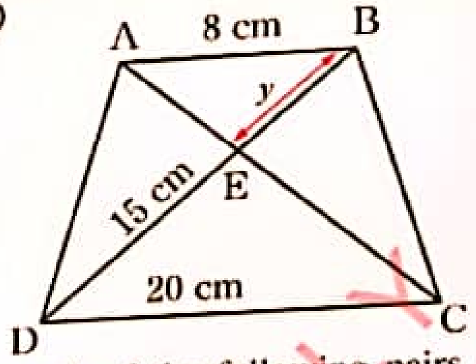
(f) If $\overline{AE} = 3$ cm, $\overline{DE} = 3$ cm, $\overline{BC} = 7$ cm, find the value of $\overline{CE}$.

12. Determine the values of x and y in the following figures:

(a)

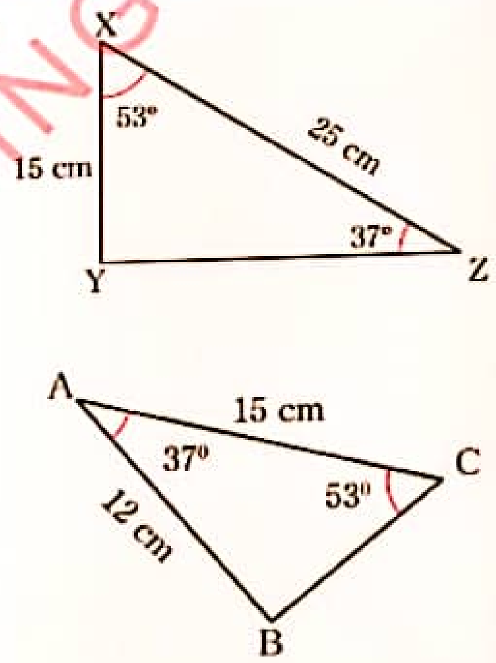


(b)

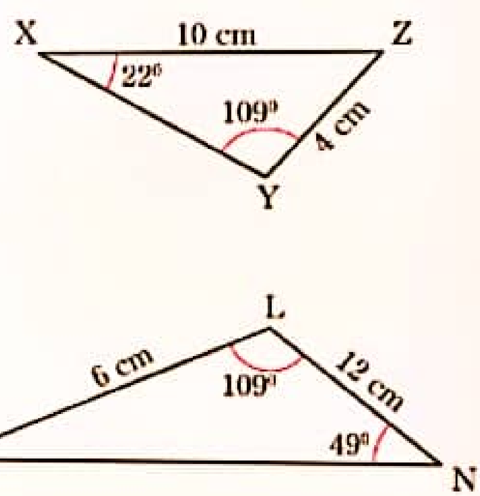


13. In each of the following pairs of figures, which triangles are similar? Calculate the lengths of the sides and angles which are not given.

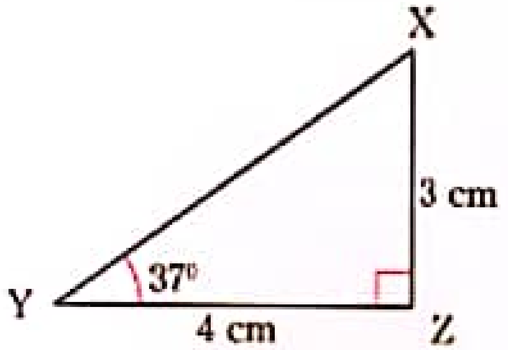
(a)

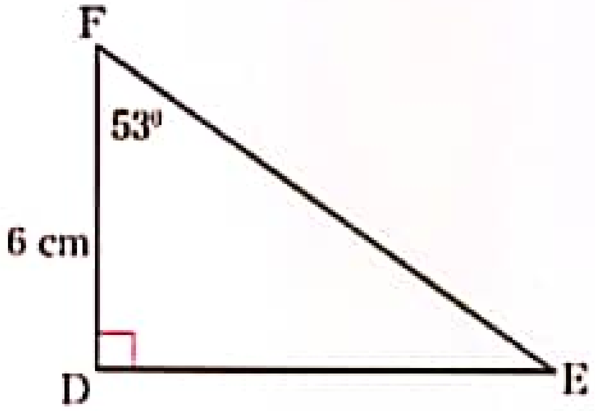


(b)

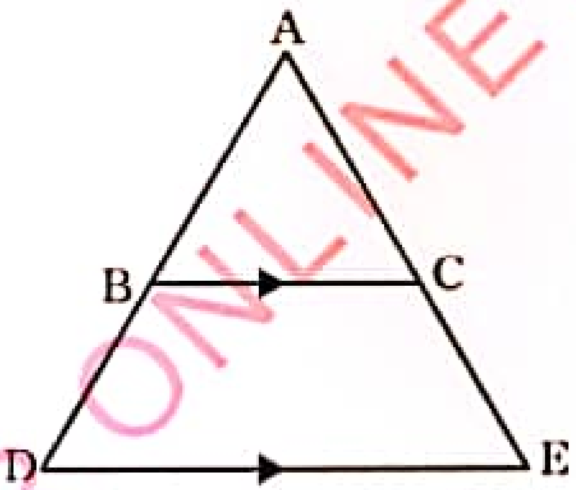


(c)

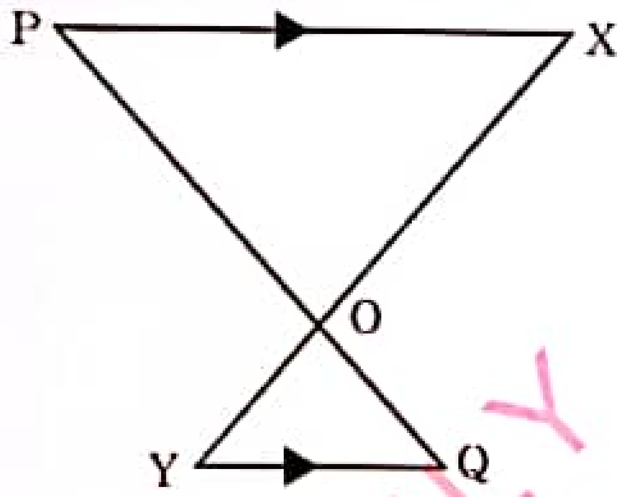




14. In the following figure, explain why ΔABC and ΔADE are similar? If $\overline{AB} = 8$ dm, $\overline{AC} = 9$ dm, $\overline{BC} = 6$ dm and $\overline{AD} = 12$ dm, calculate the values of $\overline{AE}$ and $\overline{DE}$.



15. Study the following figure and answer the questions that follow.



(a) Which triangle is similar to ΔYOQ?

(b) Given that $\overline{OP} = 4$ m, $\overline{OX} = 7$ m, $\overline{PX} = 6$ m, $\overline{YQ} = 4.5$ m, calculate the values of $\overline{OY}$ and $\overline{OQ}$.

16. A triangle ABC is defined such that $\overline{CD}$ bisects $A\hat{C}B$.

(a) Prove that $\frac{\overline{AD}}{\overline{AC}} = \frac{\overline{BD}}{\overline{BC}}$.

(b) If $\overline{AC} = 36$ cm and $\overline{BC} = 28$ cm, find length of side AD if $\overline{AB} = 16$ cm.

17. In triangle EFG, $\overline{HL}$ is drawn inside the triangle parallel to $\overline{EF}$. Prove that $\frac{\overline{FL}}{\overline{LG}} = \frac{\overline{EH}}{\overline{HG}}$ and hence find FL if H = 35 cm, HG = 44 cm, and LG = 36 cm.

Chapter Four

Algebra

Introduction

Skills in solving problems systematically, logically, and making connections between variables are essential in daily life. Such skills can be developed and applied by learning algebra. In this chapter, you will learn about binary operations, quadratic expressions, and solving quadratic equations. The competencies developed will enable you to solve daily life problems such as budgeting, comparing prices, planning for a journey and figuring proportions in cooking. Also, algebra is useful in other subjects and fields such as science, engineering and technology and many other applications.

Think

The role algebra plays in real life.

Binary operations

A binary operation is a rule for combining two quantities to produce a unique quantity from the same set. It makes use of symbols that represent one or more operations.

The symbols used in binary operations are not standard, this means, they do not represent a specific operation. A symbol may be used in a certain operation in an expression, and yet have a different meaning in another expression.

A binary operation may be denoted by symbols such as Δ and $*$, depending on the instructions given for the operation. The instructions may be given in words or by symbols. Binary operations may involve the basic arithmetic operations such as addition (+), subtraction (–), multiplication (×), and division (÷) depending on the given definition. Engage in Activity 4.1 to learn more on binary operations.

(b) How changing the number of apples or oranges affects the total fruit count?

2. If an apple is sold at TShs 950 and an orange is sold for Tshs 200, calculate the total cost of buying all the fruits.

Example 4.1

If $a * b = 5a - b$, find the value of $6 * 9$.

Solution

From $a * b = 5a - b$, it implies that,

$$6 * 9 = 5 \times 6 - 9$$
$$= 30 - 9$$
$$= 21$$

Therefore, $6 * 9 = 21$.

Example 4.2

If $a * b = a^2 - b$, find the value of y given that $4 *(2 * y) = 4$.

Solution

Given $a * b = a^2 - b$.

Start with the operation in the brackets.

$$4 *(2 * y) = 4 *(2^2 - y)$$
$$= 4 *(4 - y)$$
$$= 4^2 - (4 - y)$$
$$= 16 - 4 + y$$
$$= 12 + y$$

Thus, $12 + y = 4$

$$y = -8$$

Therefore, $y = -8$.

Example 4.3

Given that $x * y = 4x + 6y$, find the value of $6 * 4$.

Solution

Given $x * y = 4x + 6y$. It follows that,

$$6 * 4 = (4 \times 6) + (6 \times 4)$$
$$= 24 + 24$$
$$= 48$$

Therefore, $6 * 4 = 48$.

Example 4.4

Given that $p * q = p^2 - q^2$, find the value of $7 * 3$.

Solution

From $p * q = p^2 - q^2$, it follows that,

$$7 * 3 = 7^2 - 3^2$$
$$= 49 - 9$$
$$= 40$$

Therefore, $7 * 3 = 40$.

Example 4.5

If $a * b = 3a + b$, find the value of $5 * 8$.

Solution

Given $a * b = 3a + b$. It follows that,

$$5 * 8 = (3 \times 5) + 8$$
$$= 15 + 8$$
$$= 23$$

Therefore, $5 * 8 = 23$.

Example 4.6

If $x * y = x + 3y$, find the value of $5 * (8 * 6)$.

Solution

Given $x * y = x + 3y$.

Start with operation in the brackets. That is, $5*(8*6) = 5*(8+3\times 6)$

$= 5*26$

$= 5+3\times 26$

$= 5+78$

$= 83$

Therefore, $5*(8*6) = 83$.

Example 4.7

If $p\Delta q = 2p+5q$, find the value of $6\,\Delta\,3$.

Solution

From $p\Delta q = 2p+5q$, it implies that,

$6\,\Delta\,3 = (2\times 6)+(5\times 3)$

$= 12+15$

$= 27$

Therefore, $6\,\Delta\,3 = 27$.

Example 4.8

If $m\Delta n = \frac{m-n}{m+n}$. Calculate the value of $3\Delta 2$.

Solution

Given $m\Delta n = \frac{m-n}{m+n}$. It follows that,

$$3\Delta 2 = \frac{3-2}{3+2}$$

$$= \frac{1}{5}$$

Therefore, $3\Delta 2 = \frac{1}{5}$.

Exercise 4.1

1. Given that $*$ and Δ are two binary operations defined as $a*(b\,\Delta\,c) = a(b-c)$.

   If $a=2$, $b=5$ and $c=4$, evaluate the value of $(a*b)\,\Delta\,(a*c)$.

2. In a certain business, the profit calculation is defined by $a*b = 5(a+2b)$. If a company uses this formula to determine its revenue based on certain factors, what would be the result when defined by $4*2$?

3. Evaluate each of the following by using the defined binary operation.

   (a) $2*(3*4)$ if $x*y = 3x+6y$.

   (b) $5*(4*3)$ if $p*q = 3p-2q$.

   (c) $9\Delta 5$ if $c\,\Delta\,d = c^2-d^3$.

   (d) $(3*6)*6$ if $m*n = mn-\frac{n}{m}$.

4. An operation Δ on integers d and k is defined as $d\,\Delta\,k = dk+5d-3k$. Suppose a company uses this operation to calculate a special index based on its data inputs. Find the index value based on the operation $3\Delta 2$.

5. An operation Δ is defined by $u\Delta v = \frac{\sqrt{uv}}{v}$. Find the value of $9\Delta 4$.

6. In a certain situation, a company uses a special calculation formula defined by $a*b=a^2+2b$ to determine production efficiency. Given that $4*(2*y)=25$, find the value of y.

7. An operation on integers a and b is defined by $a*b=3a^3+2b$, find the value of $(3*2)*6$.

8. Given that, $u\Delta v=u^v-v^u$, find the value of $2\ \Delta\ 3$.

9. If $(a*b)=a^2+b$, find the value of x given that $4*(2*x)=25$.

10. (a) If $p*q=p+q+pq$. Find the value of $2*(1*3)$.

(b) If $a*b=a^2+b^2$. Find the value of $(2*3)*4$.

Transposition of formulae

A formula is an equation which shows how variables are related. For example, the formula which is used to find the circumference, C of a circle is given by $C=\pi d$, where C is expressed in terms of π and d. In this case, C is the subject of the formula. It is also possible to express d as the subject of the formula, that is, $d=\frac{C}{\pi}$.

The procedure of expressing a formula in different ways is called transposition of formula. The following are some examples of formulae:

(a) $A=lb$

(b) $V=\pi r^2h$

(c) $I=\frac{PRT}{100}$

(d) $A=\frac{1}{2}(a+b)h$

(e) $T=2\pi\sqrt{\frac{l}{g}}$

(f) $y=mx+c$

(g) $s=ut+\frac{1}{2}at^2$

Example 4.9

Given that $y=mx+c$, make m the subject of the formula.

Solution

Given the formula $y=mx+c$. (i)

Subtract c from both sides of equation (i) to get $y-c=mx$. (ii)

Divide equation (ii) by x throughout to get $m=\frac{y-c}{x}$

Therefore, $m=\frac{y-c}{x}$, provided $x\neq 0$.

Example 4.10

The volume, V of a cylinder with a base of radius r and height h is given by $V=\pi r^2h$. Make r the subject of the formula.

Solution

Given the formula $V=\pi r^2h$. (i)

Divide both sides of equation (i) by πh to get $\frac{V}{\pi h} = r^2$. (ii)

Apply the square root to both sides of equation (ii) to get $\sqrt{\frac{V}{\pi h}} = r$.

Therefore, $r = \sqrt{\frac{V}{\pi h}}$.

Example 4.11

Given that $T = 2\pi\sqrt{\frac{l}{g}}$, make l the subject of the formula.

Solution

Given the formula $T = 2\pi\sqrt{\frac{l}{g}}$. (i)

Divide both sides of equation (i) by 2π to obtain $\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$. (ii)

Squaring both sides of equation (ii) gives $\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$. (iii)

Multiplying both sides of equation (iii) by g gives $g\left(\frac{T}{2\pi}\right)^2 = l$.

But $g\left(\frac{T}{2\pi}\right)^2 = \frac{gT^2}{4\pi^2}$.

Therefore, $l = \frac{gT^2}{4\pi^2}$.

Example 4.12

The simple interest, I on principal, P for time, T years and at the rate of $R\%$ per annum is given by the formula $I = \frac{PRT}{100}$. Make P the subject of the formula.

Solution

Given $I = \frac{PRT}{100}$. (i)

Multiplying both sides of equation (i) by 100 gives $100I = PRT$. (ii)

Divide both sides of (ii) by RT to get $\frac{100I}{RT} = P$.

Therefore, $P = \frac{100I}{RT}$.

Exercise 4.2

In questions 1 to 13 make the given letter the subject in each formula

	Formula	Letter
1.	$I = \frac{PRT}{100}$	R
2.	$C = 2\pi r$	r
3.	$A = \frac{1}{2}bh$	h
4.	$E = \frac{w}{pv}$	p
5.	$A = P + \frac{PRT}{100}$	P
6.	$A = 2\pi r(r+h)$	h
7.	$T = \frac{3}{2}\sqrt{l}$	l

8.	$V = \frac{1}{2}\pi r^2 h$	r
9.	$S = \frac{1}{2}at^2$	t
10.	$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$	f
11.	$P = w\left(\frac{1+a}{1-a}\right)$	a
12.	$T = 2\pi\sqrt{\frac{l}{g}}$	g
13.	$C = \frac{5}{9}(F-32)$	F

Quadratic expressions

An expression whose highest exponent (degree) of the variable is 2 is called a quadratic expression.

The following are examples of quadratic expressions:

(i) $4z^2 + 3$, the highest exponent (or degree) of z is 2.

(ii) $6y^2 + y$, the highest exponent of y is 2.

(iii) $3n^2 - 2n + 1$, the highest exponent of n is 2.

In general, a quadratic expression has the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are real numbers. The term ax^2 is called the quadratic term, where a is the coefficient of x^2. The term bx is called the linear term (middle term), where b is the coefficient of x and c is the constant term.

For example, in the quadratic expression $4x^2 - 6x + 7$, the coefficient of x^2 is 4, the coefficient of x is -6 and the constant is 7.

Activity 4.2: Deducing quadratic expressions from real life

1. A garden designer plans to design different gardens in a compound. During an investigation, he noticed that the length of each garden is 4 m longer than its width. Deduce the simplest possible expression of the area of the garden by expanding the factors.
2. Explore various sources to learn more about how you can expand such results.
3. Make a presentation to demonstrate how you arrived at your conclusion.

From activity 4.2, it can be observed that, quadratic expressions can be derived from the product of two linear expressions. For instance in Figure 4.1, if the width of a rectangle is $(y + 1)$ unit and its length is $(2y + 3)$ unit, then the area is $(2y + 3)(y + 1)$ square units.

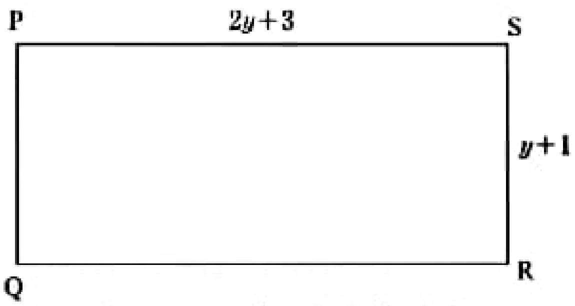


Figure 4.1: *Rectangle PQRS which demonstrate an area as a product of factors*

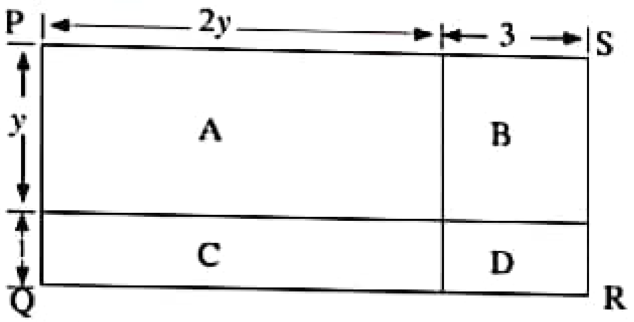


Figure 4.2: *Regions of rectangle PQRS*

When the rectangle PQRS is subdivided into four rectangles A, B, C and D, as shown in Figure 4.2, the following can be deduced:

Area of region $A = 2y \times y = 2y^2$ square units.

Area of region B
$= 3 \times y = 3y$ square units.

Area of region C
$= 1 \times 2y = 2y$ square units.

Area of region D
$= 3 \times 1 = 3$ square units.

The total Area of regions A, B, C, and D

$= (2y^2 + 3y + 2y + 3)$ square units.

$= (2y^2 + 5y + 3)$ square units.

From Figure 4.1, the total area is $(2y+3)(y+1)$ square units.

Thus, $(2y+3)(y+1) = 2y^2 + 5y + 3$.

The expression $2y^2 + 5y + 3$ is called a quadratic expression expanded from $(2y+3)(y+1)$. This expansion can also be done by multiply each term of $2y+3$ by $y+1$ as follows:

$$(2y+3)(y+1) = 2y(y+1) + 3(y+1)$$
$$= 2y^2 + 2y + 3y + 3$$
$$= 2y^2 + 5y + 3$$

Thus, $2y^2 + 5y + 3$ is the expanded form of $(2y+3)(y+1)$.

Therefore, $(2y+3)(y+1) = 2y^2 + 5y + 3$.

Example 4.13

Expand $(z+2)(2z-3)$.

Solution

$$(z+2)(2z-3) = z(2z-3) + 2(2z-3)$$
$$= 2z^2 - 3z + 4z - 6$$
$$= 2z^2 + z - 6$$

Therefore,

$(z+2)(2z-3) = 2z^2 + z - 6$.

Example 4.14

Find the coefficient of n and n^2 in the expansion of $(n+9)(n+3)$.

Solution

$$(n+9)(n+3) = n(n+3) + 9(n+3)$$
$$= n^2 + 3n + 9n + 27$$
$$= n^2 + 12n + 27$$

Therefore, the coefficients of n and n^2 are 12 and 1, respectively.

Example 4.15

Expand $(6x+5)^2$.

Solution

$$(6x+5)^2 = (6x+5)(6x+5)$$
$$= 6x(6x+5) + 5(6x+5)$$
$$= 36x^2 + 30x + 30x + 25$$
$$= 36x^2 + 60x + 25$$

Therefore, $(6x+5)^2 = 36x^2 + 60x + 25$.

Note that, from the general quadratic expression ax^2+bx+c, where a, b and c are real numbers.

(a) If $a=0$, the expression becomes linear. For example, $6x+4$, in this case, $a=0$, $b=6$ and $c=4$.

(b) If $b=0$, the expression is a quadratic but without the middle term. For example, $4x^2+8$, in this case, $a=4$, $b=0$ and $c=8$.

(c) If $c=0$, the expression is a quadratic but without the constant term. For example, $10x^2-2x$, in this case, $a=10$, $b=-2$ and $c=0$.

(d) The basic identities of quadratic expressions are as follows:

(i) $(a+b)^2=a^2+2ab+b^2$

(ii) $a^2-b^2=(a+b)(a-b)$

(iii) $(x+a)(x+b)=x^2+(a+b)x+ab$

Exercise 4.3

In questions 1 to 12, write the expanded form of each of the given expressions:

1. $(2x+3)(x+1)$
2. $(4x-3)(2x+1)$
3. $y(y+6)$
4. $(3y+4)^2$
5. $(7n-2)^2$
6. $(a+b)^2$
7. $(a-b)^2$
8. $(4x-3)(2x-1)$
9. $(3p+2q)(6p-2q)$
10. $(x+y)(x-y)$
11. $(x+2)(y-3)$
12. $(-3+y)(y+4)$
13. Find the area of a triangle whose base is $(6x+1)$ cm and height is $(4x-2)$ cm. Write the answer in the expanded form.
14. Find the amount of money required to buy $7y$ items if each item costs $(6y+5)$ shillings. Give the answer in the expanded form.
15. Given three consecutive whole numbers, write the expanded form of the product of the first and the third if the second number is n.
16. Simplify $(4a-6)(2a+5)-(2a+5)(4a-3)$.
17. The width of a rectangular field is 20 m shorter than its length y metres. Write down an expression for the area A.
18. A rope which is 42 cm long is divided into two pieces to make two squares. Write an expression for the sum of the areas.
19. A neighbour has a picture which measures 10 cm by 20 cm. He wants to make a frame of constant width which will enclose the picture. Write an expression for the area of the frame.

Factorisation of quadratic expressions

Factorisation is the process of writing an expression as a product of its factors. Numbers and algebraic expressions can be expressed in terms of factors.

For instance, 10 can be expressed as a product of 2 and 5 or 1 and 10.

The expression x^2-x-12 can be expressed as a product of two linear factors $x-4$ and $x+3$. That is, $x^2-x-12=(x-4)(x+3)$.

Quadratic expressions can be factored using three main methods which are splitting the middle term, difference of two squares and perfect squares.

Factorisation by splitting the middle term

Factorisation by splitting the middle term involves the splitting of the middle term into two terms. For instance, in the quadratic expression ax^2+bx+c, the middle term is bx. To split the middle term, the following steps can be used:

(i) Find two numbers whose sum is equal to b and whose product is equal to ac.

(ii) To find the two numbers in (i), list all the factors of ac and determine a pair whose sum is b.

Example 4.16

Factorize the expression $2x^2+7x+6$ by splitting the middle term.

Solution

The coefficients are 2, 7 and the constant term is 6. That is, $a=2$, $b=7$ and $c=6$. So $ac=2\times6=12$.

Find the pair of factors of 12 whose sum is 7.

The pairs of factors of 12 are

The sum of 1 and 12 is 13

The sum of -1 and -12 is -12

The sum of 2 and 6 is 8

The sum of -2 and -6 is -8

The sum of 3 and 4 is 7

The sum of -3 and -4 is -7

Therefore, the correct choice is 3 and 4. So the terms are $3x$ and $4x$.

$$\begin{aligned}2x^2+7x+6&=2x^2+4x+3x+6\\&=(2x^2+4x)+(3x+6)\\&=2x(x+2)+3(x+2)\\&=(x+2)(2x+3)\end{aligned}$$

Therefore, $2x^2+7x+6=(x+2)(2x+3)$.

Example 4.17

Factorize $6x^2-11x+4$ by splitting the middle term.

Solution

Since $ac=24$ and $b=-11$, find a pair of factors of 24 whose sum is -11.

The pairs of factors of 24 are: -1 and -24, -2 and -12, -3 and -8, -6 and -4.

Therefore, the correct choice is -3 and -8 since $-3+-8=-11$ and $(-3)\times(-8)=24$.

Thus, $-11x=-3x-8x$.

It follows that,

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$
$$= (6x^2 - 3x) - (8x - 4)$$
$$= 3x(2x - 1) - 4(2x - 1)$$
$$= (2x - 1)(3x - 4)$$

Therefore,

$$6x^2 - 11x + 4 = (2x - 1)(3x - 4).$$

Example 4.18

Factorize $2x^2 + x - 10$ by splitting the middle term.

Solution

The correct choice of the pair of factors of –20 is – 4 and 5; Thus, $x = -4x + 5x$.

So that,

$$2x^2 + x - 10 = 2x^2 - 4x + 5x - 10$$
$$= (2x^2 - 4x) + (5x - 10)$$
$$= 2x(x - 2) + 5(x - 2)$$
$$= (x - 2)(2x + 5)$$

Therefore,

$$2x^2 + x - 10 = (x - 2)(2x + 5).$$

Example 4.19

A building company is designing a rectangular playing ground whose total area is defined by the expression $x^2 + 6x + 9$, where x in metres is the length of playing ground. Deduce the possible dimensions of the playing ground.

Solution

The correct choice of the pair of factors of 9 is 3 and 3. Thus, $6x = 3x + 3x$.

So that, $x^2 + 6x + 9 = x^2 + 3x + 3x + 9$

$$= (x^2 + 3x) + (3x + 9)$$
$$= x(x + 3) + 3(x + 3)$$
$$= (x + 3)(x + 3)$$
$$= (x + 3)^2$$

Thus, the quadratic expression has two factors which are identical $x^2 + 6x + 9 = (x + 3)^2$.

Therefore, the playing ground is a square of length $(x + 3)$ m.

Exercise 4.4

In questions 1 to 9, factorize the expressions by splitting the middle term:

1. $x^2 + 3x + 2$
2. $6y^2 + 11y + 4$
3. $2x^2 - 17x + 8$
4. $y^2 - 7y + 6$
5. $3d^2 - 2d - 8$
6. $3 - 5a + 2a^2$
7. $c^2 - 18c + 45$
8. $12x^2 + 27x - 39$
9. $x^2 - 7xy + 12y^2$
10. Maria wants to design a garden and needs to create a rectangular flower bed with dimensions that fit a specific area. The area of the bed can be modeled by the quadratic expression $3y^2 - 11y + 10$ m^2. Determine the possible lengths of the sides of the flower bed.

11. The area of a rectangular floor is defined by the expression $(4t^2+5t+1)$ square metre. Use this expression to express the possible dimensions of the floor.
12. John is trying to simplify the expression $-1+x+2x^2$ to determine how to distribute his resources more efficiently. To help him, factorize the expression to better understand the optimal allocation.

Factorization by the **difference of two** squares

An expression in the form a^2-b^2 is called the difference of two squares. Engage in Activity 4.3 to explore more about quadratic expressions which exist as a difference of two squares.

Activity 4.3: Exploring the difference of two squares

1. Determine the product of $(a+b)$ and $(a-b)$ with the identities a^2-b^2 and give your conclusion.
2. Use the results in task 1 to simplify the expression of the form $(ax)^2-b^2$ and note down your observations.
3. Use numbers and letters to create other similar expressions.
4. Confirm your findings by consulting other relevant sources such as books and the internet and share your findings.

Example 4.20

Factorize a^2-25b^2.

Solution

Write a^2-25b^2 as a difference of two squares.

$$a^2-25b^2=a^2-5^2b^2$$
$$=a^2-(5b)^2$$
$$=(a-5b)(a+5b)$$

Therefore, $a^2-25b^2=(a-5b)(a+5b)$.

Example 4.21

Factorize x^2-81.

Solution

Write x^2-81 as a difference of two squares. That is,

$$x^2-81=x^2-9^2$$
$$=(x+9)(x-9)$$

Therefore, $x^2-81=(x+9)(x-9)$.

Example 4.22

Find the exact value of $50001^2-49999^2$

Solution

$$50001^2-49999^2$$
$$=(50001+49999)(50001-49999)$$
$$=100000\times 2$$
$$=200000$$

Therefore, $50001^2-49999^2=200000$.

Factorization of perfect squares

A quadratic expression whose factor are identical is called a perfect square.

Consider $a^2+2ab+b^2=(a+b)^2$ and $a^2-2ab+b^2=(a-b)^2$. The two quadratic identities, $(a+b)^2$ and $(a-b)^2$ are called perfect squares.

Engage in Activity 4.4 to explore more on perfect squares quadratic equations.

Activity 4.4: Exploring the perfect square expressions

1. Use the splitting of the middle term method to factorise the expressions x^2+6x+9 and $x^2-8x+16$, and compare the results with the identities $(a+b)^2$ and $(a-b)^2$.
2. Study the nature of the middle terms in task 1 after splitting and note any unique features.
3. Use the findings from tasks 1 and 2, and varieties of relevant sources to study the characteristics of $(a+b)^2$ and $(a-b)^2$ hence generalize your findings.

From Activity 4.4, it can be observed that when factorising perfect square expressions, the middle term will always split into two equal terms which reflect the $2ab$ part of $a^2+2ab+b^2$ and the $-2ab$ part of $a^2-2ab+b^2$.

Example 4.23

Factorize x^2-6x+9 and write the answer in the form of $(x+a)^2$.

Solution

$$x^2-6x+9=(x-3)(x-3)$$
$$=(x-3)^2$$

Therefore, $x^2-6x+9=(x-3)^2$.

Example 4.24

Factorize $a^2+\frac{6}{5}a+\frac{9}{25}$.

Solution

Write $a^2+\frac{6}{5}a+\frac{9}{25}=(a+...)^2$

(the first term in the bracket must be a)

The second term in the bracket is obtained by taking the square root of the constant term.

That is, $\sqrt{\frac{9}{25}}=\frac{3}{5}$.

Therefore, $a^2+\frac{6}{5}a+\frac{9}{25}=\left(a+\frac{3}{5}\right)^2$.

Note:

1. In $a^2+2ab+b^2=(a+b)^2$, the square of the sum of two quantities is equal to the sum of their squares plus twice their product.
2. In $a^2-2ab+b^2=(a-b)^2$, the square of the difference of two quantities is equal to the sum of their squares minus twice their product.
3. In all perfect squares, the constant term is the square of half the coefficient of the linear (middle) term.
4. If ax^2+bx+c is a perfect square then $b^2=4ac$.

Exercise 4.5

In question 1 to 6, factorize the quadratic expressions by splitting the middle term:

1. x^2+6x+8
2. x^2+3x+2
3. $2y^2+9y+7$
4. $4u^2+9u+5$
5. $12m^2+37m+21$
6. $22a^2+25a+7$

In question 7 to 14, factorize the quadratic expressions by perfect square:

7. $c^2+\frac{8}{9}c+\frac{16}{81}$
8. $g^2+22g+121$
9. y^2-4y+4
10. $k^2+34k+289$
11. $w^2-16w+64$
12. $4r^2-12r+9$
13. $9x^2-6x+1$
14. $4+9a^2-12a$

In question 15 to 20, factorize the quadratic expressions by using the method of difference of two squares.

15. $9x^2-121$
16. x^2-100
17. $4x^2-y^2$
18. $64a^2-25$
19. $1-a^2b^2$
20. $121a^2-64x^2$
21. Simplify $(a+b)^2-3c(a+b)$.
22. Factorize $-6+3w^2-7w$ by splitting the middle term.
23. If $kx^2+24x+16$ is a perfect square, find the value of k.
24. Factorize x^2-y^2 and hence use the results to evaluate the value of $(5452)^2-548\times548$.
25. A certain company devised a cost-saving tool expressed as x^2-y^2. Data on cost-saving among individuals were collected and expressed as 983^2-17^2. Use the cost savings tool to find how much the company saved on a bulk purchase discount.
26. A local farmer wants to calculate the exact difference in the area between two fields, one measuring 1008^2 square metres and the other measuring 992^2 square metres. Find the difference between the areas of the two fields.

Quadratic equations

In the previous section, you learned about quadratic expressions. A quadratic equation is formed when a quadratic expression is equal to a specific value. Quadratic equations take the general form $ax^2+bx+c=0$. The solutions of such an equation are also known as roots or zeros. There are also several methods for solving quadratic equations, including factorization, completing the square, and the general quadratic formula.

Solving quadratic equations by factorization

To solve quadratic equations by factorisation, the following steps can be used:

1. Write the equation in standard form: $ax^2+bx+c=0$.
2. Rewrite the quadratic expression ax^2+bx+c as a product of two linear factors.
3. Set each factor equal to zero and solve for x.

> **Note:**
> If a quadratic equation is expressed as a product of two linear factors, say $(x+a)(x+b)=0$, where a and b are constants, then $x+a=0$ or $x+b=0$ or both factors are equal to zero. This is called Zero factor theorem which states that, if a product of two or more factors is zero, then at least one of the factors must be zero.

Example 4.25

Solve the equation $(x+4)(x-3)=0$.

Solution

If $(x+4)(x-3)=0$, then either

$x+4=0$ or $x-3=0$.

Therefore, $x=-4$ or $x=3$.

Example 4.26

Solve the quadratic equation

$3x^2-x=0$.

Solution

$3x^2-x=0.$

$x(3x-1)=0$

Either, $x=0$ or $3x-1=0$.

Therefore, $x=0$ or $x=\frac{1}{3}$.

Example 4.27

Solve the quadratic equation

$10x^2+9x+2=0$.

Solution

From $10x^2+9x+2=0$, it implies that $a=10$, $b=9$, $c=2$. It follows that

$ac=10\times 2=20$.

Factors of $ac=20$ are 1, 2, 4, 5, 10, 20.

Among the factors, 4 and 5 give a product of 20 and a sum of 9.

Now, use 4 and 5 to split the middle term:

$10x^2+5x+4x+2=0$

$5x(2x+1)+2(2x+1)=0$

$(2x+1)(5x+2)=0$

Either $2x+1=0$ or $5x+2=0$.

Therefore, $x=-\frac{1}{2}$ or $x=-\frac{2}{5}$.

Example 4.28

Solve for t if $t^2+6t+8=0$.

Solution

Given $t^2+6t+8=0$. Split the middle term to get,

$t^2+4t+2t+8=0.$

$t(t+4)+2(t+4)=0$

$(t+4)(t+2)=0$

Either $t+4=0$ or $t+2=0$.

Therefore, $t=-4$ or $t=-2$.

Example 4.29

Solve the quadratic equation $x^2-9=0$.

Solution

Given $x^2-9=0$. It follows that

$$x^2-3^2=0$$

$$(x+3)(x-3)=0$$

Either $x+3=0$ or $x-3=0$.

Therefore, $x=-3$ or $x=3$.

Example 4.30

Solve the quadratic equation

$2h^2-7h=39$.

Solution

Given $2h^2-7h-39=0$. It implies that

$$2h^2+6h-13h-39=0$$

$$2h(h+3)-13(h+3)=0$$

$$(h+3)(2h-13)=0$$

Either $h+3=0$ or $2h-13=0$.

Therefore, $h=-3$ or

$$h=\frac{13}{2}.$$

Example 4.31

Solve the quadratic equation $9x^2-4=0$.

Solution

Given $9x^2-4=0$. It follows that

$$9x^2-4=0$$

$$(3x)^2-2^2=0$$

$$(3x+2)(3x-2)=0$$

Either, $3x+2=0$ or $3x-2=0$

Therefore, $x=-\frac{2}{3}$ or $x=\frac{2}{3}$.

Example 4.32

Solve the equation $y^2-2y-24=0$.

Solution

Given $y^2-2y-24=0$. It follows that

$$y^2-6y+4y-24=0$$

$$y(y-6)+4(y-6)=0$$

$$(y-6)(y+4)=0$$

Either $y-6=0$ or $y+4=0$

Therefore, $y=6$ or $y=-4$.

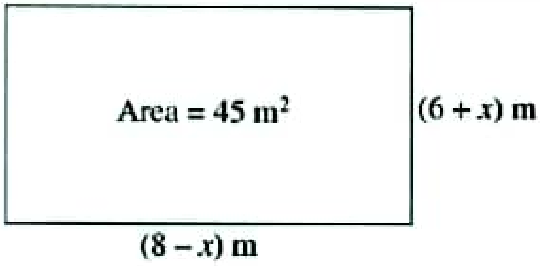
Example 4.33

A rectangular garden is 6 metres wide and 8 metres long. What length should be added to the shorter side and reduced from the longer side to form a rectangular garden with an area of 45 square metres?

Solution

Let x be the added length to the shorter side. The sides of the rectangular garden will be:

Width $=(6+x)$ m and length $=(8-x)$ as shown in the following figure.



Area of the new rectangular garden is $(8 - x)(6 + x) = 45$. It implies that

$$(8-x)(6+x)=45$$
$$48+8x-6x-x^2=45$$
$$x^2-2x-3=0$$
$$x^2-3x+x-3=0$$
$$x(x-3)+1(x-3)=0$$
$$(x-3)(x+1)=0$$

Either $x-3=0$ or $x+1=0$

Therefore, $x=3$ or $x=-1$.

But length cannot be negative.
Hence, $x=-1$ is rejected.
Thus, $x=3$ metres.

Therefore, 3 metres should be added to the shorter side.

Exercise 4.6

Solve the following quadratic equations by factorization method.

1. (a) $-x^2+4x-3=0$
   (b) $(2x+3)^2=0$
   (c) $-3x^2+12x-9=0$
   (d) $x^2-16=0$
   (e) $x^2=-2x-1$
   (f) $(x+4)^2=0$
   (g) $x^2-5x+6=0$
   (h) $2x^2+7x+3=0$
   (i) $x^2-6x+9=0$
   (j) $x^2+5x+6=0$

2. (a) $2x^2-7x+3=0$
   (b) $3x^2+8x+5=0$
   (c) $x^2-4x=0$
   (d) $2x^2+6x=0$
   (e) $x^2-7x=0$
   (f) $x^2+4x+4=0$
   (g) $4x^2-9=0$
   (h) $9x^2-25=0$
   (i) $(x-2)^2=0$
   (j) $4x^2-12x+9=0$
   (k) $3x^2-8x+5=0$

3. (a) $4x^2-49=0$
   (b) $9y^2-6y+1=0$
   (c) $y^2-4=0$
   (d) $a^2-8a+16=0$
   (e) $25a^2-9=0$
   (f) $9x^2+12x+4=0$
   (g) $16b^2=49$
   (h) $4d^2-20d+25=0$
   (i) $x^2=1$
   (j) $x^2=10x-25$
   (k) $(x-3)^2-25=0$
   (l) $\frac{x^2}{4}=4$

(m) $(x-5)^2 = 25$

(n) $b^2 + \frac{2b}{5} + \frac{1}{25} = 0$

(o) $c^2 + \frac{2}{7}c + \frac{1}{49} = 0$

(p) $(x-9)^2 - 36 = 0$

(q) $9x^2 = 12x - 4$

(r) $\frac{1}{4}a^2 + a + 1 = 0$

(s) $4r - 1 = 4r^2$

(t) $x^2 - 5x + \frac{25}{4} = 0$

4. The product of two consecutive whole numbers is 42. Find the numbers.
5. A projectile is launched from the ground with an initial velocity of 20 m/s. Its height h after t seconds is given by the equation $h(t) = -5t^2 + 20t$. Find the time when the projectile will hit the ground.
6. A garden has a rectangular area of length 3 metres more than its width. The area of the garden is 40 square metres. Find the dimensions of the garden.
7. The difference between two positive numbers is 8 and the products of the numbers is 105. Find the smaller number.
8. The base of the triangle is 5 cm less than its height. If its area is 33 cm^2, find the length of the base.
9. The perimeter of a rectangular garden is 60 m and its area is 209 m^2. Find the dimensions of the garden.
10. The concentration y in mg/l of a drug in the bloodstream is given by the equation $y = -\frac{t^2}{10} + 2t$, , where t is the time in hours administration. After how many hours will the concentration reach 10 mg/l?

Solving quadratic equations by completing the square

In the previous section, a method for solving quadratic equations by splitting the middle term was discussed. However, not all quadratic equations can be factored. For instance, in case of $x^2 + 5x + 14 = 0$, no two whole numbers add to 5 and multiply to give 14. For such equations, methods such as completing the square and quadratic formula can be used.

Completing the square is a technique used to solve quadratic equations by transforming the left-hand side of the equation into a perfect square. The steps involved are demonstrated in the following examples.

Example 4.34

What must be added to $x^2 + 10x$ to make the expression a perfect square?

Solution

The term to be added must be the square of half the coefficient of x.

The coefficient of x is 10.

Half of 10 is $\frac{1}{2}(10) = 5$.

The square of 5 is 25.

