Basic Mathematics

for Secondary Schools

Student's Book

Form Four

THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION,
SCIENCE AND TECHNOLOGY

Certificate of Approval

Title of Publication: [illegible]
Publisher: [illegible]
Author: [illegible]
ISBN: [illegible]

[illegible]

Tanzania Institute of Education

Published 2021

ISBN: 978 - 9987 - 09 - 289 - 5

Tanzania Institute of Education
P. O. Box 35094
Dar es Salaam – Tanzania

Mobile numbers: +255 735 041 170/ +255 735 041 168

E-mail: director.general@tie.go.tz
Website: www.tie.go.tz

Table of Contents

Acknowledgements

Tanzania Institute of Education (TIE) would like to acknowledge the contributions of all the organisations and individuals who participated in designing and developing this textbook. In particular, TIE wishes to thank the University of Dar es Salaam (UDSM), the University of Dodoma (UDOM), Sokoine University of Agriculture (SUA), Ardhi University (ARU), school quality assurance offices, teachers colleges, and secondary schools.

Besides, the following categories of individuals are also acknowledged:

Writers: Mr Kasabago A. Tresphory (TIE), Mr Elikana E. Manyilizu (MoEST) and Dr Ahmada O. Ali (ARU)

Editors: Dr Makungu S. Mwanzalima (UDSM), Dr Michael H. Mkwizu (SUA), Dr Aziza J. Iddi (SUA), Dr Alfred K. Hugo (UDOM), Dr Jairos K. Shinzeh (UDOM), Dr Emmanuel M. Deogratias (UDOM) and Dr Monica G. Kung'aro (UDOM)

Designer: Ms Eva G. Kwai

Illustrator: Mr Godlove S. Kyando (TIE)

Coordinator: Mr Kasabago A. Tresphory (TIE)

TIE also appreciates the contribution of secondary school teachers and students who participated in the trial phase of the manuscript. Likewise, the Institute would like to thank the Ministry of Education, Science and Technology for facilitating the writing and printing of this textbook.

Dr Aneth A. Komba
Director General
Tanzania Institute of Education

Preface

This book, *Basic Mathematics for Secondary Schools* is written specifically for Form Four students in the United Republic of Tanzania. It is prepared in accordance with the 2005 Basic Mathematics Syllabus for Secondary Schools, Form I – IV, issued by the then Ministry of Education and Vocational Training.

The book consists of eight chapters, namely Coordinate geometry, Areas and perimeters, Three – dimensional – figures, Probability, Trigonometry, Vectors, Matrices and transformations and Linear programming. Each chapter comprises of activities, project, and exercises. You are encouraged to do all the activities, exercises, and project together with any other assignment provided by your teacher. Doing so, will promote the development of the intended competencies.

Tanzania Institute of Education

Chapter One

Coordinate geometry

Introduction

Coordinate geometry is the study of geometry which describes the link between geometry and algebra through graphs involving curves and lines. It is the part of geometry where the position of points on the plane is located by using coordinates. In this chapter, you will learn how to locate the coordinate of a point on the xy – plane, find the gradient of a straight line, derive a general form of linear equations, solve problems involving linear equations, determine the coordinates of a midpoint of a line segment, and solve problems involving midpoints. You will also learn how to derive a distance formula, calculate the distance between two points on a plane, and solve problems involving parallel and perpendicular lines. The competencies developed will be applied in various real life situations such as land surveying, map projection, location of air transport, forecasting, in building constructions, human resource management, farming, and many other applications.

Gradient of a straight line

Activity 1.1: Naming the figure, quadrants, and finding gradient of the straight line

Materials: Graph papers, pencils, rulers, and marker pens

In a group or individually, perform this activity using the following steps:

1. Given two points A(2, 5) and B(4, 5), locate the points on the xy – plane. Then join the two points.
2. Draw a vertical line which is perpendicular bisector to line AB, then, name point C in the first quadrant at any point on the drawn vertical line.
3. Draw the straight lines joining points A and C, and B and C.
   (a) What is the name of a figure formed?
   (b) In which quadrant does the figure drawn belong?
4. Using the same xy – plane, locate the points D(1, –2), E(–8, –7), and F(–8, –2).
5. Join the points given in step 4.
   (a) What is the name of the figure formed?
   (b) In which quadrant does the figure drawn belong?

6. Give a conjecture between the figure formed in 3(a) and 5(a).
7. Using the figure formed in 5(a) find the gradient of the straight line joining points D and E.
8. Share your findings with your neighbours through discussion for more inputs.

Consider the straight line joining the points A(x_1, y_1) and B(x_2, y_2) as shown in Figure 1.1

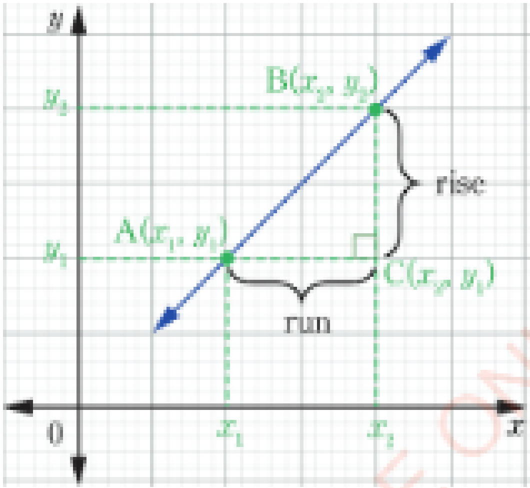


Figure 1.1: *The straight line joining two points*

The movement from point A to point B is equivalent to the horizontal movement from point A to point C followed by the vertical movement from point C to point B.

The vertical movement is often referred to as a vertical change (or rise), and the horizontal movement referred as a horizontal change (or run). The steepness of the straight line is called the gradient (or slope). It is the ratio of the vertical change (or rise) to the horizontal change (or run).

Therefore, from Figure 1.1, the gradient (or slope) m of a straight line AB is the ratio of the vertical change to the horizontal change, that is:

$$m = \frac{\text{Vertical change}}{\text{Horizontal change}} = \frac{\text{change in } y - \text{coordinates}}{\text{change in } x - \text{coordinates}} = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Example 1.1

Find the gradient of the straight line joining the points A(4, 8) and B(5, –2).

Solution

Let, $(x_1, y_1) = (4, 8)$ and $(x_2, y_2) = (5, -2)$

Using the formula for the gradient of a straight line;

$$m = \frac{\text{change in } y - \text{coordinates}}{\text{change in } x - \text{coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - 8}{5 - 4}$$

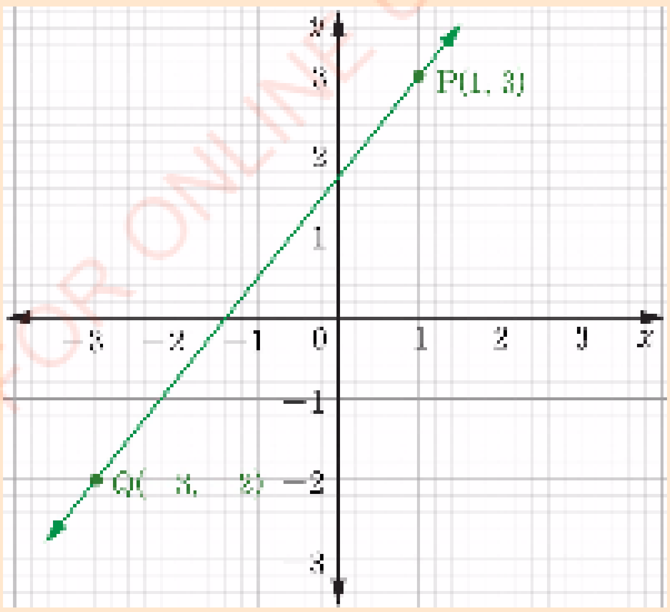
$$= -\frac{10}{1}$$

$$= -10$$

Therefore, the gradient of the straight line AB is –10.

Example 1.2

Calculate the gradient of the line PQ in the following figure.



Solution

From the given figure, choose the points P(1, 3) and Q(–3, –2):

Let, $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (-3, -2)$

Using the formula for the gradient of a straight line;

$$m = \frac{\text{change in } y - \text{coordinates}}{\text{change in } x - \text{coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - 3}{-3 - 1} = \frac{-5}{-4}$$

$$m = \frac{5}{4}$$

Therefore, the gradient of the line PQ is $\frac{5}{4}$.

Equation of a straight line

Suppose that two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on a straight line. Then, we want to know the existing relationship between x and y coordinates of a point $N(x, y)$ such that a straight line from point $N(x, y)$ joins $\overline{PQ}$. If $x_1 \neq x_2$, then, point N will join line $\overline{PQ}$ to form straight line only if the gradient of $\overline{PN}$ is the same as the gradient of $\overline{PQ}$ (See Figure 1.2).

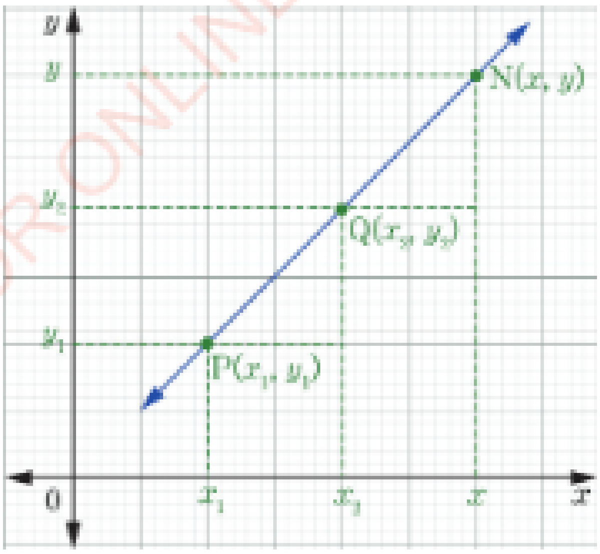


Figure 1.2: *The straight line joining the points*

The gradient of $\overline{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = m$ (1)

The gradient of $\overline{PN} = \frac{y - y_1}{x - x_1} = m$ (2)

Equations (1) and (2) gives the same results.

From equation (2) $y - y_1 = m(x - x_1)$

Making y the subject gives,

$y = m(x - x_1) + y_1$

On opening brackets, results to $y = mx - mx_1 + y_1$

On introducing the brackets, results to $y = mx + (y_1 - mx_1)$

If we let $c = y_1 - mx_1$, then $y = mx + c$. Here, m is the gradient and c is the y – intercept. Similarly, using equation (1), the same results can be obtained.

The x – intercept is found by setting $y = 0$ and solving for x in the equation

$y = mx + c$.

$0 = mx + c$ which gives,

$x = -\frac{c}{m}$.

Thus, the x – intercept is $-\frac{c}{m}$.

Therefore, the equation of a straight line joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $y = m(x - x_1) + y_1$ or $y = mx + c$.

Example 1.3

Find the equation of the straight line which passes through the point (5, –2) whose gradient is $\frac{3}{2}$.

Solution

Let, P(x, y) be any point on the straight line.

Let, $(x_1, y_1) = (5, -2)$ be a particular point on a straight line, and $m = \frac{3}{2}$.

Using the formula for gradient of a straight line;

$$m = \frac{y - y_1}{x - x_1}$$

$$\frac{3}{2} = \frac{y-(-2)}{x-5}$$

$$\frac{3}{2} = \frac{y+2}{x-5}$$

Cross multiplication gives;

$$2(y+2) = 3(x-5)$$

Opening brackets gives;

$$2y + 4 = 3x - 15$$

$$2y = 3x - 15 - 4$$

$$2y = 3x - 19$$

$$\frac{2y}{2} = \frac{3x}{2} - \frac{19}{2}$$

Therefore, the equation of a straight line is $y = \frac{3x}{2} - \frac{19}{2}$.

Alternative solution

Let, $(x_1, y_1) = (5, -2)$ and $m = \frac{3}{2}$.

Using the formula for the equation of a straight line;

$$y = m(x - x_1) + y_1$$

On substituting the values gives;

$$y = \frac{3}{2}(x-5) + (-2)$$

$$y = \frac{3x}{2} - \frac{15}{2} - 2$$

$$y = \frac{3x}{2} - \frac{19}{2}$$

Therefore, the equation of a straight line is $y = \frac{3x}{2} - \frac{19}{2}$.

Example 1.4

Verify that $y - 7 = 3(x - 3)$ represents an equation of the straight line through the point (3,7), then, determine its gradient and y – intercept.

Solution

Check whether the equation $y - 7 = 3(x - 3)$ passes through the point (3, 7).

Substituting the point (3, 7) into the equation

$y - 7 = 3(x - 3)$ gives; $7 - 7 = 3(3 - 3)$

$$0 = 0$$

Therefore, the given equation passes through the point (3, 7).

From $y - 7 = 3(x - 3)$

$$y = 3(x - 3) + 7$$

$$y = 3x - 2$$

Comparing $y = 3x - 2$ with $y = mx + c$ gives $m = 3$ and $c = -2$.