Therefore, 25 must be added to $x^2 + 10x$ to make it a perfect square.

Exercise 4.7

1. To each of the following expressions add a term which will make it a perfect square and write the result in the form $(x+k)^2$.

(a) x^2-12x (b) $a^2+\frac{3}{2}a$

(c) $x^2+\frac{7}{2}x$ (d) x^2-4x

(e) x^2-x (f) x^2+5x

(g) p^2+12p (h) $n^2+\frac{4}{3}n$

(i) $t^2+\frac{t}{2}$ (j) x^2+3x

2. Solve each of the following quadratic equations by completing the square.

(a) $x^2+2x-15=0$

(b) $4v^2-8v+3=0$

(c) $x^2-11x-3=0$

(d) $6-2c-c^2=0$

(e) $x^2-7x+11=0$

(f) $-3s^2-6s+1=0$

(g) $11-a-a^2=0$

(h) $p^2+11=6p$

(i) $10x^2+8x-2=0$

(j) $3h^2-24h-3=0$

(k) $5s^2-15s+5=0$

Solving quadratic equations by using quadratic general formula

The method of solving quadratic equations by completing the square can be used to derive the general formula for solving quadratic equations as follows.

From, $ax^2+bx+c=0$, $a\neq 0$.

Divide each term by a to get,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Subtract $\frac{c}{a}$ from both sides to obtain,

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation.

That is, $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$.

Factorize the left-hand side and simplify the right-hand side to obtain,

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Take the square root of both sides to get,

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides to get,

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or

$x = \frac{-b-\sqrt{b^2-4ac}}{2a}$; where $a \neq 0$.

If $ax^2+bx+c=0$; where $a \neq 0$, then

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ provided that

$b^2 \geq 4ac$ is the general quadratic formula.

Example 4.40

Solve $6x^2+11x+3=0$ using the quadratic formula.

Solution

Comparing $ax^2+bx+c=0$ with $6x^2+11x+3=0$ gives, $a=6$, $b=11$, and $c=3$.

Substitute these values into the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

$$x = \frac{-11 \pm \sqrt{(11)^2-4\times6\times3}}{2\times6}$$

$$x = \frac{-11 \pm \sqrt{121-72}}{12}$$

$$x = \frac{-11 \pm \sqrt{49}}{12}$$

$$= \frac{-11 \pm 7}{12}$$

$$x = -\frac{4}{12} \text{ or } x = -\frac{18}{12}$$

Therefore, $x = -\frac{1}{3}$ or $x = -\frac{3}{2}$.

Example 4.41

Solve $5x^2-6x-1=0$ using the quadratic formula.

Solution

Given $5x^2-6x-1=0$, then $a=5$, $b=-6$ and $c=-1$.

By using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2-4\times5\times(-1)}}{2\times5}$$

$$x = \frac{6 \pm \sqrt{36+20}}{10} = \frac{6 \pm \sqrt{56}}{10}$$

Either, $x = \frac{6+\sqrt{56}}{10}$ or $x = \frac{6-\sqrt{56}}{10}$

Therefore, $x = \frac{3+\sqrt{14}}{5}$ or $x = \frac{3-\sqrt{14}}{5}$.

Example 4.42

Solve the quadratic equation $-400k^2+317k-60=0$ by using the quadratic formula.

Solution

Given $-400k^2+317k-60=0$.

Compare the given quadratic equation with the standard form $ax^2+bx+c=0$, hence, $a = -400$, $b = 317$, and $c = -60$.

By using the quadratic formula,

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute the values of a, b, and c,

$$k = \frac{-(317) \pm \sqrt{(317)^2 - 4(-400)(-60)}}{2(-400)}$$

$$k = \frac{-317 \pm \sqrt{100\ 489 - 96\ 000}}{-800}$$

$$k = \frac{-317 \pm \sqrt{4\ 489}}{-800} = \frac{-317 \pm 67}{-800}$$

Either, $k = \frac{-317 + 67}{-800}$ or $k = \frac{-317 - 67}{-800}$

Therefore, $k = \frac{5}{16}$ or $k = \frac{12}{25}$.

Example 4.43

Juma bought a certain number of mangoes for 3,600 shillings. If each mango had been sold for 50 shillings less, he could have bought six more mangoes for the same amount of money. How many mangoes did he buy?

Solution

Let x be the number of mangoes bought. The price of each mango was $\frac{3\ 600}{x}$ shillings. Six more mangoes correspond to $(x + 6)$ mangoes.

Hence, each mango would cost $\frac{3\ 600}{x+6}$ shillings.

This price per mango is less than the previous one by 50 shillings. That is,

$$\frac{3\ 600}{x} - \frac{3\ 600}{x+6} = 50.$$

$$3\ 600(x+6) - 3\ 600x = 50x(x+6)$$

$$3\ 600x + 21\ 600 - 3\ 600x = 50x^2 + 300x$$

$$21\ 600 = 50x^2 + 300x$$

Simplify the equation by dividing each term by 50:

$x^2 + 6x - 432 = 0$, implying that $a = 1$, $b = 6$, and $c = -432$. From the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times (-432)}}{2 \times 1}$$

$$= \frac{-6 \pm \sqrt{36 + 1728}}{2}$$

$$= \frac{-6 \pm \sqrt{1764}}{2}$$

$$= \frac{-6 \pm \sqrt{1764}}{2}$$

$$= \frac{-6 \pm 42}{2}$$

Either, $x = \frac{-6 + 42}{2}$ or $x = \frac{-6 - 42}{2}$

$x = 18$ or $x = -24$

Checking: With 18 mangoes each costs 200 shillings. Six more mangoes each costs 150 shillings.

The difference is (200 – 150) shillings = 50 shillings.

Therefore, the number of mangoes bought was 18 because it is impossible to have a negative number of mangoes.

Exercise 4.8

Solve the following quadratic equations using the quadratic formula

1. $x^2 - 4x + 3 = 0$
2. $x^2 + 3x + 1 = 0$
3. $3x^2 - 6x - 2 = 0$
4. $2x^2 + 3x - 2 = 0$
5. $2x^2 - 7x + 3 = 0$
6. $(x+3)^2 = 10$
7. $3x^2 - 2x - 2 = 0$
8. $-22n^2 - 15n + 27 = 0$
9. $3x(x-1) = 4$
10. $x(x+3) = (x-1)(2x-1)$
11. $2x^2 - x - 3 = 0$
12. $-145p^2 + 97p - 6 = 0$
13. $x^2 + 2x - 3 = 0$
14. $-12m^2 + 36m - 27 = 0$
15. $400 + 20t - t^2 = 0$
16. $2e^2 + e = 6$
17. $6x^2 + 12x = 0$
18. $\frac{x^2}{\left(1\frac{1}{2} - x\right)(2 - x)} = 4$
19. Find two consecutive numbers whose product is 132.
20. Find two consecutive even numbers whose product is 80.
21. The ages of a man and his son are 35 and 9 years, respectively. How many years ago was the product of their ages 87 years?
22. A square garden of side 20 metres is surrounded by a path whose area is the same as that of the garden. Find the width of the path.
23. A piece of wire 40 cm long is cut into two parts and each part is then bent into a square. If the sum of the areas of these squares is 68 square centimetres, find the lengths of the two pieces of wire.

Chapter summary

1. A binary operation is rule for combining two quantities to produce a unique quantity from the same set.
2. An expression whose highest power of the variable is 2 is called a quadratic expression.
3. Factorisation is the process of writing an expression as a product of two or more factors.
4. A quadratic equation is formed when a quadratic expression is equal to a specific value, typically zero.
5. The solutions of quadratic equations are known as roots.

Revision exercise 4

1. Solve each of the following quadratic equations by the factorization method:
   (a) $x^2 + 3x = 0$
   (b) $3x^2 - 15x = 0$
   (c) $x = 3x^2$
   (d) $2x^2 = 3x$
   (e) $x(5 - x) = 0$

(f) $7x^2 - 3x = 0$

(g) $x^2 + 3x - 40 = 0$

(h) $3x^2 - 7x - 6 = 0$

(i) $12x^2 + 13x + 3 = 0$

(j) $x^2 + 3x + 2 = 0$

(k) $x^2 - 10x + 24 = 0$

(l) $2x^2 - x - 6 = 0$

(m) $3x + 2 = 9x^2$

(n) $-3x^2 + 11x - 10 = 0$

(o) $4x^2 = 25$

(p) $y^2 - 36 = 0$

(q) $(x - 8)^2 = 36$

(r) $4x^2 = 20x - 25$

(s) $9y^2 - 6y + 1 = 0$

(t) $(x + 3)^2 - 49 = 0$

2. Solve each of the following quadratic equations by completing the square:

(a) $x^2 + 6x + 7 = 0$

(b) $x^2 - 11x + 1 = 0$

(c) $x^2 = 7x - 7$

(d) $2x^2 - 10x + 7 = 0$

(e) $m^2 + 5m = 1$

(f) $p^2 - 10p + 5 = 0$

(g) $2b^2 = 8b + 11$

(h) $3a^2 - 12a = 2$

(i) $5n^2 = 20n + 28$

(j) $c^2 - 8c + 13 = 0$

3. Use the quadratic formula to solve each of the following equations:

(a) $4x^2 - 4x + 1 = 0$

(b) $5x^2 + 12x + 3 = 0$

(c) $(3x - 2)(2x - 5) = 5x(x - 2)$

(d) $3x^2 = 4x + 4$

(e) $2x^2 - 5x + 2 = 0$

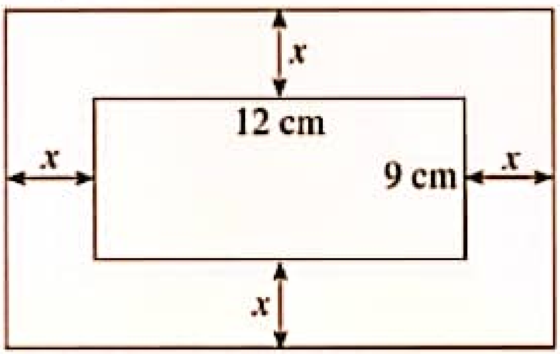
(f) $5x^2 - x - 18 = 0$

4. A man is 4 times as old as his son. In 4 years to come the product of their ages will be 520. Find the son's present age.

5. Sada has 1 800 shillings to buy pencils. There are two types of pencils whose prices differ by 40 shillings. If she buy the cheaper type she will get 12 more pencils than if she buy the expensive type. What are the prices of the two types of pencils?

6. Find two consecutive numbers such that the sum of their squares is equal to 145.

7. A picture measures 12 cm by 9 cm and is surrounded by a frame whose area is 100 square centimetres. Find the value of x as shown in the following figure.



8. When 6 is divided by a certain number, the result is the same as when 5 is added to the number and that sum divided by 6. Find the number.

9. Find a whole number such that twice its square is 11 more than 21 times the number.

10. A piece of wire 56 cm long is bent to form a rectangle of area 171 cm². Find the dimensions of the rectangle.

11. Given that $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, write v as the subject of the formula and calculate v when $f=8.1$ and $u=5.4$.

12. Solve each of the following equations.

(a) $3x-\frac{2}{x}=4$ (d) $2x-\frac{1}{x}=-1$

(b) $1-\frac{1}{x}=-5x$ (e) $x+\frac{2}{x}=3$

(c) $3+\frac{1}{x^2}=-\frac{5}{x}$ (f) $\frac{x+3}{1-x}=\frac{9}{x}$

13. Use the defined binary operation to evaluate each of the following:

(a) If $a\otimes b=ab+a+b$, find $3\otimes 4$.

(b) If $a*b=(a+b)(a-b)+2$, find $2*7$.

14. Make t the subject of the following formulae.

(a) $V=u+at$

(b) $S=ut+\frac{1}{2}at^2$

(c) $y(2t+1)=t+3$

(d) $\frac{x}{x+t}+5=t$

15. Re-arrange each formula to express the indicated variable on the right side in terms of the other variables.

(a) $V^2=u^2+2as$, u

(b) $S=Vt-\frac{1}{2}at^2$, t

(c) $A=2\pi r^2+2\pi rh$, r

(d) $E=\frac{1}{2}mv^2+mgh$, v

(e) $\frac{s}{2t-5}+5=3t$, t

(f) $T=\frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}$, $\frac{v}{c}$

(g) $V=\frac{V_0}{\sqrt{r^2-1}}$, r

16. Juma and Maria can plough together a garden for 3 hours. If Juma works alone, he will take 8 hours less than Maria to lough the same garden. Find the time each person will use to plough the garden.

17. Members of the STEM group at a certain school decided to go for a study tour. The cost for hiring a minibus is Tshs 300,000 which must be equally shared among the members. However, two students from the social group joined the group and each student was supposed to pay Tshs 5,000 less. How many members are in the STEM group?

18. The sides of a right angleds triangle are $(x+1)$ cm, $(x+2)$ cm, and $(x+5)$cm. Use these information to show that $x^2-4x-20=0$.

Chapter Five

Exponents and radicals

Introduction

Numbers can be expressed in different ways, each of which illustrates how mathematical problems can be approached and solved. Simplifying complex problems with challenging equations and solving exponential such as growth and electrical networks may require an understanding of exponents and radicals. In this chapter, you will learn the laws of exponents, perform operations on radicals, and rationalize denominators. The competencies developed will enable you to express numbers in short form and apply the knowledge to describe scientific phenomena such as population growth, magnitude of earthquakes, rates of spread of diseases and rate of growth or decay of bacteria, and many other real-life applications.

Think

Solving mathematical expressions and equations involving powers and roots without the concept of radicals and exponents.

Numbers can be expressed in different ways depending on the context and how they are used. Some of these ways include the use of exponents and radicals. Engage Activity 5.1 to explore different ways of expressing numbers.

Activity 5.1: Expressing numbers in different ways

1. Choose five whole numbers and express them in different ways you know. You can consult various sources such as the internet to learn such ways.
2. Study any unique pattern and provide reasons regarding your observations.
3. Share your conclusion with other students through a method of your choice.

Exponents

From Activity 5.1, one may have explored various ways used to express numbers. One of the common methods of expressing numbers is by multiplying the same or different numbers several times. Study the following examples.

(i) $3 = 1 \times 3$

(ii) $4 = 2 \times 2$

(iii) $8 = 2 \times 2 \times 2$

(iv) $12 = 2 \times 2 \times 3$

(v) $81 = 3 \times 3 \times 3 \times 3$

These are expanded form of numbers.

From the given examples, the same numbers that are multiplied several times can be expressed in short form. Some numbers can be expressed in short form by counting the number of times the same number is multiplied and writing the result on the top right hand of the multiplied number. For example:

(i) $4 = 2 \times 2 = 2^2$ since 2 is multiplied repeatedly two times.

(ii) $8 = 2 \times 2 \times 2 = 2^3$ since 2 is multiplied repeatedly three times.

(iii) $12 = 2 \times 2 \times 3 = 2^2 \times 3$ since 2 is multiplied repeatedly two times, and three is multiplied once.

(iv) $81 = 3 \times 3 \times 3 \times 3 = 3^4$ since 3 is multiplied repeatedly four times.

Numbers expressed in the forms 2^2, 2^3 and 3^4 are called powers. The numbers which are multiplied several times are called bases, and the number of times any number is multiplied by itself is called exponent. For example, 3^5:

3^5 is the power, where

3 is the base and

5 is the exponent.

Likewise, 3^5 is read as the "fifth power of 3" or "3 raised to exponent five" and 2^3 is read as the "third power of 2" or "2 raised to exponent 3".

Engage in Activity 5.2 to explore more about exponential numbers.

Activity 5.2: Identifying features of powers, base, and exponent of the numbers

1. Choose any counting number greater than 1 and multiply it several times.
2. Write the product in the power form and identify the power, base, and exponent.
3. Share your work and justify your answers.

Generally, a^n is a number written in exponential form, where a is the base and n is the exponent. The number a^n is called a power.

Example 5.1

Write each of the following expressions in exponential form:

(a) $5 \times 5 \times 5 \times 5$

(b) $k \times k \times k \times k \times k$

(c) $(-3) \times (-3) \times (-3) \times (-3) \times (-3)$

(d) $m \times m \times m \times \cdots \times m$ *(n times)*

Solution

(a) $5 \times 5 \times 5 \times 5 = 5^4$

(b) $k \times k \times k \times k \times k = k^5$

(c) $(-3) \times (-3) \times (-3) \times (-3) \times (-3) = (-3)^5$

(d) $m \times m \times m \times \ldots \times m = m^n$

Example 5.2

Write each of the following expressions in power form:

(a) $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6$

(b) $\left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right)$

(c) $10 \times 10 \times 10 \times 10 \times 10 \times 10$

(d) $(-7) \times (-7) \times (-7)$

Solution

(a) $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^8$

(b) $\left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right) \times \left(-\frac{3}{4}\right)^4$

(c) $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$

(d) $(-7) \times (-7) \times (-7) = (-7)^3$

Example 5.3

Identify the power, base and exponent in each of the following exponential numbers:

(a) 4^3 (b) $(-10)^6$ (c) x^n

Solution

(a) 4^3 is the power, 4 is the base and 3 is the exponent.

(b) $(-10)^6$ is the power, -10 is the base and 6 is an exponent.

(c) x^n is the power, x is the base and n is an exponent.

Example 5.4

Evaluate each of the following powers:

(a) 5^4 (b) $(-5)^3$

(c) $(-7)^2$ (d) 2^6

Solution

(a) $5^4 = 5 \times 5 \times 5 \times 5 = 625$

(b) $(-5)^3 = (-5) \times (-5) \times (-5) = -125$

(c) $(-7)^2 = (-7) \times (-7) = 49$

(d) $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

Example 5.5

Express each of the following numbers in exponential form using the indicated base.

(a) 64 in base 4

(b) 2701 in base 7

(c) -125 in base -5

Solution

(a) $64 = 4 \times 4 \times 4 = 4^3$

(b) $2401 = 7 \times 7 \times 7 \times 7 = 7^4$

(c) $-125 = (-5) \times (-5) \times (-5) = (-5)^3$

Engage in Activity 5.3 to explore more about properties of bases to their exponents

Activity 5.3: Analysing properties of powers

1. Write some numbers in power forms with negative or positive exponents and bases.
2. Use a calculator or any other mathematical software to analyse their effects in the final answer.
3. Write your conclusion and share your findings.

Exercise 5.1

1. Indentify the base and exponent in each of the following exponential numbers:

(a) 6^{17} (b) 7^4

(c) $(-10)^{19}$ (d) 3^{800}

(e) 6^2 (f) 50^0

(g) y^{25} (h) 17^6

(i) $\left(\frac{5}{6}\right)^9$ (j) $\left(\frac{1}{2}\right)^{17}$

(k) 19^{101} (l) $(-75)^8$

(m) $(x + y)^n$ (n) $(7 + x)^8$

2. Express each of the following in expanded form:

(a) 7^2 (b) $(-3)^3$

(c) 10^3 (d) 2^6

(e) 9^1 (f) $(-y)^3$

(g) $(-99)^5$ (h) $\left(\frac{9}{10}\right)^5$

(i) $(-0.35)^4$ (j) $(0.67)^3$

3. Write each of the following in exponential form and hence identify the base and the exponent:

(a) $7 \times 7 \times 7 \times 7$

(b) $(-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2)$

(c) $14 \times 14 \times 14$

(d) 19

(e) $(x + b)(x + b)(x + b)(x + b)$

(f) $(-r) \times (-r) \times (-r) \times (-r) \times (-r) \times (-r) \times (-r)$

(g) $50 \times 50 \times \cdots \times 50$ (30 times)

(h) $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

(i) $(a + b)(a + b)$

(j) $\frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8} \times \frac{w}{8}$

(k) $0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3$

(l) $v \times v \times v$

4. Find the exact value of each of the following exponential numbers:

(a) 5^2 (b) 7^2

(c) 20^3 (d) 12^2

(e) 1^3 (f) $(-9)^2$

(g) 30^3 (h) $(-2)^3$

(i) $(-3)^3$ (j) 2^5

(k) $(-5)^3$ (l) 10^3

5. Express each of the following numbers in exponential form using the given base:

(a) 25 in base 5

(b) 36 in base 6

(c) 1,728 in base 12

(d) 16 in base 2

(e) 1,000,000,000,000 in base 10

6. Use two different ways to express each of the following numbers as powers:

(a) 16

(b) 64

(c) 81

(d) 625

(e) 1,000

7. Use a calculator to find:

(a) 8^{-1}

(b) $\frac{1}{8}$

(c) 5^{-3}

(d) $\frac{1}{5^3}$

(e) 3^{-4}

(f) $\frac{1}{3^4}$

8. Simplify, then use a calculator to check the answer:
   (a) $(-1)^{13}$
   (b) $(-1)^{5}$
   (c) $(-1)^{2}$
   (d) $-(-2)^{6}$
   (e) $(-2)^{6}$
   (f) $-(-5)^{4}$
   (g) $(-5)^{4}$
9. Write each of the following in power form whose base ia a prime number choice:
   (a) 2×2^{b}
   (b) 27^{a}
   (c) $\frac{2^{h}}{4}$
   (d) $\frac{2^{m}}{2^{-m}}$
   (e) 8×2^{b}
   (f) 5×125^{k}
   (g) $\left(\frac{1}{243}\right)^{3}$

Laws of exponents

There are three groups of exponents namely positive, negative, and zero exponents. Simplification of the exponents are usually based on the four laws called multiplication, division, power, and zero power.

Multiplication law for exponents

Consider the product of two exponential numbers having the same base with positive exponents such as $5^3 \times 5^2$. The exponential numbers can be written in expanded form as follows:

$5^2 = 5 \times 5$ and $5^3 = 5 \times 5 \times 5$

Therefore,

$$5^2 \times 5^3 = \underbrace{(5\times5)}\times\underbrace{(5\times5\times5)}$$
$$= \underbrace{5\times5\times5\times5\times5}$$

Count how many times 5 is multiplied by itself. The answer is five times, which gives the exponent. Thus, 5 is multiplied by itself five times. The exponent 5 can also be obtained by adding the exponents from each base. That is,

$5^2 \times 5^3 = 5^{2+3} = 5^5$.

Similarly, if a is any number, then

$$a^4 \times a^2 = (a\times a\times a\times a)\times(a\times a)$$
$$= a^{4+2}$$
$$= a^6$$

Therefore, $a^4 \times a^2 = a^6$

> Generally, if x is any non-zero number with positive exponents m and n, then $x^m \times x^n = x^{m+n}$. This is called the multiplication law of exponents.

Example 5.6

Simplify each of the following by using multiplication law of exponents:

(a) $7^6 \times 7^8$

(b) $6^4 \times 6^3 \times 6^4$

(c) $\left(\frac{5}{9}\right)^3 \times \left(\frac{5}{9}\right)^4$

(d) $(0.45)^2 \times (0.45)^4 \times (0.45)^{12}$

Solution

By using multiplication law of exponents, it implies that,

(a) $7^6 \times 7^8 = 7^{6+8} = 7^{14}$

(b) $6^4 \times 6^3 \times 6^4 = 6^{4+3+4} = 6^{11}$

(c) $\left(\frac{5}{9}\right)^3 \times \left(\frac{5}{9}\right)^4 = \left(\frac{5}{9}\right)^{3+4} = \left(\frac{5}{9}\right)^7$

(d) $(0.45)^2 \times (0.45)^4 \times (0.45)^{12} = (0.45)^{2+4+12} = (0.45)^{18}$

Suppose the expression $(2^4)^3$ is to be written as a single exponent, then it can be written in expanded form as follows:

$$\begin{aligned}(2^4)^3 &= 2^4 \times 2^4 \times 2^4\\ &= 2^{4+4+4}\\ &= 2^{4\times3}\\ &= 2^{12}\end{aligned}$$

Therefore, $(2^4)^3 = 2^{4\times3} = 2^{12}$.

Similarly, consider a number $(a^2)^4$, where a is any real number. The number can be written as a single exponent as follows:

$$\begin{aligned}(a^2)^4 &= a^2 \times a^2 \times a^2 \times a^2\\ &= a^{2+2+2+2}\\ &= a^{2\times4}\\ &= a^8\end{aligned}$$

Therefore, $(a^2)^4 = a^{2\times4} = a^8$.

Generally, $(x^m)^n = x^{mn}$, where m and n are positive integers and x is any real number. This law is known as the power law.

Example 5.7

Simplify each of the following by using power law of exponents:

(a) $(3^6)^3$

(b) $\left(\left(\frac{3}{7}\right)^6\right)^2$

(c) $(0.7^7)^3$

Solution

(a) $(3^6)^3 = 3^{6\times3} = 3^{18}$

(b) $\left(\left(\frac{3}{7}\right)^6\right)^2 = \left(\frac{3}{7}\right)^{6\times2} = \left(\frac{3}{7}\right)^{12}$

(c) $(0.7^7)^3 = (0.7)^{7\times3} = 0.7^{21}$

Furthermore, the expression $(7 \times 5)^3$, can be written in expanded form as follows:

$$\begin{aligned}(7\times5)^3 &= (7\times5)\times(7\times5)\times(7\times5)\\ &= 7\times5\times7\times5\times7\times5\\ &= 7\times7\times7\times5\times5\times5\\ &= 7^3\times5^3\end{aligned}$$

Therefore, $(7\times5)^3 = 7^3\times5^3$.

Similarly, if a and b are real numbers, then $(a\times b)^4$ can be expanded as follows:

$$\begin{aligned}(a\times b)^4 &= (a\times b)\times(a\times b)\times(a\times b)\times(a\times b)\\ &= (a\times a\times a\times a)\times(b\times b\times b\times b)\\ &= a^4\times b^4\end{aligned}$$

Therefore, $(a\times b)^4 = a^4\times b^4$.

Generally, $(a\times b)^n = a^n\times b^n$, where a and b are real numbers and n is a positive integer is known as the power law of a product.

Example 5.8

Simplify each of the following by using the power law of a product.

(a) $(7^2\times c^3)^4$

(b) $(4\times b^2\times c^3)^3$

(c) $(0.7^4\times 0.3^2)^4$

Solution

(a) $$\begin{aligned}(7^2\times c^3)^4 &= 7^{2\times4}\times c^{3\times4}\\ &= 7^8\times c^{12}\\ &= 7^8c^{12}\end{aligned}$$

(b) $$\begin{aligned}(4\times b^2\times c^3)^3 &= 4^{1\times3}\times b^{2\times3}\times c^{3\times3}\\ &= 4^3\times b^6\times c^9\\ &= 4^3b^6c^9\end{aligned}$$

(c) $(0.7^4 \times 0.3^2)^4$

$$(0.7^4 \times 0.3^2)^4 = (0.7)^{4\times4} \times (0.3)^{2\times4}$$

$$= (0.7)^{16} \times (0.3)^8.$$

Example 5.9

Write each of the following expressions by grouping together the letters with the same exponents:

(a) d^4t^4

(b) $a^5 \times b^4$

(c) $3cd^2 \times 5c^5d^4$

Solution

(a) $d^4t^4 = (dt)^4$

(b) $a^5 \times b^4 = a \times a^4 \times b^4$

$= a \times (ab)^4$

$= a(ab)^4$

(c) $3cd^2 \times 5c^5d^4 = 3 \times 5 \times c \times c^5 \times d^2 \times d^4$

$= 15 \times c^6d^6$

$= 15(cd)^6.$

Exercise 5.2

1. Simplify each of the following expressions and write the answers in power form:

(a) $10^2 \times 10^2$

(b) $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^5$

(c) $r^4 \times r^8$

(d) $\left(\frac{7}{6}\right)^2 \times \left(\frac{7}{6}\right)^{12}$

(e) $3a^2b^3 \times 4a^4b^5$

(f) $\left(\frac{17}{20}\right)^2 \times \left(\frac{17}{20}\right)^3 \times \left(\frac{17}{20}\right)$

(g) $16^2 \times 16^4 \times 16^4$

(h) $\left(\frac{3}{4}\right)^2 \times \left(\frac{3}{4}\right)^3$

(i) $10^3 \times 10^0$

(j) $(0.58)^4 \times (0.58)^4 \times (0.58)^{16}$

(k) $3^{14} \times 3^{17} \times 3^1$

(l) $10^p \times 10^r \times 10^{10}$

(m) $8^2 \times 8^2 \times 8^2 \times 8^3$

(n) $(0.1)^p \times (0.1)^{20} \times (0.1)^3$

(o) $b^x \times b^y \times b^z$

(p) $k^t \times k^r \times k^k$

(q) $x^2 \times x^2 \times x^2$

(r) $10^t \times 10^p \times 10^s$

2. Write each of the following expressions in the simplest powers:

(a)	$(4^2)^4$	(b)	$(a^3)^2$
(c)	$(b^4)^3$	(d)	$(18^1)^{20}$
(e)	$(71^3)^2$	(f)	$(a^2)^5$
(g)	$(19^6)^1$	(h)	$(23^2)^n$
(i)	$(m^2)^{17}$	(j)	$(x^{12})^6$
(k)	$(2^3)^7$	(l)	$\left(\left(\frac{3}{4}\right)^4\right)^4$
(m)	$(10^a)^b$	(n)	$(x^{21})^3$

3. Write each of the following expressions so that each base is raised to a single exponent:

(a) $(4\times3)^2$

(b) $(2a)^2$

(c) $(3\times7)^3$

(d) $(2x^2)^2$

(e) $(2t^2)^3$

(f) $(10p^2)^3$

(g) $5(mn)^2$

(h) $3(a^2b)^3$

(i) $(2a^3b^4)^2$

(j) $(7x)^2\times(7x)^{64}\times(7x)^3$

(k) $(xy^4)(x^2y^2)^2$

(l) $(a^2m^3)\times(a^4b^7)$

4. Write each of the following expressions as a single exponential number:

(a) $4^3\times5^3$

(b) $a^{17}\times b^{17}$

(c) $(2a)^5\times a^5$

(d) $12k^{23}\times t^{23}$

(e) $2a^7\times b^7$

(f) $3^2\times3^4$

(g) $3^4\times2^4$

(h) $12^{20}\times12^{21}$

(i) $4a^4\times4b^4$

(j) $5p^3\times3s^3$

(k) $q^{-5}\times\left(\frac{1}{2}t\right)^{-5}$

(l) $4^{-3}\times4^5$

Division law of exponents

Consider the following example of dividing exponential numbers with the same base like $7^5\div7^3$. The solution can be obtained as follows:

$$7^5\div7^3=\frac{7^5}{7^3}$$

$$=\frac{7\times7\times7\times7\times7}{7\times7\times7}$$

$$=7\times7$$

$$=7^2$$

Alternatively, if the bases of the numerator and denominator are the same, then subtract the denominator exponent from numerator exponent.

That is,

$$\frac{7^5}{7^3}=7^{5-3}$$

$$=7^2$$

Similarly, $\frac{a^7}{a^4}=\frac{a\times a\times a\times a\times a\times a\times a}{a\times a\times a\times a}$,

where $a\neq0$

$$=a^{7-4}$$

$$=a^3$$

Therefore, $\frac{a^7}{a^4}=a^{7-4}=a^3$.

Generally, $\frac{x^m}{x^n}=x^{m-n}$, where $x\neq0$.

That is when exponential numbers of the same base are divided, subtract the exponent of the divisor from the exponent of the dividend.

Example 5.10

Simplify each of the following expressions:

(a) $\frac{3^{20}}{3^{12}}$

(b) $\frac{a^{28}}{a^{17}}$

(c) $\frac{(0.47)^{10}}{(0.47)^{6}}$

Solution

(a) $\frac{3^{20}}{3^{12}} = 3^{20-12}$

$= 3^{8}$

(b) $\frac{a^{28}}{a^{17}} = a^{28-17}$

$= a^{11}$

(c) $\frac{(0.47)^{10}}{(0.47)^{6}} = (0.47)^{10-6}$

$= (0.47)^{4}$

When exponential numbers of different bases with the same exponents are divided, the number can be expressed as a single exponent. If a and b are any real numbers such that $b \neq 0$, then:

$$\frac{a^4}{b^4} = \frac{a \times a \times a \times a}{b \times b \times b \times b}$$

$$= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$$

$$= \left(\frac{a}{b}\right)^4$$

Generally, $\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n$, where x, y and n are real numbers and $y \neq 0$. This is a power of a quotient.

Example 5.11

Write each of the following expressions in the form of $\left(\frac{x}{y}\right)^n$:

(a) $\frac{13^2}{9^2}$

(b) $\frac{(0.5)^4}{(0.3)^4}$

(c) $\frac{16}{3^2}$

Solution

(a) $\frac{13^2}{9^2} = \left(\frac{13}{9}\right)^2$

(b) $\frac{(0.5)^4}{(0.3)^4} = \left(\frac{0.5}{0.3}\right)^4$

(c) $\frac{16}{3^2} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$

Exercise 5.3

Simplify each of the following expressions:

1. $\frac{x^5}{x^2}$
2. $\frac{2^{7000}}{2^{6998}}$
3. $\frac{625}{5^4}$
4. $a^4 \div \left(2a^2 + 4a^2\right)$

5. $\frac{a^5 b^3}{ab^2}$

6. $\left(\frac{k^3m^4n^6}{km^2n^3}\right)^3$

7. $\left(\frac{20a^3b}{5ab}\right)^2 \times \left(\frac{5a^3b}{15a^2b^5}\right)^5$

8. $\frac{15a^3b^2c^2}{5a^3bc}$

9. $\frac{48a^4b^5}{8a^2b^3}$

10. $6r^3 \div \frac{1}{4r}$

11 $\frac{\left(3^2\times 2^{15}\times 3^{10}\times 25\right)^3}{\left(3^{16}\times 2^{16}\times 5^2\right)^2}$

12. $\frac{(3mn)^3}{8(mn)^3}$

13. $\frac{\left(3^2\times a^3\times b^4\times c^2\right)^3}{\left(14\times c^2\times a^2\times b^3\right)^2}$

14. $\frac{\left(17^3\times(4x)^2\right)^3}{\left((2x)^2\times 17^2\right)^3}$

15. $\frac{(a+b)^2}{a^2-b^2}$

Zero exponent

An expression $\frac{7^3}{7^3}$ can be simplified by using the division law of exponents as follows:

$$\frac{7^3}{7^3}=\frac{7\times7\times7}{7\times7\times7}=1 \qquad (1)$$

Alternatively, $\frac{7^3}{7^3}=7^{3-3}=7^0$. (2)

From equations (1) and (2), it follows that

$7^0 = 1$

Similarly,

if $m \neq 0$, then $\frac{m^4}{m^4}=\frac{m\times m\times m\times m}{m\times m\times m\times m}=1$ (3)

Thus, $\frac{m^4}{m^4}=m^{4-4}=m^0$ (4)

Therefore, $m^0 = 1$ by equations (3) and (4).

> Generally, if x any non-zero number, then $x^0 = 1$. In this case 0^0 is not defined.

Negative exponents

The expression $\frac{8^3}{8^5}$ can be simplified using the division law of exponents as follows:

$$\frac{8^3}{8^5}=\frac{8\times8\times8}{8\times8\times8\times8\times8}$$

$$=\frac{1}{8\times8}$$

$$=\frac{1}{8^2}$$

Therefore, $\frac{8^3}{8^5}=\frac{1}{8^2}$. (1)

Also, $\frac{8^3}{8^5}=8^{3-5}=8^{-2}$ (2)

Comparing equations (1) and (2), gives

$8^{-2}=\frac{1}{8^2}$.

Similarly, $\frac{k^5}{k^8}=\frac{k\times k\times k\times k\times k}{k\times k\times k\times k\times k\times k\times k\times k}$

$$=\frac{1}{k\times k\times k}$$

$$=\frac{1}{k^3}$$

Thus, $\frac{k^5}{k^8}=k^{-3}$, for $k\neq 0$.

Therefore, $k^{-3}=\frac{1}{k^3}$.

In general, $x^{-n}=\frac{1}{x^n}$ and $x^n=\frac{1}{x^{-n}}$, for $x\neq 0$.

When $n=1$, $x^{-n}=x^{-1}=\frac{1}{x^1}=\frac{1}{x}$.

If $x\neq 0$, then $\frac{1}{x}$ is called the reciprocal of x, which can also be written as x^{-1}.

Example 5.12

Express the following exponential numbers using positive exponents:

(a) 4^{-3} (b) a^{-7}

(c) $\frac{1}{19^{-2}}$

Solution

(a) $4^{-3}=\left(4^{-1}\right)^3$

$$=\left(\frac{1}{4}\right)^3$$

$$=\frac{1}{4^3}$$

(b) $a^{-7}=\left(a^{-1}\right)^7$

$$=\left(\frac{1}{a^1}\right)^7$$

$$=\frac{1}{a^7}$$

(c) $\left(\frac{1}{19^{-2}}\right)=\frac{1}{\left(\frac{1}{19}\right)^2}=\frac{1}{\frac{1}{19^2}}=19^2$

Example 5.13

Express the following powers by using negative exponents;

(a) $\left(\frac{1}{4}\right)^3$ (b) $(0.75)^{16}$

(c) $\frac{1}{17^2}$

Solution

(a) $\left(\frac{1}{4}\right)^3=\left(4^{-1}\right)^3=4^{-3}$

(b) $(0.75)^{16}=\frac{1}{(0.75)^{-16}}$

(c) $\frac{1}{17^2}=17^{-2}$

Example 5.14

Simply each of the following expressions by writing your answers in positive exponents:

(a) $\frac{x^5}{x^7}$ (b) $\frac{a^4b^{17}}{a^{10}b^{11}}$

Solution

(a) $\frac{x^5}{x^7}=x^{5-7}$

$$=x^{-2}$$

$$=\frac{1}{x^2}$$

Therefore, $\frac{x^5}{x^7}=\frac{1}{x^2}$.

(b) $\frac{a^4b^{17}}{a^{10}b^{11}} = \frac{a^4}{a^{10}}\times\frac{b^{17}}{b^{11}}$

$= a^{4-10}b^{17-11}$

$= a^{-6}b^{6}$

$= \left(\frac{b^6}{a^6}\right)$

$= \left(\frac{b}{a}\right)^6$

Therefore, $\frac{a^4b^{17}}{a^{10}b^{11}} = \left(\frac{b}{a}\right)^6$.

Example 5.15

Simplify each of the following expressions and give your answers in negative exponents:

(a) $\frac{3^9}{3^{16}}$

(b) $\frac{(0.7)^6}{(0.7)^2}$

(c) $\left(-2a^2b^{-2}c\right)^3\left(3ab^4c^5\right)\left(xyz\right)^0$

Solution

(a) $\frac{3^9}{3^{16}} = 3^{9-16}$

$= 3^{-7}$

Therefore, $\frac{3^9}{3^{16}} = 3^{-7}$

(b) $\frac{(0.7)^6}{(0.7)^2} = (0.7)^{6-2}$

$= (0.7)^4$

$= \frac{1}{(0.7)^{-4}}$

Therefore, $\frac{(0.7)^6}{(0.7)^2} = \frac{1}{(0.7)^{-4}}$

(c) $\left(-2a^2b^{-2}c\right)^3\left(3ab^4c^5\right)\left(xyz\right)^0$

$= \frac{\left(-2a^2c\right)^3}{b^6}\times\left(3ab^4c^5\right)\times 1$

$= \frac{-2^3\times a^6\times c^3\times 3\times a\times b^4\times c^5}{b^6}$

$= \frac{-8\times 3\times a^7\times b^4\times c^8}{b^6}$

$= -\frac{2^3\times 3^1\times a^7\times c^8}{b^2}$

$= -\frac{b^{-2}}{2^{-3}\times 3^{-1}\times a^{-7}\times c^{-8}}$

Therefore, $(-2a^2b^{-2}c)^3(3ab^4c^5)(xyz)^0 = -\frac{b^{-2}}{2^{-3}\times 3^{-1}\times a^{-7}\times c^{-8}}$

Example 5.16

Find the reciprocal of each of the following and simplify the answers:

(a) 19 (b) $\frac{1}{9}$ (c) 0.2

(d) 14^2 (e) $\left(ab^2\right)^4$ (f) $\frac{1}{7^2}$

(g) $(0.25)^{-2}$

Solution

(a) The reciprocal of 19 is $\frac{1}{19}$.

(b) The reciprocal of $\frac{1}{9}$ is 9.

(c) The reciprocal of 0.2 is $\frac{1}{0.2} = \frac{10}{2} = 5$.

(d) The reciprocal of $14^2 = \frac{1}{14^2} = \frac{1}{196}$.

(e) The reciprocal of $\left(ab^2\right)^4 = \frac{1}{\left(ab^2\right)^4} = \frac{1}{a^4b^8}$

(f) The reciprocal of $\frac{1}{7^2} = 7^2 = 49$.

(g) The reciprocal of $(0.25)^{-2} = (0.25)^2 = 0.0625$.

Exercise 5.4

1. Simplify each of the following expressions and write the answer in positive or zero exponent:

(a) $\frac{x^4}{x^7}$

(b) $\frac{(3x^4y^2)^3}{9x^2y^4}$

(c) $\frac{(-2x^7y^4)^{\frac{1}{2}}}{4x^3y^{15}}$

(d) $\frac{(2u^3v^2)^4}{(4u^2v)^2}$

(e) $\frac{(5a^2b^3)^2 \times (3a^{-1}b)^3}{(15ab^2)^2}$

(f) $\frac{(6m^4n^{-2})^2 \times (3m^{-2}n^3)^3}{(18mn^{-1})^2}$

2. Simplify each of the following expressions and write the answer in negative exponent:

(a) $\frac{2^3 \times 4^4}{(2^5)^2}$

(b) $(x^2y^2z^{-1})(x^{-1}yz^4)(x^{-3}y)$

(c) $\frac{3\pi^2r^2h^4}{\pi r^3h}$

(d) $\frac{(2a^5b^{-3})^3 \times (4a^{-3}b^2)^4}{(8a^4b^{-2})^3}$

(e) $\frac{(12a^3b^{-4})^{-3} \times (4a^{-2}b^8)^3}{(48a^6b^{-2})^2}$

(f) $\frac{(6m^4n^{-2})^2}{(3m^2n^{-4})3}$

3. Write the reciprocals of each of the following numbers in simplest form.

(a) 14

(b) $\frac{1}{8}$

(c) 100^6

(d) r^4

(e) $\frac{1}{2^8r^8}$

(f) π^{-4}

(g) 0.125

(h) $\frac{13}{29}$

(i) 0.25

(j) 0.75

Exponential equations

All mathematical equations which involve exponents are called exponential equations. For example, $x^p = y^q$ is an exponential equation where x and y are bases and p and q are exponents. The laws of exponents are usually used when solving equations that involve exponents. The following rules should be considered when solving exponential equations:

(a) If the bases are the same, then equate the exponents. That is, if $m^x = m^y$, then $x = y$.

(b) If the exponents are equal then equate the bases. That is, if $a^n = b^n$, then $a = b$.

Example 5.17

Find the value of n in each of the following equations:

(a) $2^{n+1} = 64$ (b) $4^n = 16$

Solution

(a) $2^{n+1} = 64$

Express 64 in power form to obtain

$2^{n+1} = 2^6$

Comparing exponents gives,

$n + 1 = 6$

$n = 5$

Therefore, $n = 5$.

(b) $4^n = 16$

Express in power form and compare exponents to obtain

$4^n = 4^2$

$n = 2$

Therefore, $n = 2$.

Example 5.18

Find the value of b in each of the following equations:

(a) $b^3 = 27$ (b) $(b + 1)^3 = 64$

Solution

(a) $b^3 = 27$

Express 27 in power form to obtain,

$b^3 = 3^3$

$b = 3$

Therefore, $b = 3$.

(b) $(b + 1)^3 = 64$

Write 64 in power form to get

$(b + 1)^3 = 4^3$

$b + 1 = 4$

$b = 4 - 1$

$b = 3$

Therefore, $b = 3$.

Example 5.19

Solve for x and y if $2^x \times 3^y = 144$.

Solution

From $2^x \times 3^y = 144$.

Writing 144 in exponential form gives,

$144 = 2^4 \times 3^2$

It follows that,

$2^x \times 3^y = 2^4 \times 3^2$

Comparing exponents of the same base gives $x = 4$ and $y = 2$.

Therefore, $x = 4$ and $y = 2$

Example 5.20

Given $4^x = 2^y$ and $3^x = 9^{y-1}$, find the value of x and y.

Solution

Simplify $4^x = 2^y$ to get $2^{2x} = 2^y$. Comparing the exponents gives

$2x = y$ (i)

$3^x = 9^{y-1}$

$= 3^{2y-2}$

Comparing exponents of the same base, gives

$x = 2y - 2$ (ii)

Solving equations (i) and (ii) simultaneously gives

$x = \frac{2}{3}$ and $y = \frac{4}{3}$.

Example 5.21

Solve for x in each of the following equations

(a) $5^x + 5^{(x-1)} = 30$.

(b) $4^x + 2^x - 20 = 0$

Solution

(a) $5^x + 5^x \times 5^{-1} = 30$

$\Rightarrow 5^x + \frac{5^x}{5} = 30$

$\Rightarrow 5 \times 5^x + 5^x = 30 \times 5$

$\Rightarrow 6 \times 5^x = 30 \times 5$

$\Rightarrow 6 \times 5^x = 6 \times 5 \times 5,$

Dividing by 6 in both sides, gives

$\Rightarrow 5^x = 5 \times 5$

$\Rightarrow 5^x = 5^2$

Comparing the exponents, gives

$x = 2$

Therefore, $x = 2.$

(b) $(2^x)^2 + 2^x - 20 = 0$

Let $2^x = y$. Thus, the equation becomes:

$y^2 + y - 20 = 0$

$(y-4)(y+5) = 0$

So, $y = 4$ or $y = -5$.

Since 2^x cannot be negative, $2^x = 4$ gives:

$2^x = 2^2$

$x = 2$

Therefore, the solution is $x = 2$.

Exercise 5.5

Solve each of the following exponential equations:

1. $3a^3 = 24$
2. $9^x = 3^x + 6$
3. $2r^3 = 16$
4. $r^{-2} = 4$
5. $2^{x+1} = 32$
6. $x^2 = 64$
7. $\left(\frac{1}{3}\right)^x = 81^{-1}$
8. $h^2 = 0.01$
9. $(y+3)^2 = 5^2$
10. $(x^4)^3 = x^{12n}$
11. $2^x = 4^{x-3}$
12. $\left(\frac{1}{2}\right)^{-3} = 8^x$
13. $(5y)^2 = 5^2$
14. $(1-x)^{2n} = (1-x)^{\frac{1}{2}}$
15. $288 = 2x^2$
16. $2^{x-1} \times 3^{y+1} = 108$
17. $\begin{cases} 27^{2x} \times 9^y = 1 \\ 2^{3x} \times 8^{4y} = \frac{1}{8} \end{cases}$
18. $2^x \times 4^{x+1} = 8^{2x-3}$
19. $8^{x+1} = 16^{x-3}$
20. $9^3 \times 27 = 3^x$
21. $5^x \times 3^y = 675$
22. $25^x - 23(5^x) - 50 = 0$
23. $4^x - 6(2^x) + 8 = 0$

Fractional exponents

Numbers in power form can be written in exponential form with fractional exponents. For example, $5^{\frac{1}{2}}$ is a power with $\frac{1}{2}$ as an exponent.

Taking the square of $5^{\frac{1}{2}}$ gives

$$\left(5^{\frac{1}{2}}\right)^2 = 5^{\frac{1}{2} \times 2} \quad \text{(by power law)}$$

$$= 5$$

Note that the only number which when squared gives 5 is $\sqrt{5}$, this implies that $5^{\frac{1}{2}} = \sqrt[2]{5}.$

Similar approach can be applied to other powers. For instsnce:

$$\left(5^{\frac{1}{3}}\right)^3 = 5^{\frac{1}{3} \times 3}$$

$$= 5$$

$$\Rightarrow 5^{\frac{1}{3}} = \sqrt[3]{5}$$

$$\left(5^{\frac{1}{4}}\right)^4 = 5^{\frac{1}{4}\times 4}$$

$$= 5$$

$$\Rightarrow 5^{\frac{1}{4}} = \sqrt[4]{5}$$

$$\left(5^{\frac{1}{5}}\right)^5 = 5^{\frac{1}{5}\times 5}$$

$$= 5$$

$$\Rightarrow 5^{\frac{1}{5}} = \sqrt[5]{5}$$

Based on the previous results, it can be concluded that if x is a positive number and n is any positive real number, then

$$\left(x^{\frac{1}{n}}\right)^n = x^1 = x \text{ and } x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Generally, $x^{\frac{1}{n}} = \sqrt[n]{x}$, for $x > 0$.

Example 5.22

Find the value of $49^{\frac{1}{2}}$.

Solution

Express 49 as a product of prime factors.

$49 = 7 \times 7 = 7^2$

Thus, $49^{\frac{1}{2}} = \left(7^2\right)^{\frac{1}{2}} = 7^{2\times\frac{1}{2}} = 7^1$

Therefore, $49^{\frac{1}{2}} = 7$.

Example 5.23

Simplify $\left(\frac{8}{27}\right)^{\frac{1}{3}}$.

Solution

Express the base in power form and simplify.

$$\left(\frac{8}{27}\right)^{\frac{1}{3}} = \left(\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\right)^{\frac{1}{3}}$$

$$= \left(\left(\frac{2}{3}\right)^3\right)^{\frac{1}{3}} = \frac{2}{3}$$

Therefore, $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$.

Example 5.24

Find the value of $(-125)^{\frac{1}{3}}$.

Solution

Express -125 as a product of prime factors.

$$-125 = (-5) \times (-5) \times (-5)$$

$$(-125)^{\frac{1}{3}} = \left((-5)^3\right)^{\frac{1}{3}}$$

$$= (-5)^{3\times\frac{1}{3}}$$

$$= -5$$

Therefore, $(-125)^{\frac{1}{3}} = -5$.

Exercise 5.6

Simplify each of the following exponential numbers:

1. $(64)^{\frac{1}{2}}$
2. $(1000)^{\frac{1}{3}}$
3. $\left(\frac{81}{625}\right)^{\frac{1}{2}}$
4. $(0.25)^{\frac{1}{2}}$
5. $(0.027)^{\frac{1}{3}}$
6. $(81)^{\frac{1}{4}}$

7. $(32)^{\frac{1}{5}}$

8. $(1.21)^{\frac{1}{2}}$

9. $\left(\frac{27}{64}\right)^{\frac{1}{3}}$

10. $\frac{(3x)^{\frac{1}{3}}.(3x)^{\frac{7}{3}}}{(3x)^{\frac{2}{3}}}$

11. $(c^{\frac{1}{2}}d^{\frac{1}{2}})(c^{\frac{1}{2}}-d^{\frac{1}{2}})$

12. $(2y)^{\frac{1}{2}}+z^{2})\left((2y)^{\frac{1}{2}}-z^{2}\right)$

13. $3(5)^{\frac{1}{2}}+7(5)^{\frac{1}{2}}$

14. $10(3)^{\frac{1}{3}}-4(3)^{\frac{1}{3}}$

15. $2+2(2)^{\frac{1}{2}}+(8)^{\frac{1}{2}}+3(2)^{\frac{1}{2}}$

Radicals

A rational exponent, $x^{\frac{1}{n}}$ which can be expressed as $\sqrt[n]{x}$ is known as a radical. In $\sqrt[n]{x}$, n is an index, $\sqrt{\ }$ is the radical symbol and x is the radicand. The symbol $\sqrt{\ }$ is also called a surd.

The expression $\sqrt[n]{x}$ is also known as the n^{th} root of x, which is a number that, when multiplied n times, gives the original number x. For example:

(a) $\sqrt{4}=(4)^{\frac{1}{2}}=\left(2^{2}\right)^{\frac{1}{2}}=2$

2 is the square root of 4.

(b) $\sqrt[3]{343}=(343)^{\frac{1}{3}}=\left(7^{3}\right)^{\frac{1}{3}}=7$

7 is the cube root of 343.

(c) $\sqrt[4]{\frac{1}{16}}=\left(\left(\frac{1}{2}\right)^{4}\right)^{\frac{1}{4}}=\frac{1}{2}$.

$\frac{1}{2}$ is the fourth root of $\frac{1}{16}$.

Note that, the square root of a negative real number does not exist in the set of real numbers.

Example 5.25

Find the square root of 196.

Solution

Factorize 196 in terms of prime factors:

$$196 = 2\times 2\times 7\times 7 = 2^2\times 7^2$$

So, $196 = 2^2\times 7^2$.

Apply the radical sign in both sides of the equation and simplify to obtain,

$$\sqrt{196}=\sqrt{2^2\times 7^2} =\sqrt{2^2}\times\sqrt{7^2} =\left(2^2\right)^{\frac{1}{2}}\times\left(7^2\right)^{\frac{1}{2}} =2\times 7 =14$$

Therefore, $\sqrt{196}=14$.

Example 5.26

Find the cube root of 216.

Solution

Write 216 as a product of prime factors:

$216 = 2\times 2\times 2\times 3\times 3\times 3 = 2^3\times 3^3$

Apply the cube root on both side, to obtain

$$\sqrt[3]{216}=\sqrt[3]{2^3\times 3^3} =\sqrt[3]{2^3}\times\sqrt[3]{3^3} =\left(2^3\right)^{\frac{1}{3}}\times\left(3^3\right)^{\frac{1}{3}} = 2\times 3=6$$

Therefore, the cube root of 216 is 6.

Example 5.27

Express $\sqrt[5]{1024}$ in its simplest form.

Solution

Write 1024 as a product of prime factors.

$1024 = 2\times2\times2\times2\times2\times2\times2\times2\times2\times2$

Apply the fifth root on both sides to obtain,

$$\begin{aligned}\sqrt[5]{1024} &= \sqrt[5]{2\times2\times2\times2\times2\times2\times2\times2\times2\times2}\\ &= \sqrt[5]{2^5\times2^5}\\ &= \left(2^5\right)^{\frac{1}{5}}\times\left(2^5\right)^{\frac{1}{5}}\\ &= 2\times2\\ &= 4\end{aligned}$$

Therefore, $\sqrt[5]{1024} = 4$.

Example 5.28

Express each of the following numbers in its simplest form.

(a) $\sqrt{20}$ (b) $\sqrt[3]{54}$

Solution

(a) $$\begin{aligned}\sqrt{20} &= \sqrt{2\times2\times5}\\ &= \sqrt{2\times2}\times\sqrt{5}\\ &= 2\times\sqrt{5}\\ &= 2\sqrt{5}\end{aligned}$$

Therefore, $\sqrt{20} = 2\sqrt{5}$.

(b) $$\begin{aligned}= \sqrt[3]{54} &= \sqrt[3]{3\times3\times3\times2}\\ &= \sqrt[3]{3\times3\times3}\times\sqrt[3]{2}\\ &= 3\times\sqrt[3]{2}\\ &= 3\sqrt[3]{2}\end{aligned}$$

Therefore, $\sqrt[3]{54} = 3\sqrt[3]{2}$

Example 5.29

Express each of the following numbers under a single radical sign:

(a) $2\sqrt{3}$ (b) $2\sqrt[3]{4}$

Solution

(a) $$\begin{aligned}2\sqrt{3} &= \sqrt{2\times2\times3}\\ &= \sqrt{12}\end{aligned}$$

Therefore, $2\sqrt{3} = \sqrt{12}$.

(b) $$\begin{aligned}2\sqrt[3]{4} &= \sqrt[3]{2\times2\times2\times4}\\ &= \sqrt[3]{32}\end{aligned}$$

Therefore, $2\sqrt[3]{4} = \sqrt[3]{32}$.

Exercise 5.7

1. Simplify each of the following expressions:

(a) $\sqrt{169}$ (b) $\sqrt{729}$

(c) $\sqrt[4]{2048}$ (d) $\sqrt[4]{2500}$

(e) $\sqrt{1024}$ (f) $\sqrt[3]{512}$

(g) $\sqrt[3]{343}$ (h) $\sqrt[3]{1000}$

(i) $\sqrt[3]{729}$ (j) $\sqrt[5]{\dfrac{t^4}{81s^{24}}}$

2. Express each of the following expressions in simplest form.

(a) $\sqrt{40}$ (b) $\sqrt{250}$

(c) $\sqrt[6]{1024}$ (d) $\sqrt[3]{54}$

(e) $\sqrt[3]{270}$ (f) $\sqrt[3]{162}$

(g) $\sqrt[3]{2000}$ (h) $\sqrt{2000}$

(i) $\sqrt[6]{1000000}$

3. Express each of the following numbers under a single radical sign:

(a) $5\sqrt{2}$ (b) $4\sqrt{11}$

(c) $3\sqrt{10}$ (d) $9\sqrt[8]{3}$

(e) $5\sqrt[3]{2}$ (f) $3\sqrt[3]{3}$

(g) $2\sqrt[7]{-1000}$ (h) $6\sqrt[4]{4}$

(i) $7\sqrt[5]{5}$

Addition and subtraction of radicals

Two or more radicals can be added or subtracted if they are alike. Radicals which are alike are those with the same indices and radicand. This means that radicals of the same index can be added or subtracted, just as it is done in algebraic expressions. The radicals $\sqrt{2}$ and $\sqrt[3]{2}$ cannot be added or subtracted because have unlike indices. Before adding or subtracting radicals, first simplify the terms if possible. Engage in Activity 5.4 to explore about addition and subtraction of radicals

Activity 5.4: Deduce the conditions for adding and subtracting radicals

1. Simplify the following radicals: $\sqrt{8}$, $\sqrt{32}$, $\sqrt{27}$, and $\sqrt{3\times2\times2}$.
2. Identify the like terms in task 1.
3. Consider the like terms and express each term as a sum of its roots.
4. For each group of like terms add the values obtained in task 3.
5. Pick any two terms of unlike radicals and try to add them as in task 4. What can you conclude?
6. Provide the condition for radicals to be added together.

Example 5.30

Simplify each of the following expressions:

(a) $\sqrt{2}+\sqrt{2}$ (b) $2\sqrt{3}+3\sqrt{3}$

(c) $2\sqrt[3]{81}+\sqrt[3]{24}$ (d) $\sqrt[4]{2}+\sqrt[4]{32}$

Solution

(a) $\sqrt{2}+\sqrt{2}=1\sqrt{2}+1\sqrt{2}=2\sqrt{2}$

(b) $2\sqrt{3}+3\sqrt{3}=5\sqrt{3}$

(c) $2\sqrt[3]{81}+\sqrt[3]{24}$

$=2\sqrt[3]{3\times3\times3\times3}+\sqrt[3]{2\times2\times2\times3}$

$=2\times\sqrt[3]{3\times3\times3}\times\sqrt[3]{3}+\sqrt[3]{2\times2\times2}\times\sqrt[3]{3}$

$=2\times3\sqrt[3]{3}+2\sqrt[3]{3}$

$=8\sqrt[3]{3}$

(d) $\sqrt[4]{2}+\sqrt[4]{32}$

$=\sqrt[4]{2}+\sqrt[4]{2\times2\times2\times2}\times\sqrt[4]{2}$

$=\sqrt[4]{2}+2\times\sqrt[4]{2}$

$=\sqrt[4]{2}+2\sqrt[4]{2}$

$=3\sqrt[4]{2}$

Example 5.31

Simplify each of the following expressions:

(a) $6\sqrt{7}-2\sqrt{7}$

(b) $6\sqrt[5]{28}-2\sqrt[5]{896}$

(c) $\sqrt[4]{32}-\sqrt[4]{2592}$

(d) $2+\sqrt[5]{16807}+\sqrt[5]{243}$

Solution

(a) $6\sqrt{7}-2\sqrt{7}=4\sqrt{7}$

(b) $6\sqrt[5]{28}-2\sqrt[5]{896}$

$$=6\sqrt[5]{28}-\left(2\sqrt[5]{2\times2\times2\times2\times2\times28}\right)$$

$$=6\sqrt[5]{28}-\left(2\times\sqrt[5]{2\times2\times2\times2\times2}\times\sqrt[5]{28}\right)$$

$$=6\sqrt[5]{28}-\left(2\times2\times\sqrt[5]{28}\right)$$

$$=6\sqrt[5]{28}-4\sqrt[5]{28}$$

$$=2\sqrt[5]{28}$$

(c) $\sqrt[4]{32}-\sqrt[4]{2592}$

$$\sqrt[4]{32}=\sqrt[4]{2\times2\times2\times2\times2}=2\sqrt[4]{2}$$

$$\sqrt[4]{2592}=\sqrt[4]{6\times6\times6\times6\times2}=6\sqrt[4]{2}$$

$$\sqrt[4]{32}-\sqrt[4]{2592}=2\sqrt[4]{2}-6\sqrt[4]{2}$$

$$=-4\sqrt[4]{2}$$

(d) $2+\sqrt[5]{16807}+\sqrt[5]{243}$

$$\sqrt[5]{16807}=\sqrt[5]{7\times7\times7\times7\times7}=7$$

$$\sqrt[5]{243}=\sqrt[5]{3\times3\times3\times3\times3}=3$$

$$2+\sqrt[5]{16807}+\sqrt[5]{243}=2+7+3$$

$$=12$$

Exercise 5.8

1. Simplify each of the following expressions.

(a) $\sqrt{27}+5\sqrt{3}$

(b) $\sqrt{44}+\sqrt{11}$

(c) $\sqrt{120}+\sqrt{1080}$

(d) $\sqrt{125}-\sqrt{45}$

(e) $\sqrt[3]{625}-\sqrt[3]{135}$

(f) $\sqrt[5]{1215}+\sqrt[5]{15625}$

(g) $\sqrt[4]{6}+\sqrt[4]{486}$

(h) $\sqrt[6]{15}+\sqrt[6]{10935}+3\sqrt[6]{15}$

(i) $\sqrt{486}-6\sqrt{6}$

(j) $2\sqrt{50}-\sqrt{8}$

(k) $5\sqrt{8}+6\sqrt{2}$

(l) $\sqrt[5]{243}-10\sqrt[5]{4096}$

(m) $5\sqrt[4]{64}-\sqrt[4]{112}$

(n) $\sqrt{128}-\sqrt{32}$

2. Simplify each of the following expressions.

(a) $2\sqrt{3}+\frac{2}{3}\sqrt{27}$

(b) $\frac{1}{2}\sqrt[3]{81}-\frac{5}{6}\sqrt[3]{648}$

(c) $6\sqrt[5]{764}+\sqrt[5]{486}$

(d) $8\sqrt{150}-2\sqrt{96}-\sqrt[3]{24}$

(e) $6\sqrt[7]{32768}-\sqrt[7]{256}+\frac{3}{4}\sqrt[7]{7}$

(f) $8\sqrt[3]{2n}-3\sqrt[3]{2n}+5\sqrt[3]{2n}$

(g) $6\sqrt{18}+3\sqrt{50}-\sqrt{45}$

(h) $-7\sqrt[3]{40}+\sqrt[3]{5}$

4. A recycling plant processes different types of waste materials. One day,

they process two batches of plastic waste. The first batch can be processed at a rate of $\sqrt[3]{81}$ tons per day, while the second batch can be processed at a rate of $2\sqrt[3]{24}$ tons per day. What is the total amount of plastic waste processed per day when the two batches are combined? Simplify your answer.

4. A hospital is studying two new medications, each with a different absorption rate in the body. The absorption rate for the first medication is $\sqrt[4]{80}$ mg/hour, and for the second medication, it is $3\sqrt[4]{5}$ mg/hour. What is the difference in absorption rates between the two medications? Simplify your answer.

Multiplication of radicals

Multiplication of radicals involves multiplying the numbers inside the radicals and then simplifying if possible. Thus, when multiplying radicals, the indices must be the same. Otherwise, it is impossible.

Generally, $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

Example 5.32

Simplify each of the following expression products:

(a) $\sqrt{3} \times \sqrt{5}$ (b) $\sqrt{2} \times \sqrt{32}$

(c) $\sqrt{20} \times \sqrt{28}$ (d) $\sqrt{12}(\sqrt{3}+\sqrt{5})$

(e) $(2\sqrt{3}-\sqrt{2}) \times (\sqrt{3}+3\sqrt{2})$

Solution

(a) $\sqrt{3} \times \sqrt{5} = \sqrt{15}$

(b) Given that $\sqrt{2} \times \sqrt{32}$, it follows,

$$\sqrt{32} = \sqrt{2\times2\times2\times2\times2} = 4\sqrt{2}$$

Thus,
$$\sqrt{2}\times\sqrt{32} = \sqrt{2}\times4\sqrt{2} = 4\times\sqrt{2}\times\sqrt{2} = 4\times2 = 8$$

(c) Given $\sqrt{20}\times\sqrt{28}$, it follows that,

$$\sqrt{20} = \sqrt{2\times2\times5} = 2\sqrt{5}$$

$$\sqrt{28} = \sqrt{2\times2\times7} = 2\sqrt{7}$$

Thus,
$$\sqrt{20}\times\sqrt{28} = 2\sqrt{5}\times2\sqrt{7} = 2\times2\times\sqrt{5}\times\sqrt{7} = 4\times\sqrt{5\times7} = 4\sqrt{35}$$

(d)
$$\sqrt{12}(\sqrt{3}+\sqrt{5}) = \sqrt{4\times3}(\sqrt{3}+\sqrt{5})$$
$$= 2\sqrt{3}(\sqrt{3}+\sqrt{5})$$
$$= 2\sqrt{3}\times\sqrt{3} + 2\sqrt{3}\times\sqrt{5}$$
$$= 2\times3+2\sqrt{15}$$
$$= 6+2\sqrt{15}$$

(e) Given $(2\sqrt{3}-\sqrt{2})\times(\sqrt{3}+3\sqrt{2})$

Expand as follows;

$(2\sqrt{3}-\sqrt{2})\times(\sqrt{3}+3\sqrt{2}) =$

$(2\sqrt{3}\times\sqrt{3})+(2\sqrt{3}\times3\sqrt{2})+(-\sqrt{2}\times\sqrt{3})$
$+(-\sqrt{2}\times3\sqrt{2})$
$=2\sqrt{9}+6\sqrt{6}-\sqrt{6}-3\sqrt{4}$
$=2\times3+5\sqrt{6}-3\times2$
$=6+5\sqrt{6}-6$
$=6-6+5\sqrt{6}$
$=5\sqrt{6}$

Therefore,
$(2\sqrt{3}-\sqrt{2})\times(\sqrt{3}+3\sqrt{2})=5\sqrt{6}$.

Exercise 5.9

Simplify each of the following radicals:

1. $\sqrt{20}\times\sqrt{5}$
2. $\sqrt{12}\times\sqrt{3}$
3. $\sqrt{45}\times\sqrt{54}$
4. $2\sqrt{3}\times\sqrt{54}$
5. $3\sqrt{10}\times3\sqrt{10}$
6. $\sqrt{32}\times\sqrt{12}$
7. $2\sqrt{3}\times5\sqrt{12}$
8. $(2\sqrt{15})^2$
9. $(\sqrt{6})^3$
10. $(2\sqrt{15})^2(\sqrt{7})^2$
11. $(\sqrt{5})^3\times(\sqrt{13})^2$
12. $(\sqrt{2}+1)^2$
13. $2\sqrt{2}\times\sqrt{10}\times\sqrt{20}$
14. $\sqrt{6}\times\sqrt{8}\times\sqrt{10}\times\sqrt{12}$
15. $\sqrt{5}\times\sqrt{24}\times\sqrt{40}$
16. $\sqrt{8}(\sqrt{2}-\sqrt{18})$
17. $\sqrt{5}(\sqrt{2}+\sqrt{18})$
18. $6\sqrt{2}(\sqrt{2}-\sqrt{18})$
19. $(2\sqrt{5}+1)\times(3\sqrt{5}+1)$
20. $(\sqrt{8}-\sqrt{2})^2$
21. $(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})$
22. $(4\sqrt{3}-\sqrt{12})(\sqrt{3}+\sqrt{2})$
23. $(2\sqrt{3}+1)^2$
24. $(2\sqrt{3}-\sqrt{5})(2\sqrt{3}+3\sqrt{5})$
25. $(\sqrt{6}-\sqrt{24})(\sqrt{2}+\sqrt{3})$
26. $\sqrt{8}(\sqrt{2}-\sqrt{18})$

27. Juma has a square garden of length $\sqrt{8}$ m. What is the perimeter and the total area of the garden?
28. A rectangular prism has dimensions $\sqrt{3}$ metres, 6 metres, and 2 metres. What is the volume of the prism?
29. The length of a rectangular farm is $\sqrt{8000}$ metres and its width is $\sqrt{2000}$ metres. Find the diagonal and the area of the farm.
30. The side of a square is scaled by a factor of $\sqrt{3}$. If the original side length was $\sqrt{5}$ metres, what is the new area of the square?

Division of radicals

Two radicals can be expressed as a fraction by writing one as the divisor and the other as the dividend. If two or more numbers under radicals have the same index, combine them under a single radical, then divide and simplify where possible. Engage in Activity 5.5 to learn more on division of radicals.

Activity 5.5: Deducing conditions for dividing radicals

1. In pairs, follow the steps given to divide $\sqrt{20}$ by $\sqrt{5}$.
2. Express $\sqrt{20} \div \sqrt{5}$ as a ratio of numerator and denominator.
3. Express the numerator and denominator in simple terms.
4. Collect together the terms with the same radicals and divide.
5. Follow the same steps to evaluate $\sqrt{625} \div 2\sqrt{25}$.

Generally, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ where a and b are positive numbers.

Example 5.33

Express the following radicals in their most simplified form:

(a) $\sqrt{75} \div \sqrt{12}$ (b) $5\sqrt{\frac{18}{50}}$

(c) $\frac{6\sqrt{5} \times 2\sqrt{3}}{\sqrt{20} \times 3\sqrt{12}}$

Solution

(a) $\sqrt{75} \div \sqrt{12} = \frac{\sqrt{75}}{\sqrt{12}}$

$$= \sqrt{\frac{75}{12}}$$

$$= \frac{\sqrt{3} + \sqrt{25}}{\sqrt{3} + \sqrt{4}}$$

$$= \sqrt{\frac{25}{4}}$$

$$= \frac{5}{2}$$

Therefore, $\sqrt{75} \div \sqrt{12} = \frac{5}{2}$.

(b) $5\sqrt{\frac{18}{50}} = 5 \times \sqrt{\frac{18}{50}}$

$$= 5 \times \frac{\sqrt{18}}{\sqrt{50}}$$

$$= 5 \times \frac{3\sqrt{2}}{5\sqrt{2}}$$

$$= 5 \times \frac{3 \times \sqrt{2}}{5 \times \sqrt{2}}$$

$$= 3$$

Therefore, $5\sqrt{\frac{18}{50}} = 3$.

(c) $\dfrac{6\sqrt{5}\times 2\sqrt{3}}{\sqrt{20}\times 3\sqrt{12}} = \dfrac{6\sqrt{5}\times 2\sqrt{3}}{\sqrt{(4\times 5)}\times 3\left(\sqrt{4\times 3}\right)}$

$$= \frac{6\sqrt{5}\times 2\sqrt{3}}{2\sqrt{5}\times 3\times 2\sqrt{3}}$$

$$= \frac{1}{1}$$

$$= 1$$

Therefore, $\dfrac{6\sqrt{5}\times 2\sqrt{3}}{\sqrt{20}\times 3\sqrt{12}} = 1$.

Example 5.34

Simplify $\sqrt[3]{27} \div \sqrt[3]{64}$.

Solution

Write the numbers under the radical symbol as in products of prime factors as follows:

$$\sqrt[3]{27} \div \sqrt[3]{64}$$

$$= \sqrt[3]{3\times 3\times 3} \div \sqrt[3]{2\times 2\times 2\times 2\times 2\times 2}$$

$$= 3 \div \left(\sqrt[3]{2\times 2\times 2}\times\sqrt[3]{2\times 2\times 2}\right)$$

$$= 3 \div (2\times 2)$$

$$= \frac{3}{4}$$

Example 5.35

Simplify $\dfrac{3\sqrt[3]{32}}{\sqrt[3]{2401}}$.

Solution

$$\frac{3\sqrt[3]{32}}{\sqrt[3]{2401}} = \frac{3\sqrt[3]{2\times 2\times 2\times 2\times 2}}{\sqrt[3]{7\times 7\times 7\times 7}}$$

$$= \frac{3\times\sqrt[3]{2\times 2\times 2}\times\sqrt[3]{2\times 2}}{\sqrt[3]{7\times 7\times 7}\times\sqrt[3]{7}}$$

$$= \frac{3\times 2\times\sqrt[3]{4}}{7\times\sqrt[3]{7}}$$

$$= \frac{6\sqrt[3]{4}}{7\sqrt[3]{7}}$$

Exercise 5.10

Simplify each of the following radical expressions:

1. $\dfrac{\sqrt{50}}{\sqrt{2}}$
2. $\sqrt{\dfrac{75}{3}}$
3. $\dfrac{\sqrt{12}}{\sqrt{50}}$
4. $\dfrac{\sqrt{121}}{\sqrt{44}}$
5. $\sqrt[4]{\dfrac{625}{81}}$
6. $\dfrac{\sqrt{72}}{\sqrt{900}}$
7. $\dfrac{2\sqrt{7}}{\sqrt{21}}$
8. $\dfrac{3\sqrt{5}}{\sqrt{45}}$
9. $\dfrac{\sqrt[3]{216}}{27}$
10. $\dfrac{3\sqrt{50}}{5\sqrt{32}}$
11. $\dfrac{\sqrt{8}\times\sqrt{2}}{\sqrt{12}\times\sqrt{3}}$
12. $= \dfrac{\sqrt{14}\times 2\sqrt{3}}{\sqrt{12}+\sqrt{56}}$
13. $\dfrac{5\sqrt{7}\times 2\sqrt{3}}{\sqrt{45}\times\sqrt{21}}$
14. $\dfrac{\sqrt{14}\times 2\sqrt{3}}{\sqrt{48}\times\sqrt{28}}$
15. $\dfrac{\sqrt{18}\times\sqrt{20}\times\sqrt{24}}{\sqrt{8}\times\sqrt{30}}$

16. A rectangular plot of land has an area of $\sqrt{5000}$ square metres. If the length of the plot is $\sqrt{125}$ metres, what is the width of the plot?
17. A box has a volume of $\sqrt{54000}$ cubic centimetres. If the height of the box is $\sqrt{300}$ centimetres, what is the area of the base?
18. The height of a tree is $\sqrt{320}$ metres. Another tree has a height of $\sqrt{180}$ metres. What is the ratio of the heights of the two trees?
19. A cylindrical tank has a volume of $\sqrt{288}$ cubic metres. If the height of the tank is $\sqrt{18}$ metres, what is the area of the base of the cylinder?

Rationalisation of denominators

Rationalisation of a denominator involves eliminating radicals from the denominator of a fraction. This makes denominator, to be a rational number. The procedure for rationalising the denominator are as follows:

(i) If the denominator is a single radical term, multiply the numerator and the denominator by the radical in the denominator.

For example, in rationalising the denominator of $\frac{a}{\sqrt{b}}$, multiplying both the numerator and denominator by $\sqrt{b}$ as follow.

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

Therefore, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$.

(ii) If the denominator is a sum or difference of two radicals, change the sign connecting the two terms. The resulting expression is called the rationalising factor, which is then multiplied in the numerator and the denominator.

For example, in rationalizing the denominator of $\frac{1}{\sqrt{a}+\sqrt{b}}$, the rationalising factor is $\sqrt{a}-\sqrt{b}$.

Multiply both the numerator and the denominator by $\sqrt{a}-\sqrt{b}$ as follows.

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

Therefore, $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

Similarly, for $\frac{1}{\sqrt{a}-\sqrt{b}}$, the rationalizing factor is $\sqrt{a}+\sqrt{b}$.

Thus, $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{1}{\sqrt{a}-\sqrt{b}} \times \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

Therefore, $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

The following table gives some options and reasons for choosing the rationalising factors:

Denominator	Rationalizing factor	Reason
$\sqrt{a}$	$\sqrt{a}$	$\sqrt{a}\times\sqrt{a}=a$
$\sqrt{a}-\sqrt{b}$	$\sqrt{a}+\sqrt{b}$	$\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=a-b$
$\sqrt{a}+\sqrt{b}$	$\sqrt{a}-\sqrt{b}$	$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$

Rationalize the denominator in each of the following expressions:

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{5}-\sqrt{3}}$

(c) $\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}}$ (d) $\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}}$

(e) $\frac{6}{\sqrt{7}-2}$

Solution

(a) $\sqrt{5}$ is a denominator and a single radical expression.

Multiply both numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{3\sqrt{5}}{5}$$

Therefore, $\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$.

(b) $$\frac{1}{\sqrt{5}-\sqrt{3}} = \frac{1}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{\sqrt{5}+\sqrt{3}}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}$$

$$= \frac{\sqrt{5}+\sqrt{3}}{5-3} = \frac{\sqrt{5}+\sqrt{3}}{2}$$

Therefore, $\frac{1}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{2}$.

(c) $$\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$$

$$= \frac{\sqrt{5}\left(\sqrt{5}-\sqrt{3}\right)}{5-3}$$

$$= \frac{5-\sqrt{15}}{2}$$

Therefore, $\frac{\sqrt{5}}{\sqrt{5}+\sqrt{3}} = \frac{5-\sqrt{15}}{2}$.

(d) $\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}}=\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}}\times\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}$

$$=\frac{(2+\sqrt{3})(\sqrt{2}+\sqrt{5})}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})}$$

$$=\frac{2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}}{2-5}$$

$$=\frac{2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}}{-3}$$

Therefore,

$$\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{5}}=\frac{2\sqrt{2}+2\sqrt{5}+\sqrt{6}+\sqrt{15}}{-3}.$$

(e) Given $\frac{6}{\sqrt{7}-2}$. The rationalizing factor is $\sqrt{7}+2$. Thus, Multiply both the numerator and the denominator by $\sqrt{7}+2$ to obtain,

$$\frac{6}{\sqrt{7}-2}=\frac{6}{\sqrt{7}-2}\times\frac{\sqrt{7}+2}{\sqrt{7}+2}$$

$$=\frac{6(\sqrt{7}+2)}{7-4}$$

$$=\frac{6(\sqrt{7}+2)}{3}$$

$$=2(\sqrt{7}+2)$$

$$=2\sqrt{7}+4$$

Therefore, $=\frac{6}{\sqrt{7}-2}=2\sqrt{7}+4.$

Example 5.37

Given that $\sqrt{3}=1.7321$ and $\sqrt{2}=1.1442$. Evaluate $\frac{1}{\sqrt{3}-\sqrt{2}}$.

Solution

Multiply both the numerator and denominator by $\sqrt{3}+\sqrt{2}$ to obtain:

$$\frac{1}{\sqrt{3}-\sqrt{2}}=\frac{1}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$=\sqrt{3}+\sqrt{2}$$

Since $\sqrt{3}=1.7321$ and $\sqrt{2}=1.1442$, it follows that

$$\frac{1}{\sqrt{3}-\sqrt{2}}=1.7321+1.4142$$

$$=3.1463$$

Exercise 5.11

1. Rationalize the denominator in each of the following expressions:

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{\sqrt{3}}{\sqrt{18}}$

(c) $\frac{1}{5+\sqrt{10}}$ (d) $\frac{1+\sqrt{2}}{\sqrt{2}}$

(e) $\frac{\sqrt{8}}{\sqrt{2}-\sqrt{3}}$ (f) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

(g) $\frac{9\sqrt{3}-\sqrt{2}}{\sqrt{5}+3\sqrt{2}}$ (h) $\frac{2\sqrt{7}}{\sqrt{10}-\sqrt{7}}$

(i) $\frac{x-2y}{\sqrt{x}+\sqrt{y}}$

2. Express each of the following in the simplest form.

(a) $\frac{5-\sqrt{7}}{3+\sqrt{7}}$

(b) $\frac{1}{3+4\sqrt{5}}$

(c) $\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}$

(d) $\frac{2}{\sqrt{5}-3}+\frac{1}{1-2\sqrt{3}}$

(e) $\frac{\sqrt{8}}{4-\sqrt{5}}+\frac{1}{7-2\sqrt{3}}$

(f) $\frac{2\sqrt{7}}{\sqrt{10}-\sqrt{7}}-\frac{1}{\sqrt{3}+\sqrt{7}}$

(g) $\left(2+\sqrt{2}\right)\div\left(\sqrt{3}-\sqrt{2}\right)$

3. The area of a rectangular Manila sheet is $\left(4+2\sqrt{3}\right)\text{m}^2$. If one side of the Manila sheet is $\left(3+2\sqrt{3}\right)\text{m}$, find the length of the other side in radical form.

4. A container has a volume of $72+18\sqrt{3}\ \text{m}^3$. If the length of the container is $6+\sqrt{3}\text{m}$ and height is $2+\sqrt{3}$ m, find the width of the container in radical form.

5. A rectangular canvas has an area of $50+14\sqrt{3}$ square metres. If one side of the canvas is $5+\sqrt{3}$ m, find the length of the other side in radical form.

Finding roots of numbers by calculators

Roots of numbers can be obtained by several methods which includes the use of exponent and radical rules, mathematical tables, and calculators. Engage in Activity 5.6 to explore the roots of numbers by using a calculator.

Activity 5.6: Determine roots of numbers by using calculators

1. Use any type of calculator to explore how to calculate roots of numbers.
2. Using a calculator of your choice, write the steps for finding a square root or a cube root of a number.
3. Copy the following table and use the procedures in task 2 to fill the blank correct to 4 significant figures.

Number	Square root	Cube root
49		
121		
64		
8.258		
56.78		
0.0287		
–225		

4. Share your results with others to illustrate how the answers are obtained.

Exercise 5.12

In questions 1 to 16, use a scientific calculator to evaluate each of the expressions correct to 4 significant figures:

1. $\sqrt{2156}$
2. $\sqrt[3]{1024}$
3. $\sqrt{267}$
4. $\sqrt{6.74}$
5. $\sqrt[5]{0.57}$
6. $\sqrt[7]{89.105}$

7. $\sqrt[8]{0.006}$ 8. $\sqrt[3]{0.0008}$

9. $\sqrt{25679}$ 10. $\sqrt{3567}$

11. $\sqrt[9]{646}$ 12. $\sqrt[6]{24}$

13. $\sqrt{154}$ 14. $\sqrt[15]{5}$

15. $\sqrt[7]{1.44}$ 16. $\sqrt[10]{\frac{99}{205}}$

17. Use a calculator to find the cube root of each of the following numbers, giveing the answer correct to 4 significant figures.

(a) 19 (b) 64

(c) 48.23 (d) 6.273

18. Use a calculator to evaluate the following numbers, giving the answer correct to 4 significant figures.

(a) $\sqrt[3]{55.94}$ (b) $\sqrt[3]{89.4}$

(c) $\sqrt[3]{1.25}$ (d) $\sqrt[3]{3.7}$

19. The volume of a solid cube is 1.674 m^3. Find the length of its side.

Chapter summary

1. The laws of exponents are as follows:

(a) $x^a \times x^b = x^{a+b}$

(b) $\frac{x^a}{x^b} = x^{a-b}$

(c) $(x^a)^b = x^{ab}$

(d) $x^{-a} = \frac{1}{x^a}$

(e) $x^{\frac{1}{a}} = \sqrt[a]{x}$

(e) $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ or $\left(\sqrt[b]{x}\right)^a$

2. The word radical means the n^{th} root, where $n = 2, 3, 4, \ldots$
For example, if $n = 2$, it implies the square root, and $n = 3$ implies the cube root.
3. Numbers having the same index and radicand can be added or subtracted.
4. In rationalizing the denominator, multiply both the numerator and denominator by the rationalizing factor.