Therefore, the gradient of the straight line is 3 and its y – intercept is –2.

Example 1.5

Find the equation of a straight line joining the points A(1, 6) and B(5, 9).

Solution

Let, P(x, y) be any other point on the straight line joining points A and B.

Let, $(x_1, y_1) = (1, 6)$ and $(x_2, y_2) = (5, 9)$.

Using the formula for the gradient of a straight line;

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{9 - 6}{5 - 1}$$

$$m = \frac{3}{4}$$

Then, using a point P(x, y) and the point $(x_1, y_1) = (1, 6)$ the equation of the straight line is given by $m = \frac{y - y_1}{x - x_1}$

$$\frac{3}{4} = \frac{y - 6}{x - 1}$$

$$y - 6 = \frac{3}{4}(x - 1)$$

$$y = \frac{3}{4}(x - 1) + 6$$

$$= \frac{3}{4}x - \frac{3}{4} + 6$$

$$= \frac{3}{4}x + \frac{21}{4}$$

Therefore, the equation of the straight line is $y = \frac{3}{4}x + \frac{21}{4}$.

The general equation of the straight line

Consider the equation of the straight line joining the pair of points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be any other point on the straight line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } m = \frac{y - y_1}{x - x_1}$$

Therefore,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1};$$

which is the required equation or $y = (x - x_1)\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1$

or $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$

or $y(x_2 - x_1) - y_1(x_2 - x_1) = x(y_2 - y_1) - x_1(y_2 - y_1)$

or $-(y_2 - y_1)x + (x_2 - x_1)y + x_1(y_2 - y_1) - y_1(x_2 - x_1) = 0$

Since x_1, y_1, x_2 and y_2 are given real numbers, then $-(y_2 - y_1)$, $(x_2 - x_1)$, and $x_1(y_2 - y_1) - y_1(x_2 - x_1)$ are real numbers too. They can be replaced by constants a, b, and c respectively. Then the equation becomes: $ax + by + c = 0$ where $a = -(y_2 - y_1)$, $b = x_2 - x_1$, and $c = x_1(y_2 - y_1) - y_1(x_2 - x_1)$.

Therefore, the general equation of the straight line is $ax + by + c = 0$.

Example 1.6

Sketch the graph of the straight line presented by $3x - 4y - 24 = 0$.

Solution

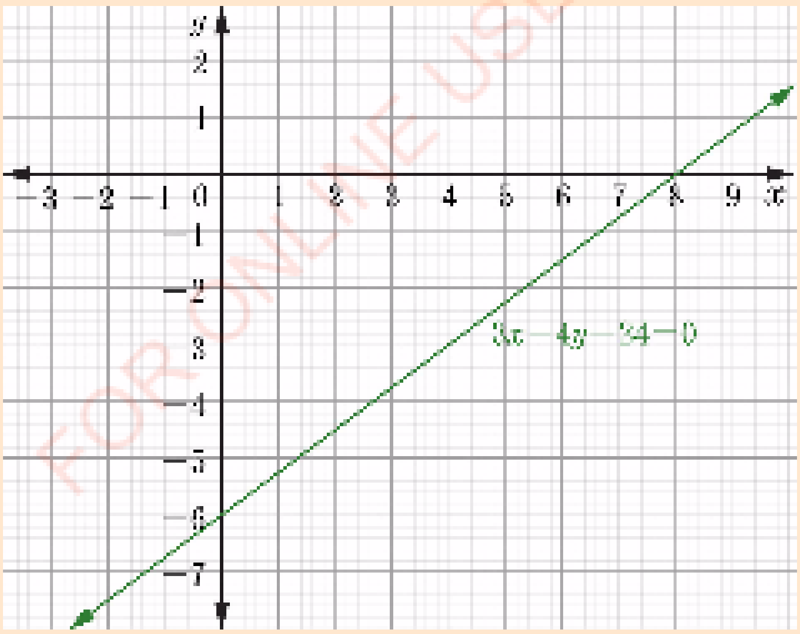
The x – intercept can be found by setting $y = 0$ in the equation. That is;

$$3x - 4(0) - 24 = 0$$
$$3x - 24 = 0$$
$$3x = 24$$
$$x = 8$$

The y – intercept can be found by setting $x = 0$ in the equation. That is;

$$3(0) - 4y - 24 = 0$$
$$-4y - 24 = 0$$
$$-4y = 24$$
$$y = -6$$

Thus, the coordinate of intercepts are (8, 0) and (0, –6). The graph is as shown in the following figure.



Gradient for the parallel lines to the y – axis

When values of x do not change; that is $x = x_1 = k$ where k is a constant, then, $x - x_1 = 0$.

Therefore, $m = \frac{y - y_1}{0}$ is not defined. In this case, the gradient is undefined.

Geometrically, lines which have undefined gradients are parallel to the y – axis.

The equations of the lines are determined by the x – intercept. If the x – intercept is 4, then the equation of the line is $x = 4$.

In general, the equation of a line whose gradient is undefined is $x = k$, where k is the x – intercept. For example, Figure 1.3 shows graphs for the equations $x = 1$, $x = 2$, and $x = 3$.

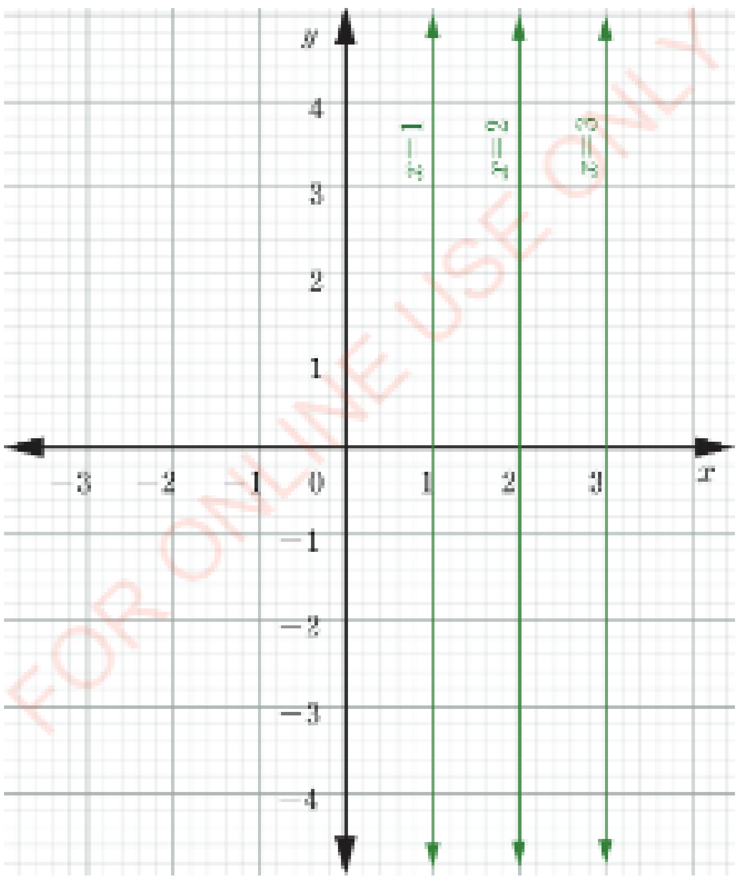


Figure 1.3: *Graph of lines parallel to the y – axis*

Gradient for the parallel lines to the x – axis

Similarly, the values of y corresponding to values of x, may not vary; so that $y = y_1 = c$ where c is a constant, then $y - y_1 = 0$.

Therefore, $m = \frac{0}{x - x_1}$. The gradient of the line is zero.

Geometrically, lines which have gradient zero are parallel to the x – axis and their equations are determined by the y–intercept.

If the y–intercept is 3, the equation of the line is $y = 3$.

In general, the equation of a line whose gradient is zero is $y = c$ where c is the y – intercept.

Figure 1.4 shows graphs for the equations $y = 1$ and $y = -2$.

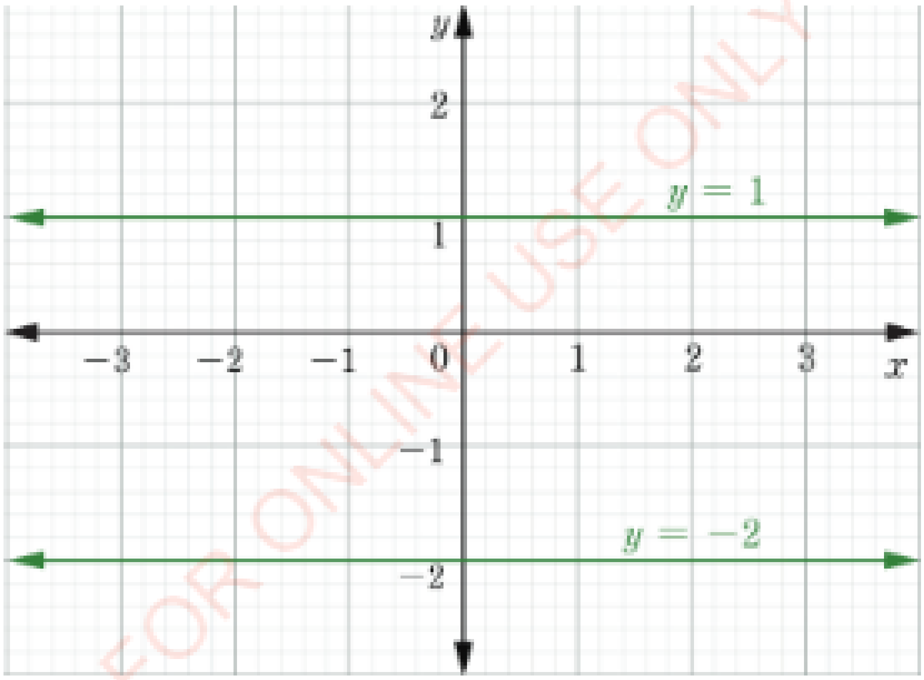


Figure 1.4: *Graph of Lines parallel to the x – axis*

Exercise 1.1

Answer the following questions:

1. Find the gradient of the straight lines joining each of the following pairs of points:
   (a) (1, 6) and (5, 7)
   (b) (3, 2) and (7, –2)
   (c) (–3, 4) and (8, 1)
   (d) (–1, –4) and (4, –3)
   (e) $\left(\frac{1}{2}, 2\right)$ and $\left(3, \frac{1}{3}\right)$
   (f) $\left(-2\frac{1}{4}, 7\right)$ and $\left(8, -2\frac{1}{2}\right)$

2. Find the equation of the straight line whose gradient is 2 and passes through the point (3, 5).

3. For each of the following conditions, find the equation of the straight line in the form of $y = mx + c$:
   (a) Passing through the point (4, 7) having gradient 3.
   (b) Passing through the points (4, 7) and (–2, –5).
   (c) Passing through the point A(4, –3) whose gradient is $\frac{2}{5}$.

4. Verify that the points (–2, 2) and (–6, 0) lie on the line joining points A(–4, 1) and B (2, 4).

5. Find the equation of the straight line in the form of $ax + by + c = 0$ joining each of the following pairs of points:
   (a) (1, 6) and (5, 9)
   (b) (3, 2) and (7, –2)
   (c) (–3, 4) and (8, 1)
   (d) (–1, –4) and (4, –3)
   (e) $\left(\frac{1}{2}, 2\right)$ and $\left(3, \frac{1}{3}\right)$
   (f) $\left(-2\frac{1}{4}, 7\right)$ and $\left(8, -2\frac{1}{2}\right)$

6. Find the equation of each of the following straight lines in the form: $ax + by + c = 0$.
   (a) The straight line joining the points (2, 4) and (–3, 1).
   (b) The straight line through the point (3, 1) with gradient $-\frac{3}{5}$.
   (c) The straight line through the point (3, –4) and which has the same gradient as the straight line $5x - 2y = 3$.

7. Determine the value of k in order that the straight line whose equation is $kx - y = 5$ passes through the point (3, 5).

8. What should be the value of t to allow the straight line represented by the equation $3x - ty = 16$ to pass through the point (5, –4)?

9. Draw the graph of the straight line represented by each of the following equations:
   (a) $14x + 7y = 25$
   (b) $3y = 4x - 12$
   (c) $15 = 3x - 5y$

10. Plot the graph of the straight line whose equation is $4x + y + 6 = 0$.

11. Find the x and y – intercepts of the straight line $3x - 2y + 10 = 0$.

12. Find the equation of a straight line with gradient $\frac{2}{3}$ and having the same y – intercept as the straight line $2x - 5y + 20 = 0$.

13. Determine the values of m and c so that the straight line $y = mx + c$ will pass through the points $(-1, 4)$ and $(3, 5)$.

14. The points $(4, -2)$ and $(-2, -5)$ lie on the straight line $\frac{x}{a} + \frac{y}{b} = 1$. Find the values of a and b.

15. Determine the equation of the straight line whose gradient is $\frac{3}{5}$ and passes through the intersection of the straight lines $3x - 2y = 2$ and $2x - 3y = 3$.