Revision exercise 5

1. Write each of the following numbers without radicals:

(a) $\sqrt{900}$

(b) $\sqrt{160\,000}$

(c) $\sqrt[3]{8 \times 27 \times 5^3}$

2. Write each of the following numbers in surd form:

(a) $7^{\frac{1}{2}}$

(b) $19^{\frac{2}{3}}$

(c) $2^{\frac{1}{3}}$

3. Simplify each of the following expressions:

(a) $\sqrt{675} + \sqrt{75}$

(b) $\sqrt{1024} + \sqrt{4}$

(c) $\sqrt[3]{8} + \sqrt{64}$

(d) $\sqrt{175} + \sqrt{28} - \sqrt{63}$

(e) $\sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}$

(f) $\sqrt{1\,000} - \sqrt{40} - \sqrt{63}$

(g) $\sqrt{25} \times \sqrt{6}$

(h) $\sqrt{75} \times \sqrt{3}$

4. Simply each of the following radicals:

(a) $\sqrt{50}$ (f) $\sqrt[5]{2048}$

(b) $\sqrt{375}$ (g) $\sqrt{729}$

(c) $\sqrt{125}$ (h) $\sqrt[3]{625}$

(d) $\sqrt[3]{250}$ (i) $\sqrt[4]{1296}$

(e) $\sqrt[4]{4096}$ (j) $\sqrt[6]{3000}$

5. Simplify each of the following radicals:

(a) $\sqrt{4y^2}$ (d) $\sqrt{xy^2}$

(b) $\sqrt{8ym^3}$ (e) $\sqrt{729a^3b^3c^3}$

(c) $\sqrt{24y^3}$

6. Rationalize the denominator in each of the following expressions and simplify:

(a) $\frac{1}{\sqrt{3}}$ (e) $\frac{2}{\sqrt{5}-1}$

(b) $\frac{1}{\sqrt{2}+1}$ (f) $\frac{\sqrt{3}+1}{2\sqrt{3}}$

(c) $\frac{1}{(\sqrt{3}+1)}$ (g) $\frac{\sqrt{6}+4}{\sqrt{6}+\sqrt{2}}$

(d) $\frac{1}{\sqrt{3}+\sqrt{2}}$

7. Given that $a = \frac{1}{\sqrt{5}}$ and $y = \frac{1-a}{1+a}$, express each the following in its simplest surd form.

(a) y (b) $\frac{1}{a} - a$

8. Expand and simplify each of the following:

(a) $(\sqrt{3}-1)^2$

(b) $(\sqrt{5}-\sqrt{2})(\sqrt{3}-\sqrt{2})$

(c) $(\sqrt{3}+1)^2$

9. Find the square root of each of the following numbers:

(a) 2 916 (b) 5 625 (c) 0.25

10. Use a calculator to evaluate each of the following (give your answer correct to 4 significant figures):

(a) $\sqrt{8900}$ (d) $\sqrt[3]{75}$

(b) $\sqrt[3]{0.0015}$ (e) $\sqrt[3]{0.009}$

(c) $\sqrt[11]{256789}$ (f) $\sqrt[9]{1256}$

11. Write each of the following expressions as a single exponent.

(a) $10^4 \times 10^{-2}$ (f) $(8^{-3})^3$

(b) $\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{4}\right)^2$ (g) $\left(\left(\frac{3}{4}\right)^4\right)^{\frac{1}{2}}$

(c) $6^0 \times 4^2$ (h) $(x^6)^2$

(d) $10^3 \div 10^{-3}$ (i) $a^4b^3a^{-2}b^{-1}$

(e) $(5^3)^2$

12. Simplify each of the following:

(a) $\dfrac{m^4n^2}{m^2n}$

(b) $\dfrac{16a^5}{4a^3}$

(c) $\dfrac{(2r^3)^2}{(2r)^3}$

(d) $24a^2b^3 \div 3a^{-2}b^{-3}$

(e) $\dfrac{27x^8y^4}{(2xy^2)^2} \times \dfrac{(4x^2y^2)^2}{(3xy)^2}$

(f) $\dfrac{(8a^5b^4)(3a^2b^5)^3}{(9ab^2)(2ab)^3}$

(g) $\dfrac{2^5 \times 9^{-2}}{27^{-3} \times 8^{-4}}$

(h) $\dfrac{(x^3y)^3(2xy)^{-2}}{4x^{-4}y^{-5}}$

13. Evaluate each of the following expressions.

(a) $\dfrac{2^{m-1}-2^{m-2}}{2^{m-1}-2^{m+2}}$

(b) $\dfrac{3^{10+x} \times 27^{3x-4}}{3^{7x}}$

(c) $\dfrac{3 \times 125^{r+1} + 9 \times 5^{3r-1}}{8 \times 5^{3r} - 5 \times 125^{r}}$

(d) $\dfrac{27^{3n} \times 3^5 \times 3^{2n}}{9 \times 3^{11n} \times 3^{-2n}}$

(e) $\dfrac{7^{n-2}+7^n}{3 \times 7^n + 4 \times 7^n}$

(f) $\dfrac{4^5 \times 6^4 \times 12^3}{30^3 \times 3^6}$

(g) $\dfrac{5^{m+2}+5^m}{3 \times 5^m - 5^m}$

(h) $\dfrac{3^{4002}-3^{4000}+8}{3^{4000}+1}$

14. Solve each of the following equations.

(a) $8^x \times 4^{x-2} = 1$

(b) $9^{-2m+1} - 3^4 = 0$

(c) $3^r + 3^{r+2} = 90$

(d) $(5^r - 125)(2^r - 16) = 0$

(e) $2^{n+4} = 0.025$

(f) $3^{x^2-2x-3} = 1$

(g) $\left(\dfrac{27}{125}\right)^{p-1} = \left(\dfrac{25}{9}\right)^{2p+1}$

(h) $2^{2a+3} - 9 \times 2^a = -1$

(i) $9^x - 12(3^x) + 27 = 0$

(j) $49^x + 1 = 2(7^x)$

15. Solve each of the following equations for the unknown parameter:

(a) $\sqrt{a+1} = 9$

(b) $\sqrt{30-b} = b$

(c) $\sqrt{5c+11} + 4 = 10$

(d) $\sqrt{d+1} + \sqrt{d-5} = 8$

16. Factorise the following exponential expression

(a) $4^x + 9(2^x) + 18$

(b) $9^x + 4(3^x) - 5$

(c) $49^x - 7^{x+1} + 12$

(d) $9^x - 4^x$

(e) $4^x - 14(2^x) + 49$

(f) $25^x - 4(5^x) + 4$

17. Without using a calculator, solve for x in each of the following equations.

(a) $4^{x+1} = \left(\frac{1}{8}\right)^x$

(b) $\frac{25^x}{5^{x-3}} = \frac{5^x}{125^{x-2}}$

(c) $\frac{3^{2x+2}}{9^{3-x}} = \frac{27^{1-2x}}{3^{2x}}$

(d) $4\left(\frac{1}{3}\right)^x = 324$

18. Show that:

(a) $\frac{(2^x \times 3^2)(4 \times 2^{3-x})}{6 \times 2^{x+1}} = \frac{24}{2^x}$

(b) $\left(\frac{\sqrt{5}}{5}(1+\sqrt{2})\right)^2 = 3 + 2\sqrt{2}$

19. Given that $\sqrt{u + v\sqrt{5}} = \frac{31}{6+\sqrt{5}}$, where u and v are integers, find the value of u and v without using a calculator.

20. Simplify each of the following expressions:

(a) $\frac{(-4x^7y^4)^3}{2x^3y^{15}}$

(b) $\frac{(8x^3y)^2}{2xy^2}$

(c) $\frac{(-5x^6y^3)^2}{5y^8}$

(d) $(-5x^2y^3)^3$

(e) $\frac{(-4x^4y^2)^2}{3y^4}$

(f) $\frac{(4x^4y)^3}{3y^4}$

Chapter Six

Logarithms

Introduction

Some real life situations such as earthquakes, population growth, and levels of acid or alkali in liquids are associated with very large or very small numbers. In such cases, describing such situations becomes difficult. However the concept of logarithm has made such descriptions easy for an individual to understand the effects of their occurrences. In this chapter, you will learn to write numbers in standard form and use laws of logarithms to solve problems. The competencies developed will enable you to simplify and find solutions to complex expressions, solve problems related to exponential growth decay, and understand the science of earthquakes and the spread of diseases, and many other applications.

Think

Describing real life situations such as magnitudes of earthquakes, the amount of acid and base in liquids, and population growth without the knowledge of logarithms.

Standard form of numbers

When a number is expressed in the form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer, it is said to be in standard form or scientific notation or standard notation. For instance, the following numbers are expressed in standard form:

(a) $290 = 2.9 \times 100 = 2.9 \times 10^2$

(b) $29 = 2.9 \times 10 = 2.9 \times 10^1$

(c) $2.9 = 2.9 \times 10^0$

(d) $0.29 = 2.9 \times 0.1 = 2.9 \times 10^{-1}$

When writing numbers in standard form, the following must be considered:

1. For numbers between 0 and 1, move the decimal point towards right until a number is between 1 and 10 is obtained and the exponent of 10 is negative.
2. For numbers greater than or equal to 10, move the decimal point left until the value is between 1 and 10, resulting in a positive exponent of 10.
3. The exponent of 10 is determined by the number of places the decimal point is moved, either to the right or left.

4. For the numbers which are between 1 and less than 10, the exponent of 10 is 0.
5. In multiplication of numbers in standard form, the exponents are added, while in division the exponents are subtracted.

Example 6.1

Write each of the following numbers in standard form:

(a) 230000000

(b) 245

(c) 0.00045

Solution

(a) $230000000 = 2.3 \times 100000000$
$= 2.3 \times 10^8$

(b) $245 = 2.45 \times 100 = 2.45 \times 10^2$

(c) $0.00045 = 4.5 \times 0.0001$
$= 4.5 \times 10^{-4}$

Example 6.2

Express each of the following numbers in ordinary decimals numerals:

(a) 4.8×10^3

(b) 3.5×10^{-6}

(c) 1.4×10^{-4}

Solution

(a) $4.8 \times 10^3 = 4.8 \times 1000$
$= 4800$

(b) $3.5 \times 10^{-6} = 3.5 \times 0.000001$

$= 0.0000035$

(c) $1.4 \times 10^{-4} = 1.4 \times 0.0001$
$= 0.00014$

Example 6.3

Simplify each of the following expressions and write the answers in standard form:

(a) $(7 \times 10^2)(8 \times 10^4)$

(b) $(7 \times 10^{-2})(8 \times 10^{-4})$

Solution

(a) $(7 \times 10^2)(8 \times 10^4)$

$= (7 \times 8) \times (10^2 \times 10^4)$

$= 56 \times 10^6$

$= 5.6 \times 10 \times 10^6$

$= 5.6 \times 10^7$

Therefore,
$(7 \times 10^2)(8 \times 10^4) = 5.6 \times 10^7$.

(b) $(7 \times 10^{-2})(8 \times 10^{-4})$

$= 7 \times 8 \times 10^{-2} \times 10^{-4}$

$= 56 \times 10^{-6}$

$= 5.6 \times 10^1 \times 10^{-6}$

$= 5.6 \times 10^{-5}$

Therefore,
$(7 \times 10^{-2})(8 \times 10^{-4}) = 5.6 \times 10^{-5}$.

Example 6.4

Evaluate each of the following expressions, giving the answers in standard form:

(a) $\dfrac{4 \times 10^8}{5 \times 10^5}$

(b) $\dfrac{4848 \times 10^{-5}}{20 \times 10^2}$

Solution

(a) $\dfrac{4\times10^{8}}{5\times10^{5}}=\dfrac{4}{5}\times\dfrac{10^{8}}{10^{5}}$

$=0.8\times10^{3}$

$=8\times10^{-1}\times10^{3}$

$=8\times10^{2}$

Therefore, $\dfrac{4\times10^{8}}{5\times10^{5}}=8\times10^{2}$.

(b) $\dfrac{4848\times10^{-5}}{20\times10^{2}}=\dfrac{4848}{20}\times\dfrac{10^{-5}}{10^{2}}$

$=242.4\times10^{-7}$

$=2.424\times10^{2}\times10^{-7}$

$=2.424\times10^{-5}$

Therefore, $\dfrac{4848\times10^{-5}}{20\times10^{2}}=2.424\times10^{-5}$.

Example 6.5

Evaluate each of the following, giving the answers in standard form:

(a) $3\times10^{4}+4.32\times10^{3}$

(b) $5\times10^{-3}+6\times10^{-5}$

(c) $(6\times10^{6})-(2\times10^{4})$

(d) $3.14\times10^{-2}-4.03\times10^{-3}$

Solution

(a) $3\times10^{4}+4.32\times10^{3}$

$=3\times10^{4}+0.432\times10^{4}$

$=(3+0.432)\times10^{4}$

$=3.432\times10^{4}$

(b) $5\times10^{-3}+6\times10^{-5}$

$=5\times10^{-3}+0.06\times10^{-3}$

$=(5+0.06)\times10^{-3}$

$=5.06\times10^{-3}$

(c) $(6\times10^{6})-(2\times10^{4})$

$=(6\times10^{6})-(0.02\times10^{6})$

$=(6-0.02)\times10^{6}$

$=5.98\times10^{6}$

(d) $3.14\times10^{-2}-4.03\times10^{-3}$

$=3.14-10^{-2}-0.403\times10^{-2}$

$=(3.14-0.403)\times10^{-2}$

$=2.737\times10^{-2}$

Exercise 6.1

1. Write each of the following numbers in standard form:

(a) 31065 (b) 95.1

(c) 9999 (d) 6

(e) $\dfrac{1}{100}$ (f) 69.03

(g) 253009115 (h) 5.41

(i) 0.0004068 (j) 7.245

(k) 1985 (l) 0.000008

(m) $\dfrac{3}{4}$ (n) 30

(o) 463.18 (p) 265000

2. Write each of the following in decimal numerals:

(a) 9.10×10^{5}

(b) 7.4×10^{-4}

(c) 3×10^{0}

(d) 26.5×10^{4}

(e) 2.74×10^{4}

(f) 4.2×10^{-3}

(g) 3.68×10^{3}

(h) 8.67×10^{-2}

(i) 2.5×10^1

(j) 9.18×10^5

(k) 4.0×10^1

(l) 1.06×10^2

3. Evaluate each of the following expressions and write the answers in standard form:

(a) $(2.25\times10^3)\times(4\times10^6)$

(b) $(2.75\times10^4)\times(8\times10^2)$

(c) $(8.5\times10^{-3})\times(2.4\times10^2)$

(d) $(25\times10^4)\times(8\times10^4)$

(e) $(222\times10^{-3})\times(5.5\times10^{-2})$

4. Evaluate each of the following expressions, giving the answer in the form $A\times 10^n$, where $1 \le A < 10$ and n is an integer:

(a) $\frac{9\times10^4}{2\times10^2}$ (b) $\frac{7\times10^6}{1.4\times10^3}$

(c) $\frac{3\times10^8}{5\times10^8}$ (d) $\frac{192\times10^{-3}}{3\times10^{-2}}$

(e) $\frac{1984\times10^{-8}}{400\times10^{-3}}$ (f) $\frac{3.5\times10^{-2}}{7\times10^3}$

(g) $\frac{125\times10^{-2}}{5\times10^2}$ (h)

5. Find the area of a circle of radius 20 cm, giving the answer in standard form (use $\pi = 3.142$).

6. The distance of the Earth from the sun is approximately 1.494×10^{11} kilometres. Write this distance in decimal numerals.

7. The diameters of certain molecules have been calculated as follows:

(a) Hydrogen: 2.47×10^{-8} cm

(b) Oxygen: 3.39×10^{-8} cm

Represent the diameters in decimals.

8. A salon has 4×10^2 clients in a month. If each stylist works 2.0×10^2 hours per month, how many clients does each stylist serve per hour if there are 4×10^1 stylists? Give your answer in standard form.

9. A farm has 2.5×10^2 acres which are planted with maize. Each acre yields 1.2×10^3 kg of maize. The farm sells 2.0×10^5 kg of maize. How many kilograms of maize are left?

10. A project requires 4.0×10^7 bricks at Tshs 1.5×10^3 per brick. The project has a budget of Tshs 1.2×10^8. How much money will remain after buying the bricks?

11. An intravenous drip delivers 2.0×10^1 ml of fluid per hour for 6.0×10^1 hours. If the patient needs a total of 1.5×10^3 ml of fluid, how much more time is required to deliver the remaining fluid?

12. A construction site has two steel beams. One weighs 3.2×10^4 kg and the other weighs 4.5×10^5 kg. What is their combined mass?

13. A doctor prescribes to a patient a dose of 2.0×10^{-3} g per day. If the patient has taken 5.5×10^{-4} g so far, how much more should the patient take to reach the full dose?

14. An earthquake releases 7.5×10^{18} joules of energy. A smaller aftershock releases 2.0×10^{17} joules of energy. What is the total energy released by both events?

15. A factory emits 5.6×10^5 kg of carbon dioxide annually. If the factory reduces its emissions by 1.4×10^4 kg in a year, what is the new emission level?

Concept of logarithms

When a number is expressed in power form, it is written as a base raised to an exponent. For instance, if $a = b^x$, then 'a' is written in terms of base 'b' raised to an exponent 'x' or 'x' is the logarithm of a to base b.

The exponent x is the number that shows how many times a base is multiplied by itself to obtain a product. Thus, x is called the logarithm of a to base b. Symbolically, this is written as $\log_b a = x$, where $a > 0$, $b \neq 0$ and $b \neq 1$. This notation is called logarithmic notation. For instance, 64 in exponential form can be expressed as $64 = 2^6$. The exponent 6 is called the logarithm of 64 to base 2, written as $6 = \log_2 64$.

Consider the following:

(i) $25 = 5^2$ is written as $2 = \log_5 25$

(ii) $1000 = 10^3$ is written as $3 = \log_{10} 1000$

(iii) $0.0001 = 10^{-4}$ is written as $-4 = \log_{10} 0.0001$

$64 = 4^3$ is written as $\log_4 64 = 3$.

$64 = 8^2$ is written as $\log_8 64 = 2$.

In general, $a = b^x$ is written as $x = \log_b a$ where a and b are positive real numbers and x is a real number, $b \neq 0$ and $b \neq 1$.

Note: When expressing the logarithm of a number, make sure to specify the base to which it refers.

Example 6.6

Express each of the following equations according to the given instruction:

(a) $\log_2 8 = 3$ in exponential form.

(b) $5^{-3} = \frac{1}{125}$ in logarithmic form.

(c) $0.1 = 10^{-1}$ in logarithmic form.

Solution

(a) $\log_2 8 = 3$ in exponential form is written as $2^3 = 8$.

(b) $5^{-3} = \frac{1}{125}$ in logarithmic form is $-3 = \log_5\left(\frac{1}{125}\right)$.

(c) $0.1 = 10^{-1}$ in logarithmic form is $-1 = \log_{10}(0.1)$.

Example 6.7

Solve for x in each of the following equations:

(a) $x = \log_{10} 100$

(b) $-5 = \log_x (0.00001)$

(c) $\log_8 x = 2$

Solution

(a) Given $x = \log_{10} 100$.

Writing in exponential form gives $100 = 10^x$. Thus, $10^2 = 10^x$

Equating exponents gives $x = 2$.

(b) Given $-5 = \log_x 0.00001$.

Writing in exponential form, gives $x^{-5} = 0.00001$.

But $0.00001 = 10^{-5}$.

Thus, $x^{-5} = 10^{-5}$.

Therefore, $x = 10$.

(c) Given $\log_8 x = 2$.

Writing in exponential form gives $8^2 = x$.

Therefore, $x = 64$.

Special cases of logarithms

The following are some special cases on logarithms of numbers:

1. If $\log_a a = x$, then $a^x = a^1$ which gives $x = 1$.
   Therefore, $\log_a a = 1$.
   Thus, $\log_{10} 10 = 1$ and $\log_2 2 = 1$.
2. If $\log_a (a)^n = x$ for a positive number a; then $a^x = a^n$, which gives $x = n$.
   Therefore, $\log_a (a)^n = n$.
3. If $a^0 = 1$, then $\log_a 1 = 0$. Thus, logarithm of 1 to any base is 0.

Base 10 logarithms

Base 10 logarithms are logarithms of numbers to base 10, also known as common logarithms. The base 10 is usually left out when writing common logarithms to base 10. For instance, instead of writing $\log_{10} 315$ it is simply written as $\log 315$. In general, $\log_{10} x$ is written as $\log x$.

The following are some logarithms of numbers which are powers of integral exponents of 10:

$\log 100 = \log 10^2 = 2$

$\log 10 = \log 10^1 = 1$

$\log 1 = \log 10^0 = 0$

$\log 0.1 = \log 10^{-1} = -1$

$\log 0.01 = \log 10^{-2} = -2$

$\log 0.001 = \log 10^{-3} = -3$

In general, $\log 10^n = n$

Exercise 6.2

1. Write each of the following expressions in logarithmic form:
   (a) $2^4 = 16$ (b) $5^2 = 25$
   (c) $3^5 = 243$ (d) $4^3 = 64$
   (e) $10^6 = 1{,}000{,}000$ (f) $3^{-2} = \frac{1}{9}$
   (g) $10^0 = 1$ (h) $13^1 = 13$
   (i) $10^{-3} = 0.001$ (j) $10 = \frac{1}{10^{-1}}$
   (k) $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$ (l) $23^{-1} = \frac{1}{23}$
2. Write each of the following expressions in exponential forms:
   (a) $\log_{11} 121 = 2$
   (b) $\log_{10} 10{,}000 = 4$
   (c) $\log_{10} 0.1 = -1$
   (d) $\log_4 2 = \frac{1}{2}$
   (e) $\log_2 0.25 = -2$
   (f) $\log_{\frac{1}{5}} \left(\frac{1}{125}\right) = 3$
3. Find the value of x in each of the following equations:
   (a) $\log_2 x = 2$
   (b) $\log_5 1 = x$
   (c) $\log_5 x = 1$
   (d) $\log_4 x = -3$
   (e) $\log_4 256 = x$
   (f) $\log_x 10 = 1$
   (g) $\log_2 \left(\frac{1}{1\,024}\right) = x$

(h) $\log_x 100{,}000{,}000 = 8$

(i) $\log_x 1{,}000 = 3$

(j) $\log_{\frac{1}{2}} 0.0625 = x$

(k) $\log_{25} x = \frac{3}{2}$

(l) $\log_x \frac{1}{27} = -3$

(m) $\log_2 256 = x$

(n) $\log_2 1 = x$

(o) $\log_2 x = \frac{7}{2}$

4. Determine the number whose logarithm to base 5 is –3.
5. Solve each of the following:
   (a) $\log_3(x+2) = 2$
   (b) $\log_2 16 = x+5$
6. Simplify $\frac{\log 125 - \log 5}{\log 25 + \log 5}$.

Laws of logarithms

The laws of logarithms, also known as logarithm rules, are fundamental identities or rules that describe how to manipulate logarithms. The following are the key laws:

1. Product rule	$\log_a(xy) = \log_a x + \log_a y$
2. Quotient rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. Power rule or Rule of exponents	$\log_a m^n = n\log_a m$
4. Roots rule	$\log_a \sqrt[m]{x^n} = \log_a x^{\frac{n}{m}} = \frac{n}{m}\log_a x$, where n and m are integers and $m \neq 0$
5. Change of base formula	$\log_a x = \frac{\log_c x}{\log_c a}$

Derivation of the laws of logarithms

The laws of logarithms can be derived as follows:

Product rule

Let $p = \log_a x$ and $q = \log_a y$ (1)

Expressing equation (1) in exponential form gives,

$x = a^p$ and $y = a^q$ (2)

From the rules of exponents, $a^p \times a^q = a^{p+q}$.

It follows that $xy = a^p \times a^q = a^{p+q}$. (3)

Expressing equation (3) in logarithmic form gives,

$$\log_a(xy) = \log_a a^{p+q}$$
$$= (p+q)\log_a a$$
$$= p+q$$

Thus, $\log_a(xy) = p+q$. (4)

Substituting equations in (1) into equation (4) gives

$\log_a(xy) = \log_a x + \log_a y$.

Therefore, $\log_a(xy) = \log_a x + \log_a y$.

Example 6.8

Evaluate each of the following.

(a) $\log_3(81\times9)$

(b) $\log_5(125\times625)$

Solution

(a) $\log_3(81\times9)=\log_3 81+\log_3 9$

$=\log_3 3^4+\log_3 3^2$

$=4\log_3 3+2\log_3 3$

$=(4\times1)+(2\times1)$

$=4+2$

$=6$

Therefore, $\log_3(81\times9)=6$.

(b) $\log_5(125\times625)=\log_5 125+\log_5 625$

$=\log_5 5^3+\log_5 5^4$

$=3\log_5 5+4\log_5 5$

$=(3\times1)+(4\times1)$

$=3+4$

$=7$

Therefore, $\log_5(125\times625)=7$.

Example 6.9

Find the value of each of the following:

(a) $\log_2(4\times8)$

(b) $\log_{10}(0.01\times100{,}000)$

Solution

(a) $\log_2(4\times8)=\log_2 4+\log_2 8$

$=\log_2 2^2+\log_2 2^3$

$=2\log_2 2+3\log_2 2$

$=(2\times1)+(3\times1)$

$=2+3$

$=5$

Therefore, $\log_2(4\times8)=5$.

(b) $\log_{10}(0.01\times100000)$

$=\log_{10}0.01+\log_{10}100000$

$=\log10^{-2}+\log10^5$

$=2\log10+5\log10$

$=(-2\times1)+(5\times1)$

$=-2+5$

$=3$

Therefore, $\log_{10}(0.01\times100{,}000)=3$.

Example 6.10

Write each of the following as a single logarithm.

(a) $\log_a 2+\log_a 7$

(b) $\log_x 4+\log_x 5$

Solution

(a) $\log_a 2+\log_a 7=\log_a(2\times7)$

$=\log_a 14$

(b) $\log_x 4+\log_x 5=\log_a(4\times5)$

$=\log_a 20$

Example 6.11

If $\log_{10}2=0.3010$, $\log_{10}3=0.4771$, and $\log_{10}5=0.6990$. Evaluate each of the following:

(a) $\log_{10}6$

(b) $\log_{10}30$

(c) $\log_{10}45$

Solution

(a) $\log_{10}6=\log_{10}2+\log_{10}3$

$=0.3010+04771$

$=0.7781$

(b) $\log_{10} 30 = \log_{10} 3 + \log_{10} 10$

$= \log_{10} 3 + 1$

$= 0.4771 + 1$

$= 1.4771$

(c) $\log_{10} 45 = \log_{10}(3 \times 3 \times 5)$

$= \log_{10} 3 + \log_{10} 3 + \log_{10} 5$

$= 2\log_{10} 3 + \log_{10} 5$

$= 2 \times 0.4771 + 0.6990$

$= 1.6532$

Power rule

Power rule states that $\log_a m^n = n\log_a m$.

Let $p = \log_a m$. (i)

It is derived as follows:

Express equation (i) in exponential form to get

$a^p = m$ (ii)

Raise both sides of equation (ii) to the power n to get,

$a^{pn} = m^n$ (iii)

Apply logarithm to base a on both sides of equation (iii) to obtain,

$\log_a \left(a^{pn}\right) = \log_a m^n$ (iv)

Simplify equation (iv) to get $pn = \log_a m^n$.

But $p = \log_a m$.

It follows that, $n\log_a m = \log_a m^n$

Therefore, $\log_a m^n = n\log_a m$.

From this rule, it follows that

$$\log_a \frac{1}{x} = -\log_a x$$

Example 6.12

Evaluate $\log_3 9^2$.

Solution

$\log_3 9^2 = 2\log_3 9$

$= 2\log_3 3^2$

$= 2(2)\log_3 3$

$= 4(1)$

$= 4$

Therefore, $\log_3 9^2 = 4$.

Example 6.13

Find the values of each of the following:

(a) $\log_4 (64)^5$

(b) $\log\ (100)^{25}$

(c) $\log\ (0.1)^6$

Solution

(a) $\log_4 (64)^5 = 5\log_4 (64)$

$= 5\log_4 4^3$

$= 5 \times 3\log_4 4$

$= 15$

Therefore, $\log_4 (64)^5 = 15$.

(b) $\log(100)^{25} = 25\log 100$

$= 25\log 10^2$

$= 25 \times 2\log 10$

$= 25 \times 2 \times 1$

$= 50$

Therefore, $\log(100)^{25} = 50$.

(c) $\log(0.1)^6 = 6\log(0.1)$

$= 6\log 10^{-1}$

$= 6\times(-1)\log 10$

$= 6\times(-1)\times 1$

$= -6$

Therefore, $\log(0.1)^6 = -6$.

Quotient rule

Let $p = \log_a x$ and $q = \log_a y$. (1)

Expressing equation (1) in exponential form gives

$x = a^p$ and $y = a^q$. (2)

From equation (2), divide x by y to obtain

$\frac{x}{y} = a^p \div a^q = a^{p-q}$ (by laws of exponents)

Thus, $\frac{x}{y} = a^{p-q}$. (3)

Express equation (3) in logarithmic form to obtain

$\log_a\left(\frac{x}{y}\right) = \log_a a^{p-q}$

$= (p-q)\log_a a$

Thus, $\log_a\left(\frac{x}{y}\right) = p - q$. (4)

Substitute the expressions for p and q from equation (1) into (4) to obtain

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Therefore, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

Example 6.14

Find the value of each of the following expressions.

(a) $\log_3\left(\frac{27}{9}\right)$.

(b) $\log_3(9 \div 243)$

(c) $\log(10 \div 0.001)$

Solution

(a) $\log_3\left(\frac{27}{9}\right) = \log_3 27 - \log_3 9$

$= \log_3 3^3 - \log_3 3^2$

$= 3\log_3 3 - 2\log_3 3$

$= 3(1) - 2(1)$

$= 1$

Therefore, $\log_3\left(\frac{27}{9}\right) = 1$.

(b) $\log_3(9 \div 243) = \log_3 9 - \log_3 243$

$= \log_3 3^2 - \log_3 3^5$

$= 2\log_3 3 - 5\log_3 3$

$= 2(1) - 5(1)$

$= -3$

Therefore, $\log_3(9 \div 243) = -3$.

(c) $\log(10 \div 0.001) = \log_{10}\left(\frac{10}{0.001}\right)$

$= \log_{10} 10 - \log_{10} 0.001$

$= \log_{10} 10^1 - \log_{10} 10^{-3}$

$= 1\log_{10} 10 - (-3)\log_{10} 10$

$= 1 + 3 = 4$

Therefore, $\log(10 \div 0.001) = 4$.

Roots law

The law states that $\log_a \sqrt[m]{x^n} = \frac{n}{m}\log_x a$

It is proved as follows:

Let $p = \log_a \sqrt[m]{x^n}$. From the law of exponents, $\sqrt[m]{x^n} = x^{\frac{n}{m}}$

It follows that $p = \log_a x^{\frac{n}{m}}$

$$= \frac{n}{m}\log_a x$$

Therefore, $\log_a \sqrt[m]{x^n} = \frac{n}{m}\log_a x$.

Example 6.15

Find the values of each of the following:

(a) $\log_9 \sqrt{729}$ (b) $\log \sqrt[3]{0.000001}$

(c) $\log \sqrt[5]{100000}$ (d) $\log_3 \sqrt{\frac{1}{27}}$

Solution

(a) $\log_9 \sqrt{729} = \log_9 (729)^{\frac{1}{2}}$

$$= \frac{1}{2}\log_9 9^3$$

$$= \frac{1}{2}\times 3\log_9 9$$

$$= \frac{3}{2}$$

Therefore, $\log_9 \sqrt{729} = \frac{3}{2}$.

(b) $\log \sqrt[3]{0.000001} = \log(0.000001)^{\frac{1}{3}}$

$$= \log(10^{-6})^{\frac{1}{3}}$$

$$= \log 10^{-\frac{6}{3}}$$

$$= \log 10^{-2}$$

$$= -2\log 10$$

$$= -2\times 1$$

Therefore, $\log \sqrt[3]{0.000001} = -2$.

(c) $\log \sqrt[5]{100000} = \log(100\,000)^{\frac{1}{5}}$

$$= \frac{1}{5}\log 100000$$

$$= \frac{1}{5}\log 10^5$$

$$= \frac{1}{5}\times 5\log 10$$

$$= \frac{1}{5}\times 5\ 1$$

$$= 1$$

Therefore, $\log \sqrt[5]{100000} = 1$.

(d) $\log_3 \sqrt{\frac{1}{27}} = \log_3 \left(\frac{1}{27}\right)^{\frac{1}{2}}$

$$= \frac{1}{2}\log_3 \left(\frac{1}{27}\right)$$

$$= \frac{1}{2}\log_3 \left(\frac{1}{3}\right)^3$$

$$= \frac{1}{2}\times 3\log_3 \left(\frac{1}{3}\right)$$

$$= \frac{3}{2}\log_3 3^{-1}$$

$$= \frac{3}{2}\times(-1)\log_3 3$$

$$= -\frac{3}{2}\times 1$$

$$= -\frac{3}{2}$$

Therefore, $\log_3 \sqrt{\frac{1}{27}} = -\frac{3}{2}$.

Example 6.16

Determine the value of x given that $\log_2 x = \log_2 2 - \log_2 3$.

Solution

Given $\log_2 x = \log_2 2 - \log_2 3$. It implies that,

$$\log_2 x = \log_2\left(\frac{2}{3}\right)$$

Thus, $x = \frac{2}{3}$.

Therefore, $x = \frac{2}{3}$.

Example 6.17

Given that $\log 2 = 0.30103$ and $\log 3 = 0.47712$. Calculate the value of $\log 48$.

Solution

$$\log 48 = \log(2^4 \times 3)$$
$$= \log 2^4 + \log 3$$
$$= 4\log 2 + \log 3$$
$$= 4(0.30103) + 0.47712$$
$$= 1.68124$$

Therefore, $\log 48 = 1.68124$.

Example 6.18

Find the value of x given that

$$\log_a x = \frac{1}{2}\log_a 4 + \frac{1}{3}\log_a 27.$$

Solution

$$\log_a x = \frac{1}{2}\log_a 4 + \frac{1}{3}\log_a 27$$
$$= \log_a 4^{\frac{1}{2}} + \log_a 27^{\frac{1}{3}}$$
$$= \log_a (2^2)^{\frac{1}{2}} + \log_a (3^3)^{\frac{1}{3}}$$
$$= \log_a 2 + \log_a 3$$
$$= \log_a (2 \times 3)$$
$$= \log_a 6$$

Thus, $\log_a x = \log_a 6$.

Therefore, $x = 6$.

Example 6.19

Simplify $\frac{\log 6}{\log 216}$.

Solution

$$\frac{\log 6}{\log 216} = \frac{\log 6}{\log 6^3}$$
$$= \frac{\log 6}{3\log 6}$$
$$= \frac{1}{3}$$

Therefore, $\frac{\log 6}{\log 216} = \frac{1}{3}$.

Exercise 6.3

1. Find the value of each of the following expressions.
   (a) $\log_3(9 \times 81)$
   (b) $\log_5(5 \times 25 \times 625)$
   (c) $\log\ (100 \div 0.0001)$
   (d) $\log_7(49 \div 343)$
2. Calculate the value of each of the following expressions.
   (a) $\log_7 49^3$

(d) $\log_2 \sqrt{8}$

(b) $\log_5 (5 \div 125)^3$

(e) $\log \sqrt[3]{1000}$

(c) $\log \sqrt[3]{0.0001}$

(f) $\log 0.001^5$

3. Simplify each of the following expressions using the laws of logarithms:

(a) $\dfrac{\log 64}{\log 4}$

(b) $\log_2 28 - \log_2 7$

(c) $\log_3 10 + \log_3 8.1$

(d) $\log 20 + \log 50$

(e) $\log 3^4 + \log\left(\dfrac{10}{81}\right)$

(f) $\dfrac{\log \sqrt[3]{4}}{\log \sqrt{4}}$

4. Without using a calculator, find the value of 2log5 + log36 – log9.

5. Use logarithmic laws to expand each of the following expressions.

(a) $\log_5\left(\dfrac{x^2 y^{-3}}{25}\right)$

(b) $\log\left(\dfrac{u^2\sqrt{v}}{zv^5}\right)$

(c) $\log\left(\dfrac{(x+y)^2}{\sqrt{z}}\right)$

6. Given that log2x = u and log2y = v, write:

(a) 2^{2u-2} in terms of x

(b) 2^{4v+1} in terms of y

7. Find the value of x in each of the following equations:

(a) $\log_a x = \log_a 5 + 2\log_a 3$

(b) $\log_a x = 3\log_a 15 - 2\log_a 15$

(c) $\log_3 x = \log_3 4 + \log_3 5$

(d) $\log x = \log 2 + \log 5$

(e) $\log x = \log 20 - \log 200 + \log 50$

(f) $\log x = \log 2 - \log 20 + \log 5$

8. Given that $\log 2 = 0.30103$, $\log 3 = 0.47712$, and $\log 5 = 0.69897$.
Find the value of:

(a) log 90

(b) $\dfrac{1}{5}\log\left(\dfrac{\sqrt{6}}{5}\right)^5$

9. If $\log y + 2\log x = 3$, express y in terms of x.

10. Express p in terms of q for each of the following:

(a) $\log q - 4\log p = 2$

(b) $\log q + \log m = \log(m - bp)$

11. Solve for x given $\log(x-2) - \log(x-1) = 0$.

12. Evaluate each of the following

(a) log9(81)

(b) $\log_6 36$

(c) log2(16)

(d) log8(64)

13. Given $\log_{10}2 = 0.30103$,

$\log_{10}3 = 0.47712$,

$\log_{10}5 = 0.69897$,

$\log_{10}7 = 0.845098$.

Evaluate the following

(a) $\log_7 5$

(c) $\log_9 50$

(b) $\log_{12} 8$

(d) $\log_4 10$

Change of base formula

The change of base for logarithms gives a way to convert a logarithm from one base to another. This is useful when evaluating logarithmic expressions of any baseby translating expressions into common logarithms.

The rule of change of base states that:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

It is proved as follows:

Let $y = \log_b x$ (i)

By definition of logarithms, equation (i) becomes:

$b^y = x$ (ii)

Taking the logarithm to base a on both sides of equation (ii) gives,

$\log_a\left(b^y\right) = \log_a x$ (iii)

Using the power rule for logarithms, equation (iii) becomes:

$y\log_a b = \log_a x$

Solving for y gives,

$$y = \frac{\log_a x}{\log_a b}$$

But $y = \log_b x$.

Therefore: $\log_b x = \dfrac{\log_a x}{\log_a b}$

Example 6.20

Evaluate each of the following:

(a) $\log_2 32$ (b) $\log_5 125$

(c) $\log_3 81$ (d) $\log_4 16$

Solution:

(a) $$\log_2 32 = \frac{\log_{10}32}{\log_{10}2} = \frac{\log_{10}2^5}{\log_{10}2} = \frac{5\log_{10}2}{\log_{10}2} = 5$$

(b) $$\log_5 125 = \frac{\log_{10}125}{\log_{10}5} = \frac{\log_{10}5^3}{\log_{10}5} = \frac{3\log_{10}5}{\log_{10}5} = 3$$

(c) $$\log_3 81 = \frac{\log_{10}81}{\log_{10}3} = \frac{\log_{10}3^4}{\log_{10}3} = \frac{4\log_{10}3}{\log_{10}3} = 4$$

(d) $$\log_4 16 = \frac{\log_{10}16}{\log_{10}4} = \frac{\log_{10}4^2}{\log_{10}4} = \frac{2\log_{10}4}{\log_{10}4} = 2$$

Example 6.21

Given,

$\log_{10}2 = 0.30103$, $\log_{10}3 = 0.47712$,

$\log_{10}5 = 0.69897$ and $\log_{10}7 = 0.845098$

Evaluate the following:

(a) $\log_3 7$ (b) $\log_5 12$

(c) $\log_7 15$ (d) $\log_6 20$

Solution

(a) $\log_3 7 = \frac{\log_{10}7}{\log_{10}3}$

$= \frac{0.845098}{0.47712} = 1.77124$

(b) $\log_5 12 = \frac{\log_{10}12}{\log_{10}5} = \frac{\log_{10}(2^2 \times 3)}{\log_{10}5}$

$= \frac{2\log_{10}2 + \log_{10}3}{\log_{10}5}$

$= \frac{2 \times 0.30103 + 0.47712}{0.69897} = 1.55396$

(c) $\log_7 15 = \frac{\log_{10}15}{\log_{10}7} = \frac{\log_{10}(3 \times 5)}{\log_{10}7}$

$= \frac{\log_{10}3 + \log_{10}5}{\log_{10}7}$

$= \frac{0.47712 + 0.69897}{0.845098} = 1.39166$

(d) $\log_6 20 = \frac{\log_{10}20}{\log_{10}6} = \frac{\log_{10}(2^2 \times 5)}{\log_{10}6}$

$= \frac{2\log_{10}2 + \log_{10}5}{\log_{10}2 + \log_{10}3}$

$= \frac{2 \times 0.30103 + 0.69897}{0.30103 + 0.47712} = 1.67995$

Logarithms of numbers

For many years, finding logarithms of numbers to base 10 has been tedious due to the use of slide rules and tables of common logarithms. For instance, to use common logarithmic tables, it was necessary to express a number in standard form, followed by identifying a mantissa and a characteristic that are then used to read logarithms of the respective numbers from a table of common logarithms.

Engage in Activity 6.1 to explore about logarithms of numbers.

Activity 6.1: Determining the logarithm of numbers using calculators

1. Explore the internet or other sources and learn different ways to determine the logarithms of numbers by using calculators (mathematical software and calculator devices).
2. Determine logarithm values of numbers in the ranges $0 < x \leq 1$, $1 < x \leq 10$, and numbers which are greater than 10.
3. Record your results in a table, compare the results from these categories and share your observations with others.

Example 6.22

Find the logarithm of each of the following numbers, correct to 4 decimal places:

(a) 1 (d) 356 (g) 75,648

(b) 0.0253 (e) 2,534 (h) 64.667

(c) −3 (f) 62.94

Solution

To find the logarithm of a number, enter the keystrokes of the logarithm to the base available in your calculator and record the answer.

Answers

(a) 0 (b) −1.5969

(c) Does not exist (d) 2.5514

(e) 3.4038 (f) 1.7989

(g) 4.8788 (h) 1.8107

Antilogarithms

An antilogarithm of a number is the inverse process of finding a number whose logarithm is known. For instance, since 1.7989 is the logarithm of 62.94, then 62.94 is called the antilogarithm of 1.7989. This can also be read as 62.94 is the antilogarithm of 1.7989 or $10^{1.7989} = 62.94$

In general, the antilogarithm of a number x is denoted as $\log^{-1} x$. Engage in Activity 6.2 to explore about antilogarithm

Activity 6.2: Determining antilogarithms by using calculators

1. Explore the internet or other sources and learn different ways to determine the antilogarithms.
2. Explore any available mathematical software or calculating devices to learn how to determine antilogarithms.
3. Choose different numbers and find their respective logarithms.
4. Use the mathematical software or calculating device to find the antilogarithms of the results in task 3.
5. Compare the numbers you have chosen in task 3 with those obtained in task 4.
6. Explain to others how you arrived at your results and any patterns you noticed.

Example 6.23

Use a calculator to find the number whose logarithm is:

(a) 3 (b) 0.7

(c) −2 (d) 1.8810

(e) 3.7515 (f) 1.2466

Solution

Enter the keystrokes to find the antilogarithm of a number based on the available calculator and record the answer.

Answers

(a) 1000 (c) 0.01 (e) 5642.87

(b) 5.01 (d) 76.03 (f) (f) 17.64

Exercise 6.4

1. Use a calculator to find the logarithm of each the following numbers: giving the answer correct to 4 decimal places.

   (a) 1.583 (b) 2.98

   (c) 4.088 (d) 8.541

   (e) 9.007 (f) 6

   (g) 3.444 (h) 7.5348

2. Use a calculator to find the logarithm of each the following numbers: giving the answer correct to 4 decimal places.

   (a) 0.682 (b) 0.008

   (c) 0.74 (d) 0.0000449

   (e) 0.031199 (f) 0.01478

   (g) 0.125 (h) 0.00981

3. Use a calculator to find the logarithm of each the following numbers giving the answer correct to 4 decimal places.

(a) 8725	(g) 20
(b) 700.1	(h) 1986
(c) 76 408	(i) 78085.9
(d) 4 300 000	(j) 354
(e) 83.7	(k) 43.657
(f) 111 327	(l) 493 000

4. Determine a number whose logarithm is: (give the answer correct to 4 significant figure)

(a) 3.6156	(e) 0.07783
(b) 1.8937	(f) −3.8567
(c) 0.4515	(g) 1.2466
(d) 8	(h) −6.7515

5. (a)If $\log x = 3.1425$, determine $\sqrt{x}$.

(b) If $\log x = 2.5543$, determine $\sqrt[4]{x}$.

(c) If $\log x = 4.3217$, find the value of $\sqrt[3]{x}$.

6. Find the value of x in each of the following:

(a) $\log x = 2.9285$

(b) $\log x = 1.6990$

Applications of Logarithms

Logarithms are used in solving problems involving exponential relationships. Logarithms are widely applied in real-life situations such as measuring pH in chemistry, calculating compound interest in finance, and determining earthquake magnitudes.

Activity 6.3: Exploring applications of logarithms in daily life

1. Using various exploration methods such as searching through the internet, reading books and interviewing experts, explore the applications of logarithms in daily life.
2. Using logarithm laws, demonstrate how real life problems are solved.
3. Share your findings with others through demonstrations and other methods of your choice.

Example 6.25

Solve the equation $7^{x+1} = 90$. Give you answer into 4 decimal places.

Solution:

Given $7^{x+1} = 90$.

Taking logarithm on both sides gives

$\log(7^{x+1}) = \log 90$

$(x+1)\log 7 = \log 90$

$x+1 = \frac{\log 90}{\log 7}$, $x = \frac{\log 90}{\log 7} - 1$

Using a calculator, $x = 1.3124$

Example 6.25

The pH of a solution is given by the formula $pH = -\log[H^+]$. If the hydrogen ion concentration, $[H^+]$ of pure water is $1\times10^{-7}M$, calculate the pH of pure water.

Solution

Given $pH = -\log[H^+]$ and $[H^+] = 1\times10^{-7}$M.

Substitute the value of the hydrogen ions into the formula to get

$$pH = -\log\left[1\times10^{-7}\right]$$

Using the laws of logarithms or calculator to determine the logarithm gives

$$pH = 7$$

Therefore, the pH of pure water is 7.

Example 6.26

The formula for determining a compound interest is given by $A = P(1+r)^t$, where A is the final amount, P is the principal, r is the annual interest rate, and t is the time in years. Find the number of years it will take for the principal to double if $r = 5\%$.

Solution

Given $A = 2P$, $r = 5\%$, $A = P(1+r)^t$

From $A = P(1+r)^t$, substitute the given values to get

$$2P = P(1+0.05)^t$$

$$2 = (1.05)^t$$

Apply logarithms on both sides and simplify to obtain,

$$\log 2 = \log(1.05)^t$$

$$\log 2 = t\log(1.05)$$

$$t = \frac{\log 2}{\log(1.05)}$$

Using a calculator to find the value of t gives $t = 14.2$.

Therefore, the principal will double after 14.2 years.

Example 6.27

The Richter scale measures the magnitude M of an earthquake based on the energy released by using the formula $M = \log\left(\frac{E}{E_0}\right)$, where E is the energy released and E_0 is the standard energy. If an earthquake recorded a magnitude of 8 with standard energy $E_0 = 10^4$ Joules, what was the energy released?

Solution

Given $M = \log\left(\frac{E}{E_0}\right)$, $E_0 = 10^4$ Joules, $M = 8$.

Substituting these values into the formula gives

$$8 = \log\left(\frac{E}{10^4}\right)$$

Applying writing into exponential form gives,

$$\frac{E}{10^4} = 10^8$$

$$E = 10^8 \times 10^4$$

$$= 10^{12}$$

Therefore, the energy released by the earthquake is 10^{12} Joules.

Exercise 6.5

Whenever required, take $e = 2.7183$.

1. Solve each of the following exponential equations giving the answers correct to 4 decimal places.
   (a) $3^x = 2^{x-1}$
   (b) $4\times1.5^x = 30$

(c) $\log_5 n = \log_4 2$

(d) $\log_m 15 = 3$

(e) $5^{x+2} - 5^x = 20$

(f) $2^{2x} - 5 \times 2^x + 6 = 0$

2. The pH of a solution is 7. Calculate the hydrogen ion concentration $[H^+]$ using the formula $[H^+] = 10^{-pH}$.

3. Solve x given $(\log_2 x)^2 + 12 = \log_2 x^7$

4. A population grows exponentially according to the formula $P(t) = P_0 e^{rt}$, where P_0 is the initial population, r is the growth rate, and t is time. If the initial population is 500 and the growth rate is 2%, how long will it take for the population to reach 1000?

5. Using the compound interest formula $A = P(1+r)^t$, calculate the time t needed for an investment to double if the interest rate $r = 4\%$ annually.

6. Rearrange the formula $L = 10\log\left(\frac{I}{I_0}\right)$ to solve for I, then calculate I if $L = 80$ dB and $I_0 = 1\times 10^{-12}$ watts/m².

7. The decay of a substance over time is given by $N(t) = N_0 e^{-\lambda t}$ where N_0 is the initial quantity, λ is the decay constant, and t is time. If $N_0 = 500$ and $N(t) = 250$ after 10 years, calculate the value of λ.

8. Using the compound interest formula $A = P(1+r)^t$, calculate the time t needed for an investment to triple its initial value if the interest rate is 5% annually.

9. The intensity level L of sound is 90 dB. Calculate the sound intensity I if $I_0 = 1\times 10^{-12}$ watts/m², given $L = 10\log\left(\frac{I}{I_0}\right)$.

10. The population of a certain bacteria grows according to the formula $P(t) = P_0 e^{rt}$. If the initial population is 200 and the population doubles in 5 hours, calculate the growth rate r.

11. Solve the following simultaneous equations

(a) $\begin{cases} 2^x = 8 \times 2^y \\ \log(2x - 3y) = 1 \end{cases}$

(b)

$$\begin{cases} \log_2(y+3) - 1 = \log_2 x \\ \log_2(x+4) + \log_2(x-4) = \log_2(y-2) \end{cases}$$

Chapter summary

1. A number in standard form is written as A × 10n, where 1 ≤ A < 10 and n is an integer.
2. A number in exponential form can be expressed in logarithmic form as shown in the following table.

Exponential form	Logarithmic form
$1000 = 10^3$	$\log_{10} 1\,000 = 3$
$1 = 10^0$	$\log_{10} 1 = 0$
$0.01 = 10^{-2}$	$\log_{10} 0.01 = -2$

Generally, if $x = a^b$, then $\log_a x = b$.

3. The laws of logarithms are:

Logarithm of a product:

$$\log_a(MN) = \log_a M + \log_a N$$

Logarithm of a quotient:

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

Logarithm of a power:

$$\log_a(M)^p = p\log_a M$$

Logarithm of identical power and base: $\log_a a = 1$

Logarithm of a root:

$$\log_a \sqrt[m]{x^n} = \frac{n}{m}\log_a x$$

4. The conversion formula from $\log_a x$ to $\log_b x$ is given by $\log_b x = \frac{\log_a x}{\log_a b}$.

5. The principles of calculating the logarithms depend on the laws of exponents.
That is;
(i) when multiplying, add logarithms,
(ii) when dividing, subtract, and
(iii) when raising to a power, multiply by the exponent.

Revision exercise 6

1. Write each of the following numbers in standard form:

(a) 8 419 000 (d) 0.000123

(b) 45.7 (e) 4

(c) 716 (f) 0.005

2. Determine the decimal numerals for each of the following:

(a) 9.15×10^5 (b) 8×10^{-3}

(c) 1.06×10^2 (d) 2.5×10^1

3. Compute each of the following, give the answer in standard form:

(a) $(8\times10^{-5})\times(27.5\times10^{15})$

(b) $(12.5\times10^{4})\times(8\times10^{-7})$

(c) $\frac{8\times10^{-3}}{5\times10^{-5}}$

(d) $\frac{1.728\times10^{2}}{1.2\times10^{2}}$

(e) $(2.5\times10^{3})\times(1.5\times10^{2})-(2.0\times10^{4})$

(f) $\frac{(1.2\times10^{2}-6\times10^{-1})}{5.0\times10^{2}}$

(g) $\frac{(1.2\times10^{3})\times(2.1\times10^{-2})}{4\times10^{2}}$

(h) $6.13\times10^{-10}+3.89\times10^{-8}$

4. Find the value of x in each of the following:

(a) $\log_8 x = 4$

(b) $\log_x\left(\frac{1}{125}\right) = -3$

(c) $\log\ x = 3$

5. Determine the value of x in each of the following:

(a) $\log(x2+3x-44)=1$

(b) $\log(2x+1)=0$

6. Determine the number whose logarithm is defined as follows:

(a) base 10 is 6
(b) base 6 is 6.

7. Given that $\log x = 8.0524$, find the value of $\log \sqrt[3]{x}$.

8. Given that $\log 2 = 0.30103$, $\log 5 = 0.69897$, and $\log 7 = 0.84510$, find the value of $\log\left(\frac{35}{2}\right)$ without using a calculator.

9. Without using calculator, determine the value of $\log 5 - \log 8 + 4\log 2$.

10. Without using a calculator, simplify each of the following:
   (a) $\frac{\log\sqrt{10}}{\log 10} \times \log 100$
   (b) $\log_4 176 - \log_4 11$

11. Use a calculator to find the logarithm of each of the following numbers:
   (a) 0.8008 (b) 724 079
   (c) 0.0002 (d) 23.9

12. Find the value of x if:
   (a) $\log x = -4.3217$
   (b) $\log x = 2.5543$
   (c) $\log_{10}\left(x^2 + 15x\right) = 2$.

13. If $\log_2\left(x^2y^3\right) = 15$ and $\log_2\left(x^4y^3\right) = 27$, find each of the following:
   (a) $\log_2 x$ (b) $\log_2 y$

14. Given that $\log_a xy^4 = 8$ and $\log_a\left(\frac{x^2}{y^3}\right) = -6$:
   (a) Write the two equation connecting $\log_a x$ and $\log_a y$.
   (b) Determine the value of $\log_a x$ and $\log_a y$.
   (c) Evaluate $\log_a\left(y\sqrt[3]{x}\right)$

15. The intensity level L in decibels (dB) is given by $L = 10\log\left(\frac{I}{I_0}\right)$, where I is sound intensity and I_0 is the reference sound intensity. If the intensity level is increased from 50 dB to 80 dB, calculate the factor by which the intensity I has increased.

16. A certain radioactive substance has a decay formula $N(t) = N_0e^{-0.2t}$. If the initial amount N_0 is 200 grams and the amount remaining is 50 grams after t years, find t.

17. Given that $\log_a b = c$ and $\log_a d = f$, express $\log_a\left(\frac{b^2d}{\sqrt{d}}\right)$ in terms of c and f.

18. The amount in a bank account grows according to the formula $A = Pe^{rt}$. If the amount in the account triples in 5 years, find the annual growth rate r assuming continuous compounding.

19. The population of species grows according to the formula $P(t) = P_0e^{kt}$. If the population doubles in t years, find the value of k in terms of t.

20. Find the value of x for each of the following equations.
   (a) $\log_{10}\left(x^2 - 4\right) = 2\log_{10}(x-2)$.
   (b) $2^{2x} - 2^x - 6 = 0$

21. Find the value of x, if

$\log_2(x-1)+\log_2(x+3)=4$.

22. Determine the value of x if:

(a) $\log_3 x+\log_3 x+1=\log_3 12$

(b) $\log x=\log\left(\frac{9}{\sqrt{25}}\right)+3\log 2-\log\left(\frac{3}{5}\right)$

23. Find the value of x in the equation $\log_3 x+\log_3(x+1)=\log_3 12$.

24. Determine the value of $\log x$ in the equation $x=\frac{a^2\sqrt[3]{b}}{cd^2}$ if $\log a=1.2$, $\log b=1.5$, $\log c=0.5$, $\log d=0.8$.

25. Given that $\log_3 x = u$, $\log_{27} y=v$, and $\frac{x}{y}=3^z$, express z in terms of u and v.

26. If $\log 2=0.36103$, $\log 3=0.47712$, $\log 5=0.69897$, and $\log 7=0.8451$, evaluate each of the following:

(a) $\log_2(18)$

(b) $\log_{15}(45)$

(c) $\log_7(200)$

(d) $\log_9(27)$

(e) $\log_8(20)$

(f) $\log_4 64$

Chapter Seven

Sets

Introduction

Objects exist in different categories and characteristics which sometimes may be difficult to describe and understand their nature. Understanding the concepts of sets simplifies the process of categorising and organising such objects based on their nature. In this chapter, you will learn the concept of a set, distinguish between types of sets, perform operations with sets, represent sets in Venn diagrams, and determine the number of elements in a set. The competencies developed will enable you to simplify, categorise, analyse, and generalise different problems arising from real-life situations more systematically.

Think

Organising and categorising data with little or no knowledge about the concepts of sets.

Concept of a set

A set is a collection of well defined objects with common and distinct features. The objects in a set are are called elements or members of the set. The members of a set are usually separated by commas and enclosed within curly brackets { }. For instance, if B is a set representing students in a class and John is a student in that class, then John is a member of set B or is an element of set B. This relationship is denoted by the symbol $\in$, which means 'a member of' or 'an element of'. That is, John $\in$ B.

For example, if $C = \{1, 2, 3\}$, then $1 \in C$, $2 \in C$ and $3 \in C$ where as $5 \notin C$ (5 is not an element of C). The number of distinct elements in set A is represented by $n(A)$. For example, if $A = \{a, e, i, o, u\}$, then $n(A) = 5$.

Engage in Activity 7.1 to explore how to form sets in real life.

Activity 7.1: Sorting items having the same characteristics

1. Gather a variety of items such as exercise books, pens, pencils, erasers, rulers, notebooks, highlighters, and paperclips.
2. Sort the items based on their specific characteristics.
3. How many different groups of items have you created?
4. Assign each group a letter: A, B, C, etc.
5. Reflect on the similarities and differences between the groups. Consider the characteristics that influenced your sorting decision.

Sets can be described in different ways. The most common ways of describing a set are: by statement form, by listing (roster form), and by a formula (set builder notation). For instance, if set A is a set of even numbers, then it can be described as follows:

(a) Statement from: A = {even numbers}

(b) By listing: A = {2, 4, 6, 8, 10, ...}

(c) Using a formula:
$A = \{x : x = 2n, \text{ where } n \in \mathbb{N}\}$ and is read as A is the set of all x such that x is an even number.

Example 7.1

Given a set
A = {odd numbers between 1 and 10}.
Describe the set A by;

(a) listing

(b) set builder notation

Solution

(a) A = {3, 5, 7, 9}

(b) $A = \{x : x = 2n - 1, \text{ where } n \in \mathbb{N} \text{ and } 2 \leq n \leq 5\}$.

Example 7.2

Describe the following sets in roster form.

(a) The set of all letters in the word "AGRICULTURE".

(b) The set of all letters in the word "DISTINGUISH".

(c) The set of all vowels in the word "EQUATION".

Solution

(a) It is observed that the distinct letters in the word "AGRICULTURE" are A, C, E, G, I, L, R, T, and U. Since the order in which the elements are written is does not matter and the repetition of elements has no effect, then the set of all letters in the word AGRICULTURE is:
{A, C, E, G, I, L, R, T, U}

(b) The word "DISTINGUISH" has the following distinct letters: D, I, S, T, N, G, U, and H. Therefore, the set of all letters in DISTINGUISH is:
{D, I, S, T, N, G, U, H}.

(c) The word "EQUATION" has the vowels: A, E, I, O, and U. Therefore, the set of all vowels in the world EQUATION is:
{A, E, I, O, U}

Exercise 7.1

In questions 1 to 3, list the elements of each of the following sets:

1. A = {x : x is an odd number less than 10}
2. B = {days of the week that begin with the letter S}
3. C = {prime numbers less than 13}

In questions 4 to 8, write each of the sets in statement form:

4. A = {1, 2, 3, 4, . . .}
5. B = {1, 3, 5, 7, 9, 11}
6. C = {a, e}
7. C = {2, 3, 5, 7, 11, 13, 17, 19}
8. E = {a, e, i, o, u}

In questions 9 to 12, write each set using a set builder notation:

9. A = {even numbers}
10. B = {numbers divisible by 5}

11. C = {natural numbers less than 20}
12. D = {perfect squares numbers}
13. Determine whether each of the following is a set and provide a reason for your answer.
    (a) {3, 5, 7, 11}
    (b) {dog, cow, stone, goat}
    (c) {a, e, i, o, u}
    (d) {Asia, Africa, Europe, Australia}
    (e) {January, Monday, Wednesday, November}
14. List the elements of each of the following sets:
    (a) P = {last three consonants of English alphabets}
    (b) D = {months beginning with the letter J}
    (c) F = {prime numbers less than 20}
15. Describe the following sets in set builder notation.
    (d) The set of all letters in the word "PROBABILITY".
    (e) The set of reciprocals of natural numbers.
    (f) B = {1, 3, 9}
16. Describe each of the following sets in roster form.
    (a) $\{x : x \in \mathbb{Z} \text{ and } |x| \leq 3\}$
    (b) $\{x : x$ is the letter in the word PROPORTION$\}$
    (c) $\{x : x = \frac{n}{n^2+1} \text{ and } 1 \leq 3, n \in \mathbb{N}\}$

Types of sets

Sets can be categorised into different types based on the number and nature of their elements. The categories include: finite, infinite, singleton, and empty sets.

Finite set

A set is finite if it contains a countable number of elements. For instance, A= {a, b, c, d, e} is a finite set because its elements can be counted.

Infinite set

A set is infinite if it contains an uncountable number of elements. When representing infinite sets, a few elements are listed followed by an ellipsis (⋯) to indicate that the list does not have an end. For instance, A = {2, 4, 6, 8, ...} is an infinite set because the entire elements can not be counted.

Empty set

An empty set is a set which has no elements. It is also called as a null or void set. For instance, the set of prime numbers between 31 and 37 is an empty set because there is no prime number between 31 and 37. An empty set is denoted by the symbol { } or Ø.

Singleton set

A singleton set is a set that contains only one element. For instance, H = {2}.

Comparison of sets

Two or more sets can be compared and classified as equivalent sets, equal sets or one is a subset of the other.

Equivalent sets

Two sets are equivalent if they have the same number of elements. Normally, the symbol "≡" is used to denote equivalent. For instance, set A = {2, 4, 6, 8} and set B = {a, b, c, d} are equivalent because they have equal number of elements. That is, $n(A) = n(B) = 4$.

Generally, if $n(A) = n(B)$, then the sets A and B are said to be equivalent. That is, $A \equiv B$.

Example 7.3

Show that A = {4, 6} and B = {goat, dog} are equivalent.

Solution

Since $n(A) = 2$ and $n(B) = 2$, then, $A \equiv B$.

Therefore, the sets A and B are equivalent.

Equal sets

Two sets, A and B are said to be equal if they are equivalent and their members are exactly the same regardless of the arrangement or order. For example, if A = {1, 2, 3, 4} and B = {3, 1, 4, 2}, then A = B since all elements of A are elements of B and all elements of B are also elements of A.

> Note: All equal sets are equivalent but not all equivalent sets are equal.

Example 7.4

If $P = \{x : x^2 - 9 = 0\}$ and $Q = \{x : x \in \mathbb{Z} \text{ and } 7 < x < 10\}$, show that $P \equiv Q$.

Solution

Given $P = \{x : x^2 - 9 = 0\}$ and $Q = \{x : x \in \mathbb{Z} \text{ and } 7 < x < 10\}$ it follows that,

$$P = \{x : x^2 - 9 = 0\}$$
$$= \{-3, 3\}$$

Thus, only -3 and 3 satisfy $x^2 - 9 = 0$.

Hence, $n(P) = 2$.

Again, $Q = \{x : x \in \mathbb{Z} \text{ and } 7 < x < 10\}$

$= \{8, 9\}$, thus $n(Q) = 2$.

It follows that, $n(P) = n(Q) = 2$.

Therefore, $P \equiv Q$.

Exercise 7.2

1. Which of the following sets are finite, infinite or empty?
   (a) A = {Nairobi, Dar es Salaam}
   (b) B = {2, 4, 6, …, 36}
   (c) $C = \{x : x \in \mathbb{N};\ x > 0\}$
   (d) D = {All goats in your class}
   (e) E = {All mango trees in the world}
   (f) $F = \{x : x \text{ is a multiple of } 3\}$
   (g) $G = \{x : x \text{ is a prime numbers}\}$
   (h) H = {1, 3, 5, 7}
   (i) I = { }
2. Which of the following pairs of sets are equivalent?
   (a) A = {a, b, c} and B = {b, c, d}
   (b) B = {Rufiji, Ruaha, Malagarasi} and C = {Lion, Leopard}
   (c) D = {a, b, c, d} and E = {a, b, c}
3. Which of the following sets are equal?
   (a) A = {a, b, c, d}, B = {d, a, b, c}, C = {a, e, i, o, u}.
   (b) H = {a, b, c, d, e}, I = {d, c, b, a}, J = {a, e, b, c, d}.

4. Which of the following sets are finite, infinite or empty:
   (a) A = {African countries in Asian continent}
   (b) B = {All the letters of the English alphabets}
   (c) D = {All months with 32 days}
   (d) E = {2, 4, 6, 8, 10}
   (e) $F = \{x : x > 20\}$
   (f) G = {1, 2, 3, 4, 5, ...}
5. Give three examples of finite, infinite, and empty sets in real life situations.
6. Show that the set of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal.
7. Which of the following sets are empty?
   $A = \{x : x^2 - 5 = 0 \text{ and } x \text{ is rational}\}$
   $B = \{x : x \text{ is an even prime number}\}$
   $C = \{x : 6 < x < 7, x \in \mathbb{N}\}$
8. Find the pairs of equal sets from the following sets, if any:
   $A = \{0\}$,
   $B = \{x > 7 \text{ and } x < 4\}$,
   $C = \{x : x - 5 = 0\}$,
   $D = \{x : x^2 = 4\}$,
   $E = \{x : x$ is a positive integer which is a root of the equation $x^2 - 2x - 15 = 0\}$,
   $F = \{x : x$ is the letter of the word REAP$\}$

Subsets

Set A is a subset of another set B if all elements of A are also elements of B. For example, if $A = \{a, b, c, e\}$ and $B = \{a, b, c, d, e\}$ then A is a subset of B because all elements of A belong to B. If A has less elements than B, then A is called a proper subset of B, denoted by $A \subset$. If A is a subset of B and the two sets are equal, then A is called an improper subset of B, written as $A \subseteq B$. If $A = B$, then $A \subseteq B$ and $A \subseteq B$.

Therefore, a subset of a set can either be proper or improper.

> Note: The empty set is a subset of any set, and a set is a subset of itself.

Example 7.5

List all the subsets of A = {a, b}.

Solution

The subsets of A = {a, b} are all possible sets which can be obtained by taking some or all elements of A.
Therefore, the subsets of set A are: { }, {a}, {b}, {a, b}.

Example 7.6

What are the subsets of B = {1, 2, 3}?

Solution

The subsets of $B = \{1, 2, 3\}$ are all possible sets which can be obtained by taking some or all elements of set B. Therefore, the subsets of set B are: { }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

The number of subsets of a given set can be obtained by using a formula 2^n, where n is number of elements of the given set. The formula is obtained by observing the pattern in forming subsets of sets as shown in the following table.

Set	Subsets	Number of subsets
{ }	{ }	$1 = 2^0$
{1}	{ }, {1}	$2 = 2^1 = 2$
{2, 3}	{ }, {2}, {3}, {2,3}	$4 = 2 \times 2 = 2^2$
{1, 2, 3}	{ }, {1}, {2}, {3}, {1,2}, {1,3}, {2, 3}, {1, 2, 3}	$8 = 2 \times 2 \times 2 = 2^3$
{1, 2, 3, … ,*n*}	{ }, {1}, {1, 2}, …, {1, 2,…, *n*}	2^n

Therefore, the formula for finding the number of subsets of a set with n elements is 2^n.

Example 7.7

What is the total number of subsets that can be formed from the set

A = {1, 2, 3, 4}

Solution

Since $n(A) = 4$, then $2^n = 2^4 = 16$.

Therefore, set A has a total of 16 subsets.

Universal set

A universal set is a set that contains the elements of all sets under consideration. It is denoted by the symbol μ or U. For instance, the set of integers contains all elements of sets of prime members, odd numbers, and whole numbers. In this context, the ste of integers is considered to be the universal set that contains all of these elements.

Disjoint sets

Two or more sets are said to be disjoint if they do not have any common elements. For instance, if A is the set of all odd numbers less than 20 and B is the set of all even numbers less than 20, then the sets A and B do not have any elements in common. Therefore, A and B are disjoint sets. Otherwise, they are joint sets.

Exercise 7.3

1. List all the subsets of each of the following sets:
   (a) A = {1}
   (b) B = ∅
   (c) C = {Tito, Juma}
2. Find the number of subsets of each of the following sets.
   (a) A = {2, 4, 6, 8}
   (b) B = {a, b, c, d, e, f, g}
   (c) { }

In questions 3 to 5, which set in each pair is a proper subset of the other? Use the symbol $\subset$.

3. A = {a, b, c, d, e, f, g, h} and B = {d, e, f}
4. A = {2, 4} and D = {2, 4, 5}
5. A = {1, 2, 3, 4, . . .} and C = {2, 4, 6, 8, . . .}

6. What is the total number of subsets that can be formed from a set with seven elements?

7. For each of the following statements write T if the statement is true and F if the statement is false:
   (a) If $G = \{4, 5, 6\}$ and $H = \{5, 6, 7, 8\}$, then $G \subset H$.
   (b) If $K = \{\ \}$ and $L = \{\text{tree, house, egg}\}$, then $L \subset K$
   (c) If $A = \{s, t, u, v\}$ and $B = \{s, t, u, v\}$, then $A \subseteq B$ and $B \subseteq A$.
   (d) If $I = \{1, 2, 3, 4, \ldots\}$ and $J = \{4, 6, 8, \ldots\}$, then, $J \subset I$.

8. If G = {cities, towns, and regions in Tanzania}, which of the following sets are subsets of G?
   A = {Nairobi, Dar es Salaam}
   B = {Dodoma, Mombasa, Mwanza}
   $C = \emptyset$
   D = {Arusha, Iringa, Bagamoyo}
   E = {Mbeya, Tunduru, Ruvuma}

9. Which of the following sets are subsets of K, where, $K = \{p, q, r, s, t, u, v, w\}$
   (a) $A = \{p, s, t, x\}$
   (b) $B = \{q, r, d, t\}$
   (c) $C = \{\ \}$
   (d) $D = \{p, q, r, s, t, u, v, w\}$
   (e) $E = \{a, b, c, d\}$
   (f) $F = \{s, v, q\}$

10. What is $n(A)$ if:
    (a) $A = \{\ \}$
    (b) $A = \{2, 3, 4, 5, 6\}$

11. Write in statement form the universal set of the following sets:
    (a) $A = \{a, b, c, d\}$
    (b) $B = \{1, 2, 3, 4\}$

12. How many subsets are there in each of the following sets?
    (a) $A = \{1, 2, 3, 4, 5\}$
    (b) $B = \{a, b, c\}$
    (c) $C = \{x : x \in \mathbb{N}, 9 \leq x < 13\}$

13. Using set notation, show which set is a subset of the other:
    (a) A = {Countries in East Africa} and B = {Tanzania, Kenya, Uganda}
    (b) $C = \{p, q, r, s, t, u, v\}$ and $D = \{s, t, u, v\}$

14. State whether or not the following sets are disjoint:
    (a) $A = \{a, b, c, d, e\}$
        $B = \{h, i, j, k\}$
    (b) $A = \{x: x \leq 3, x \in \mathbb{N}\}$
        $B = \{0\}$
    (c) $A = \{x: 0 \leq x \leq 4, x \in \mathbb{N}\}$
        $B = \{x: -2 < x \leq 1, x \in \mathbb{Z}\}$

15. Let $A = \{1, 2, 3, 6, 7, 9\}$. Determine each of the following statement whether it is true or false:
    (i) $2 \in A$
    (ii) $\{1, 2, 3\} \subset A$
    (iii) $\emptyset \subset A$
    (iv) $\{6, 9\} \not\subset A$

Operations on sets

It is possible to combine two or more sets under specific conditions to obtain a new set. Engage in Activity 7.2 to experience set operations in real life.

Activity 7.2: Exploring set operations in daily life

1. Collect simple household items such as lids of bottles, coloured buttons, and fruits.
2. Organise and compare sets of items by combining them. Identify common items and items which are not in the whole set, and then distinguish the sets.
3. Represent the sets in a method of your choice and justify your categories.

In Activity 7.2, one has performed set operations in different ways by categorizing and comparing, combining and identifying common items.

Set operations are categorized into four types namely: union, intersection, complement, and set difference. These operations involve identifying elements in each set and determining the number of elements involved.

Union of sets

The union of two or more sets is the set that contains all the elements of each set, without any repetition. The union of sets is represented by the cup symbol "∪". Thus, $A \cup B$ denotes that an element x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$ or x is in both A and B. The keyword for the union of sets is "or".

Example 7.8

Find $A \cup B$ if $A = \{a, b, c, d, e, f\}$ and $B = \{a, e, i, o, u\}$.

Solution

Given $A = \{a, b, c, d, e, f\}$ and
$B = \{a, e, i, o, u\}$. It follows that,

$$A \cup B = \{a, b, c, d, e, f, i, o, u\}$$

Therefore, $A \cup B = \{a, b, c, d, e, f, i, o, u\}$.

Intersection of sets

The intersection of sets, A and B is the set which contains all the elements that are common to both sets. It is denoted by the cap symbol "∩". Thus, the intersection of sets A and B is written as $A \cap B$, which means, $x \in A \cap B$ if $x \in A$ and $x \in B$. The keyword for intersection is "and".
For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$, then $A \cap B = \{1, 3, 5\}$.

Example 7.9

Find $A \cap B$ if $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d, e, f\}$ and state if A and B are disjoint sets or not.

Solution

Given $A = \{a, e, i, o, u\}$ and
$B = \{a, b, c, d, e, f\}$, it follows
that $A \cap B = \{a, e\}$

Therefore, A and B are joint sets.

Example 7.10

Find $A \cap B$ if $A = \{a, e, i\}$ and $B = \{b, c, f\}$. State if A and B are disjoint sets or not.

Solution

Given $A = \{a, e, i\}$ and $B = \{b, c, f\}$. The sets A and B do not have common elements. Thus, A and B are disjoint sets. Therefore, $A \cap B = \varnothing$.

Complement of a set

If A is a subset of a universal set, then the complement of set A is the set that contains all the elements that are in the universal set but not in A. It is denoted by A' or A^c. For instance, if $U = \{a, b, c, d, ..., z\}$ and $A = \{a, b\}$, then $A' = \{c, d, e, ..., z\}$.

Example 7.11

Given $\mu = \{15, 45, 135, 275\}$ and $A = \{15\}$, find A'.

Solution

$A' = \{45, 135, 275\}$

Example 7.12

Given $\mu = \{a, e, i, o, u\}$ and $B = \{e, i\}$, find B'.

Solution

The set B' contains all elements which are not in B.

Therefore, $B' = \{a, o, u\}$.

Example 7.13

Given $\mu = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$,

$A = \{2, 3, 5, 7, 11, 13\}$, and

$B = \{2, 4, 6, 8, 10, 12, 14\}$, find:

(a) $A' \cup B'$ (b) $A \cap B'$

Solution

From the given sets, it implies that,

$A' = \{1, 4, 6, 8, 9, 10, 12, 14\}$

$B' = \{1, 3, 5, 7, 9, 11, 13\}$.

(a) $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

(b) $A \cap B' = \{3, 5, 7, 11, 13\}$

Number of elements in a set

The number of elements in a finite set is found by counting each element in the set. The number of elements in a set is also known as the cardinality of the set. For example, you might want to find the number of elements in the union, intersection, or complements of finite sets. If the set is infinity, then its cardinality is infinities. Engage in Activity 7.3 to explore how to determine the number of elements in a set.

Activity 7.3: Exploring the number of elements in a set

1. Consider the universal set $\mu = \{x : x \leq 10,\ x \in \mathbb{N}\}$, $A = \{x \text{ is a prime number}\}$, and $B = \{x \text{ is an odd number}\}$.
   List all the elements of μ, A, B, $A \cap B$, $A \cup B$, $A \cap B'$, $A \cup B$, and $A' \cap B$.
2. Explore different sources such as offline and online libraries to learn how to find the number of elements in a set.
3. Use the knowledge acquired in task 2 to find the number of elements of sets in task 1.
4. Study the number of elements of the sets A, B, $A \cap B$, and $A \cup B$ in task 3, then deduce any unique relationship.
5. Study the number of elements in μ, A, and A' and deduce any unique relationship.
6. State the number of elements of $A' \cap B$, $A \cap B$, and A, and deduce any relationship between the number of elements of the sets.
7. Based on your observations in tasks 4 to 6, write mathematical statements to generalize your findings and share with others.

From Activity 7.3, one may have learned that, the number of elements of a set can be obtained by counting elements in the given set. The number of elements of a set A is denoted by $n(\text{A})$ and read as "the number of elements in set A".

For any two sets A and B, the following are the relationships between the two sets.

1. $n(\text{A} \cup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})$
2. $n(\mu) = n(\text{A}) + n(\text{A}')$
3. $n(\text{A}) = n(\text{A} \cap \text{B}') + n(\text{A} \cap \text{B})$
4. $n(\text{A}') = n(\text{A}' \cap \text{B}) + n(\text{A}' \cap \text{B}')$

Example 7.14

A total of 40 people attending a seminar were asked to describe the source of energy between gas and charcoal they use in cooking. Among them, 16 said they use gas, 25 use charcoal and 6 use neither gas nor charcoal. How many participants use both gas and charcoal?

Solution

Let $\mu = \{\text{people attended the seminar}\}$,

$\text{G} = \{\text{people who use gas}\}$, and

$\text{C} = \{\text{people who use charcol}\}$.

Thus, $n(\mu) = 40$, $n(\text{G}) = 16$, $n(\text{C}) = 25$,

and $n(\text{G} \cup \text{C})' = 6$.

Required to find $n(\text{G} \cap \text{C})$

From $n(\text{G} \cup \text{C}) = n(\mu) - n(\text{G} \cup \text{C})'$

$$= 40 - 6$$
$$= 34$$

But $n(\text{G} \cup \text{C}) = n(\text{G}) + n(\text{C}) - n(\text{G} \cap \text{C})$.

Rearranging the equation gives,

$$n(\text{G} \cap \text{C}) = n(\text{G}) + n(\text{C}) - n(\text{G} \cup \text{C})$$
$$n(\text{G} \cap \text{C}) = 16 + 25 - 34$$
$$= 7$$

Therefore, 7 participants used both gas and charcoal.

Example 7.15

In a community of 500 people, 300 are in the climate action group and 250 are in a poverty alleviation group. If 180 are in both groups, how many people are in each of the following categories?

(a) Climate action group only.

(b) Poverty alleviation group only.

(c) At least in one group.

(d) Neither of the two groups.

Solution

Let: $\mu = \{\text{people in a community}\}$

$\text{C} = \{\text{people in climate action group}\}$

$\text{P} = \{\text{people in poverty alleviation group}\}$

Given $n(\mu) = 500$, $n(\text{C}) = 300$,

$n(P) = 250$, and $n(C \cap P) = 180$.

(a) Required to find $n(\text{C} \cap \text{P})'$.

From $n(\text{C}) = n(\text{C} \cap \text{P}) + n(\text{C} \cap \text{P}')$.

Thus, $300 = 180 + n(\text{C} \cap \text{P}')$

$$n(\text{C} \cap \text{P}') = 300 - 180$$
$$= 120$$

Therefore, 120 people are only in climate action group.

(b) Required to find $n(\text{C}' \cap \text{P})$.

$$n(\text{P}) = n(\text{C} \cap \text{P}) + n(\text{C}' \cap \text{P}).$$
$$n(\text{C}' \cap \text{P}) = 250 - 180$$
$$= 70$$

Therefore, 70 people are in poverty reduction group.

(c) Required to find $n(\text{C} \cup \text{P})$.

$$n(\text{C} \cup \text{P}) = n(\text{C}) + n(\text{P}) - n(\text{C} \cap \text{P})$$
$$= 300 + 250 - 180$$
$$= 370$$

Therefore, 370 people are at least in one group.

(d) Required to find $n(C \cup P)'$.

$$n(\mu) = n(C \cup P) + n(C \cup P)'$$
$$500 = 370 + n(C \cup P)'$$
$$n(C \cup P)' = 500 - 370$$
$$= 130$$

Therefore, 130 people are in neither of the the two groups.

Example 7.16

In a village with 400 households, 240 have access to clean water and 220 have good sanitation. If 90 households have no access to clean water and good sanitation, how many households have:

(a) Both clean water and good sanitation.

(b) Only one of these services.

Solution

Let $\mu = \{\text{households in a village}\}$,

W = {households with acces to clean water},

$S = \{\text{households with sanitation}\}$.

Given $n(\mu) = 400$, $n(W) = 240$,

$n(S) = 220$, and $n(W \cup S)' = 90$.

(a) Required to find $n(W \cap S)$.

$$n(\mu) = n(W \cup S) + n(W \cup S)'$$
$$400 = n(W \cup S) + 90$$
$$n(W \cup S) = 400 - 90$$
$$= 310$$

But $n(W \cup S) = + n(W) + n(S) - n(W \cap S)$

$$310 = 240 + 220 - n(W \cap S)$$
$$n(W \cap S) = 240 + 220 - 310$$
$$= 460 - 310$$
$$= 150$$

Therefore, 150 households have both clean water and good sanitation.

$$n(W) = n(W \cap S) + n(W \cap S')$$
$$240 = 150 + n(W \cap S')$$
$$n(W \cap S') = 240 - 150$$
$$= 90 \text{ (have clean water only)}$$

But $n(S) = n(W \cap S) + n(W' \cap S)$.

$$220 = 150 + n(W' \cap S)$$
$$n(W' \cap S) = 220 - 150$$
$$= 70 \text{ (have sanitation only)}$$

Households with only one service = clean water only + sanitation only

$$= 90 + 70$$
$$= 160$$

Therefore, 160 have only one of these services.

Exercise 7.4

From question 1 to 10 find the union and intersection of the given sets.

1. A = {5, 10, 15}, B = {15, 20}
2. A = { }, B = {14, 16}
3. A = {first five letters of the English alphabet}, B = {a, b, c, d, e}
4. A = {counting numbers}, B = {prime numbers}
5. A = {cup, spoon}, B = {cup, plate}
6. A = {All multiples of 5 less than 30}, B = {All multiples of 10 less than 30}
7. A = {All prime factors of 42}, B = {All prime factors of 15}
8. A = {All even numbers less than 10}, B = {All multiples of 3 less than 12}
9. A = {64, 81, 100, 121}, B = {64, 81, 144}.

10. $A = \{a, b, c, d\}$, $B = \{d, e\}$, and $C = \{\ \}$

11. Find the intersection of the sets, $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$.

12. If $\mu = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, and $B = \{1, 3, 7\}$, find A' and B'.

13. If μ = {mango, orange, tomato, cabbage}, A' = {mango, tomato}, list the elements of A.

14. If $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 4, 6, 8, 10\}$, list the element of $A \cup B$.

15. If μ = {red, green, blue, black, white}, find A' if A = {green, white, black}.

16. Given that $\mu = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Find each of the following:
    (a) $A' \cap B$ (b) $n(A \cap\ ')$
    (c) $B' \cup A'$ (d) $B \cup A'$
    (e) $(A \cup B)'$

17. Given the universal set $\mu = \{0, 1, 2, 3, 4, 5, 6, 7\}$. If $A = \{0, 1, 3\}$ and $B = \{5, 6, 7\}$. Write T for a true statement and F for a false statement in each of the following:
    (a) $A' = B$
    (b) $A' \cup B' = \{1, 2, 3, 4, 5, 6, 7\}$
    (c) $A' \cap B' = (A \cup B)'$
    (d) $n(A \cap B) = 1$
    (e) $n(A' \cup B') = 9$
    (f) $n(A \cap B') + n(A \cap B) = n(A)$

18. For each of the following pairs of sets, find their union and intersection:
    (a) $A = \{2, 4, 6\}$, $F = \{4, 6, 8, 10\}$
    (b) $Y = \{x : 2 < x \leq 8, x \in \mathbb{N}\}$, $W = \{x : 7 \leq x < 11, x \in \mathbb{N}\}$

19. Given that $\mu = \{a, b, c, d, e, f\}$, $A = \{a, b, c\}$ and $B = \{a, d, e, f\}$.
    Find each of the following:
    (a) $A \cap B$ (b) $A \cup B$
    (c) $A' \cap B$ (d) $A \cap B'$
    (e) $A' \cap B'$ (f) $A' \cup B'$

20. Suppose $\mu = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, and $B = \{2, 4\}$. Find
    (a) $A' \cup B$ (b) $(A')'$
    (c) $(A \cap B)'$

21. Suppose $A = \{2, 4, 6, 8\}$ and $A' = \{5, 9\}$, list the elements of μ.

22. Two sets A and B are subsets of a universal set μ such that $n(\mu) = 15$, $n(A) = 6$, and $n(B') = 4$. Find:
    (a) $n(A')$ (b) $n(B)$

23. If $\mu = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5$, $A = \{-2, 0, 2, 4, 5\}$, and $B = \{-4, -3, -2, 0, 1, 3\}$:
    (a) List the elements of each of the following sets:
        (i) A' (ii) B'
        (iii) $A' \cup B'$ (iv) $A \cap B$
        (v) $A \cap B'$

(b) Find: (i) $n(A' \cup B')$

(ii) $n(A' \cap B)$

(iii) $n(A \cap B)$

24. In a certain street with 200 houses, 170 houses have electricity and 145 have tiled floors. How many houses have both electricity and tiled floors? Assuming that each house house has a tiled floor or electricity or both.