16. Find the equations of the straight lines subject to each of the following conditions:
   (a) $m = -\frac{5}{2}$, $c = 4$
   (b) $m = 0$, $c = -1$
   (c) gradient $= -\frac{6}{5}$ and passing through the point $(2, 2)$.

17. Find the equation of the straight line through the point $(0, 5)$ with gradient zero.

18. Find the equation of the line through the point $(2, 0)$ with undefined gradient.

19. Determine the gradient and the y – intercept for each of the following lines:
   (a) $3x - y = 12$
   (b) $7x - 2y + 3 = 0$
   (c) $ax + by + c = 0$
   (d) $\frac{x}{a} + \frac{y}{b} = 1$

20. Find the equation of the straight line joining the points $(0, 4)$ and $(3, 0)$.

FOR ONLINE USE ONLY
DO NOT DUPLICATE

The midpoint of a line segment

Activity 1.2: Finding the midpoint of a line segment

Materials: Graph papers, ruler, pencils, and eraser

In a group or individually, perform this activity by using the following steps:

1. Locate points A(5, 5) and B(–1, 5) on the graph paper. Draw $\overline{AB}$.
2. Hold the paper up to the line and fold the paper so that points A and B match exactly. Crease the paper slightly.
3. Open the paper and put a point where the crease intersect $\overline{AB}$. Label this midpoint C.
4. Repeat the first three steps using points X(–3, 3) and Y(1, 7).

   Label the midpoint Z. Then, answer the following questions:

   (a) find the coordinates of the point C.

   (b) find the coordinates of the point Z.

   (c) study the coordinates of points A, B, and C. Write a rule that relates these coordinates and then, use points X, Y, and Z to verify your answers.
5. Share your findings with your neighbour for more inputs.

The midpoint of the line segment is a point which divides the line segment into two equal parts. Let P be a point with coordinates (x_1, y_1) and R with coordinates (x_2, y_2). It is required to find coordinates of the point Q (x, y), where Q is the midpoint of $\overline{PR}$ (See Figure 1.5).

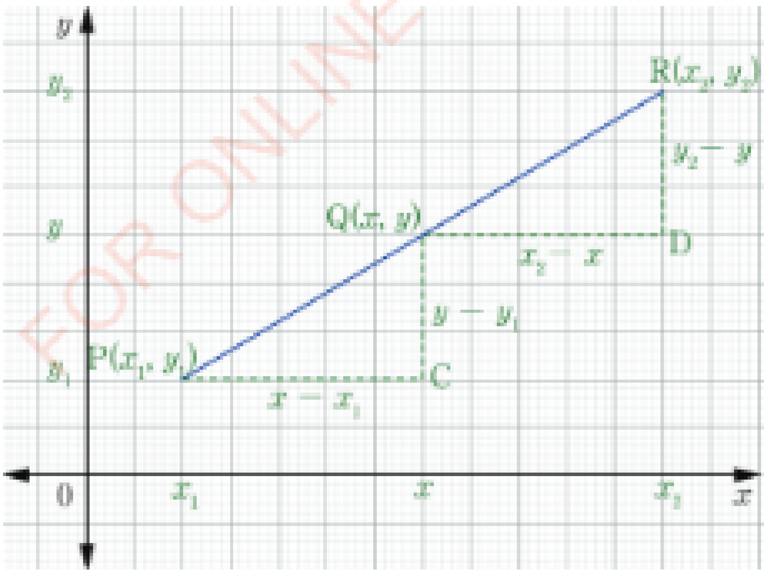


Figure 1.5: *The midpoint of a straight line*

Considering angles of ΔPCQ and ΔQDR.

$\Delta PCQ \sim \Delta QDR$ (RHS – Similarity Theorem)

Thus, $\frac{\overline{PC}}{\overline{QD}} = \frac{\overline{PQ}}{\overline{QR}} = \frac{\overline{QC}}{\overline{RD}}$

Using $\frac{\overline{PC}}{\overline{QD}} = \frac{\overline{PQ}}{\overline{QR}}$ gives $\frac{x - x_1}{x_2 - x} = \frac{\overline{PQ}}{\overline{QR}}$

But, Q is the midpoint of $\overline{PR}$, then $\overline{PQ} = \overline{QR}$

Then, $\frac{\overline{PC}}{\overline{QD}} = \frac{\overline{PQ}}{\overline{PQ}}$, $\frac{\overline{PC}}{\overline{QD}} = 1$

$\frac{x - x_1}{x_2 - x} = \frac{1}{1}$

or $x - x_1 = x_2 - x$

$2x = x_1 + x_2$

Hence, $x = \frac{x_1 + x_2}{2}$.

Using $\frac{\overline{PQ}}{\overline{QR}} = \frac{\overline{QC}}{\overline{RD}}$ gives $\frac{1}{1} = \frac{y - y_1}{y_2 - y}$

or $y_2 - y = y - y_1$

$y_1 + y_2 = 2y$

Hence, $y = \frac{y_1 + y_2}{2}$.

Therefore, the coordinates of Q are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Generally, the midpoint formula of the line segment joining the points $P(x_1, y_1)$ and $R(x_2, y_2)$ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 1.7

Find the coordinates of the midpoint of the line joining the points (–4, 7) and (1, –18).

Solution

The midpoint formula is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ where $(x_1, y_1) = (-4, 7)$

and $(x_2, y_2) = (1, -18)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-4+1}{2}, \frac{7+(-18)}{2}\right)$$

$$= \left(-\frac{3}{2}, -\frac{11}{2}\right)$$

Therefore, the coordinates of the midpoint are $\left(-\frac{3}{2}, -\frac{11}{2}\right)$.

Example 1.8

Find the equation of the straight line in the form of $ax+by+c=0$ passing through the point (5, 3) and the midpoint of the line joining the points (–2, –4) and (–6, –8).

Solution

The midpoint formula is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ where;

$(x_1, y_1) = (-2, -4)$ and $(x_2, y_2) = (-6, -8)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-2-6}{2}, \frac{-4-8}{2}\right)$$

$$= (-4, -6)$$

The gradient of the straight line joining points (–4, –6) and (5, 3)

is $m = \frac{3-(-6)}{5-(-4)} = \frac{9}{9} = 1$.

The equation of the straight line passing through the point (5, 3) is $\frac{y-3}{x-5} = 1$

$y-3 = x-5$

$x-y-2 = 0$.

Therefore, the equation of a straight line is $x-y-2=0$.

Exercise 1.2

Answer the following questions:

1. Find the coordinates of the midpoint joining each of the following pairs of points:
   (a) (8, 1) and (2, 5)
   (b) (0, 0) and (12, 3)
   (c) (–3, 5) and (9, 1)
   (d) (–2, 8) and –4, –2)
   (e) (0, 3) and (5, –2)
   (f) $\left(-2\frac{1}{4}, 7\right)$ and $\left(8, -2\frac{1}{2}\right)$

2. If the line from the point $(-4, y_1)$ to the point $(x_2, -3)$ is bisected at the point (1, –1), find the values of y_1 and x_2.

3. The midpoint of a line segment is (–2, 7) and one of its end point is (1, 7). Find the other end point.

4. The midpoint of the sides of a triangle are (2, 0), $\left(4, -3\frac{1}{2}\right)$, and $\left(6, \frac{1}{2}\right)$. Find the vertices of the triangle if one of them is (4, 3).

5. Given that the vertices of the triangle are (–4, 2), (–10, –6), and (2, –3). Find the coordinates of the midpoints of each side.

6. A parallelogram has vertices A(–2, 4), B(3, 1), C(6, 4), and D(1, 7). Find the coordinates of the midpoint of the diagonals. Hence, show that the diagonals bisect each other.

7. Three vertices of a parallelogram ABCD are A(–1, 3), B(2, 7), and C(5, –7). Find the coordinates of vertex D using the principle that the diagonals bisect each other.

8. Show that the lines joining the midpoint of the opposite sides of the quadrilateral ABCD with vertices A(5, –2), B(3, 4), C(–2, 3), and D(–5, –7) bisect each other.

9. Find the equation of the line through the point (2, 3) and the midpoint of (1, 2) and (5, 2).

Distance between two points on a plane

Activity 1.3: Measuring the distance between two points

Materials: Rope, meter ruler, pencil and writing pad.

In a group or individually, perform the following tasks:

1. Measure and record the distance between the edges of the blackboard on top, at the centre and at the bottom.
2. Give the conjecture about the answers obtained in task 1.
3. Share your findings with your friends for more inputs.

In Figure 1.6, suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points so that the distance $\overline{PQ}$ is to be determined in terms of x_1, y_1, x_2, and y_2.

Figure 1.6: *Distance between points on the xy – plane*

Join $\overline{PQ}$ and draw dotted lines to join $\overline{AP}$ and $\overline{AQ}$ as shown in Figure 1.6 and then, $\overline{PA} = x_2 - x_1$ and $\overline{QA} = y_2 - y_1$. ΔPAQ is a right – angled triangle.

Applying Pythagoras' theorem to ΔPAQ gives:

$$(\overline{PQ})^2 = (\overline{PA})^2 + (\overline{AQ})^2$$

or $$(\overline{PQ})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore, $\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Generally, the distance (d) between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the plane is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 1.9

Find the distance between the points (–1, 7) and (4, –5).

Solution

Taking $(-1, 7)$ as (x_1, y_1) and $(4, -5)$ as (x_2, y_2), the required distance d is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(4 - (-1))^2 + (-5 - 7)^2}$$

$$= \sqrt{(4 + 1)^2 + (-5 - 7)^2}$$

$$= \sqrt{(5)^2 + (-12)^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

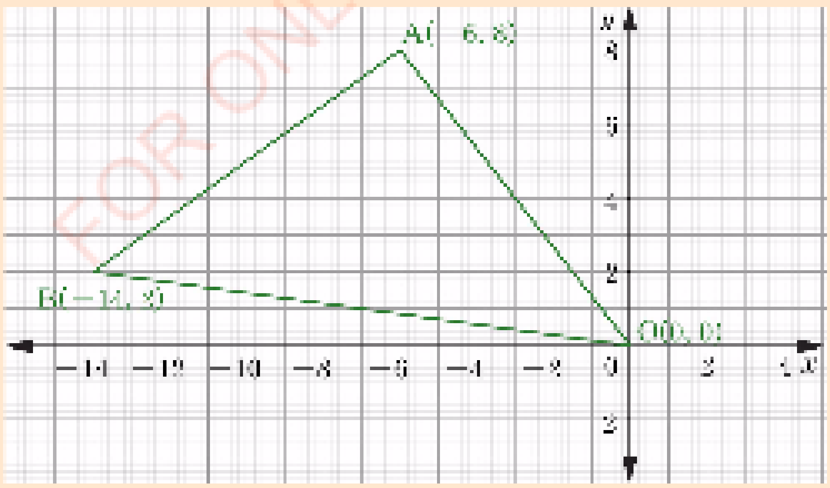
$$= 13.$$

Therefore, the distance between the points is 13 units.

Example 1.10

Determine whether the triangle with vertices O(0, 0), A(–6, 8), and B(–14, 2) is equilateral, isosceles or scalene.

Solution



Let A, B, and O represent the vertices as shown in the figure.

$$\overline{AO} = \sqrt{(-6-0)^2 + (8-0)^2} \text{ units}$$

$$= \sqrt{36+64} = 10 \text{ units}$$

$$\overline{BO} = \sqrt{(-14-0)^2 + (2-0)^2} \text{ units}$$

$$= \sqrt{196+4} = 10\sqrt{2} \text{ units}$$

$$\overline{AB} = \sqrt{(-14-(-6))^2 + (2-8)^2} \text{ units}$$

$$= \sqrt{64+36} = 10 \text{ units}$$

Thus, $\overline{AB} = \overline{AO} = 10$ units.

Therefore, the triangle with the vertices (0, 0), (–6, 8), and (–14, 2) is an isosceles triangle because two sides are equal.

Example 1.11

Find the coordinates of the point P on the y – axis, which is the same distance from points Q(3, 4) and R(1, –2).