Venn diagrams

A set can be represented by a diagram called a Venn diagram. A Venn diagram is named after a British Mathematician, John Venn. In drawing Venn diagrams, rectangles represent universal sets, while ovals or circles represent subsets. For instance, the set A = {a, b, c} can be represented in a Venn diagram as shown in Figure 7.1.

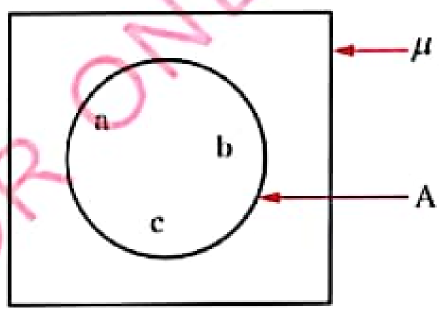


Figure 7.1: *Venn diagram for set A*

In this case, μ represents the universal set and A is a subset of μ. If two sets have elements in common, their ovals overlap. For instance, if A = {a, b, c} and B = {a, b, d}, then the representation of the sets A and B in a Venn diagram is as shown in Figure 7.2.

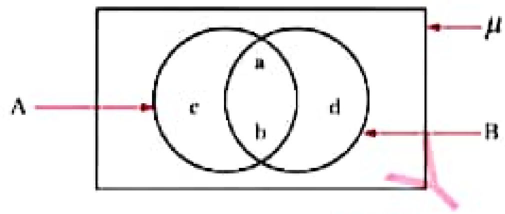


Figure 7.2: *Joint sets in a Venn diagram*

If the sets are disjoint, the circles or ovals do not overlap. For instance, if A = {a, b} and B = {1, 2}, the Venn diagram for the two disjoint sets is shown in Figure 7.3.

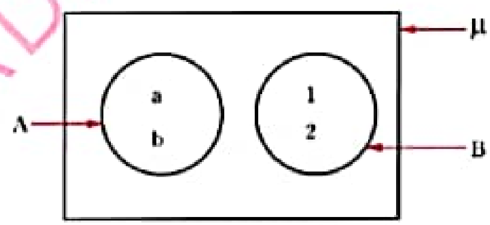


Figure 7.3: *Disjoint sets in a Venn diagram*

Representation of the complement of a set in a Venn diagram

If A is a subset of a universal set μ, then the complement of A may be represented in a Venn diagram as shown in Figure 7.4, where A′ is represented by the shaded region.

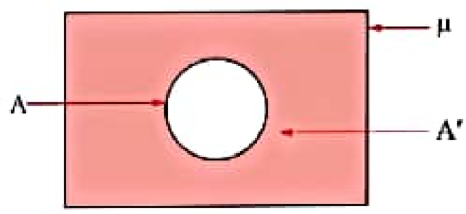


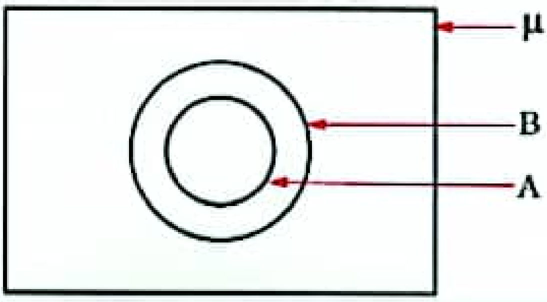
Figure 7.4: *Venn diagram showing complement of a set.*

Example 7.17

If A is a proper subset of B, represent the two sets in a Venn diagram.

Solution

Since all elements of et A are in set B, then $A \subset B$. Therefore, the Venn diagram for $A \subset B$ is as follows.



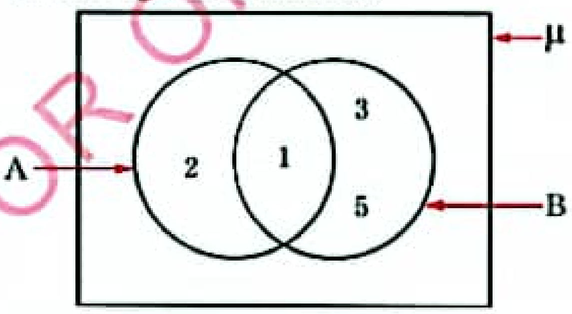
Example 7.18

Represent $A \cup B$ in a Venn diagram, given that

$A = \{1, 2\}$ and $B = \{1, 3, 5\}$.

Solution

Since 1 is a common element to both sets, then the sets are represented in a Venn diagram as follows.



Example 7.19

Given $\mu = \{a, b, c, d, e\}$ and $A = \{b, c, e\}$.

(a) Represent the information in a Venn diagram.

(b) Determine $n(A)$, $n(A')$ and $n(\mu)$.

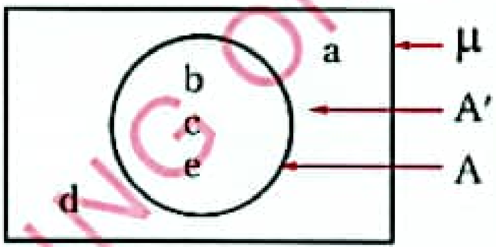
(c) Find $n(A) + n(A')$.

(d) Comment on the result in $n(\mu)$ and $n(A) + n(A')$.

Solution

The Venn diagram representing the given data is shown in the following figure.

(a)



(b) From the Venn diagram $n(A) = 3$, $n(A') = 2$, and $n(\mu) = 5$.

(c) $n(A) + n(A') = 5$

(d) The number of elements in a universal set is equal to the sum of the number of elements in set A and the number of elements in the set A'.

That is, $n(\mu) = n(A) + n(A')$.

Example 7.20

Given the sets, $\mu = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 4, 5\}$, and $B = \{1, 2, 4, 6\}$. Show that $(A \cup B)' = A' \cap B'$ by using a Venn diagram.

Solution

Given that

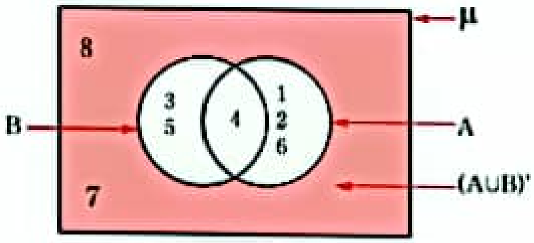
$\mu = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 4, 5\}$, and $B = \{1, 2, 4, 6\}$.

The complement of the two sets A and B are; $A' = \{1, 2, 6, 7, 8\}$ and $B' = \{3, 5, 7, 8\}$.

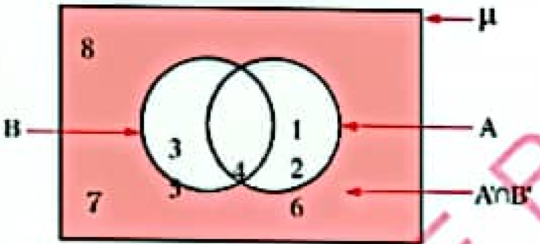
But $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

Thus, $(A \cup B)' = \{7, 8\}$

The set $(A \cup B)'$ is shown in the following Venn diagram.



Similarly, $A' \cap B' = \{7, 8\}$. The set $A' \cap B'$ is shown in the following Venn diagram.



From the two Venn diagrams, it follows that $(A \cup B)' = A' \cap B'$.

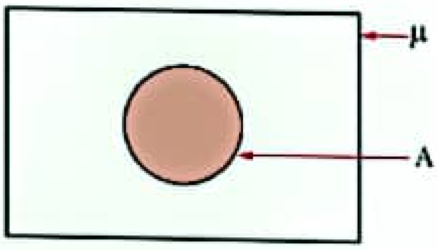
Therefore, $(A \cup B)' = A' \cap B'$.

Example 7.21

Represent $A \cap \mu$ in a Venn diagram and shade the required region.

Solution

The set $A \cap \mu$ is represented in in the following Venn diagram.



Therefore, $A \cap \mu = A$.

Example 7.22

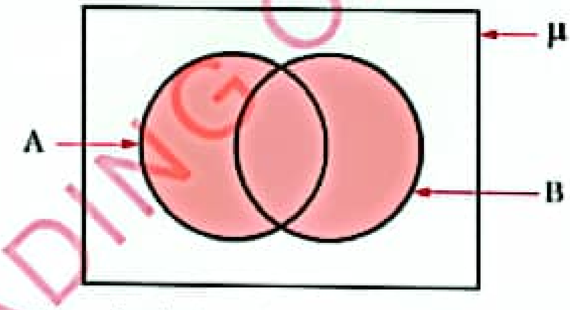
If A and B are two sets with some common elements, represent each of the following sets in a Venn diagram.

(a) $A \cup B$

(b) $A \cap B$

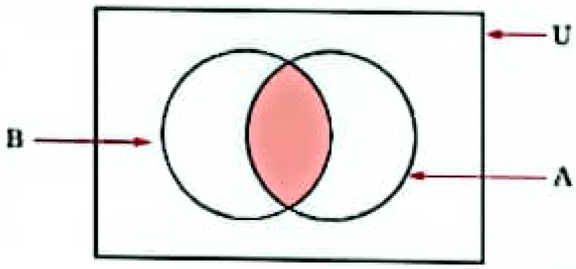
Solution

(a) $A \cup B$



The shaded region represents $A \cup B$.

(b) $A \cap B$



The shaded region represent $A \cap B$.

Example 7.23

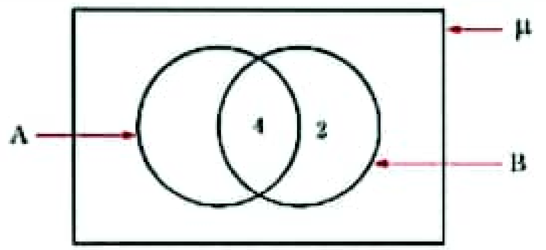
If A and B are two sets such that $n(A \cap B) = 4$, $n(A \cup B) = 6$, and $n(A) = 4$:

(a) How many elements are there in B?

(b) Between A and B, which set is a subset of the other?

Solution

Using a Venn diagram, the two sets can be represented as follows:



(a) Elements of B only = 6 − 4 = 2.
Therefore, B has 4 + 2 = 6 elements.

(b) $A \subset B$. This is because all elements in set A are contained in set B.

Example 7.24

In a class of 120 students, 40 study English, 60 study Kiswahili, and 30 study both Kiswahili and English. Find the number of students who study:

(a) English only.

(b) Neither English nor Kiswahili.

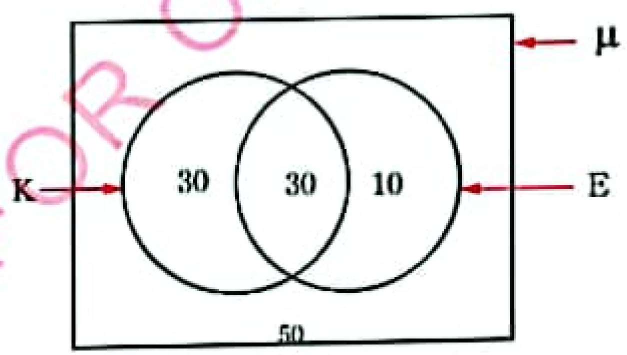
Solution

Let μ = {All students in the school}

E = {All students who study English}

K = {All students who study Kiswahili}

Representation of the three sets in a Venn diagram is as follows.



Given $n(E \cap K) = 30$, $n(E) = 40$, and $n(K) = 60$.

(a) The number of students who study English only is given by:

$$n(E \cap K') = n(E) - n(E \cap K)$$
$$= 40 - 30 = 10$$

Therefore, 10 students study English only.

(b) The number of students who study Kiswahili only is given by:

$$n(K \cap E') = n(K) - n(E \cap K)$$
$$= 60 - 30 = 30$$

The number of students who study English or Kiswahili or both are:

$$30 + 30 + 10 = 70$$

Therefore, the number of students who study neither Kiswahili nor English are:
120 − 70 = 50 students.

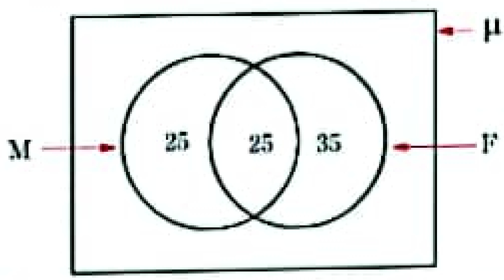
Example 7.25

In a certain school, 50 students eat meat, 60 eat fish, and 25 eat both meat and fish. Assuming that every student eats meat or fish, find the total number of students in the school.

Solution

Let μ = {All students in the school},
M = {All student who eat meat}, and
F = {All students who eat fish}.

The Venn diagram representing these information is as follows:



From the Venn diagram, total number of students = 25 + 25 + 35 = 85.

Therefore, there are 85 students in the school.

In general, the number of elements in two sets is connected by the formula:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The formula can be verified using a Venn diagram as follows:

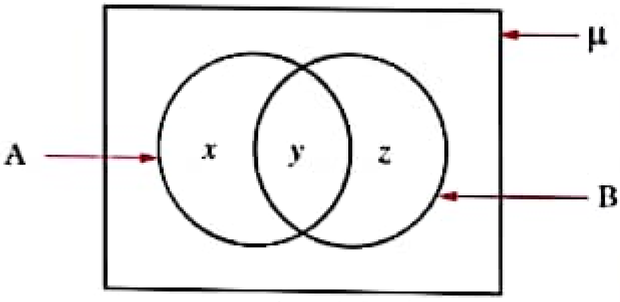


Figure 7.5

Let the number of elements in each region be x, y, and z as shown in Figure 7.5. The following equations can be obtained:

$n(A) = x + y$

$n(B) = z + y$

$n(A \cap B) = y$

$n(A \cup B) = x + y + z$

$n(A) + n(B) = (x + y) + (y + z)$

$\qquad = (x + y + z) + y$

$n(A) + n(B) = n(A \cup B) + n(A \cap B)$

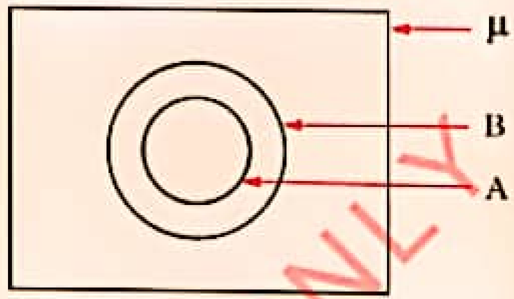
Therefore,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

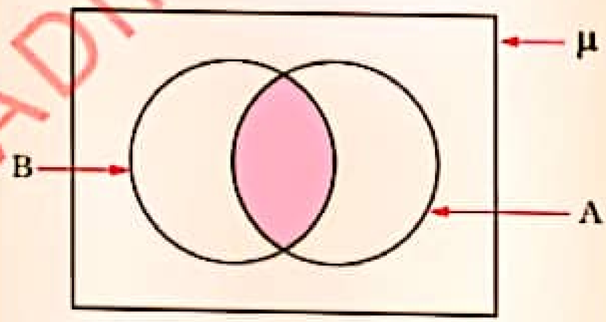
Exercise 7.5

1. Represent each of the following in a Venn diagram:
   (a) $A = \{a, b, c, d\}$
   (b) $A = \{a, b, c\}$ and $B = \{a, b, c\}$
   (c) $A = \{1, 2, 3\}$ and $B = \{4, 6, 8\}$

2. Use the following Venn diagram to describe the relationship between the sets A and B.



3. By using set notation, write the set represented by the shaded region in the following Venn diagram.



4. Assuming that the sets A and B have some elements in common, draw a Venn diagram to show the set $A' \cap B$.

5. If A and B are joint sets, represent $A' \cup B'$ in a Venn diagram.

6. In a Venn diagram, shade the region representing the set $A \cup A'$.

7. If $n(A \cup B) = 30$, $n(A) = 14$, and $n(A \cap B) = 6$, use a Venn diagram to find $n(B)$.

8. Draw Venn diagrams and shade the regions representing each of the following sets:

(a) $A \cup B$

(b) $A \cap B$

(c) $A \cup B'$

9. If $n(B) = 4$, $n(A \cup B) = 4$, and $n(A \cap B) = 0$, how many members are there in set A?

10. If $n(A) = 8$ and $n(A \cap B) = 8$, what is the relationship between the sets A and B?

11. Given $n(A \cap B) = 6$, $n(A \cup B) = 10$, and $n(B) = 6$.
    (a) Find the number of members in set A.
    (b) Between A and B, which set is a subset of the other?

12. Given $n(A \cap B) = 0$ and $n(A \cup B) = 4$. What is the relationship between sets A and B?

13. In a class, 15 students play basketball, 11 play netball, and 6 play both basketball, and netball. How many students are there in the class if every student plays at least one game?

14. In a school of 160 students, 50 have bread for breakfast and 80 have sweet potatoes. Assuming that none takes both bread and sweet potatoes, how many students have neither bread nor sweet potatoes for breakfast?

15. In a class of 20 students, 12 students study English but not History, 4 study History but not English, and 1 studies neither English nor History. How many students in the class study History?

16. There are 24 men at a meeting, 12 are farmers, 18 are soldiers, and 8 are both farmers and soldiers.
    (a) How many men are either farmers or soldiers?
    (b) How many men are neither farmers nor soldiers?

17. In a class of 30 students, 20 students take Physics, and 12 take both Chemistry and Physics. How many students take Chemistry, if 8 students take neither Physics nor Chemistry?

18. In a certain meeting 30 people drank Pepsicola, 60 drank Coca-Cola, and 25 drank both Pepsicola and Coca-Cola. Assuming that each person took Pepsicola or Coca-Cola, how many people were there at the meeting?.

9. Every woman in a certain club owns a landrover or a bicycle. If 23 women own landrovers, 14 own bicycles, and 5 own both a landrover and a bicycle, how many women are there in the club?

20. A survey of 160 households, showed that 95 kept cats and 80 kept dogs. How many households kept both a cat and a dog?

21. A survey among 90 men who had spent a day in volleyball or football showed that 60 played volleyball and 22 played both volleyball and football. How many men played football?

22. In a village, all the people speak Kiswahili or English or both. Among them, 97% speak Kiswahili and 64% speak English.
    (a) What percentage of people speak both Kiswahili and English?
    (b) What percentage of people speak English only?

23. In a class of 36 students, everyone studies Biology or Physics or both. If 9 students study both subjects and 12 study Physics but not Biology, how many students study Biology but not Physics?

Chapter summary

1. A set is a group or collection of objects that have common characteristics.
2. A set is finite if it has a countable number of elements.
3. A set is infinite if it has uncountable number of elements.
4. An empty set is a set with no elements.
5. Equivalent sets have equal number of elements.
6. Equal sets have the same elements and an equal number of elements.
7. A subset of a set contains some or all elements of another set.
8. A universal set is the set which contains all elements of all sets under consideration.
9. A complement set consists of all members which are not in the given set, but are elements in a universal set.
10. A Venn diagram is a pictorial representation of sets or relationships between sets.
11. A union of two sets is a set which is formed when the members of the two sets are collected together without repetition.
12. The intersection of two sets is a set formed by taking all elements that are common in both sets.
13. A singleton is a set that contains only one element.

Revision exercise 7

1. If $A = \{a, b, c\}$, which of the following statements are true?
   (a) $A \in A$ (b) $d \in A$
   (c) $c \in A$ (d) $b \in A$

2. If $A = \{a, b, c, d\}$, which of the following statements represents the set A?
   (a) A set of four letters of the English alphabet.

(b) Some of the sets of the first four letters of the English alphabet.

(c) The set of the first four letters of the English alphabet.

(d) The set of the four consonants of the English alphabet.

3. Describe the set A = {1, 4, 9, 16, 25} in statement form.

4. List the elements of the set given by $A = \{x : x$ is a counting number less than $8\}$.

5. Which of the following sets is an empty set?

(a) A = {Ø} (b) B = {0}

(c) R = Ø (d) S = { }

6. Which of the following sets are equal or equivalent?

A = {1, 2, 3, 4, 5}

B = {1, 4, 7, 10, 15}

C = {cars, ships, aeroplanes}

D = {cars, ships, aeroplanes}

7. Which of the following sets are finite or infinite?

$A = \{x : x$ is a prime number$\}$

B = {all letters of the English alphabet}

C = {all students in your school}

D = {1, 4, 9, 16, ...}

E = {1, 4, 9, 16, 25, ..., 81}

8. If A = {a, b}, use symbols to write *a* and *b* as elements of A.

9. How many subsets are there in A = {a, b, c, d, e, f, g}.

10. List all the subsets of A = {2, 4, 6}.

11. In the following pairs of sets, name the set which is a subset of the other:

(a) A = {a, b, c} and B = {e, d, b, c, a}

(b) C = {Tom, John, Idrisa} and D = {John, Tom}

12. If A = {a, b, c}, B = {4, a, 1}, and D={b, a}, which of the following statements are true?

(a) $A \subset B$ (b) $B \subset D$

(c) $A \subset D$ (d) $D \subset A$

13. If $A = \{a, b, c, d\}$ and $B = \{\ \}$ find:

(a) $A \cup B$

(b) $A \cap B$

14. Let A be the set of the first ten counting numbers and B be the set of the first four prime numbers. Find:

(a) $A \cup B$ (b) $A \cap B$

15. If μ = {1, 2, 3, 4,5} and A′ = {2, 3, 5} list the elements of A.

16. If μ = {a, b, c, d, e}, B = {c, d}, and A = {a, b, c}, find the following:

(a) $A' \cap B'$

(b) $A \cup (B \cap A)$

17. Draw a Venn diagram to show the region representing each of the following sets:

(a) $A' \cup B$ (b) $B' \cap A'$

(c) $A \cap B$ (d) $\mu \cup (A' \cap B)$

18 If A={All even numbers} space and B={All perfect squares}, draw a Venn diagram to illustrate the relationship between A and B.

19. Draw a Venn diagram to represent each of the following sets, if A and B are joint sets.

    (a) $A' \cup B'$

    (b) $A' \cap B'$

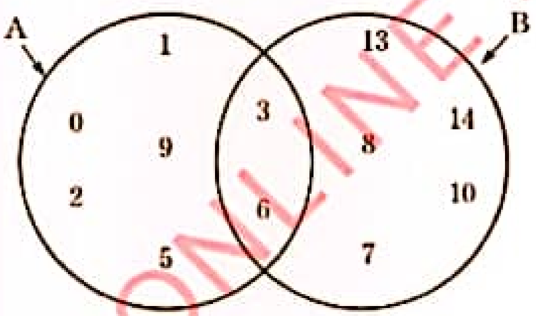
    (c) $A \cap B'$

    (d) $A' \cap B$

    (e) $(A \cup B)'$

20. In a group of 29 tourists from different countries, 17 went to Manyara National Park, 13 went to Mikumi National Park, and 8 went to neither Mikumi nor Manyara National Park. How many tourists went to both places?

21. Study the following figure and answer the questions that follow.



    (a) List all members of sets A and B.

    (b) Find $n(A \cup B)$ and $n(A \cap B)$.

    (c) How many elements are there in A only?

22. In an interview at a railway station, 50 travellers reported that last month they had been to Tanga, 48 had been to Arusha and 36 had been back to Arusha and Tanga. Using a Venn diagram, find the number of travellers that were interviewed.

23. In a survey of a certain school, 80 students had participated in a vaccination program only, while 60 had participated in a health program only, and 120 had participated in both. It was further found that 90 had not participated in either program. Find;

    (a) The total number of students who were vaccinated.

    (b) The number of students who participated in the health program.

    (c) The number of students who participated in vaccination or a health program or both.

    (d) The total number of students in the survey.

24. A school is assessing the participation of students in sports and academic club extracurricular activities. Out of the 120 students surveyed, 70 are involved in sports and 80 in academic clubs. The school's data shows that 50 students are involved in both activities. Calculate the following:

    (a) The number of students involved in either sports or academic clubs.

    (b) The number of students involved in sports only.

    (c) The number of students involved in academic clubs only.

    (d) The number of students not involved in either activity.

25. Out of 31 women participants in a gender equality seminar, some said they advocate women's rights and some have received higher education. If 22 participants advocate women's rights, 20 have received higher education and 5 neither advocate women's rights nor have received higher education. Calculate the number of participants who advocate women's rights and have higher education.

26. In a group of 60 people, 27 like coffee and 42 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?

27. There are 35 students in an art class and 57 students in a dance class. Find the number of students who are either in art or dance classes;

(a) When two classes meet at different hours and 12 students are enrolled in both activities.

(b) When two classes meet at the same hour.

28. In a workforce of 80 employees, 15 have undertaken a skills development program and 25 are currently employed. If 10 employees have undergone skills development training and are employed:

(a) Draw a Venn diagram to represent the given information.

(b) Use the Venn diagram in (a) to find:

(i) The number of employees who have undertaken a skill development program and are not currently employed.

(ii) The number of employees who have either have taken a skill development program nor employed.

Chapter Eight

Trigonometry

Introduction

Accurate measurements such as distances, angles, and navigation would be difficult without having the knowledge of trigonometry. In this chapter, you will learn trigonometric ratios, trigonometric ratios of special angles, and perform calculations that involve angles of elevation and depression. The competencies developed will enable you to solve various problems related to building constructions, designing, navigation, and many other applications.

Think

Imagine what would happen in a world without global positioning systems and other navigation technologies.

Trigonometric ratios

Sine, cosine, and tangent are the basic trigonometric ratios abbreviated as 'sin', 'cos', and 'tan'. These are referred to as ratios because they are defined by ratios of the sides of a right-angled triangle. Other trigonometric ratios derived from these basic ratios are secant, cosecant, and cotangent. These are reciprocals of the three basic trigonometric ratios.

Activity 8.1: Measuring heights using shadows

1. Explore the basic trigonometric ratios (sine, cosine, and tangent) and their real life applications.
2. On a sunny day, place a long stick vertically on the ground, then measure the height and length of its shadow.
3. Measure the shadow of a tall object such as a tree or building and use similar triangles to estimate its height.
4. Share your findings with the class, explaining your process and justifying your results.

Consider the right-angled triangle ABC shown in Figure 8.1.

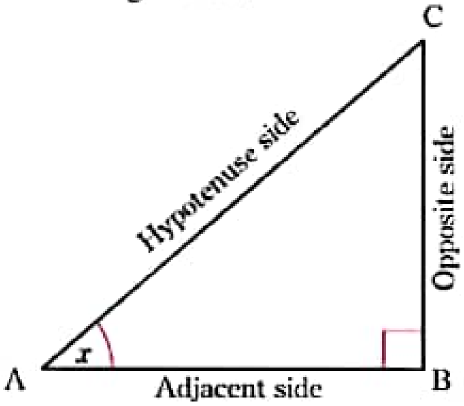


Figure 8.1: *Right-angled triangle ABC*

From ΔABC, in Figure 8.1, $C\hat{A}B = x$, $\overline{AC}$ is the lenght of the hypotenuse side, with respect to angle x, $\overline{AB}$ is the length of adjacent side, and $\overline{BC}$ is the length of opposite side.

Consider similar triangles shown in Figure 8.2.

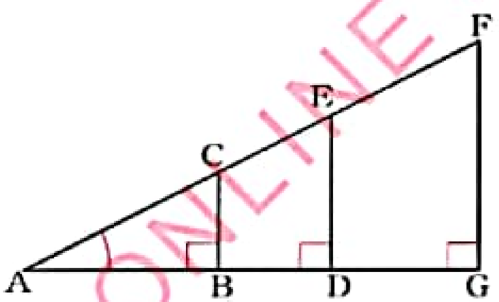


Figure 8.2 *Similar triangles*

From Figure 8.2, since the triangles are similar, it follows that

$$\frac{\overline{CB}}{\overline{AB}} = \frac{\overline{ED}}{\overline{AD}} = \frac{\overline{FG}}{\overline{AG}} = t,$$

where t is a constant ratio. This constant ratio is called the tangent of the angle at the vertex A and is written in short as $\tan A$.

Similarly, $\frac{\overline{BC}}{\overline{AC}} = \frac{\overline{DE}}{\overline{AE}} = \frac{\overline{GF}}{\overline{AF}} = s,$

where s is a constant ratio. This constant ratio is called the sine of the angle at the vertex A and is written in short a $\sin A$.

Likewise, $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AE}} = \frac{\overline{AG}}{\overline{AF}} = c,$

where c is a constant ratio. This constant ratio is called the cosine of the angle at vertex A and is written in short as $\cos A$.

From Figure 8.1 and 8.2, it can be deduced that, the lengths of the sides of a right-angled triangle are used to define the trigonometric ratios as follows:

$$\tan A = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}} = \frac{\overline{BC}}{\overline{AB}}$$

$$\sin A = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}} = \frac{\overline{BC}}{\overline{AC}}$$

$$\cos A = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}} = \frac{\overline{AB}}{\overline{AC}}$$

The following mnemonic is a useful way of remembering these definitions.

SO	TO	CA
H	A	H

Using the mnemonic, it follows that,

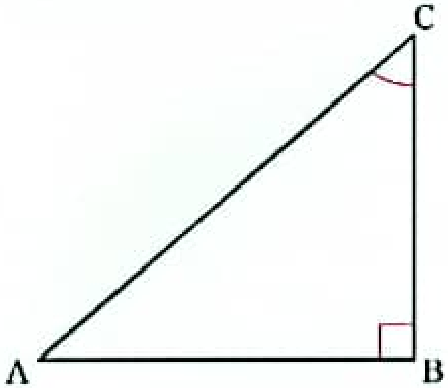
$S = \frac{O}{H}$ is a definition of sin A,

$T = \frac{O}{A}$ is a definition of tan A,

$C = \frac{A}{H}$ is a definition of cos A.

Example 8.1

Study the following triangle and answer the questions that follow.

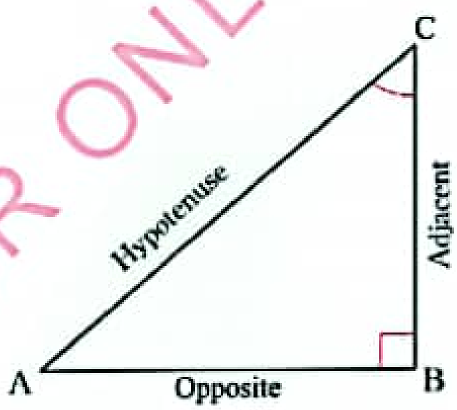


(a) Use angle C from ΔABC to label the hypotenuse, opposite, and adjacent sides.

(b) Use the triangle to write the trigonometric ratios for angles A and C.

Solution

(a) With reference to angle C, the triangle ABC is labelled as shown in the following figure.



From the figure, $\overline{AB}$ is the opposite side, $\overline{BC}$ is the adjacent, and $\overline{AC}$ is the hypotenuse.

(b) $\sin A = \frac{\overline{BC}}{\overline{AC}}$, $\cos A = \frac{\overline{AB}}{\overline{AC}}$,

$\tan A = \frac{\overline{BC}}{\overline{AB}}$

$\sin C = \frac{\overline{AB}}{\overline{AC}}$, $\cos C = \frac{\overline{BC}}{\overline{AC}}$, and

$\tan C = \frac{\overline{AB}}{\overline{BC}}$.

Example 8.2

Given that $\sin X = \frac{\overline{YZ}}{\overline{XZ}}$, draw a right-angled triangle that represents this information and determine each of the following ratios:

(a) sin Z

(b) cos Z

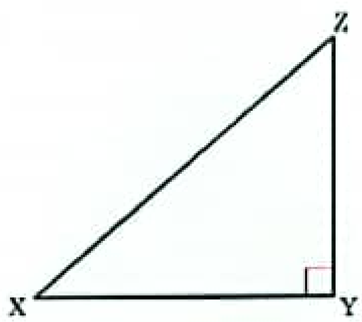
(c) tan Z

(d) cos X

(e) tan X

Solution

The respective triangle is shown in the following figure.



(a) $\sin Z = \frac{\overline{XY}}{\overline{XZ}}$

(b) $\cos Z = \frac{\overline{YZ}}{\overline{XZ}}$

(c) $\tan Z = \frac{\overline{XY}}{\overline{YZ}}$

(d) $\cos X = \frac{\overline{XY}}{\overline{XZ}}$

(e) $\tan X = \frac{\overline{YZ}}{\overline{XY}}$

Example 8.3

In a right-angled triangle ABC,

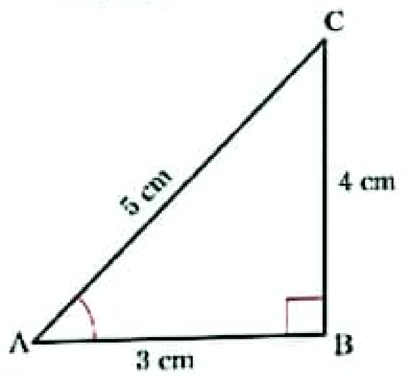
$\overline{AB} = 3\text{cm}$, $\overline{BC} = 4\text{cm}$, and $\overline{AC} = 5\text{cm}$.

Find the value of each of the following:

(a) sin A (b) cos A (c) tan A

Solution

The given information is summarized in the following figure.



From the figure, triangle ABC is a right-angled triangle such that, $A\hat{B}C = 90°$.

It follows that:

(a) $\sin A = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse side}}$

$= \dfrac{\overline{BC}}{\overline{AC}} = \dfrac{4\text{ cm}}{5\text{ cm}}$

Therefore, $\sin A = \dfrac{4}{5}$.

(b) $\cos A = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse side}}$

$= \dfrac{\overline{AB}}{\overline{AC}} = \dfrac{3\text{ cm}}{5\text{ cm}}$

Therefore, $\cos A = \dfrac{3}{5}$.

(c) $\tan A = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$

$= \dfrac{\overline{BC}}{\overline{AB}} = \dfrac{4\text{cm}}{3\text{cm}}$

Therefore, $\tan A = \dfrac{4}{3}$.

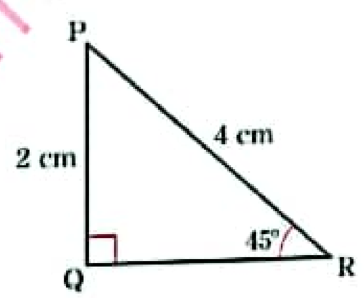
Example 8.4

In a triangle PQR, angle PQR is a right-angle. If $\overline{RP} = 4$ cm and $\overline{PQ} = 2$ cm, find the value of each of the following:

(a) $\overline{QR}$ (b) tan R (c) cos R

Solution

Consider the following figure.



(a) Apply the Pythagoras theorem as follows:

$\overline{QR}^2 + 2^2 = 4^2$

$\overline{QR}^2 = 16 - 4 = 12$

$\overline{QR} = \sqrt{12} = 2\sqrt{3}$

Therefore, $\overline{QR} = 2\sqrt{3}$ cm.

(b) $\tan R = \dfrac{\text{Length of opposite side}}{\text{Length of adjacent side}}$

$= \dfrac{\overline{PQ}}{\overline{QR}}$

$= \dfrac{2\text{cm}}{2\sqrt{3}\text{cm}} = \dfrac{1}{\sqrt{3}}$

Therefore, $\tan R = \dfrac{\sqrt{3}}{3}$.

(c) $\cos R = \dfrac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$

$$= \frac{\overline{QR}}{\overline{PR}} = \frac{2\sqrt{3}\text{ cm}}{4\text{ cm}}$$

$$= \frac{\sqrt{3}}{2}$$

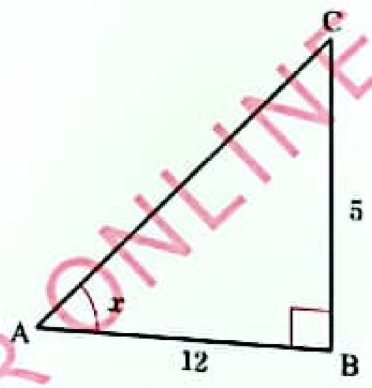
Therefore, $\cos R = \dfrac{\sqrt{3}}{2}$.

Example 8.5

If $\tan x = \dfrac{5}{12}$, find the value of $\cos x$.

Solution

By definition, $\tan x = \dfrac{5}{12}$ can be represented as shown in the following triangle.



Using Pythagoras' theorem, the length of the hypotenuse side is given by;

$$\overline{AC} = \sqrt{144 + 25}$$

$$= \sqrt{169} = 13 \text{ units.}$$

Thus, $\cos x = \dfrac{\overline{AB}}{\overline{AC}} = \dfrac{12}{13}$

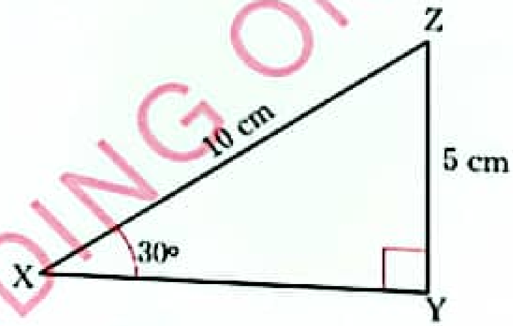
Therefore, $\cos x = \dfrac{12}{13}$.

Example 8.6

If XYZ is a triangle such that $X\hat{Y}Z = 90°$, $\overline{ZY} = 5$ cm, $Z\hat{X}Y = 30°$, and $\overline{ZX} = 10$ cm, find the value of $\sin 30°$.

Solution

Consider the following figure. From the figure, it implies that that



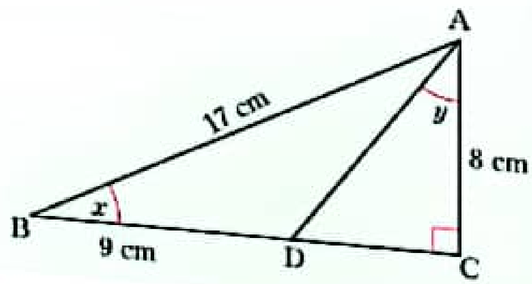
$$\sin 30° = \frac{\overline{ZY}}{\overline{ZX}}$$

$$= \frac{5\text{cm}}{10\text{cm}} = 0.5$$

Therefore, $\sin 30° = 0.5$.

Example 8.7

Use the information provided in the following figure to calculate the value of $\cos x + \sin y$.



Solution

From ΔABC, apply the Pythagoras theorem as follows:

$$17^2 = (\overline{BC})^2 + 8^2$$

$(\overline{BC})^2 = 17^2 - 8^2$

$\overline{BC} = \sqrt{17^2 - 8^2}$

$= 15\text{cm}$

But, $\overline{DC} = \overline{BC} - \overline{DB}$.

Thus, $\overline{DC} = 15\text{cm} - 9\text{cm}$

$= 6 \text{ cm}$

Thus, $\cos x = \dfrac{\overline{BC}}{\overline{AC}}$.

$\cos x = \dfrac{15}{17}$

From ΔADC, apply Pythagoras theorem.

$\overline{AD}^2 = 6^2 + 8^2$

$\overline{AD} = \sqrt{100}$

$= 10 \text{ cm}$

Thus, $\sin y = \dfrac{\overline{DC}}{\overline{AD}}$.

$\sin y = \dfrac{6}{10}$

$= \dfrac{3}{5}$

Hence, $\cos x + \sin y = \dfrac{15}{17} + \dfrac{3}{5}$.

$= \dfrac{126}{85}$

Therefore, $\cos x + \sin y = \dfrac{126}{85}$.

Exercise 8.1

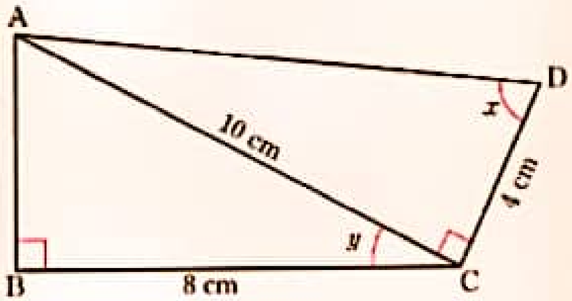
1. In a triangle LMN, $L\hat{N}M = 90°$, $\overline{LM} = 10$ cm, $\overline{MN} = 6$ cm, and $\overline{LN} = 8$ cm. Find:
   (a) tan M (b) sin M (c) cos M

2. In a triangular plot ABC, $B\hat{A}C = 90°$, $\overline{AB} = 8$ m, $\overline{AC} = 15$ m, and $\overline{BC} = 17$ m. Find the value of each of the following:
   (a) sin C (b) tan C (c) cos C

3. In a triangle RST, $R\hat{S}T = 90°$, $\overline{RS} = 4$cm, and $\overline{TS} = 3$cm.. Find:
   (a) $\overline{TR}$ (b) cos R (c) sin R

4. A rectangular field is 100 m long and 50 m wide. If one of its diagonals makes an angle x with the length, find the value of $\tan x$.

5. Use the following figure to find:
   (a) $\tan x$
   (b) $\sin y$



6. If $\sin x = \dfrac{4}{5}$, find the value of:
   (a) $\tan x$ (b) $\cos x$

7. If $\cos x = \dfrac{15}{17}$, find the value of:
   (a) $\sin x$ (b) $\tan x$

8. Given that $\sin 30° = \frac{1}{2}$, find the value of $\cos 30°$.

9. In a triangle ABC, $\overline{AB} = c$, $\overline{BC} = a$, $\overline{AC} = b$, and $A\hat{B}C = 90°$. Find in terms of a, b or c each of the following:

   (a) sin C (b) cos C (c) tan C

   (d) $\frac{\sin C}{\cos C}$

   (e) The relationship between tan C and $\frac{\sin C}{\cos C}$

10. In a right-angled triangle, the length of the hypotenuse is 10 cm and one of its angles is 70°. Calculate the length of the shortest side (use $\cos 70° = 0.342$).