Solution

Since P is a point on the y – axis, its coordinates are of the form (0, y) (refer the figure that follows)

$$\overline{QP} = \sqrt{(0-3)^2 + (y-4)^2}$$

$$\left(\overline{QP}\right)^2 = (-3)^2 + (y-4)^2$$

$$= 9 + y^2 - 8y + 16$$

$$= y^2 - 8y + 25$$

$$\left(\overline{RP}\right)^2 = (0-1)^2 + (y-(-2))^2$$

$$= 1 + (y+2)^2$$

$$= 1 + y^2 + 4y + 4$$

$$= y^2 + 4y + 5$$

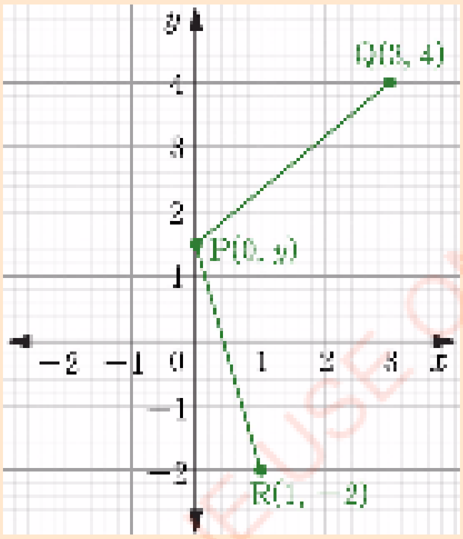
Since, $\overline{QP} = \overline{RP}$, then $(\overline{QP})^2 = (\overline{RP})^2$

or $y^2 - 8y + 25 = y^2 + 4y + 5$

$$y^2 - 8y = 5 - 25 + y^2 + 4y$$

$$-12y = -20$$

$$y = \frac{-20}{-12} = \frac{5}{3} = 1\frac{2}{3}$$



Therefore, the coordinates of the point P are $\left(0, 1\frac{2}{3}\right)$.

Exercise 1.3

Answer the following questions:

1. Find the distance of the line segment joining each of the following pairs of points:
   (a) $(3,2)$ and $(8,14)$
   (b) $(1,3)$ and $(4, 7)$
   (c) (p, q) and (r, s)
   (d) $(1, 2)$ and $(5, 2)$
   (e) $(-2, -3)$ and $(-6, -9)$
   (f) $\left(-2\frac{1}{4}, -11\right)$ and $\left(8, -2\frac{1}{2}\right)$
2. Find the distance of the point $(-15, 8)$ from the origin.
3. The points P, Q, and R have coordinates $(5, -3)$, $(-6, 1)$, and $(1, 8)$ respectively. Show that PQR is an isosceles triangle.
4. By using the distance formula determine whether the triangles with the following vertices are isosceles, equilateral, scalene or right angled:
   (a) $(1,0)$, $(-1,0)$, $(8,14)$
   (b) $(4,5)$, $(0, -2)$, $(-3,1)$
   (c) $(2, -2)$, $(1, 5)$, $(-1, -1)$
   (d) $(-3, -2)$, $(-1, 6)$, $(2, 1)$
5. The distance between the points $(-3, 5)$ and $(x, 2)$ is 5. Find the value of x.
6. The line joining the points $(2, 6)$ and $(6, 2)$ is the base of an isosceles triangle. If the x–coordinate of the vertex is -3, find its y–coordinate.
7. Find the coordinates of the point on the y – axis which is equidistant from the points $(8, 4)$ and $(6, 6)$.
8. Find the coordinates of the point on the x – axis which is equidistant from the points $(-6, 3)$ and $(1, 4)$.
9. Find the coordinates of the vertex D of the quadrilateral A$(0, 4)$, B$(-4, 7)$, C$(-7, 3)$, and D if $\overline{AD} = \overline{BC}$ and D is on the x – axis.
10. Show that each of the following vertices form a right–angled triangle. Hence, determine its area.
    (a) $(4, 4)$, $(-, 1)$, and $(2, -4)$
    (b) $(10, 5)$, $(3, 2)$, and $(6, -5)$

FOR ONLINE USE ONLY
DO NOT DUPLICATE

Parallel and perpendicular lines

Parallel lines

Two lines which do not meet when produced infinitely are called parallel lines and the angle between them is zero.

Suppose in Figure 1.7, L_1 and L_2 are parallel lines. Then, from any point C on L_1, CED is drawn perpendicular to x – axis so that it cuts L_2 at point E and x – axis at point D.

Since ΔBDE and ΔADC are equiangular, then

$\Delta BDE \sim \Delta ADC$ (RHS – Similarity Theorem)

Then, $\frac{\overline{CD}}{\overline{ED}} = \frac{\overline{AD}}{\overline{BD}}$

Cross multiplication gives;

$\overline{CD}.\overline{BD} = \overline{AD}.\overline{ED}$

Re – arrangement gives;

$\frac{\overline{CD}}{\overline{AD}} = \frac{\overline{ED}}{\overline{BD}}$

But, $\frac{\overline{CD}}{\overline{AD}}$ = The gradient of $L_1 = m_1$

And $\frac{\overline{ED}}{\overline{BD}}$ = The gradient of $L_2 = m_2$

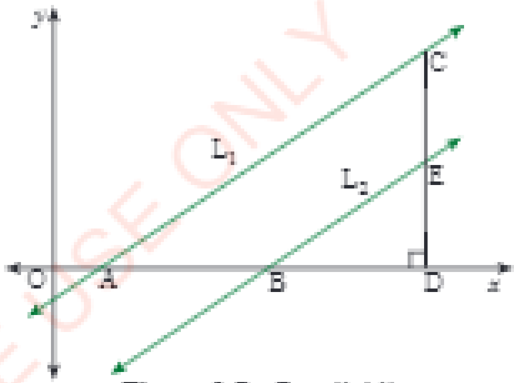


Figure 1.7: *Parallel lines*

Thus, the gradient of L_1 = the gradient of L_2 or $m_1 = m_2$.

Therefore, if two lines are parallel, they have the same gradient. The statement is also true, when two distinct lines have the same gradient, then they are parallel.

Example 1.12

Find the value of x so that the points A(1, 3), B(–2, –3), and C(x, 7) lie on the same straight line.

Solution

Because points A, B, and C are collinear, it implies that $\overline{BA}$ and $\overline{BC}$ have the same gradient.

The gradient of $\overline{BA} = \frac{3-(-3)}{1-(-2)} = \frac{6}{3} = 2$

The gradient of $\overline{BC} = \frac{7-(-3)}{x-(-2)} = \frac{10}{x+2}$

Then, the gradient of $\overline{BC}$ = the gradient of $\overline{BA}$

$$\frac{10}{x+2} = 2$$

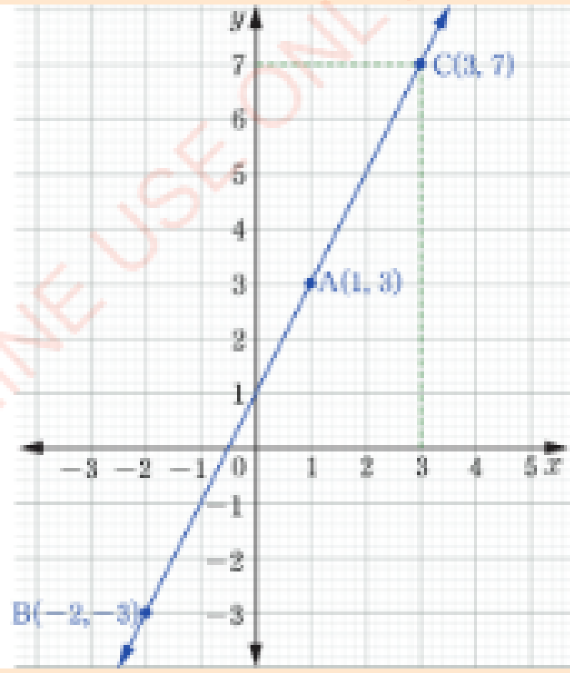
$$10 = 2(x+2)$$

$$2x = 10 - 4$$

$$2x = 6$$

$$x = 3$$

Therefore, the value of x is 3.



Example 1.13

A line through the point (3, 1) is parallel to the line $16x - 4y = 23$. Find the equation of the line.

Solution

Find the gradient of the line:

$16x - 4y = 23$

Make y the subject:

$$\frac{-4y}{-4} = \frac{-16x}{-4} + \frac{23}{-4}$$

$$y = 4x - \frac{23}{4}$$

Parallel lines have equal gradients.

Let the gradient of the first line be m_1 and of the other line be m_2 respectively,

then, $m_1 = m_2$ which gives;

$m_1 = 4$, $m_2 = 4$.

Now, using the formula $m = \dfrac{y - y_1}{x - x_1}$

where $(x_1, y_1) = (3, 1)$, $m = 4$

$$4 = \frac{y - 1}{x - 3}$$

cross multiplication gives;

$$y - 1 = 4(x - 3)$$
$$y - 1 = 4x - 12$$
$$y = 4x - 12 + 1$$
$$y = 4x - 11$$

Therefore, the equation of the line is $y = 4x - 11$.

Exercise 1.4

Answer the following questions:

1. Determine whether $\overline{AB}$ is parallel to $\overline{PQ}$ in each of the following cases:
   (a) A(6, –4), B(8, 4), P(7, 1), Q(6, 5)
   (b) A(–2, 0), B(1, 0), P(2, 5), Q(6, 17)
   (c) A(8, –5), B(11, –3), P(1, 1), Q(–3, 7)
   (d) A(3, 1), B(7, 3), P(–3, 2), Q(1, 0)
2. Show that A(–3, 1), B(1, 2), C(0, –1) and D(–4, 2) are vertices of a parallelogram.
3. Find the equation of the straight line through the point (6, 2) and parallel to the line whose equation is $x + 3y - 13 = 0$.
4. One vertex of a parallelogram is (5, 5) and the equations of its two sides are $2x + 3y = 7$ and $x - 3y + 4 = 0$. Find the equations of the other two sides and coordinates of the vertices.

5. Find the equation of a straight line which passes through point A(–3, 4) and is parallel to the line with equation $3x + 4y - 15 = 0$.

6. Show that the following equations represent the sides of a parallelogram:

   $3x + 4y - 12 = 0,$

   $2x - 5y - 10 = 0,$

   $6x + 8y + 13 = 0,$ and

   $6x - 15y + 25 = 0.$

7. State the conditions that the line represented by $Ax + By + C = 0$ should:
   (a) pass through the origin.
   (b) be parallel to the x – axis.
   (c) be parallel to the y – axis.
   (d) cut off equal intercepts on the axes

8. The midpoints of the adjacent sides of the quadrilateral with vertices A(5, –2), B(3, 4), C(–2, 3) and D(0, –3) are joined. By comparing gradients of the resulting figure, give the geometrical identification of the figure.

9. Find the coordinates of the point on the x – axis so that the line joining it and the point (5, –2) is parallel to a line whose gradient is $\frac{3}{4}$.

10. Write in a simplified form the condition that the line joining points A(3, 2) and P(x, y) is parallel to the line joining the points B(–1, –3) and C(4, 7).

11. Find the value of y so that the points A(3, y), B(0, 4), and C(–2, 8) are collinear.

12. Find the value of x so that the straight line joining the points (x, –5) and (2, 1) is parallel to the straight line joining the points (1, 5) and (–2, –1).

Perpendicular lines

Two lines are perpendicular if they intersect at right angle. Suppose that two lines L_1 and L_2 with slopes m_1 and m_2 respectively, are perpendicular to each other as shown in Figure 1.8.

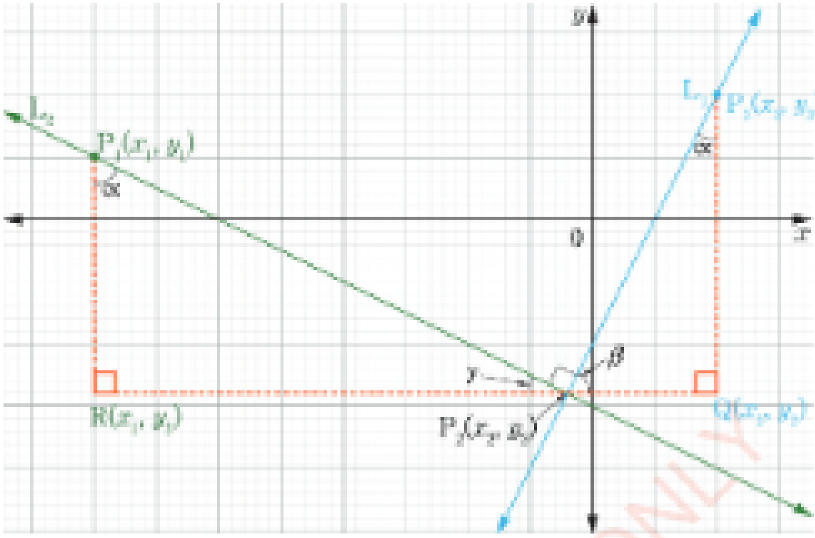


Figure 1.8: *The graph of perpendicular lines*

Choose points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $R(x_1, y_2)$, and $Q(x_3, y_2)$. Also α, β, and γ are the Greek letters for alpha, beta, and gamma respectively which represents the degree measures of the angles. Then:

$\alpha + \beta = 90°$ (complementary $\angle$s)

$\alpha + \gamma = 90°$ (complementary $\angle$s)

Thus, $\beta = \gamma$.

Hence, $\Delta P_2QP_3 \sim \Delta P_1RP_2$

$$\frac{\overline{P_2Q}}{\overline{QP_3}} = \frac{\overline{P_1R}}{\overline{RP_2}}$$

$$\frac{x_3 - x_2}{y_3 - y_2} = \frac{y_1 - y_2}{x_1 - x_2} \qquad (1)$$

But, gradient of $L_1 = m_1 = \frac{y_3 - y_2}{x_3 - x_2}$ (2)

And the gradient of $L_2 = m_2 = \frac{y_1 - y_2}{x_1 - x_2}$ (3)

Then, substituting (2) and (3) in (1) gives,

$$\frac{1}{m_1} = -m_2 \text{ or } 1 = -m_1m_2$$

Hence, $m_1m_2 = -1$

Therefore, if two lines with gradients m_1 and m_2 are perpendicular then $m_1m_2 = -1$.