11. In a right-angled triangle, the tangent of one of the acute angles is 1. How are the measures of the two legs related to each other?

12. If α is an acute angle and $\sin\alpha = 0.75$, find the values of $\cos\alpha$ and $\tan\alpha$.

13. Given that $\sin\theta = \frac{4}{9}$, find the values of $\cos\theta$ and $\tan\theta$.

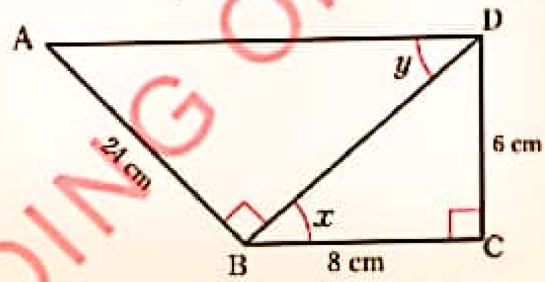
14. In a right-angled triangle ACB, $A\hat{C}B = 90°$, $\overline{AC} = 4$ cm, and $\overline{BC} = 3$ cm.

   (a) How are the angles A and B related?

   (b) Find the value of sin A, cos A, sin B, cos B and deduce the relationship between the ratios of different angles.

   (c) Find the values of $\sin^2 A + \cos^2 A$ and $\sin^2 B + \cos^2 B$ and make a conclusion about these relationships.

15. In the following figure, evaluate $\sin x + \tan y$.



Trigonometric ratios of special angles

Special angles are angles whose trigonometric ratios can be found by using simple ratios. The special angles are 30°, 45°, 60°, and 90°.

Consider the equilateral triangle ABC in Figure 8.3. Let the length of each side be 2 units. The altitude from vertex A bisects $\overline{BC}$ at D.

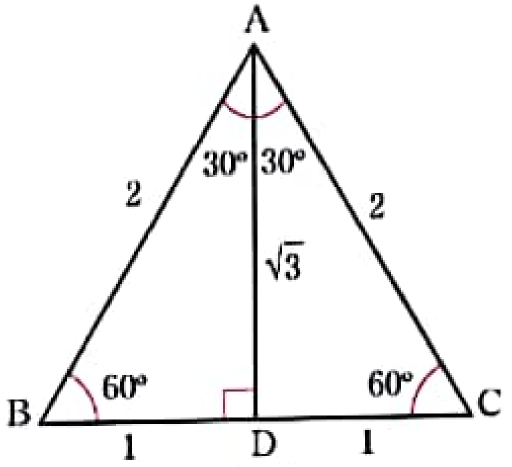


Figure 8.3: *A triangle with special angles 30° and 60°*

From Figure 8.3, it follows that

$\overline{BD} = \overline{DC} = 1$ unit.

Apply the Pythagoras' theorem:

$$\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2$$

$$1^2 + \overline{AD}^2 = 2^2$$

$$\overline{AD}^2 = 4 - 1 = 3$$

$\overline{AD} = \sqrt{3}$ units.

Thus, the following trigonometric ratios are obtained.

$$\sin 60° = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{2},$$

$$\tan 60° = \frac{\overline{AD}}{\overline{BD}} = \frac{\sqrt{3}}{1} = \sqrt{3},$$

$$\cos 60° = \frac{\overline{BD}}{\overline{AB}} = \frac{1}{2},$$

$$\sin 30° = \frac{\overline{BD}}{\overline{AB}} = \frac{1}{2},$$

$$\tan 30° = \frac{\overline{BD}}{\overline{AD}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

$$\cos 30° = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{2}.$$

Consider an isosceles triangle PQR in Figure 8.4 in which the base angle is 45° and $\overline{PQ} = \overline{RQ} = 1$ unit.

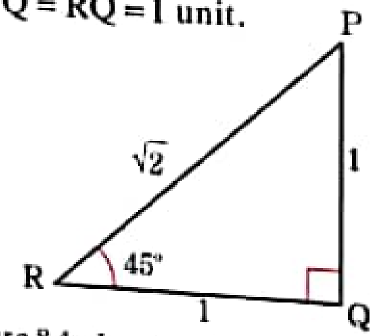


Figure 8.4: *Isosceles triangle with special angle measuring 45°*

Using the Pythagoras' theorem, it implies that, $\overline{PR} = \sqrt{1^2 + 1^2} = \sqrt{2}$ units.

The following trigonometric ratios are obtained.

$$\tan 45° = \frac{\overline{PQ}}{\overline{RQ}} = \frac{1}{1} = 1,$$

$$\sin 45° = \frac{\overline{PQ}}{\overline{PR}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\cos 45° = \frac{\overline{RQ}}{\overline{PR}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Values of sin 90°, cos 90°, and tan 90°

Consider the right-angled triangle ABC in Figure 8.5.

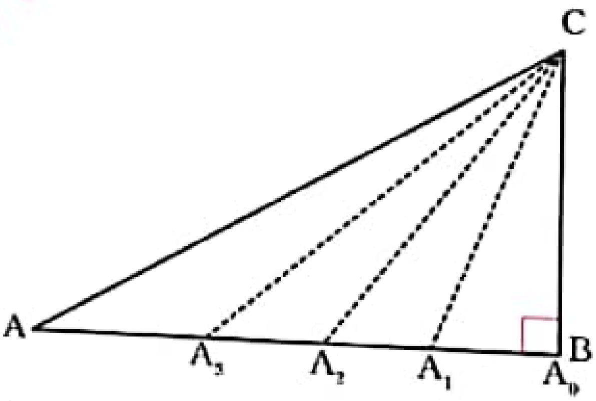


Figure 8.5: *Trigonometric ratios of angle 90°*

In Figure 8.5, if $\overline{AC}$ approaches $\overline{A_0C}$, then $C\hat{A}A_0$ approaches 90°. The hypotenuse side $\overline{AC}$ and the opposite side $\overline{BC}$ overlap so that $\overline{AC} = \overline{BC}$. Also, the adjacent side $\overline{AB} = \overline{AA_0} = 0$. Thus,

$$\sin 90° = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}}$$

$$= \frac{\overline{BC}}{\overline{AC}} = \frac{\overline{BC}}{\overline{BC}}$$

$$= 1$$

$$\cos 90^\circ = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$$

$$= \frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AA_0}}{\overline{BC}}$$

$$= \frac{0}{\overline{BC}} = 0, \text{ if } \overline{BC} \neq 0$$

$$\tan 90^\circ = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}}$$

$$= \frac{\overline{BC}}{\overline{AB}} = \frac{\overline{BC}}{\overline{AA_0}}$$

$$= \frac{\overline{BC}}{0} = \text{undefined}$$

Values of sin 0°, cos 0°, and tan 0°

Consider the right-angled triangle in Figure 8.6.

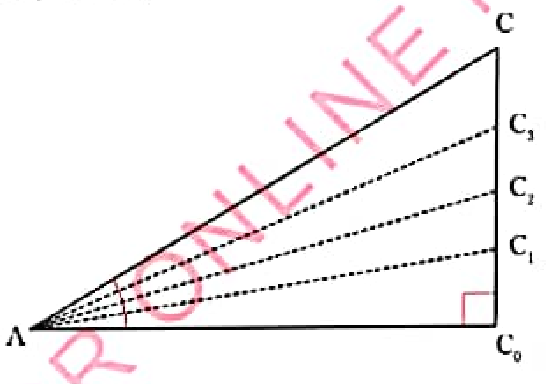


Figure 8.6: *Movement of* $\overline{AC}$ *towards* $\overline{AC_0}$

In Figure 8.6, if $\overline{AC}$ collapses to $\overline{AC_0}$, then $C\hat{A}C_0$ becomes 0°. The hypotenuse and adjacent sides overlap, that is $\overline{AC} = \overline{AB}$. This means that the opposite side becomes zero, $\overline{BC} = \overline{CC_0} = 0$. Thus,

$$\sin 0^\circ = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}}$$

$$= \frac{0}{\overline{AC}}$$

$$= 0$$

Therefore, $\sin 0^\circ = 0$.

Also,

$$\cos 0^\circ = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}}$$

$$= \frac{\overline{AB}}{\overline{AC}} = 1$$

Therefore, $\cos 0^\circ = 1$.

Similarly,

$$\tan 0^\circ = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}}$$

$$= \frac{0}{\overline{AB}}$$

$$= 0$$

In this case, the opposite side shrinks to zero so that the hypotenuse overlaps with the adjacent side.

Therefore, $\tan 0^\circ = 0$.

The results for all special angles are summarised in Table 8.1.

Table 8.1: *Trigonometric ratios of special angles*

x	$\sin x$	$\cos x$	$\tan x$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

Example 8.8

Find the exact value of $2\sin 60° + \cos 30°$.

Solution

Using the values in Table 8.1, it follows that;

$\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

Thus,

$$2\sin 60° + \cos 30° = 2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}.$$

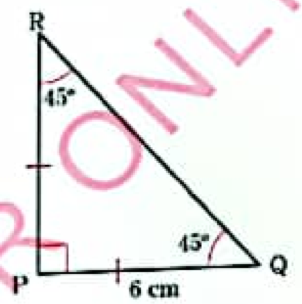
Therefore, $2\cos 60° + \cos 30° = \frac{3\sqrt{3}}{2}$.

Example 8.9

An isosceles triangle PQR is such that $P\hat{Q}R = 45°$ and $R\hat{P}Q = 90°$. If $\overline{PQ} = 6\text{cm}$, find the length of $\overline{RQ}$, giving the answer in radical form.

Solution

Consider the triangle PQR.



From the ΔPQR, it follows that

$$\cos 45° = \frac{6}{RQ}$$

$$\overline{RQ} = \frac{6}{\cos 45°}$$

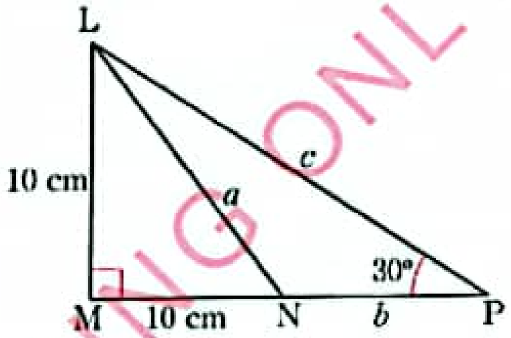
$$= \frac{6}{\frac{\sqrt{2}}{2}}$$

$$= 6\sqrt{2}$$

Therefore, $\overline{RQ} = 6\sqrt{2}$ cm.

Example 8.10

Find the values of a, b, and c in the triangle LMP.



Solution

In ΔLMN, apply the Pythagoras' theorem. That is,

$$a^2 = 10^2 + 10^2$$

$$a^2 = 200$$

$$a = 10\sqrt{2}\text{ cm}$$

In ΔLMP, it implies that,

$$\tan 30° = \frac{\overline{LM}}{\overline{MP}}$$

Thus, $\frac{\sqrt{3}}{3} = \frac{10}{MP}$.

$$\overline{MP} = \frac{10 \times 3}{\sqrt{3}}$$

$$= 10\sqrt{3}\text{ cm}$$

But $\overline{MP} = 10 + b$.

$$10\sqrt{3} = 10 + b$$

Thus, $b = 10\left(\sqrt{3} - 1\right)$ cm.

In ΔLMN, it follows that,

$\sin 30° = \frac{\overline{LM}}{\overline{LP}}$

Thus, $\frac{1}{2} = \frac{10\text{ cm}}{c}$.

$c = 2 \times 10 = 20\text{cm}$

Therefore, $c = 20\text{cm}$

Exercise 8.2

1. Find the exact value of each of the following expressions:
   (a) $\sqrt{3}\tan 60°$
   (b) $\sqrt{2}(\cos 45° + \cos 0°)$
   (c) $2(\sin 30° + \cos 0°)$
   (d) $\frac{\sqrt{3}}{3}\cos 30° - 2\tan 45°$

2. If $\sin x = \frac{1}{2}$, find the following:
   (a) The value of x.
   (b) $\tan x$, given that x is an acute angle.

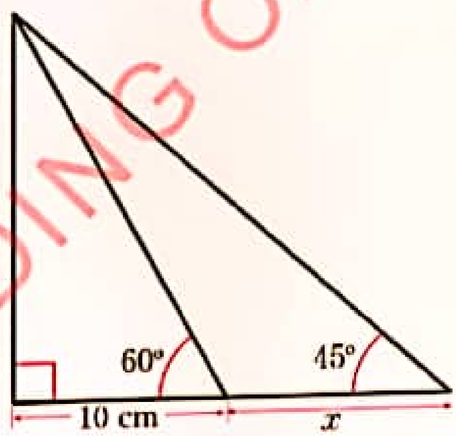
3. A ladder leans against a vertical wall and makes an angle 60° with the wall. If the highest point of the ladder is 4 m from the ground, find the length of the ladder.

4. Without using a calculator, simplify each of the following:
   (a) $\cos 45° \sin 45°$
   (b) $\sin 60° (\tan 30° - \cos 30°)$

5. Find the value of each of the following:
   (a) $\frac{\sin 60°}{\cos 60°}$
   (b) $\frac{\sin 30°}{\cos 30°}$

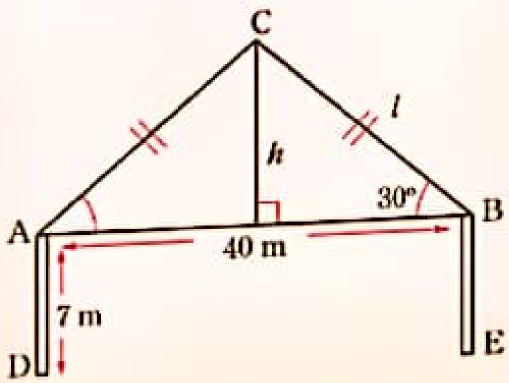
6. Evaluate each of the following expressions:
   (a) $\frac{\sin 30° + \cos 60°}{2}$
   (b) $(\sin 60°)^2 \cos 60°$
   (c) $\sin 45° + 2\cos 60°$

7. Find the value of x in the following figure.



8. The diagonal of a rectangular plot of land is 4 m long and makes an angle of 30° with its length. Find:
   (a) The length and width of the rectangular plot.
   (b) The length of a wire required to enclose the plot.

9. The roof of a godown is placed on top of the walls 7 m high as shown in the following figure.



(a) Find the length of the slants (l) of the roof.

(b) How far above the ground is the highest point of the roof?

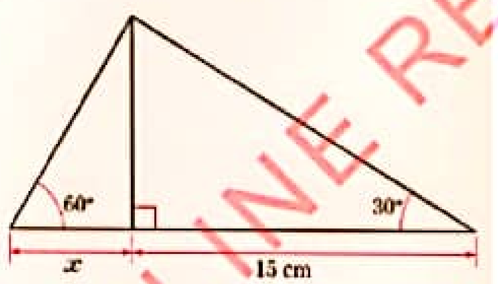
10. One end of a rope of length 38 m is tied to the top of a flagpole and another end is fixed to a point on the ground. If the angle the rope makes with the flagpole is 45°, find:

(a) The height of the flagpole.

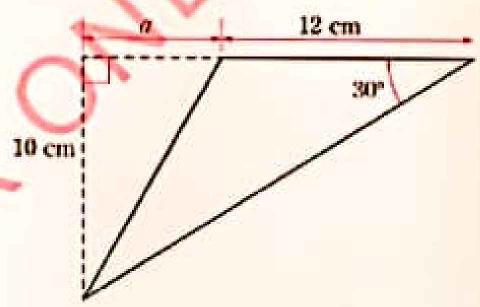
(b) The distance between the fixed point on the ground and the base of the flagpole.

11. Find the value of the unknown in parameter each of the following figures.

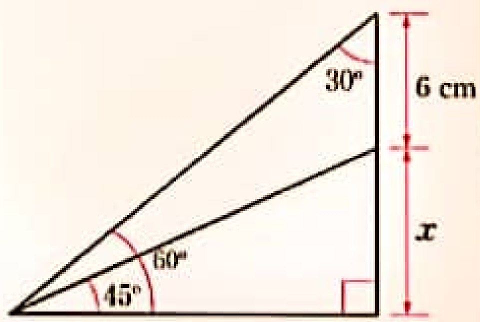
(a)



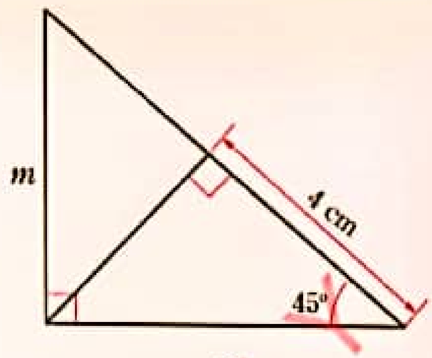
(b)



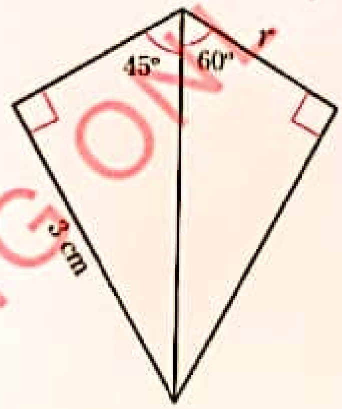
(c)



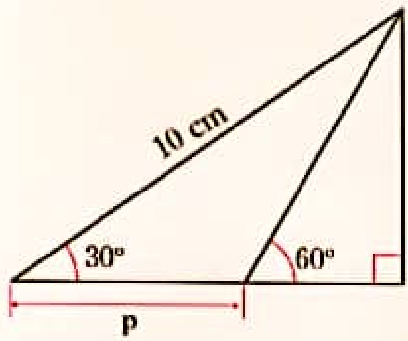
(d)



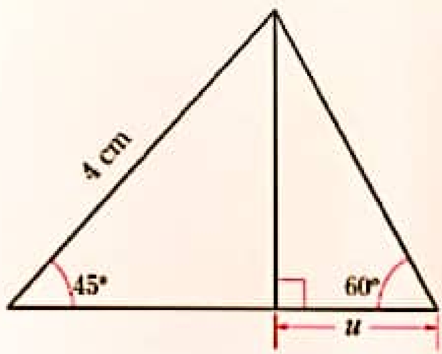
(e)



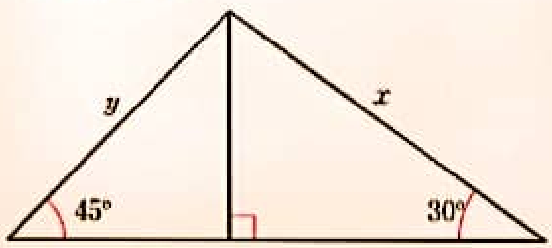
(f)



(g)



12. Use the following figure to express y in terms of x



Trigonometric ratios of any angle

To understand trigonometric ratios of any angle, it is important to go beyond right-angled triangles. By using the unit circle and the Cartesian coordinate system, these ratios can be defined for any angle, making it easier to explore relationships between angles.

Activity 8.2: Introducing the relationship of sides, lengths, and angles of a triangle

1. Using a scale, draw three right-angled triangles PQR, ABC, and MNT such that $\overline{PQ} = 4\text{cm}$, $\overline{QR} = 3\text{ cm}$, $\overline{PR} = 5\text{cm}$, $\overline{AB} = 8\text{cm}$, $\overline{BC} = 15\text{cm}$, $\overline{AC} = 17\text{cm}$, $\overline{MN} = 6\text{cm}$, $\overline{NT} = 8\text{cm}$, and $\overline{MT} = 10\text{ cm}$.
2. Calculate $\frac{\overline{PQ}}{\overline{PR}}$, $\frac{\overline{AB}}{\overline{AC}}$, and $\frac{\overline{MN}}{\overline{MT}}$.
3. Calculate $\frac{\overline{QR}}{\overline{PR}}$, $\frac{\overline{BC}}{\overline{AC}}$, and $\frac{\overline{NT}}{\overline{MT}}$.
4. Calculate $\frac{\overline{PQ}}{\overline{QR}}$, $\frac{\overline{AB}}{\overline{BC}}$, and $\frac{\overline{MN}}{\overline{NT}}$.
5. Use a protractor to measure $P\hat{R}Q$, $A\hat{C}B$, and $M\hat{T}N$.
6. Calculate the values of sine, cosine and tangent of angles in task 5 and compare the results with the ratios in tasks 2, 3, and 4, respectively.
7. Use a protractor to measure $B\hat{A}C$, $Q\hat{P}R$, and $N\hat{M}T$.
8. Repeat task 6 using the angles obtained in task 7.
9. Comment on the results obtained in tasks 6 and 8.
10. Share your observations on the relevance of the relationship between angles and sides of a right-angled triangle.

Consider a circle of unit radius subdivided into four congruent sectors by the coordinate axes whose origin is at the centre of the circle as shown in Figure 8.7.

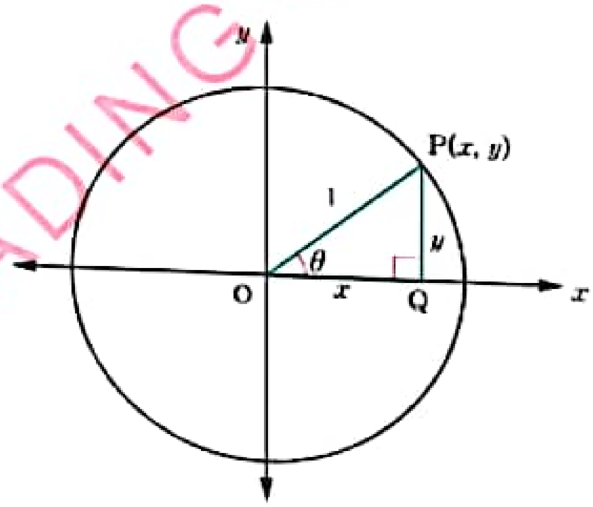


Figure 8.7: *An acute angle in a unit circle*

Let θ be any acute angle $(0° < \theta < 90°)$ located in the first quadrant and P be a point on the circle with coordinates (x, y), where $\overline{OP}$ is the radius of the unit circle.

The trigonometric ratios in this circle can be obtained by using the sides of a right-angled triangle OPQ as follows:

$$\sin\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{y}{1} = y,$$

$$\cos\theta = \frac{\overline{OQ}}{\overline{OP}} = \frac{x}{1} = x,$$

$$\tan\theta = \frac{\overline{QP}}{\overline{OQ}} = \frac{y}{x}.$$

Figure 8.7 shows that all the trigonometric ratios are positive as per corresponding x and y axes.

In Figure 8.8, θ is an obtuse angle ($90^\circ < \theta < 180^\circ$) located in the second quadrant. The trigonometric ratios of θ are related to the trigonometric ratios of acute angle $180^\circ - \theta$.

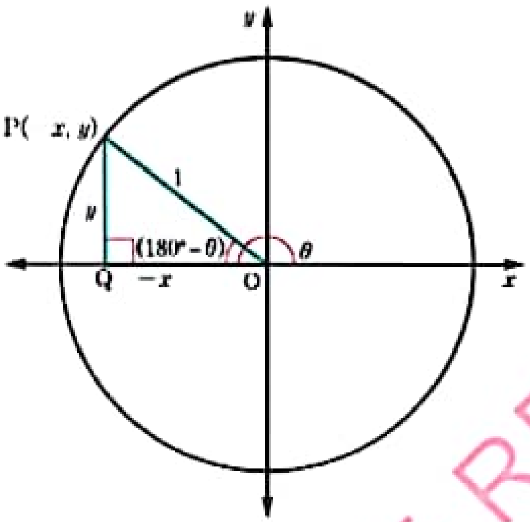


Figure 8.8: *Reflex angle in a unit circle*

The trigonometric ratios in the second quadrant can be obtained by using the sides of a right-angled triangle OQP as follows:

$$\sin\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{y}{1} = y = \sin(180 - \theta)$$

$$\cos\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{-x}{1} = -x = -\cos(180 - \theta)$$

$$\tan\theta = \frac{\overline{QP}}{\overline{OQ}} = \frac{y}{-x} = -\tan(180 - \theta)$$

From Figure 8.8, it can be observed that, the trigonometric ratio of sine in the seocond quadrant is positive, while those of cosine and tangent are negative as per corresponding x and y axes.

In Figure 8.9, θ is a reflex angle ($180^\circ < \theta < 270^\circ$), which is in the third quadrant.

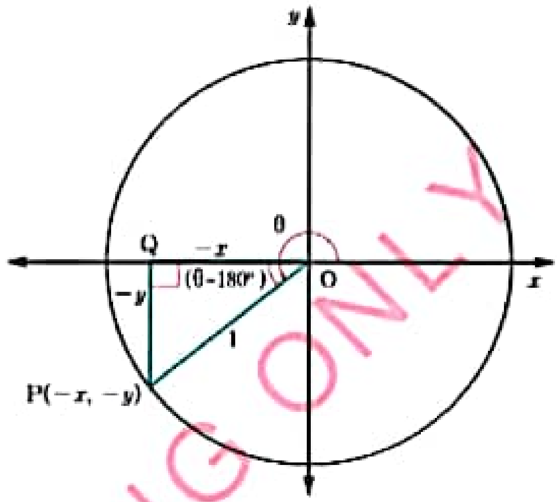


Figure 8.9: *Reflex angle in a unit circle*

The trigonometric ratios of θ are related to the trigonometric ratios of an acute angle $\theta - 180^\circ$. The trigonometric ratios in the third quadrant can be obtained by using the sides of the right-angled triangle OQP as follows:

$$\sin\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{-y}{1} = -y = -\sin(\theta - 180^\circ)$$

$$\cos\theta = \frac{\overline{OQ}}{\overline{OP}} = \frac{-x}{1} = -x = -\cos(\theta - 180^\circ)$$

$$\tan\theta = \frac{\overline{QP}}{\overline{OQ}} = \frac{-y}{-x} = \frac{y}{x} = \tan(\theta - 180^\circ)$$

Figure 8.9 shows that, the trigonometric ratios of sine and cosine in the third quandrant are negative, while for the tangent is positive as per corresponding x and y axes.

In Figure 8.10, θ is a reflex angle ($270^\circ < \theta < 360^\circ$) which is in fourth quadrant. The trigonometric ratios of θ are linked to the trigonometric ratios of an acute angle $360^\circ - \theta$.

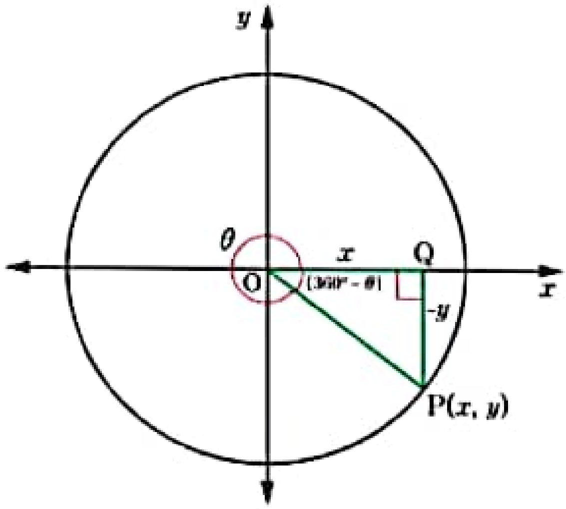


Figure 8.10: *A reflex angle in a unit circle*

The trigonometric ratios in the fourth quadrant can be obtained by using the sides of the right-angled triangle OQP as follows:

$$\sin\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{-y}{1} = -y = -\sin(360° - \theta)$$

$$\cos\theta = \frac{\overline{OQ}}{\overline{OP}} = \frac{x}{1} = x = \cos(360° - \theta)$$

$$\tan\theta = \frac{\overline{QP}}{\overline{OQ}} = -\frac{y}{x} = -\tan(360° - \theta)$$

Figure 8.10 shows that, the trigonometric ratios of sine and tangent in the fourth quadrant are negative, while for the cosine is positive as per corresponding x and y axes.

Signs of trigonometric ratios

Trigonometric ratios can be positive or negative depending on the size of the angle or the quadrant in which the angle is found.

The results obtained are summarised in Figure 8.11. These results are helpful in determining whether sine, cosine, and tangent of an angle is positive or negative.

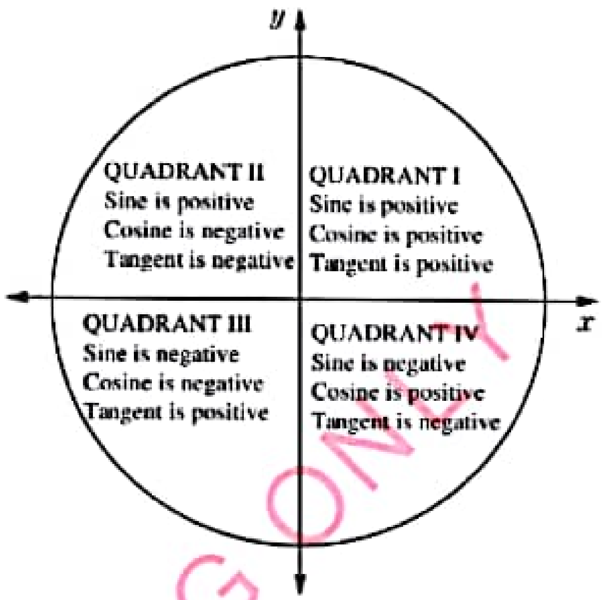


Figure 8.11: *Signs of trigonometric ratios*

Example 8.11

Write the signs of each of the following trigonometrical ratios:

(a) $\sin 170°$ (c) $\tan 310°$

(b) $\cos 240°$ (d) $\sin 300°$

Solution

(a) 170° lies in the second quadrant, hence sin 170° is positive.

(b) 240° lies in the third quadrant, hence cos 240° is negative.

(c) 310° lies in the fourth quadrant, hence tan 310° is negative.

(d) 300° lies in the fourth quadrant, hence sin 300° is negative.

Example 8.12

Express each of the following in terms of an acute angle:

(a) $\cos 165°$ (c) $\tan 95°$

(b) $\sin 317°$ (d) $\tan 258°$

Solution

(a) 165° is in the second quadrant, then

$\cos 165° = -\cos(180° - 165°)$
$= -\cos 15°$

(b) 317° is in the fourth quadrant, then

$\sin 317° = -\sin(360° - 317°)$
$= -\sin 43°$

(c) 95° is in the second quadrant, then

$\tan 95° = -\tan(180° - 95°)$
$= -\tan 85°$

(d) 258° is in the third quadrant, then

$\tan 258° = \tan(258° - 180°)$
$= \tan 78°$

Positive and negative angles

Angles may be positive or negative depending on the direction in which the angle is measured. Figure 8.2 gives the clockwise and anticlockwise measurements of an angle.

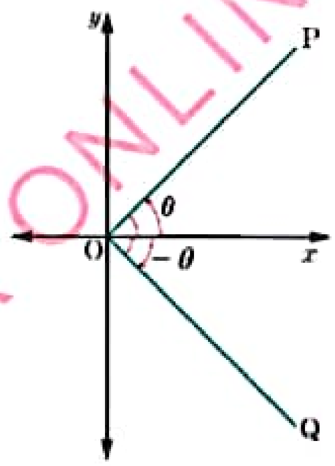


Figure 8.12: *Measurements of angles in a clockwise and anticlockwise directions*

In Figure 8.12, it can be deduced that, angles measured in the clockwise direction from the positive x-axis are negative. Angles measured in the anticlockwise direction from the positive x-axis are positive.

Figures 8.13 and 8.14 illustrate how positive and negative angles can be located in the four quadrants. The corresponding positive and negative angles whose trigonometric ratios are the same can easily be found.

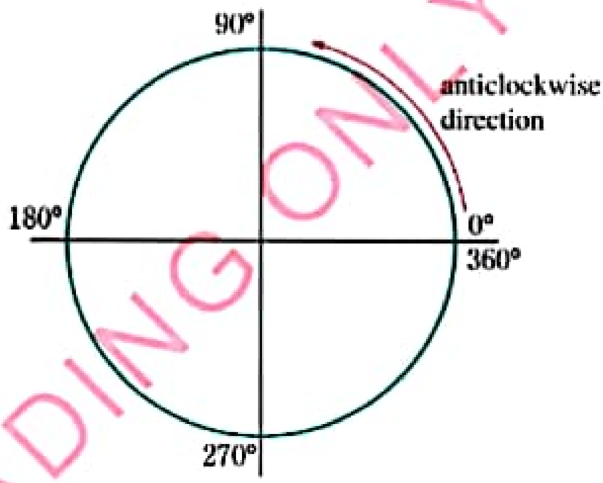


Figure 8.13: *Positive angles in a unit circle*

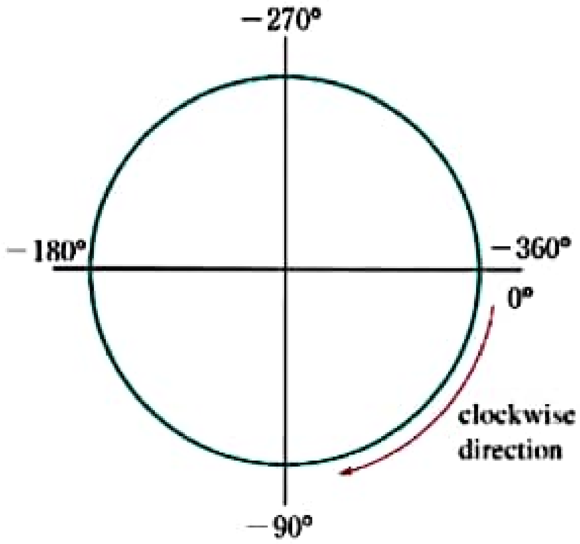


Figure 8.14: *Negative angles in a unit circle*

If θ is positive, the negative angle corresponding to θ is $(-360+\theta)$. If θ is negative, the positive angle corresponding to θ is $(360+\theta)$.

Note: $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, and $\tan(-\theta) = -\tan\theta$.

Example 8.13

Find the positive or negative angle corresponding to each of the following angles:

(a) 273° (c) 304°

(b) −210° (d) −115°

Solution

(a) The negative angle corresponding to 273° is (−360° + 273°) = −87°.

(b) The positive angle corresponding to −210° is (360° − 210°) = 150°.

(c) The negative angle corresponding to 304° is (−360° + 304°) = −56°.

(d) The positive angle corresponding to −115° is (360° − 115°) = 245°.

Example 8.14

Find the sine, cosine, and tangent of each of the following angles:

(a) 144° (b) −231° (c) −70° (d) 310°

Solution

(a) $\sin 144° = \sin(180° - 144°)$

$= \sin 36° = 0.5878$

$\cos 144° = -\cos(180° - 144°)$

$= -\cos 36° = -0.8090$

$\tan 144° = -\tan(180° - 144°)$

$= -\tan 36° = -0.7265$

(b) $\sin(-231°) = \sin(360° - 231°)$

$= \sin 129°$

$= \sin(180° - 129°)$

$= \sin 51° = 0.7771$

$\cos(-231°) = -\cos 51° = -0.6293$

$\tan(-231°) = -\tan 51° = -1.2349$

(c) $\sin(-70°) = \sin(360° - 70°)$

$= \sin 290°$

$= -\sin(360° - 290°)$

$= -\sin 70° = -0.9397$

$\cos(-70°) = \cos 70° = 0.3420$

$\tan(-70°) = -\tan 70° = -2.7475$

(d) $\sin(310°) = -\sin(360° - 310°)$

$= -\sin 50° = -0.7660$

$\cos(310°) = \cos 50°$

$= 0.6428$

$\tan(310°) = -\tan 50°$

$= -1.1918$

Exercise 8.3

1. Write the signs of each of the following trigonometric ratios:
   (a) cos 160°
   (b) sin 310°
   (c) tan 75°
   (d) sin 220°
   (e) cos 355°
   (f) tan 190°

2. Express each of the following in terms of sine, cosine or tangent of an acute angle:
   (a) cos 308°
   (b) sin 217°
   (c) tan 175°
   (d) sin 333°
   (e) cos 268°
   (f) tan 103°

3. Express each of the following in terms of sin 50°:
   (a) sin 130°
   (b) sin 230°
   (c) sin 310°

4. Express each of the following in terms of cos 20°:
   (a) cos 160° (b) cos 200°
   (c) cos 340°
5. Express each of the following in terms of tan 40°:
   (a) tan 140° (b) tan 220°
   (c) tan 320°
6. Without using a calculator, find the value of each of the following:

   (a) $\dfrac{\sin(-150)\cos 315}{\tan 300}$

   (b) $\dfrac{\tan(-30)\cos 60}{\sin(-45)}$

Relationship between trigonometric ratios

Consider ΔABC as shown in Figure 8.15.

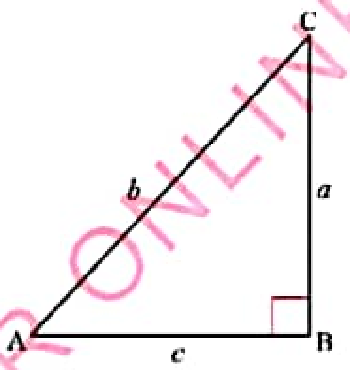


Figure 8.15: *Right-angled triangle ABC*

In Figure 8.15, the angles A and C are complementary. That is,

A + C = 90°

Thus, C = 90° − A.

But $\sin A = \dfrac{a}{b}$ and $\cos C = \dfrac{a}{b}$.

Thus, $\sin A = \cos C = \cos(90° - A)$.

Hence, $\sin A = \cos(90° - A)$.

Ttherefore, the sine of an angle is equal to the cosine of its complement and vice versa. From Figure 8.15, it follows that

$$\frac{\sin A}{\cos A} = \frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$$

But $\tan A = \dfrac{a}{c}$.

Hence, $\dfrac{\sin A}{\cos A} = \tan A$.

Also, $\sin^2 A = \dfrac{a^2}{b^2}$ and $\cos^2 A = \dfrac{c^2}{b^2}$.

$$\sin^2 A + \cos^2 A = \frac{a^2}{b^2} + \frac{c^2}{b^2} = \frac{a^2 + c^2}{b^2}.$$

But $a^2 + c^2 = b^2$ (Pythagoras' theorem)

Thus, $\sin^2 A + \cos^2 A = \dfrac{b^2}{b^2} = 1$.

Therefore, $\sin^2 A + \cos^2 A = 1$.

Generally, for any angle θ, the corresponding trigonometrical identity is $\sin^2\theta + \cos^2\theta = 1$.

Example 8.15

Given that $\sin\theta = \dfrac{4}{9}$, find $\cos\theta$ and $\tan\theta$ for $0° \le \theta \le 90°$.

Solution

Given $\sin\theta = \dfrac{4}{9}$.

From $\sin^2\theta + \cos^2\theta = 1$,

$$\cos^2\theta = 1 - \sin^2\theta$$

But $\sin^2\theta = \dfrac{16}{81}$.

$$\cos^2\theta = 1 - \frac{16}{81} = \frac{65}{81}$$

Hence, $\cos\theta = \pm\sqrt{\dfrac{65}{81}} = \pm\dfrac{\sqrt{65}}{9}$.

Since θ is acute, it follows that

$$\cos\theta = \frac{\sqrt{65}}{9}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4}{9} \div \frac{\sqrt{65}}{9} = \frac{4\sqrt{65}}{65}$$

Therefore, $\cos\theta = \frac{\sqrt{65}}{9}$ and $\tan\theta = \frac{4\sqrt{65}}{65}$

Example 8.16

Given that $\propto$ and β are complementary angles and $\sin\alpha = \frac{5}{13}$, find $\tan\beta$.

Solution

Since $\propto$ and β are complementary angles, it implies that

$$\propto + \beta = 90°$$

$$\propto = 90° - \beta.$$

Hence, $\sin\propto = \cos(90° - \propto)$

$$= \cos\beta = \frac{5}{13}.$$

But $\sin^2\beta = 1 - \cos^2\beta$.

$$\sin^2\beta = 1 - \frac{25}{169}$$

$$= \frac{144}{169}$$

Thus, $\sin\beta = \pm\frac{12}{13}$. Since β is an acute angle, then $\sin\beta = \frac{12}{13}$.

Hence, $\tan\beta = \frac{\sin\beta}{\cos\beta}$

$$\tan\beta = \frac{12}{13} \div \frac{5}{13}.$$

$$= \frac{12}{5}$$

Therefore, $\tan\beta = \frac{12}{5}$.

Example 8.17

If $\sin\theta = 0.9397$ and $\cos\theta = 0.3420$, without using a calculator find the value of $\tan\theta$.

Solution

Given $\sin\theta = 0.9397$ and

$$\cos\theta = 0.3420$$

From, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$= \frac{0.9397}{0.3420}$$

$$= 2.748$$

Therefore, $\tan\theta = 2.748$.

Exercise 8.4

Without using calculators, answer the following questions:

1. Given that $\cos\theta = \frac{4}{5}$, find the value of $\sin\theta$.
2. If $\tan\propto = \frac{\sqrt{2}}{5}$, find the value of $\sin(90° - \propto)$.
3. If $\propto$ and β are complementary angles and $\tan\propto = \frac{\sqrt{6}}{3}$, find the value of $\tan\beta$.
4. If $\sin A = 0.9733$ and $\cos A = 0.2250$, find the value of tan A.
5. Given that $\cos\theta = 0.9272$ and $\tan\theta = 0.4040$. Find the value of $\sin\theta$.
6. If $\tan\theta = \frac{\sqrt{3}}{4}$, find the value of $\sin\theta$.
7. Given that $\tan\theta = 0.75$, find the value of $\cos\theta$.

8. If $\cos x = \frac{p}{q}$ and $\tan x = \frac{r}{p}$, find the value of $\sin x$.

9. If $\tan \beta = \frac{r}{\sqrt{s}}$ and $\tan \alpha = \frac{\sqrt{s}}{r}$, show that $\cos \beta = \sqrt{\frac{s}{r^2 + s}}$, where α and β are complementary angles.