Example 1.14

Find the equation of the straight line through the point $P(-2, 5)$ and perpendicular to the line whose equation is $6x - 7y = 4$.

Solution

The equation of the straight line $6x - 7y = 4$ can be written as $y = \frac{6}{7}x - \frac{4}{7}$.

The gradient of the line is $\frac{6}{7}$.

Thus, $m_1 = \frac{6}{7}$

since, $m_1 m_2 = -1$

then, $\frac{6}{7}m_2 = -1$

$m_2 = -\frac{7}{6}$.

Since the required line passes through the point $(-2, 5)$, its equation is:

$$\frac{y-5}{x-(-2)} = -\frac{7}{6}$$

$$\frac{y-5}{x+2} = -\frac{7}{6}$$

Cross multiplication gives;

$6y - 30 = -7x - 14$

$6y + 7x = 16$

or $7x + 6y - 16 = 0$

Therefore, the equation of a straight line is $7x + 6y = 16$.

Example 1.15

Show that $A(-3, 2)$, $B(5, 6)$, and $C(7, 2)$ are vertices of a right – angled triangle.

Solution

Gradient of $\overline{AB} = \frac{6-2}{5+3} = \frac{4}{8} = \frac{1}{2}$ Gradient of $\overline{BC} = \frac{2-6}{7-5} = \frac{-4}{2} = -2$

Since the gradients of $\overline{AB}$ and $\overline{BC}$ are negative reciprocals and their product gives -1, then, triangle ABC is right–angled at B.

Therefore, $A(-3, 2)$, $B(5, 6)$, and $C(7, 2)$ are vertices of a right – angled triangle.

Exercise 1.5

Answer the following questions:

1. State the gradient of the line which is perpendicular to each of the straight lines with gradients:
   (a) $\frac{3}{4}$ (b) -5
   (c) 3 (d) $-\frac{A}{B}$

2. Find the coordinates of a point having its x–coordinate -2 so that the line joining it to the point $(-3, 1)$ is perpendicular to the line through the point $(2, 7)$ and $(5, -1)$.

3. For each of the following right–angled triangles having vertices at A, B, and C, determine two sides that are perpendicular to each other.
   (a) $A(3, 2)$, $B(5, -4)$, and $C(1, -2)$
   (b) $A(7, 8)$, $B(7, 2)$, and $C(2, 2)$
   (c) $A(0, 9)$, $B(-4, -1)$, and $C(3, 2)$.

4. Find the equation of the straight line passing through the point $(6, 2)$ and perpendicular to the line joining the points $P(3, -1)$ and $Q(-2, 1)$.

5. Find the equation of the straight line which is a perpendicular bisector to the line joining the points $(2, 0)$ and $(6, 0)$.

6. Find the coordinates of the foot of the perpendicular line from the point $(4, -2)$ to the line $2x - 3y - 4 = 0$.

7. Show that $(4, 7)$, $(0, 4)$, $(3, 0)$, and $(7, 0)$ are the vertices of a square.

8. Find the equation of a line perpendicular to $3x - 11y - 4 = 0$ and passing through the point $(-3, 3)$.

9. Find the equation of the line which passes through the point $A(1, 7)$ and is perpendicular to the line joining the points $B(4, 4)$ and $C(-2, 3)$.

10. Find the coordinates of the foot of the perpendicular line from the point $(4, 3)$ to the line $8x + 6y = 25$.

Chapter Summary

1. Gradient or slope m of a line is the steepness of the line segment. It is the ratio of change in y–coodinates and change in x–coordinates for the given points $A(x_1, y_1)$ and $B(x_2, y_2)$.
   That is
   $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.
2. The midpoint formula of a line segment is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ for the given points $A(x_1, y_1)$ and $B(x_2, y_2)$.
3. The distance d between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the formula
   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
4. Parallel lines have the same gradient. That is $m_1 = m_2$ where m_1 is the gradient of line 1 and m_2 is the gradient of line 2.
5. The product of the gradients of two perpendicular lines is negative one. That is $m_1 m_2 = -1$ where m_1 is the gradient of line 1 and m_2 is the gradient of line 2.
6. The equation of a straight line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $y = m(x - x_1) + y_1$ or $y = mx + c$.
7. The general equation of the straight line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $ax + by + c = 0$.

Revision exercise 1

Answer the following questions:

1. In which quadrants are each of the following points located?
   (a) (2, 4) (b) (4, –2)
   (c) (–2, –3) (d) (–2, 1)
2. Find the equation of the straight line joining the points (6, 3) and (5, 8).
3. Show that the points (5, 8) and (6, 7) are equidistant from the point (–2, 0).
4. Find the gradient of each of the following pairs of points:
   (a) (3, 3) and (1, 4)
   (b) (–2, –3) and (–4, –6).
5. Find the equation of the straight line through the point (3, –2) which is:
   (a) parallel to the line whose equation is $2x + 5y = 17$.
   (b) perpendicular to the line whose equation is $2x + 5y = 17$.
6. Find the equations of the following straight lines, in the form: $ax + by + c = 0$.
   (a) The line joining the points (2, 4) and (–3, 1).
   (b) The line through the point (3, 1) parallel to the line whose equation is $3x + 5y = 6$.

(c) The line through the point $(3, -4)$ perpendicular to the line whose equation is $5x - 2y = 3$.

7. The points A, B, and C have coordinates $(7, 0)$, $(3, -3)$, and $(-3, 3)$ respectively. Find the coordinates of the point Q, R, and S of the midpoint of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively.

8. Prove that the four points $(4, 0)$, $(7, -3)$, $(-2, -2)$, and $(-5, 1)$ are the vertices of a parallelogram and find the equations of its diagonals.

9. Find the equation of the straight line which is parallel to the line whose equation is $x + 4y - 1 = 0$ and which passes through the point $(4, -3)$.

10. Find the distance of a straight line joining the points $(-2, 1)$ and $(6, 7)$.

11. A perpendicular line from the point $(-2, 1)$ intersects the line at the point $(-1, 3)$. Find its distance to the line.

12. Show that a quadrilateral with the points A$(1, -3)$, B$(6, -2)$, C$(3, 4)$, and D$(-2, 3)$ is a parallelogram.

13. The point $(5, -7)$ is the vertex of a right angle of a right – angled triangle whose hypotenuse lie along the line $6x - 13y = 39$. A second vertex of the triangle is $(0, -3)$. Find the remaining vertex.

14. Find the coordinates of the point on the line presented by the equation $3x - 3y + 7 = 0$ which is equidistant from the points $(-4, -8)$ and $(7, -1)$.

15. Find the coordinates of the point on the x – axis such that the line joining it to the point $(3, -1)$ forms a right angle with the line through the points $(3, -1)$ and $(-5, -5)$.

16. The vertices of a triangle are A$(2, 4)$, B$(-3, 1)$ and C$(5, -7)$. Find the coordinates of the foot of the perpendicular drawn from A to the line $\overline{BC}$.

17. The points A$(1, -1)$, B$(4, -1)$ and C$(1, 3)$ are the vertices of a right – angled triangle. If the right angle is at A, find the area of the triangle.

Project 1

1. Plane A is parallel to plane B, and plane B is parallel to plane C. Is plane A parallel to plane C? Explain your answer.
2. Describe objects in buildings around your environment that illustrates your response in question 1.

Chapter Two

Areas and perimeters

Introduction

Area is the surface occupied by a flat shape or surface of an object. Perimeter is the lengh of a closed shape. In this chapter, you will learn how to derive and apply the formula of the area of any triangle, rhombus, parallelogram, square, rectangle, and trapezium in solving problems. You will also learn how to derive and apply the formula of finding a length of a side of a regular polygon, find the perimeter and the area of a regular polygon, the circumference and the area of a circle from inscribed regular polygon. Furthermore, you will learn how to find the ratio of areas of similar polygons and solve problems related to ratio or similar polygons. The competencies developed will be applied in various fields such as in business, building constructions, cooking when the volume of containers for carrying liquids, solids or gaseous must be known. The competencies developed will also be applied in fencing off an area, building a swimming pool, and many other applications.

Areas of any triangle

Activity 2.1: Finding the area of a triangle

In a group or individually, perform the following tasks:

1. Use a graph paper to draw a right–angled triangle of any convenient size.
   (a) By counting squares inside the triangle, estimate the area of a triangle.
   (b) What is the length of its height?
   (c) What is the length of its base?
   (d) Calculate the area of the triangle.
2. Discuss with your friends how the surface areas and volumes of various things can be used in real life.

Consider a right – angled triangle with base b and height h as shown in Figure 2.1.

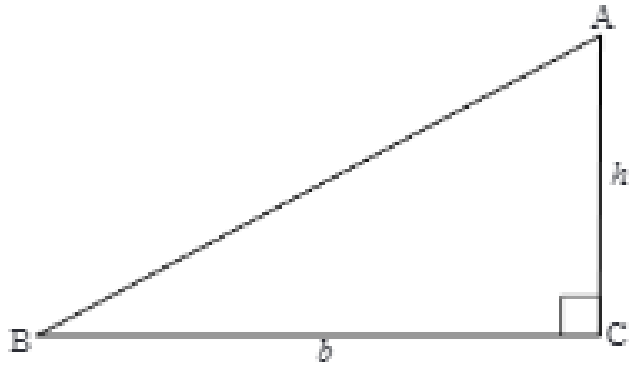


Figure 2.1: *The right – angled triangle*

The area of triangle ABC is given by $A = \frac{1}{2}bh$.

This is the formula for finding the area of any right – angled triangle with base b and height h.

Another type of a triangle is the one with altitude which lies within the triangle as shown in Figure 2.2.

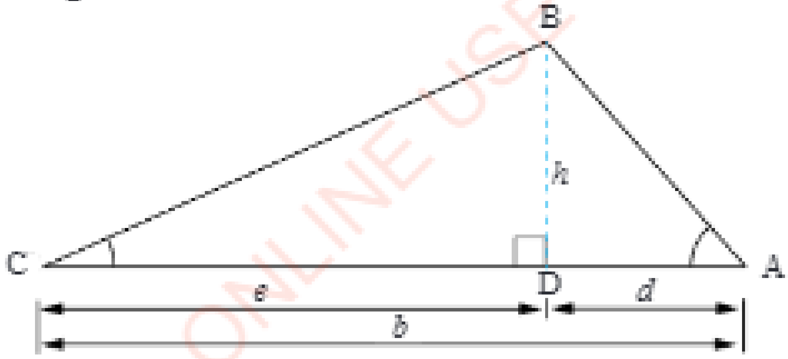


Figure 2.2: *Triangle with altitude lying within the triangle*

Area of ΔABC = Area of ΔBCD + Area of ΔADB

$$= \frac{1}{2}eh + \frac{1}{2}dh$$

$$= \frac{1}{2}h(e + d)$$

Therefore, the area of $\Delta ABC = \frac{1}{2}bh$, since $e + d = b$.

The case where the altitude of a triangle lies outside the triangle (Figure 2.3), the area can be found as follows:

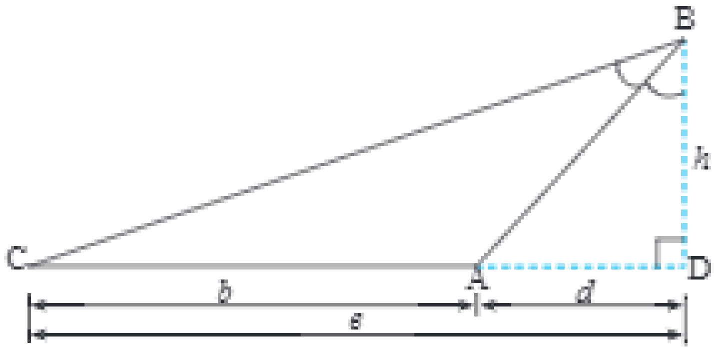


Figure 2.3: *Triangle with altitude lying outside the triangle*

$$\begin{aligned}\text{Area of } \Delta ABC &= \text{Area of } \Delta BCD - \text{Area of } \Delta ADB\\ &= \frac{1}{2}eh - \frac{1}{2}dh\\ &= \frac{1}{2}h(e-d)\\ &= \frac{1}{2}bh, \text{ since } e-d=b.\end{aligned}$$

Therefore, the area of $\Delta ABC = \frac{1}{2}bh$.

In each case the formula is the same.

Therefore, the formula for the area of a triangle with base b and corresponding height h is $\frac{1}{2}bh$.

The knowledge of trigonometrical ratios can also be used to find another formula for the area of any triangle. Two triangles with altitude lying within the triangle and outside the triangle respectively are shown in Figures 2.4 and 2.5.

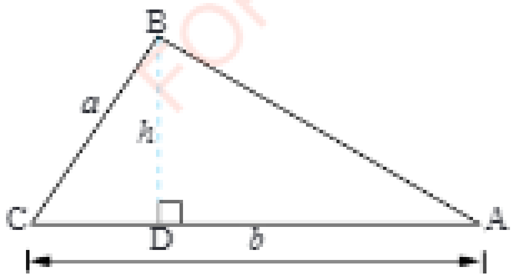


Figure 2.4: *Triangle with altitude lying within the triangle*

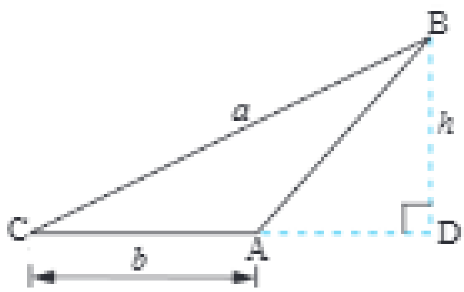


Figure 2.5: *Triangle with altitude lying outside the triangle*

Using Figure 2.4,

$\sin\hat{C} = \frac{h}{a}$; $h = a\sin\hat{C}$

Area of $\Delta ABC = \frac{1}{2}bh = \frac{1}{2}ba\sin\hat{C}$

Therefore, area of $\Delta ABC = \frac{1}{2}ab\sin\hat{C}$.

Similarly, using Figure 2.5, $\sin\hat{C} = \frac{h}{a}$; $h = a\sin\hat{C}$

Area of $\Delta ABC = \frac{1}{2}bh = \frac{1}{2}ba\sin\hat{C}$

Therefore, area of $\Delta ABC = \frac{1}{2}ab\sin\hat{C}$.

By using similar method, it can be shown that the area of triangle ABC with sides a, b, and c, (Figure 2.6) is given by:

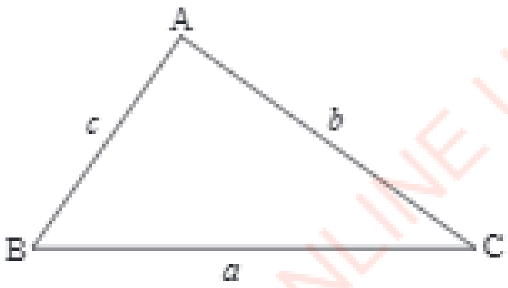


Figure 2.6: *A triangle with sides and angles*

Area of $\Delta ABC = \frac{1}{2}ac\sin\hat{B}$

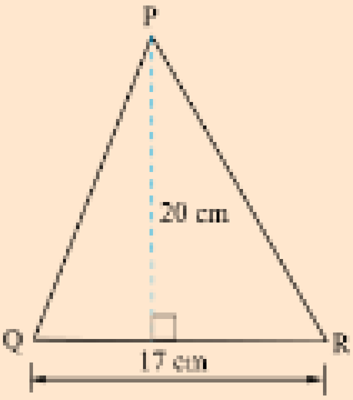
Area of $\Delta BCA = \frac{1}{2}ab\sin\hat{C}$

Area of $\Delta BAC = \frac{1}{2}bc\sin\hat{A}$

Generally, the area of a triangle is half the product of the lengths of two sides and the sine of the included angle.

Example 2.1

Calculate the area of a triangle shown in the following figure:



Solution

Let, height $(h) = 20$ cm

base $(b) = 17$ cm

The area of $\Delta PQR = \frac{1}{2}bh$

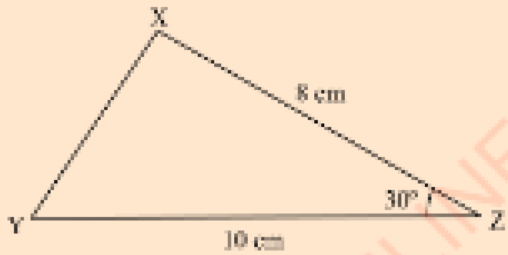
$= \frac{1}{2} \times 17 \text{ cm} \times 20 \text{ cm}$

$= 170 \text{ cm}^2$

Therefore, the area of $\Delta PQR = 170 \text{ cm}^2$.

Example 2.2

Calculate the area of a triangle shown in the following figure:



Solution

Let, $x = 10$ cm, $y = 8$ cm and $Y\hat{Z}X = 30°$

Area of $\Delta YZX = \frac{1}{2}xy \sin \hat{Z}$

$= \frac{1}{2} \times 10 \text{ cm} \times 8 \text{ cm} \times \sin 30°$

$= \frac{1}{2} \times 10 \times 8 \times \frac{1}{2} \text{ cm}^2$

$= 20 \text{ cm}^2$

Therefore, the area of $\Delta YZX = 20 \text{ cm}^2$.

Area of a trapezium

Activity 2.2: Estimating the area of a trapezium and a parallelogram

In a group or individually, perform the following tasks:

1. Draw and label trapezium ABCD on grid or graph paper. The bases and altitude of the trapezium can have any convenient measure of your choice. Draw the height and label it h. Label the bases as b_1 and b_2.
2. Draw another trapezium FGHI whose bases and height are correspondingly congruent to trapezium ABCD.
3. Arrange the trapeziums ABCD and FGHI so that two of the congruent legs are adjacent to each other. Then:
   (a) deduce the length of the parallelogram in terms of b_1 and b_2.
   (b) derive the area of trapezium ABCD.
   (c) derive the area of trapezium FGHI.
   (d) find the sum of areas of the trapeziums obtain in (b) and (c) above.
   (e) derive the area of parallelogram.
   (f) what is the relationship between the expressions for the area of trapeziums and parallelogram.
   (g) share your findings with your neighbours.

Figure 2.7 represents a trapezium ABCD where b_1 and b_2 are bases and h is the height.

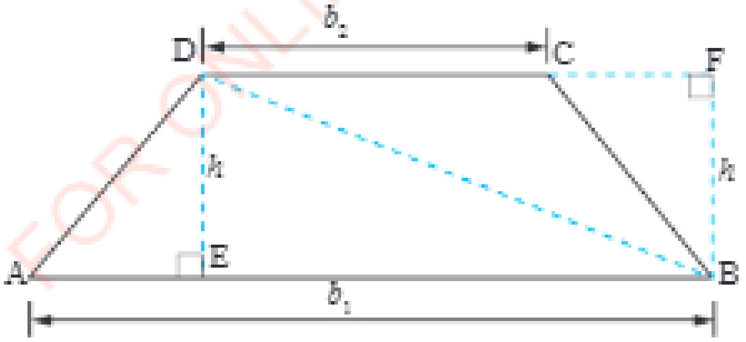


Figure 2.7: *A trapezium*

To find the area of the trapezium, first divide it into two triangles ABD and BCD with the same height h.

The bases are $\overline{AB} = b_1$ and $\overline{CD} = b_2$

The area of $\Delta ABD = \frac{1}{2} b_1 h$

The area of $\Delta BCD = \frac{1}{2} b_2 h$

The area of a trapezium ABCD = Area of ΔABD + Area of ΔBCD

$$= \frac{1}{2} b_1 h + \frac{1}{2} b_2 h$$

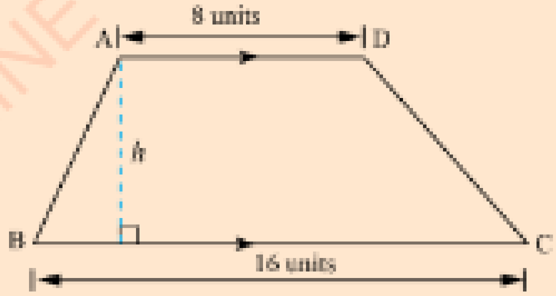
$$= \frac{1}{2} (b_1 + b_2) h$$

Therefore, the area of a trapezium ABCD = $\frac{1}{2}(b_1 + b_2)h$.

Generally, the area of a trapezium is equal to half the product of the sum of the lengths of parallel sides (bases) and the perpendicular distance between them (height).

Example 2.3

Calculate the height of a trapezium with area 84 square units and bases of 16 units and 8 units as shown in the following figure:



Solution

Let, b_1 = 16 units

b_2 = 8 units

A = 84 square units

h = ?

Area of trapezium $= \frac{1}{2}(b_1 + b_2)h$

$$84 = \frac{1}{2}(16 + 8)h$$

$12h = 84$, hence $h = 7$ units

Therefore, the height of the trapezium is 7 units.

Example 2.4

Given that the height of a trapezium is 16m and the length of one base is 28 m. Calculate the dimension of the other base if its area is 480 m².

Solution

Given $h = 16\text{m}$, $b_1 = 28\text{m}$, $A = 480\text{m}^2$ and $b_2 = ?$

Area of the trapezium $= \frac{1}{2}(b_1 + b_2)h$

$$480\text{ m}^2 = \frac{1}{2}(28\text{ m} + b_2) \times 16\text{ m}$$

$$480\text{ m}^2 = (28\text{m} + b_2) \times 8\text{ m}$$

$$28\text{ m} + b_2 = 60\text{ m}$$

$$b_2 = (60 - 28)\text{ m}$$

$$= 32\text{m}$$

Therefore, the dimension of the other base is 32 m.

Area of a parallelogram

The formula for the area of a parallelogram can be obtained from the formula for finding the area of a trapezium. The lengths of the bases for a parallelogram are equal. Consider the parallelogram ABCD shown in Figure 2.8.

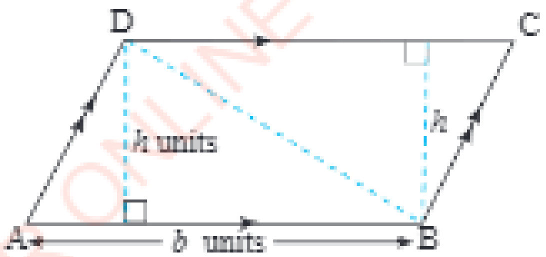


Figure 2.8: *Parallelogram with bases and perpendicular heights*

The area of a parallelogram ABCD $= \frac{1}{2}(\overline{AB} + \overline{DC})h$ square units.

Since, $\overline{AB} = \overline{DC} = b$ units, then

the area of a parallelogram ABCD $= \frac{1}{2}(2 \times b \text{ units})h$ units

$= bh$ square units

Hence, the area of parallelogram ABCD = bh square units.

Therefore, the area of a parallelogram is equal to the product of the length of the base and the perpendicular height.

Example 2.5

The base of the parallelogram is thrice its height. If the area is 675 cm², find:

(a) the height of the parallelogram.

(b) the base of the parallelogram.

Solution

Let, b and h denotes the base and height of a parallelogram respectively.

(a) Given that $b = 3h$ and $A = 675\text{ cm}^2$

Area of the parallelogram = base × height

$$A = bh$$
$$= 3h \times h$$
$$675\text{ cm}^2 = 3h^2$$
$$h^2 = \frac{675\text{ cm}^2}{3}$$
$$= 225\text{ cm}^2$$
$$h = 15\text{ cm}$$

Therefore, the height of the parallelogram is 15 cm.

(b) Given that $b = 3h$
$$= 3 \times 15\text{ cm}$$
$$= 45\text{ cm}$$

Therefore, the base of the parallelogram is 45 cm.

Area of a rhombus

The area of a rhombus can be obtained by using the formula for finding the area of a parallelogram because a rhombus is a special type of parallelogram whose sides are equal. Figure 2.9 shows a rhombus with base b and height h.

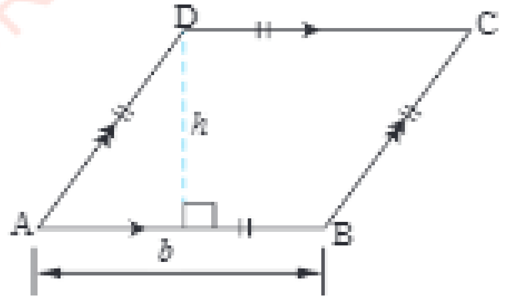


Figure 2.9: *The height and base of a rhombus*

The area of a rhombus ABCD = base × height.

Another formula for the area of a rhombus can be obtained by using the lengths of the diagonals. Consider the rhombus ABCD with diagonals $\overline{AC}$ and $\overline{BD}$ as shown in Figure 2.10.