10. What will be the sine and cosine of an acute angle whose tangent is $2\frac{2}{5}$?

11. If α and β are complementary angles and $\sin \alpha = \frac{\sqrt{3}}{2}$, find the value of $\sin \beta$.

12. If $\sin \alpha = \frac{5}{8}$, find the value of $\cos \alpha$.

13. Given that $\cos \lambda = \frac{6}{7}$, find the value of $\tan \lambda$.

14. Given that $\sin x = -\frac{3}{8}$, find the value of $\cos x$.

15. If $\cos \alpha = -\frac{1}{3}$, find the value of $\tan \alpha$.

16. If $\sin \theta = 0.9848$, find the value of $\cos \theta$.

17. Find the value of tan A if $\sin A = 0.8192$.

18. If $\cos P = 0.3746$, find the value of sin P.

19. If $\cos Q = 0.9063$, find the value of tan Q.

20. Given that $\cos \theta = 0.3090$, find the value of $\sin \theta$ and $\tan \theta$.

21. If A and B are acute angles, and $\sin A = \cos B$, show that $A + B = 90°$

Calculating trigonometric ratios using a calculator

For many years, mathematicians have been obtaining values of trigonometric ratios from trigonometric tables. This method is tedious and may lead to errors in calculations. With emerging technologies, the values of trigonometric ratios are now easily obtained through the use of calculators and other mathematical softwares.

Activity 8.3: Calculating values of trigonometric ratios using calculators

1. Use various sources such as the internet, offline libraries, or interviews to learn how to determine values of trigonometric ratios from calculators.
2. Study the calculator and record the main steps required to find values of trigonometric ratios.
3. Use a calculator to complete the following table with values correct to 4 significant figures.

Angle θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°			
15°			
23.6°			
45°			
130°			
90°			
263°			
–263°			

4. Study the values and record the pattern you have observed from sine, cosine, and tangent values in the table.
5. Justify your reasoning by demonstrating how you have arrived at your answers.

Inverses of trigonometric ratios

Given a trigonometric ratio, say x, the corresponding angle can be found using the inverse trigonometric functions, denoted as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. For example, if $\cos 0° = 1$, then $\cos^{-1}(1) = 0°$. It is read as "the inverse of cosine 1 is zero". This applies similarly to other ratios, such as the inverse of sine and the inverse of tangent.

The inverse trigonometric ratios can be determined using a calculator, graphical methods, and other techniques. Participate in Activity 8.4 to learn how to determine the inverse of trigonometric ratios using a calculator.

Activity 8.4: Determining inverse trigonometric ratios using a calculator

1. Explore different sources including the internet to learn how to use a calculator to determine inverse trigonometric ratios.
2. Write all the necessary steps to be followed and use the knowledge obtained in Task 1 to complete the following table.

$\sin x = 0.5$	$x =$
$\cos x = 0.5344$	$x =$
$\tan \theta = 1.4071$	$\theta =$
$\sin y = 0.834$	$y =$
$3\sin \alpha = 2$	$\alpha =$
$\cos \beta = 2.0784$	$\beta =$
$\tan 3x = 4.3260$	$x =$
$\cos \theta = \tan 25°$	$\theta =$

3. Explain the process you have used to arrive at your conclusion.

Example 8.18

If $\tan x = 1.4071$, find the value of x.

Solution

Given $\tan x = 1.4071$, it follows that;

$$x = \tan^{-1}(1.4071)$$
$$= 54°36'$$

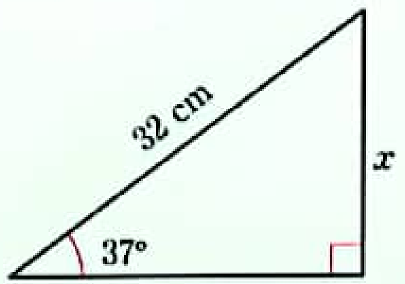
Therefore, $x = 54°36'$.

Note: The values of sine and cosine of an angle cannot be greater than 1, but values of tangents of angles can be greater than 1.

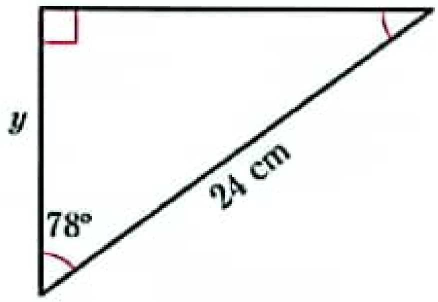
Example 8.19

Find the values of x and y, correct to 3 significant figures in each of the following figures.

(a)



(b)



Solution

(a) From the given figure, it implies that

$$\frac{x}{32 \text{ cm}} = \sin 37°$$

$$x = 32 \text{ cm} \times \sin 37°$$

$$x = 19.2580 \text{ cm}$$

Therefore, $x = 19.3$ cm.

(b) From the given figure, it follows that

$$\frac{y}{24 \text{ cm}} = \cos 78°$$

$$y = 24 \text{ cm} \times \cos 78°$$

$$y = 4.9898 \text{ cm}$$

Therefore, $y = 4.99$ cm.

Example 8.20

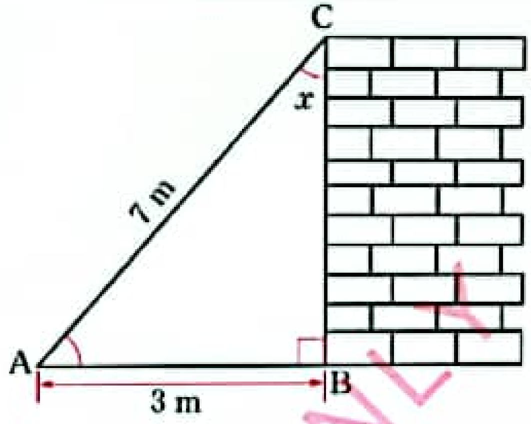
A ladder 7 m long leans against a vertical wall so that the distance between the foot of the ladder and the wall is 3 m. Find the following:

(a) The angle which the ladder makes with the wall.

(b) The height above the ground at which the upper end of the ladder touches the wall.

Solution

Consider the following figure.



Let $\overline{AC}$ be the length of the ladder and $\overline{AB}$ be the distance along the horizontal ground.

From the figure, $\overline{AB} = 3$ m and $\overline{AC} = 7$ m.

(a) $\sin x = \frac{\overline{AB}}{\overline{AC}} = \frac{3 \text{ m}}{7 \text{ m}} = \frac{3}{7}$

$$x = \sin^{-1}\left(\frac{3}{7}\right)$$

$$x = 25° \, 23'$$

Therefore, the ladder makes an angle of 25° 23′ with the wall.

(b) $\cos 25°23' = \frac{\overline{CB}}{\overline{AC}}$

$$= \frac{\overline{CB}}{7 \text{ m}}$$

$$\overline{CB} = 7 \text{ m} \times \cos 25°23' = 6.3245 \text{m}$$

Therefore, the ladder reaches 6.32 m up the wall (3 significant figures).

Exercise 8.5

1. Use a scientific calculator to find the value in each of the following trigonometric ratios correct to 4 significant figures.

(a) sin 56°	(d) cos 64° 15′
(b) tan 36°	(e) sin 26° 11′
(c) cos 2°	(f) tan 70°

2. If y is an acute angle, use a calculator to find the value of y in each of the following:

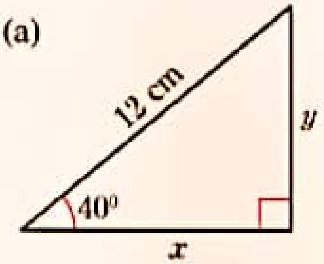
   (a) $\cos y = 0.2034$ (d) $\sin y = 0.8952$

   (b) $\sin y = 0.5975$ (e) $\cos y = \frac{2}{5}$

   (c) $\tan y = 1.5000$ (f) $\tan y = \frac{1}{3}$

3. Find the values of x and y in each of the following, giving your answer correct to 1 decimal place.

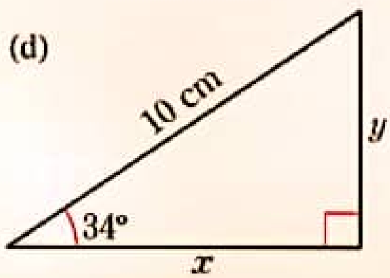
(a)



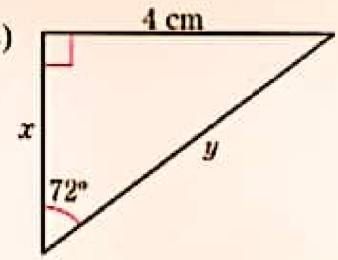
(b)

(c)

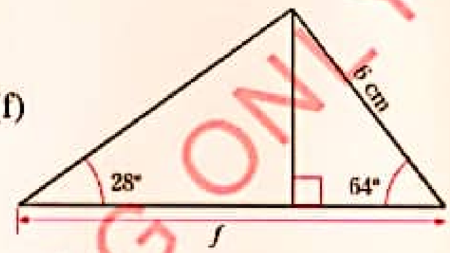
(d)



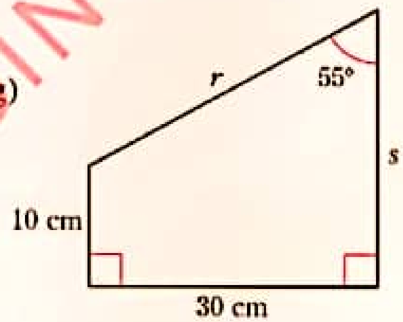
(e)



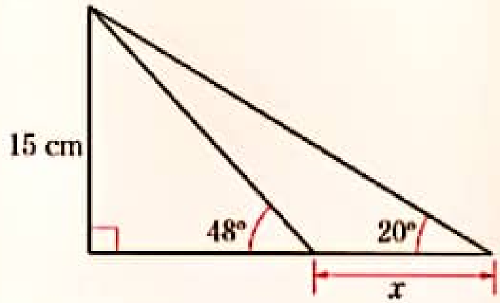
(f)



(g)

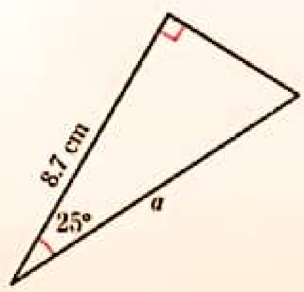


(h)



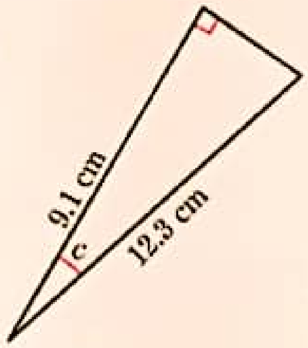
4. In each of the following triangles, find the values of the unknown letters:

(a)

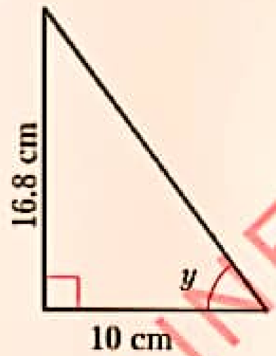


(b)

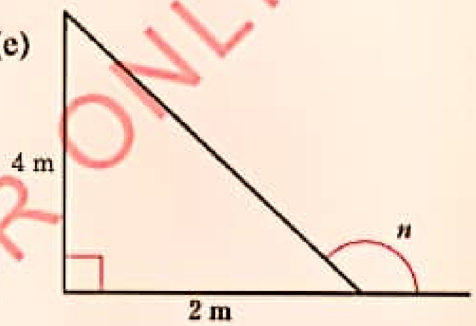
(c)



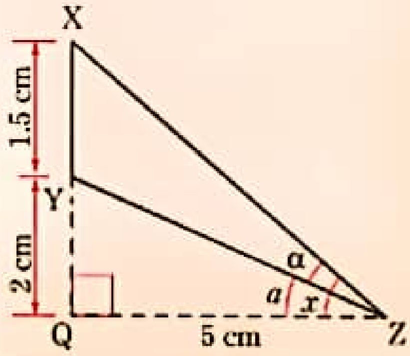
(d)



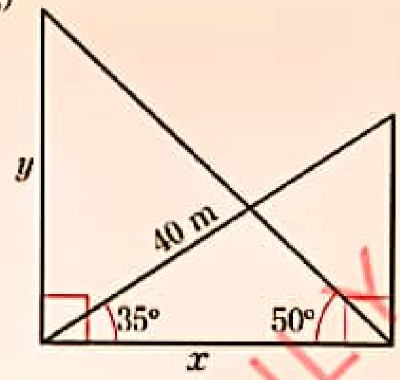
(e)



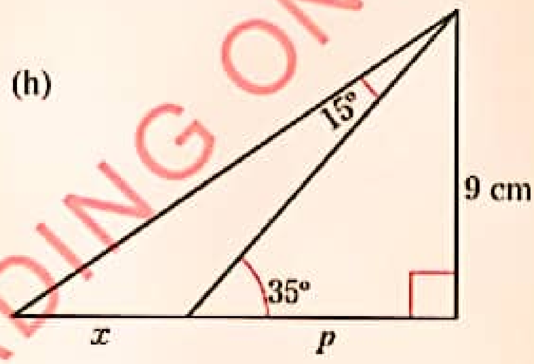
(f)



(g)

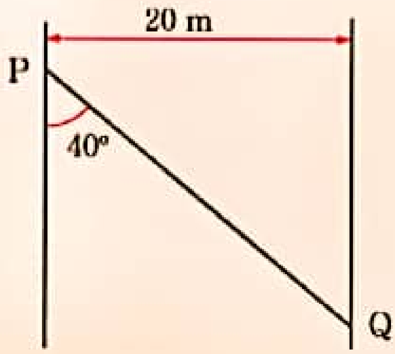


(h)

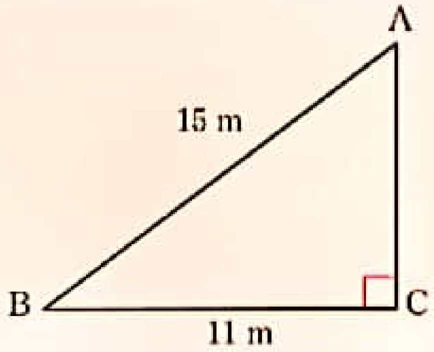


5. A water tap can be reached from Juma's house by walking 100 m North and then 30 m East. Find the bearing (direction) of the water tap from Juma's house.

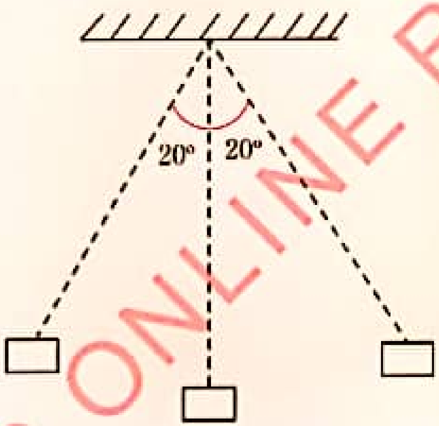
6. A river with parallel banks is 20 m wide. Find the distance $\overline{PQ}$ if P and Q are two points on either side of the river as shown in the following figure.



7. An inclined iron rod AB of length 15 m has a wire tied at A so as to be used to lift loads at point C which is 11 m from B as shown in the following diagram. Find the angle that the iron rod makes with this wire.



8. A 30 cm long pendulum swings back and forth through an angle of 20° on each side as shown in the following figure. How high does the lower end of the pendulum rise?



9. The diagonal of a rectangle makes an angle of 39° with the longer side. Find the width of the rectangle if its length is 50 cm.

10. A car which was originally at sea level traveled a distance of 400 m uphill inclined at an angle of 28° to the horizontal.

   (a) Find the height the car attained above sea level after travelling 400 m.

   (b) Find the horizontal distance travelled correct to the nearest metre.

11. The depth of a straight underground mine tunnel is 240 m. The end of the mine tunnel is 200 m horizontally from the entrance. What angle does the tunnel make with the horizontal?

12. A technician is designing a ramp to load a heavy machine onto a platform. The platform is 1.5 m high and the ramp needs to be at an inclination of 20° to ensure safe loading. Determine the length required and the horizontal distance it will cover.

The angles of elevation and depression

When observing an object, you can either observe it horizontally if the object and line of sight are at the same level. Observing it downward if the observer is above the object and observing it upwards if the observer is below the observed object. To experience these scenarios, engage in Activity 8.5.

Activity 8.5: Studying different angles formed when observing objects

1. At your place, choose an object to observe which is at the same level as your eyes. For instance, a person whose height is the same as yours and you look at his eyes.
2. In the second scenario, observe an object which you need to look downward. For instance, looking at a coin on the floor while standing.

3. In the third scenario, observe an object which is at a position which you need to look upwards. For instance, looking at the top of the building.
4. Study and reflect on all three cases and discover features such as angles formed, and possibilities of determining distances between the observer and the objects without measurements.
5. Use the internet or other relevant sources to study and justify your observations and share your findings through visual diagrams.

In Activity 8.5, it can be deduced that observing an object below or above the horizontal level, the line of sight forms an angle below or above. An angle formed between a horizontal level and a line of sight by observing an object downward is called an angle of depression. The angle formed by observing an object upwards is called an angle of elevation. Figure 8.16 illustrates the angles of depression and elevation.

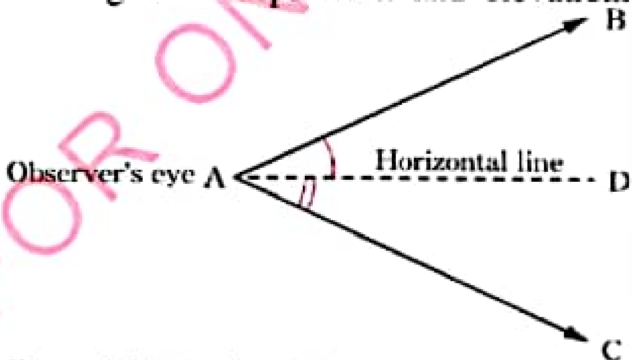


Figure 8.16: *Angles of elevation and depression*

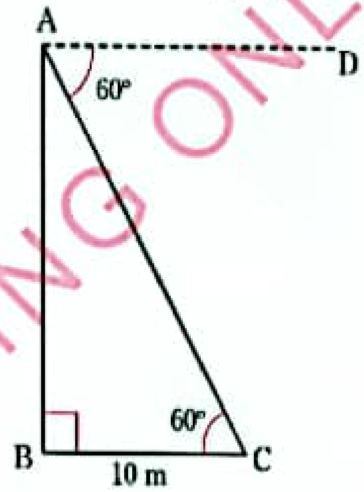
In Figure 8.16, if B and C are objects to be observed, $\hat{CAD}$ is the angle of depression and $\hat{BAD}$ is the angle of elevation. The line AB and AC are lines of sights and the line AD represents the horizontal level.

Example 8.21

From the top of a tower, the angle of depression of a point on the ground 10 m away from the base of a tower is 60°. How high is the tower?

Solution

Consider the following figure.



Let A be the top point of the tower, C be the point of observation and B be the base of the tower. Thus, $\tan 60° = \frac{\overline{AB}}{10 \text{ m}}$. It follows that,

$$\overline{AB} = 10 \text{ m} \times \tan 60°$$

$$\overline{AB} = 10 \text{ m} \times 1.7321$$

$$\overline{AB} = 17.321 \text{ m}$$

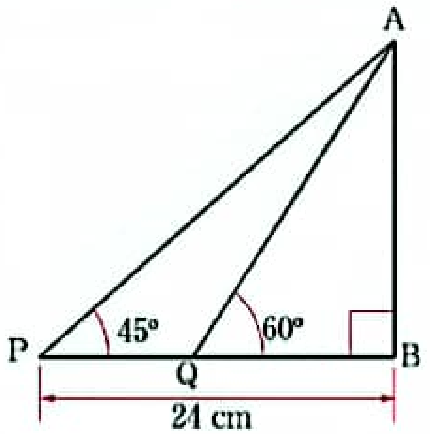
Therefore, the height of the tower is 17.32 m.

Example 8.22

Two pegs, P and Q are on level ground. Both pegs lie due west of a flag post. The angle of elevation of the top of the flag post from P is 45° and from Q is 60°. If P is 24 m from the foot of the flag post, find PQ.

Solution

Consider the following figure.



Let $\overline{AB}$ be the height of the flag post. It implies that

$$\frac{\overline{AB}}{24\text{m}} = \tan 45°$$

$$\overline{AB} = 24\text{m} \times 1$$

$$= 24\text{m}$$

By using the triangle AQB, it follows that

$$\frac{\overline{AB}}{\overline{QB}} = \tan 60°$$

Thus, $\frac{24\,\text{m}}{\overline{QB}} = \sqrt{3}$

$$\overline{QB} = \frac{24\,\text{m}}{\sqrt{3}}$$

$$= \frac{24\sqrt{3}\,\text{m}}{3}$$

$$= 8\sqrt{3}\,\text{m}$$

It follows that,

$$\overline{PQ} = \left(24 - 8\sqrt{3}\right)\text{ m}$$

$$= 8\left(3 - \sqrt{3}\right)\text{ m}$$

$$= 10.144\text{ m}$$

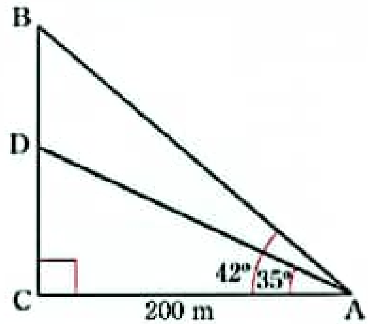
Therefore, $\overline{PQ} = 10.14$ m.

Example 8.23

A mobile phone mast is erected on a hilltop. The angle of elevation of the base of the mast from a point on the ground 200 m from the base of the hill is 35°, while the angle of elevation from the same point to the top of the mast is 42°. Find the height of the mast.

Solution

The given information is translated into the following figure.



In ΔABC, it implies that,

$$\tan 42° = \frac{\overline{BC}}{200\,\text{m}}$$

$$\overline{BC} = 200 \tan 42°\,\text{m}$$

$$= 180.08\text{ m}$$

In ΔADC, it implies that,

$$\tan 35° = \frac{\overline{DC}}{200\,\text{m}}$$

$$\overline{DC} = 200 \tan 35°\,\text{m}$$

$$= 140.04\text{ m}$$

Height of the mast, $\overline{DC}$ is given by;

$$\overline{DB} = \overline{BC} - \overline{DC}$$
$$= 180.08 \text{ m} - 140.04 \text{ m}$$
$$= 40.04 \text{ m}$$

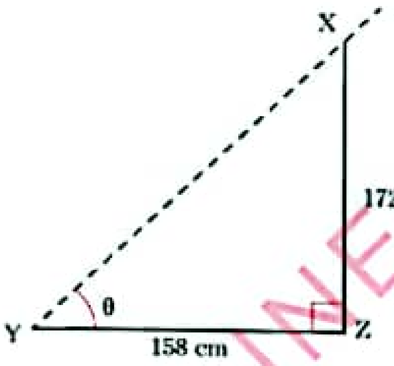
Therefore, the height of the mast is 40 m correct to 2 significant figures.

Example 8.24

A man who is 172 cm tall, notices that his shadow measures 158 cm in length. Find the angle of elevation of the sun.

Solution

Consider the following figure.



From the figure, $\overline{XZ}$ represents the height of a man, $\overline{YZ}$ represents the length of his shadow and θ is the angle of elevation of the sun. It follows that,

$$\tan\theta = \frac{\overline{XZ}}{\overline{YZ}}$$
$$= \frac{172 \text{ cm}}{158 \text{ cm}}$$
$$= 1.089$$
$$\theta = \tan^{-1}(1.089)$$
$$= 47^\circ\ 26'$$

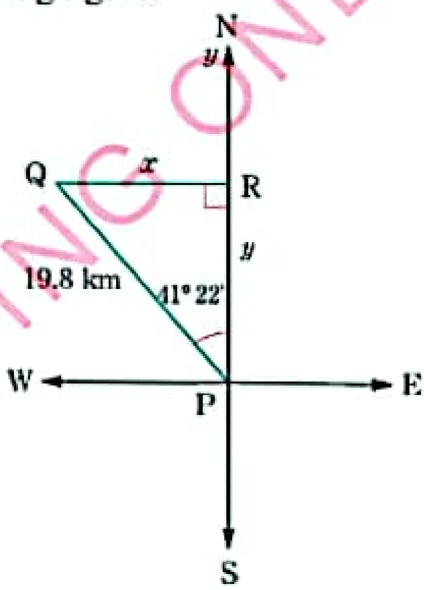
Therefore, the angle of elevation of the sun is $47^\circ\ 26'$.

Example 8.25

Petro starts from point P, travelling 19.8 km in the direction N 41° 22′W. How far has he travelled west and north, respectively?

Solution

The information is presented in the following figure.



Let x and y be the distances in km due west and north of P, respectively.

Using a right-angled triangle PRQ, It follows that

$$\sin 41^\circ\ 22' = \frac{x}{19.8 \text{km}}$$
$$x = 19.8 \text{ km} \times \sin 41^\circ 22'$$
$$= 13.09 \text{ km}$$

Similarly,

$$\cos 41^\circ\ 22' = \frac{y}{19.8 \text{km}}$$
$$y = 19.8 \text{ km} \times \cos 41^\circ 22'$$
$$= 14.86 \text{ km}$$

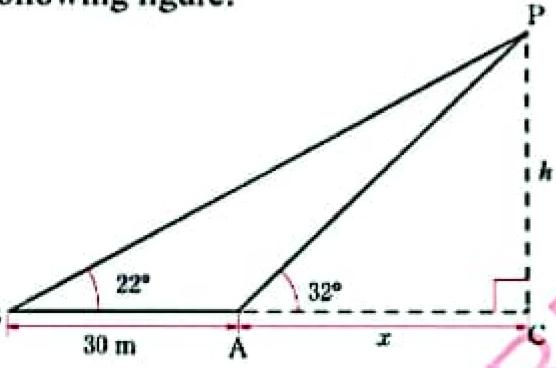
Therefore, Mr. Petro travelled 13.09 km west of P and 14.86 km north of P.

Example 8.26

From point A, Fatuma observes the angle of elevation of the top of a church tower to be 32°. Moving 30 m further away to point B on the same horizontal level as the bottom of the tower C, she observes the angle of elevation to be 22°. Find the distance $\overline{AC}$ and the height of the tower.

Solution

Present the information as shown in the following figure.



Let h be the height of the tower and x be the distance $\overline{AC}$ as shown in the figure. It follows that,

$$\tan 32° = \frac{h}{x} \text{ and } \tan 22° = \frac{h}{x+30}$$

Thus, $h = x \tan 32°$ (i)

Similarly,

$$h = (x + 30) \tan 22°$$

$$h = x \tan 22° + 30 \tan 22° \quad \text{(ii)}$$

Equating equations (i) and (ii) gives,

$$x \tan 32° = x \tan 22° + 30 \tan 22°$$

$$x (\tan 32° - \tan 22°) = 30 \tan 22°$$

$$x = \frac{30 \tan 22°}{\tan 32° - \tan 22°}$$

$$= \frac{30 \times 0.4040}{0.6249 - 0.4040}$$

$$= \frac{12.12}{0.2209}$$

$$= 54.88$$

Substituting the value of x in equation (i) gives,

$$h = 54.88 \times \tan 32°$$

$$= 54.88 \times 0.6249$$

$$= 34.29$$

Therefore, the distance $\overline{AC}$ is 54.88 m and the height of the tower is 34.29 m.

Exercise 8.6

1. A man whose eye is 180 cm above the ground is standing 8 m from a tree 7 m tall. What is the angle of elevation of the top of the tree from his eye?
2. The length of the shadow of a 16 m tree is 8 m. What is the size of the angle of elevation of the sun?
3. Find the height of a tower if the angle of elevation of the top at a point 20 m from its foot is 34°.
4. A tree casts a 60 m shadow when the angle of elevation of the Sun is 25°. How tall is the tree?
5. The angle of elevation of the top of a tree from a point on the ground 30 m from the base of the tree is 37°. Find the height of the tree.
6. From the top of a cliff 80 m high, two boats are seen in a direction due west. Find the distance between the boats if their angles of depression

from the top of the cliff are 45° and 30°. Find the actual distance of the boat which is furthest from the top of the cliff.

7. The angle of elevation of the top of a building 24 m high is observed from the top and from the bottom of a vertical ladder, and found to be 45° and 60°, respectively. Find the height of the ladder.

8. From point P on the level ground, the angle of elevation of the top of a flag post is 60°. If the height of the flag post is 39 m, how far from the base is point P?

9. From a balcony of a building 10 m high, Mwaka observes a person moving away from the base of the building at an angle of depression 5°42′. After 5 minutes, the angle of depression was 2°20′. Calculate the speed of the man in m/min.

10. From the top of a building 20 m high, a man watches people walking along the street. If the angle of depression of the foot of a passer-by is 50°, how far is the passer-by from the foot of the building?

11. The angle of elevation of the top of a tower from a point on the ground 70 m from the foot of the tower is 60°. Calculate the height of the tower.

12. Sarah is looking at a stone on the ground from an upstairs window. The level of Sarah's eyes above the ground is 5.2 m. The stone is 2.5 m horizontally from the wall of the house. Calculate the angle of depression of the stone from Sarah's eyes.

13. Musa stood at the top of a vertical cliff 60 m high and observed a boat at sea level at an angle of depression of 40°. Calculate the distance of the boat from the base of the cliff.

14. A crime investigation officer discovered a bullet embedded in the wall 3 m above the floor. It was also found that the bullet had entered the wall at an angle 13°25'. How far from the wall was the bullet fired if the gun was held 1.5 m above the floor?

15. Masanja is building a wooden ramp to allow people who use wheelchairs easier access to the public library. The ramp must be 3 metres tall. Find the angle of elevation if the ramp begins 30 m away from the library.

16. At a point 182 m from the foot of a tower on a level road, the angle of elevation of the top of the tower is 36°44′. Find the height of the tower.

17. From the top of a cliff 35 m high, the angles of depression of two boats lying in a line due east of the cliff are 27° and 23°. Find the distance between the boats.

18. The angle of depression of a boat from a cliff 25 m high is 12°. Find the distance of the boat from the bottom of the cliff.

Chapter summary

1. Trigonometric ratios

$$\sin A = \frac{\text{Length of opposite side to angle A}}{\text{Length of hypotenuse side}}$$

$$\cos A = \frac{\text{Length of adjacent side to angle A}}{\text{Length of hypotenuse side}}$$

$$\tan A = \frac{\text{Length of opposite side to angle A}}{\text{Length of adjacent side to angle A}}$$

2. Trigonometric ratios for special angles:

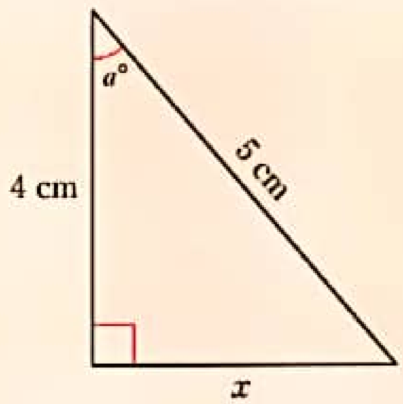
x	$\sin x$	$\cos x$	$\tan x$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined
180°	0	−1	0
270°	−1	0	undefined
360°	0	1	0

3. Relationship between trigonometric ratios for angle ∝.
   (a) $\sin\alpha = \cos(90° - \alpha)$
   (b) $\cos\alpha = \sin(90° - \alpha)$
   (c) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
   (d) $\sin^2\alpha + \cos^2\alpha = 1$
   (e) $\sin(-\alpha) = -\sin\alpha$
   (f) $\cos(-\alpha) = \cos\alpha$
   (g) $\tan(-\alpha) = -\tan\alpha$

4. Angle of elevation and depression:
   (a) Angle of elevation is the angle obtained when the line of sight is above a horizontal line.
   (b) Angle of depression is the angle is obtained when the line of sight is below a horizontal line.

Revision exercise 8

1. Use a calculator to find each of the following:
   (a) sin 60°
   (b) cos 28° 28′
   (c) tan 63° 48′
2. Use a calculator to find the value of y in each of the following:
   (a) $\tan y = 0.9036$
   (b) $\cos y = 0.2554$
   (c) $\sin y = 0.4971$
3. In the following figure, find the values of a and x.



4. A pendulum bob is hanging at the end of a string which is 18 cm long. Find the vertical height through which the bob rises and falls as the pendulum swings through 30° in each side of the vertical.

5. When the angle of elevation of the sun is 30°, the shadow of a post is 6 m longer than when it is 60°. Find the height of a post.

6. The angle of elevation of the top of a church tower from a point due East and 96 m away from its base is 30°. From another point due West of the church tower the angle of elevation of the top is 60°. Find the distance of the later point from the base of the church tower.

7. A point A is 289 m from point C on a bearing N32°W, point B is 450 m from point C on a bearing N58°E. Find the distance from A to B.

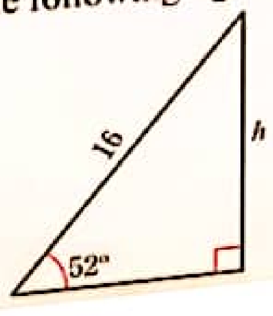
8. When the angle of elevation of the sun is 55°, a tower casts a shadow 20 m long. Find the height of the tower.

9. A kite is flying directly over a straight path 100 m long. The angle of elevation of the kite from one end of the path is 35°, if the angle of elevation of the kite from the other end of the path is 55°, how high is the kite?

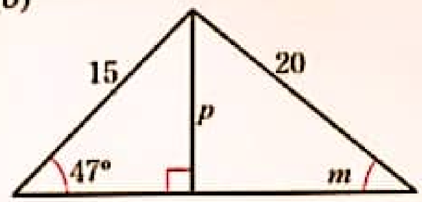
10. A boy finds that the angle of elevation of the top of a tree from a point on the ground is 25°. He walks in a straight line 30 m closer to the foot of the tree. The angle of elevation of the top is now 50°. How high is the tree?

11. Find the values of the letters in each of the following figures.

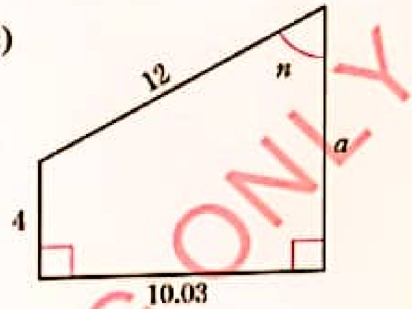
(a)



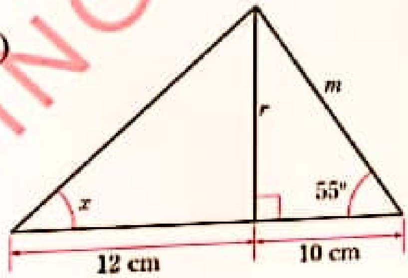
(b)



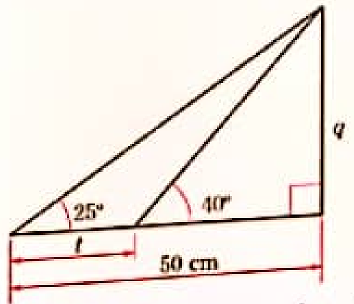
(c)



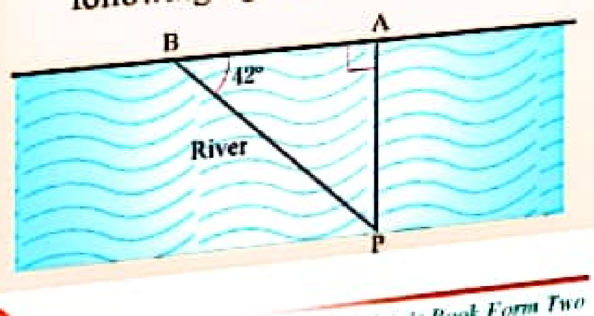
(d)



(e)



12. An observer is at a point A on the bank of a river. The foot of a coconut tree is at point P directly on the opposite bank. A distance AB of 27 m is measured along the bank so that $B\hat{A}P$ is a right angle and $A\hat{B}P = 42°$, as shown in the following figure.



Find:

(a) The width of the river.

(b) The height of the coconut tree if the angle of elevation of the top of the coconut tree is 22° from A.

(c) The angle of elevation of the top of the coconut tree from B.

(d) The distance from B to the top of the coconut tree.

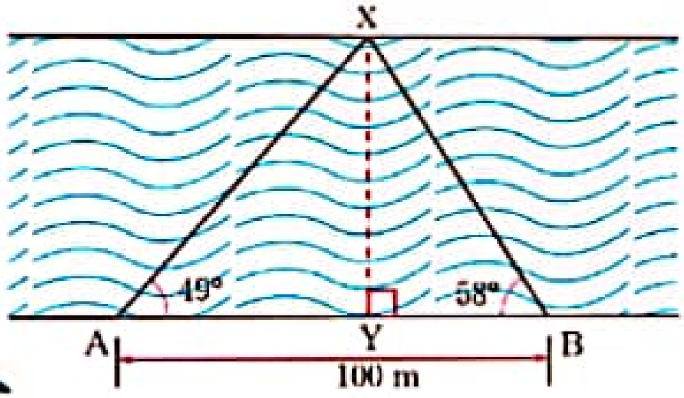
13. A ladder of length 12 m leans against a vertical wall.

(a) If it makes an angle of 28° with the wall, how far up the wall does it reach?

(b) If it reaches 10 m up the wall, what angle does it make with the horizontal?

14. Juma notices that the angle of elevation of the top of a mango tree is 32°. Walking 11 metres in a direction towards the tree he observes that the angle of elevation is 45°. Find the height of the tree.

15. Anna wishes to find the width $\overline{XY}$ of the river as shown in the following figure. She measures a distance $\overline{AB}$ = 100 m along the bank of the river. She observes that a point X on the other bank of the river is such that $X\hat{A}B = 49°$ and $X\hat{B}A = 58°$.



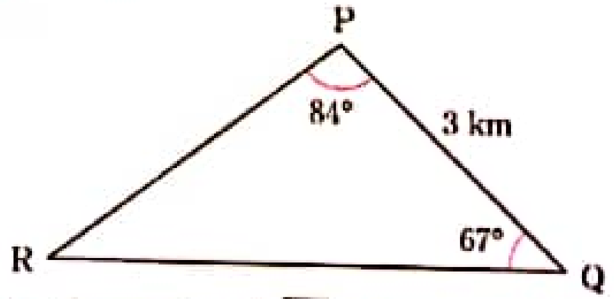
What width of the river did Anna obtain?

16. Two boats, A and B are due south of a cliff. The two boats are 890 m apart. The angles of elevation at the top of the cliff of the two points A and B are 26°14′ and 17°56′, respectively. Find the height of the cliff.

17. A boat at point R is observed from two points P and Q which are 3 km apart as shown in the following figure. If $P\hat{Q}R = 84°$ and $P\hat{Q}R = 67°$, find:

(a) $\overline{PR}$

(b) $\overline{QR}$

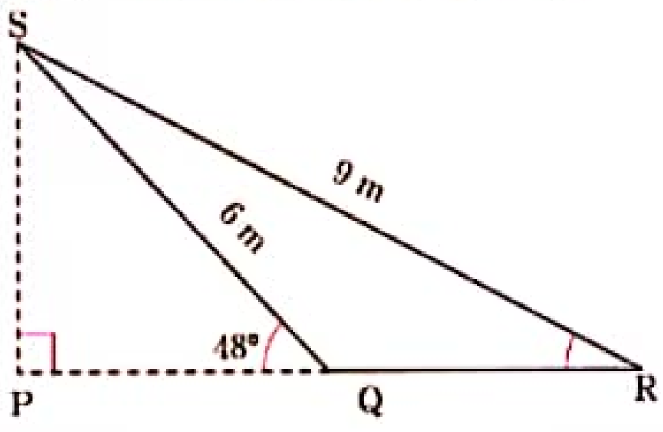


18. A railway signal $\overline{SP}$ is supported by two chains $\overline{SQ}$ and $\overline{SR}$ of length 6 m and 9 m, respectively as shown in the following figure. If the angle of elevation of the top of the signal from point Q is 48°, find:

(a) $S\hat{R}Q$.

(b) The distance QR.

(c) The height of the signal SP.



19. Evaluate each of the following expressions without using a calculator:

(a) $\dfrac{\tan 60° \sin 30°}{\sin 45°}$

(b) $\dfrac{\sin 30° \cos 60°}{\tan 30°}$

(c) $\dfrac{\sin 45° \tan 60°}{\cos 30°}$

(d) $2 \cos 135° - \sin 30°$

(e) $\tan(-300°) - \tan 120°$

(f) $\sin 225° + \cos 45°$

20. If α and β are complementary angles and $\sin \alpha = \dfrac{\sqrt{3}}{5}$, find the value of:

(a) $\cos \alpha$ (b) $\tan \beta$

21. Using a calculator, find the value of each of the following:

(a) $\sin 192°$ (d) $\sin(-15°)$

(b) $\cos 224°$ (e) $\cos(-129°)$

(c) $\tan 321°$ (f) $\tan(-310°)$

22. Find the angles between $0°$ and $360°$ which satisfy each of the following equations:

(a) $\sin \theta = -0.2468$

(b) $\cos \theta = 0.3579$

(c) $\tan \alpha = -2.356$.

23. Find the angles between $-360°$ and $360°$ which satisfy each of the following equations:

(a) $\sin \theta = 0.1234$

(b) $\cos \theta = -0.5678$

(c) $\tan \theta = 0.3546$.

24. If P is the point $(-3, 8)$, find the sine, cosine, and tangent of the obtuse angle between $\overline{OP}$ and the x-axis, where O is the point at $(0, 0)$.

25. Express each of the following in terms of an acute angle:

(a) $\sin 238°$

(b) $\cos(-263°)$

(c) $\tan(-36°)$.

26. From a certain point X, Hamisi observes that the angle of elevation of the top of a tall building to be $40°$. Moving 50 m further away to a point Y on a level road, he notices the angle of elevation to be $29°$. Find:

(a) The distance of Y from the bottom of the building.

(b) The height of the building.

27. The angles of elevation of a balloon from two points, A and B that are 0.3 km apart are $62°$ and $48°$, respectively, as shown in the following figure. If the balloon is vertically above the line AB, find its distance above the line AB.