Since the diagonals of a rhombus bisect each other at right angles, area of a rhombus ABCD can be found as follows:

Area of ΔABC = Area of ΔADC

$$= \frac{1}{2}\left(\overline{AC}\right) \times \frac{1}{2}\left(\overline{DB}\right)$$

Area of $\Delta ABC = \frac{1}{4}\left(\overline{AC} \times \overline{DB}\right)$

Area of a rombus ABCD

$$= (\text{Area of } \Delta ABC) \times 2$$

$$= \frac{1}{4}\left(\overline{AC} \times \overline{DB}\right) \times 2$$

$$= \frac{1}{2}\left(\overline{AC} \times \overline{DB}\right) \text{ square units}$$

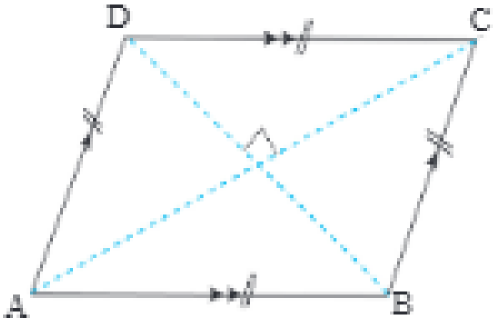


Figure 2.10: *A rhombus with bases and diagonals*

Therefore, the area of a rhombus is equal to half the product of the lengths of the diagonals.

Example 2.6

Find the area of a rhombus whose diagonals have lengths 13.5cm and 18.5 cm.

Solution

Let, l_1 = 13.5 cm and l_2 = 18.5 cm

The area of the rhombus = $\frac{1}{2}$ (product of the lengths of the diagonals)

$$= \frac{1}{2} l_1 l_2$$

$$= \frac{1}{2} \times 13.5 \text{ cm} \times 18.5 \text{ cm}$$

$$= 124.875 \text{ cm}^2$$

Therefore, the area of the rhombus is 124.875 cm².

Example 2.7

If the area of a rhombus is 125 cm^2 and its corresponding height is 12.5 cm, then calculate the corresponding base.

Solution

Given the area $A = 125\ cm^2$, the height $h = 12.5$ cm and the base $b = ?$

The area of the rhombus = base × height

$$125\ cm^2 = b \times 12.5\ cm$$

$$b = \frac{125\ cm^2}{12.5\ cm}$$

$$b = 10\ cm$$

Therefore, the corresponding base is 10 cm.

Area of a rectangle

In Figure 2.11, the rectangle PQRS is divided into two congruent triangles PQS and RSQ by the diagonal $\overline{SQ}$.

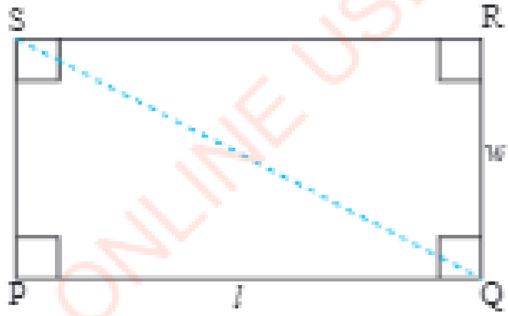


Figure 2.11: *A rectangle with its diagonal*

$$\text{Area of } \Delta PQS = \text{Area of } \Delta RSQ = \frac{1}{2}\left(\overline{PQ} \times \overline{PS}\right)$$

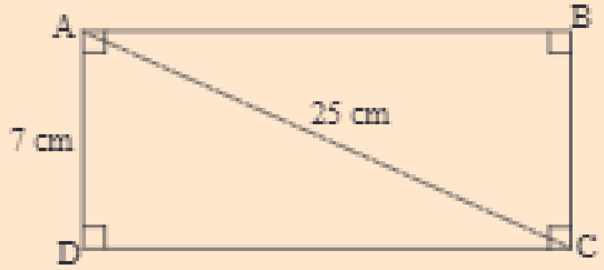
$$\text{Area of rectangle PQRS } = \frac{1}{2}\left(\overline{PQ} \times \overline{PS}\right) \times 2 = \overline{PQ} \times \overline{PS}.$$

If PQ is length l and PS is width w, then the area A is equal to the product of the length and width. That means:

$$A = lw.$$

Example 2.8

Calculate the area of a rectangle shown in the following figure:



Solution

The length $\overline{DC}$ of a rectangle can be found by using the Pythagoras theorem:

$$\left(\overline{AD}\right)^2 + \left(\overline{DC}\right)^2 = \left(\overline{AC}\right)^2$$

Given $\overline{AD} = 7$ cm and $\overline{AC} = 25$ cm

$$(7\text{ cm})^2 + \left(\overline{DC}\right)^2 = (25\text{ cm})^2$$

$$\left(\overline{DC}\right)^2 = (625 - 49)\text{ cm}^2$$

$$= 576\text{ cm}^2$$

$$\overline{DC} = 24\text{ cm}$$

The area of the rectangle = length × width

$= 24\text{ cm} \times 7\text{ cm}$

$= 168\text{ cm}^2$

Therefore, the area of the rectangle is 168 cm².

Area of a square

Since a square is a special rectangle in which all the sides are equal, its area can be determined by the formula for the area of a rectangle.

Therefore, area of a square is length times length or $A = l^2$ square units (since length = width = l units).

Alternatively, the area of a square can be determined using known lengths of diagonals. Consider the square ABCD with diagonals $\overline{AC}$ and $\overline{BD}$ as shown in Figure 2.12.

Since diagonals of a square bisect each other at right angles, area of triangle ABC is equal to area of triangle ADC.

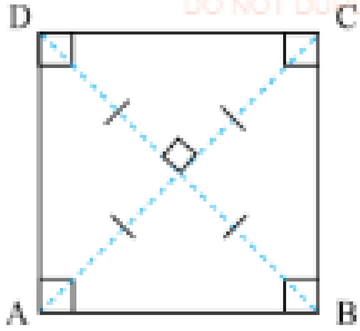


Figure 2.12: *Bisecting diagonals of a square*

$$\text{Area of } \Delta ABC = \text{Area of } \Delta ADC = \frac{1}{2}\left(\overline{AC}\right) \times \frac{1}{2}\left(\overline{DB}\right)$$

$$= \frac{1}{4}\left(\overline{AC} \times \overline{DB}\right)$$

$$\text{Area of a square ABCD} = 2 \times \frac{1}{4}\left(\overline{AC} \times \overline{DB}\right) = \frac{1}{2}\left(\overline{AC} \times \overline{DB}\right)$$

$$= \frac{1}{2}\left(\overline{AC}\right)^2, \text{ since } \overline{DB} = \overline{AC}.$$

Therefore, the area of a square is equal to half the product of the lengths of the diagonals, or the area of a square is half the length of a diagonal square.

Example 2.9

Calculate the area of a square with the diagonals of length 8.8 cm.

Solution

Let, d represents the length of the diagonal

Given $d = 8.8$ cm

The area of the square $= \frac{1}{2}d^2$

$= \frac{1}{2} \times 8.8 \text{ cm} \times 8.8 \text{ cm}$

$= 38.72 \text{ cm}^2$

Therefore, the area of the square is 38.72 cm^2.

Example 2.10

A woman own a square plot of land 112.5 m long. She wants to plant trees in the entire plot. Each tree requires 6.25 m² of land. How many trees can she plant?

Solution

Given length $l = 112.5$ m. Area, A of the land for each tree = 6.25 m²

$$\text{The area of a square plot} = \text{length} \times \text{length}$$
$$= 112.5 \text{ m} \times 112.5 \text{ m}$$
$$= 12\ 656.25 \text{ m}^2$$

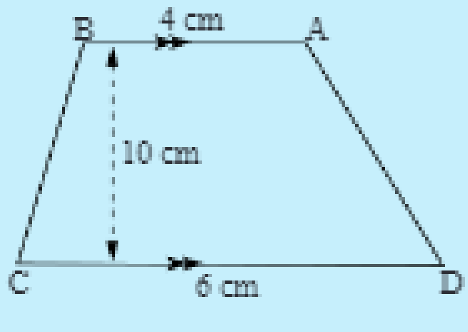
$$\text{Number of trees to be planted} = \frac{\text{the area of a square plot}}{\text{the area of each tree of land}}$$
$$= \frac{12\ 656.25 \text{ m}^2}{6.25 \text{ m}^2} = 2\ 025$$

Therefore, she can plant 2 025 trees.

Exercise 2.1

Answer the following questions:

1. The area of a triangle with base 12 cm is 36 cm². Find the corresponding height of the triangle.
2. One side of a right – angled triangle is 24 units long. Its area is 64 square units.
   (a) How long is the other side?
   (b) How long is the hypotenuse?
3. Calculate the area of a rhombus whose diagonals are 12 dm and 10 dm.
4. If the area of a parallelogram ABCD is 30 cm² and $\overline{BC} = 5$ cm, calculate the corresponding height.
5. Calculate the area of trapezium ABCD in the following figure:

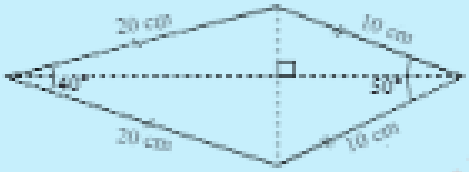


6. The dimensions of a the hall floor are 20 metres by 15 metres. The height of the hall is 10 metres. Calculate:
   (a) the area of the floor.
   (b) the total surface area of the walls.

7. The area of triangle ABC is 140 cm². If $\overline{AB} = 20$ cm, $\overline{AC} = 14$ cm, then find $B\hat{A}C$.

8. The area of triangle PQR is 60 square centimetres. If $\overline{PQ} = 10$ cm, find $\overline{PR}$ given that the included angle is 150°.

9. What is the area of paper required to make a kite shown in the following figure?



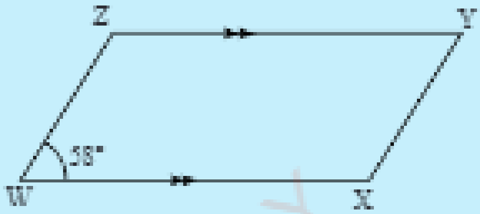
10. Find the area of the trapezium whose bases are 10 cm and 16 cm and whose height is 12 cm.

11. ABCD is a parallelogram in which $\overline{AB} = \overline{AD} = 10$ cm and $B\hat{A}D = 60°$. Calculate the area of the parallelogram.

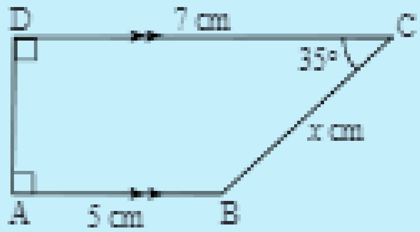
12. In triangle ABC, $C\hat{A}B = 90°$, $\overline{AB} = 8$ cm, $\overline{AC} = 6$ cm.
   Calculate:
   (a) the area of the triangle.
   (b) the length of the perpendicular line from A to $\overline{BC}$.

13. A man owns a square plot of land 75m long. He wants to plant trees in the entire plot. Each tree requires 5 m² of land. How many trees can he plant?

14. The following figure shows a parallelogram WXYZ.
   If $\overline{WZ} = 7$ cm, $\overline{WX} = 9$ cm, $Z\hat{W}X = 58°$,
   find the area of the parallelogram.



15. The following figure shows a trapezium ABCD.
   Calculate:
   (a) the value of x.
   (b) the area of the trapezium ABCD.



16. Find the area of trapezium TVWZ with vertices, $T(-3, 4)$, $V(3, 4)$, $W(6, -1)$ and $Z(-5, -1)$.

Perimeter of a regular polygon

Activity 2.3: Deriving the formula for the perimeter of a regular polygon

In a group or individually, perform the following tasks:

1. In pieces of paper, draw a circle of any convenient radius.
2. Draw a regular five sided polygon inscribed in a circle.
3. Write the formula for the perimeter of a regular polygon.
4. Measure the value of each angle at the centre and give comments.
5. Share your findings with your neighbours.

The sum of the lengths of the sides of a polygon is known as perimeter denoted by p. A polygon is inscribed in a circle if each of its vertices lies on the circle. If the lengths of the sides of the polygon are the same, the polygon is an inscribed regular polygon. Figure 2.13 shows an inscribed regular pentagon of length of side S.

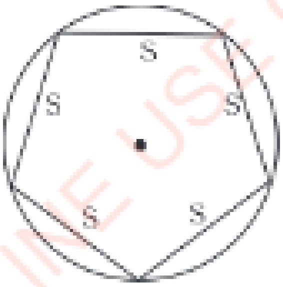


Figure 2.13: *Regular pentagon inscribed in a circle*

A regular polygon with any number of sides n (larger than 2) can be inscribed in a circle. For example, to construct an inscribed regular pentagon (5 sides), draw rays intersecting the circle in five points from the centre of the circle.

Each angle at the centre will measure $\frac{360^\circ}{5}$ or 72°. If the points of intersection on the circle are connected by a line segments, the figure formed will be an inscribed regular pentagon as shown in Figure 2.14.

Figure 2.14: *Regular pentagon inscribed in a circle*

Consider a regular polygon of n sides inscribed in a circle of radius r and centre O. Let $\overline{AB}$ be a side of the polygon and $\overline{OX}$ the perpendicular from O to $\overline{AB}$ as shown in Figure 2.15. $\overline{OX}$ is the bisector of angle AOB.

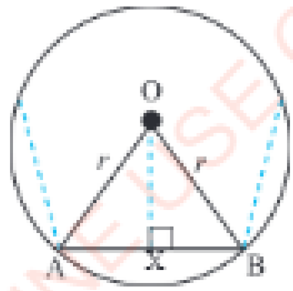


Figure 2.15: *Regular polygon inscribed in a circle of radius r*

Since the regular polygon has n sides, $A\hat{O}B = \frac{360°}{n}$

But, $A\hat{O}X = B\hat{O}X$ and $A\hat{O}X + B\hat{O}X = A\hat{O}B$.

Therefore, $A\hat{O}X = \frac{1}{2}\left(\frac{360°}{n}\right) = \frac{180°}{n}$.

Let the length of the side of the regular polygon $\overline{AB}$ be s. Then, $\overline{AX} = \frac{1}{2}s$.

From ΔAOX, $\sin\left(\frac{180°}{n}\right) = \frac{\overline{AX}}{\overline{AO}}$

$$\sin\left(\frac{180°}{n}\right) = \frac{\frac{1}{2}s}{r}$$

$$s = 2r\sin\left(\frac{180°}{n}\right)$$

Therefore, the length of side s of a regular polygon with n sides and radius r is given by:

$$s = 2r\sin\left(\frac{180°}{n}\right).$$

Another formula is by expressing $2r$ as diameter d of the circle. Thus,

$$s = d\sin\left(\frac{180°}{n}\right).$$

The formula for a perimeter of a regular polygon of n sides can be derived from the formula for the length of a side of a regular polygon.

For n – sided regular polygon with length of each side s, the perimeter is given by;

$$p = ns$$

$$= n\left[2r\sin\left(\frac{180°}{n}\right)\right]$$

$$= 2nr\sin\left(\frac{180°}{n}\right)$$

Therefore, the perimeter p of a regular polygon with n sides inscribed in a circle of radius r is given by:

$$p = 2nr\sin\left(\frac{180°}{n}\right).$$

Another formula is by expressing $2r$ as diameter d of the circle.

Therefore, $p = nd\sin\left(\frac{180°}{n}\right).$

Example 2.11

Find the length of one side of a regular nine – sided polygon inscribed in a circle of radius 10 cm.

Solution

Given the radius $r = 10$ cm, $n = 9$

Recall the formula for the length of a side of a regular polygon:

$$s = 2r\sin\left(\frac{180^\circ}{n}\right)$$

$$s = 2 \times 10 \text{ cm} \times \sin\left(\frac{180^\circ}{9}\right)$$

$$= 2 \times 10 \text{ cm} \times \sin 20^\circ$$

$$= 20 \text{ cm} \times \sin 20^\circ$$

$$= 20 \text{ cm} \times 0.3420$$

$$= 6.84 \text{ cm}$$

Therefore, the length of one side of a regular nine–sided polygon inscribed in a circle is 6.84 cm.

Example 2.12

Find the radius of a circle which inscribes a regular hexagon with perimeter 50 dm.

Solution

Given the perimeter $p = 50$ dm and $n = 6$

Recall the formula for the perimeter of a regular polygon:

$$p = 2nr\sin\left(\frac{180^\circ}{n}\right)$$

$$50 \text{ dm} = 2 \times 6 \times r\sin 30^\circ$$

$$r = \frac{50 \text{ dm}}{12\sin 30^\circ}$$

$$= \frac{50 \text{ dm} \times 2}{12} = 8.33 \text{ dm}$$

Therefore, the radius of a circle is 8.33 dm.

Exercise 2.2

Answer the following questions:

1. Find the length of a side and perimeter of a regular nine – sided polygon inscribed in a circle of radius 5 cm.
2. Find the length of a side and perimeter of a regular 12 – sided polygon inscribed in a circle of radius 3 units.
3. Show that the length of a side and radius of a circle inscribing a regular hexagon are equal.
4. The ratio of radii of circles inscribing two regular hexagons is $\frac{7}{5}$. The perimeter of the larger hexagon is 63 cm. Find the perimeter, the side, and radius of the smaller one.
5. Find the radius of a circle which inscribes an equilateral triangle with perimeter 24 cm.
6. Find the radius of a circle which inscribes a square with perimeter 32 cm.
7. Find the radius of a circle which inscribes a regular hexagon with perimeter 48 cm.
8. The radius of a circle is 12 cm. Find the perimeter of the regular polygon inscribed in a circle if the polygon is a:
   (a) triangle.
   (b) quadrilateral.
9. Find the length of a side and perimeter of a regular decagon inscribed in a circle of radius 6 cm.
10. Find the length of a side and perimeter of regular nonagon inscribed in a circle of radius 9 units.
11. Find the ratio of the perimeter of a regular hexagon to the inscribed radius circle. Show that the answer is the same for all regular hexagons no matter how big or how small the polygon.
12. Find the value of one interior angle of a regular polygon inscribed in a circle of radius 5 cm given that the length of one side is $5\sqrt{2}$ cm.

Area of a regular polygon

Consider a regular polygon of n sides inscribed in a circle of radius r and centre O. If each vertex is connected to O, the polygonal region is divided into n triangles. One of the triangles is shown in Figure 2.16 as AOB where $\overline{AB}$ is one side of the polygon.

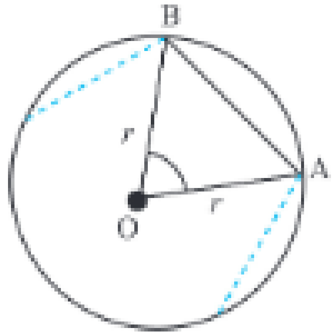


Figure 2.16: *Regular polygon inscribed in a circle of radius* r

Then, $B\hat{O}A = \frac{360^\circ}{n}$

$$\text{Area of } \Delta OAB = \frac{1}{2} \times \left(\overline{OA} \times \overline{OB}\right) \times \sin\left(\frac{360^\circ}{n}\right)$$

$$= \frac{1}{2} r^2 \sin\left(\frac{360^\circ}{n}\right)$$

$$\text{Area of the regular polygon } = n \times \text{Area of } \Delta OAB$$

$$= n \times \frac{1}{2} \times r^2 \sin\left(\frac{360^\circ}{n}\right)$$

$$= \frac{1}{2} nr^2 \sin\left(\frac{360^\circ}{n}\right)$$

Therefore, the area A of a regular polygon of n sides inscribed in a circle of radius r is given by:

$$A = \frac{1}{2} nr^2 \sin\left(\frac{360^\circ}{n}\right).$$

Example 2.13

Find the area of a regular polygon of 15 sides inscribed in a circle of radius 4 cm .

Solution

Given; $n = 15$, radius $r = 4$ cm

Recall the formula for the area of regular polygon:

$$A = \frac{1}{2}nr^2 \sin\left(\frac{360^\circ}{n}\right)$$

$$= \frac{1}{2} \times 15 \times 4^2 \text{ cm}^2 \times \sin\left(\frac{360^\circ}{15}\right)$$

$$= (120 \times \sin 24^\circ) \text{ cm}^2$$

$$= (120 \times 0.4067) \text{ cm}^2$$

$$= 48.804 \text{ cm}^2$$

Therefore, the area of a regular polygon of 15 sides inscribed in a circle is 48.804 cm^2.

Example 2.14

The area of a regular 6 – sided plot of a land inscribed in a circular track of radius r is 720 m^2. Find the radius of the track.

Solution

Given the area A = 720 m^2 and $n = 6$

Recall the formula for the area of a regular polygon:

$$A = \frac{1}{2}nr^2 \sin\left(\frac{360^\circ}{n}\right)$$

$$= \frac{1}{2} \times 6 \times r^2 \times \sin\left(\frac{360^\circ}{6}\right)$$

$$720 \text{ m}^2 = 3r^2 \sin 60^\circ$$

$$r^2 = \frac{720 \text{ m}^2}{3 \times 0.866} = 277.14 \text{ m}^2$$

$$r = 16.65 \text{ m}$$

Therefore, the radius of the track is 16.65 m.

Exercise 2.3

Answer the following questions:

1. Find the area of a regular nine – sided polygon inscribed in a circle with radius 5 cm.

2. Find the area of a regular 12 – sided polygon inscribed in a circle with radius 3 cm.

3. Prove that the area A of an equilateral triangle inscribed in a circle with radius r is $A = \frac{(3\sqrt{3})}{4} r^2$. Use the formula to find the area of the equilateral triangle inscribed in a circle with radius 2 dm.

4. Prove that the area A of a square inscribed in a circle with radius r is given by $A = 2r^2$. Use this formula to find the area of a square inscribed in a circle with radius 4 cm.

5. Find the area of a regular pentagon inscribed in a circle of radius 8 dm.

6. Find the area of a regular decagon inscribed in a circle of diameter 20 cm.

7. The area of a regular octagon is 100 cm². Find the radius of the circle inscribing it.

8. Two concentric regular hexagons are such that the radius of the circle inscribing the larger is twice that of the smaller. If the radius of the smaller is 5 cm, find the area between the two polygons.

9. Find the area of a regular polygon having 360 sides inscribed in a circle of radius 10 cm.

10. Find the area of a regular octagon inscribed in a circle of radius 2.5 cm.

11. Find the area of a 20 – sided regular polygon inscribed in a circle of radius 6 cm.

12. Find the area of a regular decagon of side 2 cm.

Circumference and area of a circle from inscribed regular polygon

Activity 2.4: Finding the number of sides of the regular polygon

In a group or individually, perform the following tasks:

1. Using manila card, draw a circle of any convenient radius.
2. Draw small sectors in a circle and colour them with two different colours in every two consecutive sectors.
3. Cut into pieces the sectors obtained in task 2. Then, arrange them in patterns.
4. Describe with reasons your observations in task 3.
5. Mention the number of sides of the regular polygon from this activity.
6. Share your findings with your neighbours.

The circumference of a circle is the distance of a circle's edge. The perimeter of a regular polygon having many sides, say 180, inscribed in a circle with radius r, is a good estimate for the circumference of the circle. The perimeter of a regular polygon inscribed in a circle of radius r is given by:

$$p = 2nr\sin\left(\frac{180^\circ}{n}\right) \text{ or } p = 2r\left(n\sin\frac{180^\circ}{n}\right)$$

If $n = 10$, $p = 2r\,(10\sin 18^\circ) = 2r \times 3.090$

If $n = 72$, $p = 2r\,(72\sin 2.5^\circ) = 2r \times 3.141$

If $n = 180$, $p = 2r\,(180\sin 1^\circ) = 2r \times 3.141$

If $n = 540$, $p = 2r\,(540\sin 20') = 2r \times 3.142$

Note: $1^\circ = 60'$

The greater the number of sides of the regular polygon, the nearer $n\sin\left(\frac{180^\circ}{n}\right)$ approaches the value of π. If the number of the sides, n of a regular polygon increases, its perimeter approaches the circumference of the circle inscribing it. Therefore, the circumference c of a circle of radius r is given by $c = 2\pi r$.

Similarly, the area of a regular polygon having many sides, say 180, inscribed in a circle with radius r, is a good estimate for the area of an inscribed circle. Area of a regular polygon inscribed in a circle of radius r is given by:

$$A = \frac{1}{2}nr^2 \sin\left(\frac{360^\circ}{n}\right)$$

$$\text{or } A = r^2\left[\frac{n}{2}\sin\left(\frac{360^\circ}{n}\right)\right]$$

If $n = 10$, $A = r^2(5\sin 36^\circ) = r^2(2.939)$

If $n = 72$, $A = r^2(36\sin 5^\circ) = r^2(3.138)$

If $n = 180$, $A = r^2(90\sin 2^\circ) = r^2(3.141)$

If $n = 540$, $A = r^2(270\sin 40') = r^2(3.142)$

The greater the number of sides of the regular polygon, the nearer $\left[\frac{n}{2}\sin\left(\frac{360^\circ}{n}\right)\right]$ approaches the value of π.

Therefore, the area of a circle of radius r is given by $A = \pi r^2$.

Example 2.15

Find the circumference of a circle of radius 14 cm $\left(\text{Use } \pi = \frac{22}{7}\right)$.

Solution

Let r represents radius of a circle.

Recall the formula for the circumference of a circle:

$$C = 2\pi r$$

$$C = 2 \times \frac{22}{7} \times 14 \text{ cm}$$

$$= 88 \text{ cm}$$

Therefore, the circumference of the circle is 88 cm.

Example 2.16

The enclosed area of Juma's circular field is 785 000 m².

Find the radius of the field (Use $\pi = 3.14$).

Solution

The enclosed area of the circular field: $A = \pi r^2$

$$785\ 000\ \text{m}^2 = 3.14 \times r^2$$

$$r^2 = \frac{785\ 000\ \text{m}^2}{3.14}$$

$$r^2 = 250\ 000\ \text{m}^2$$

$$r = \sqrt{250\ 000\ \text{m}^2}$$

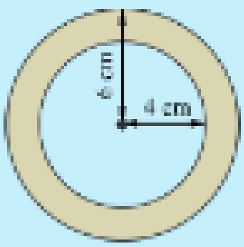
$$r = 500\ \text{m}$$

Therefore, the radius of the field is 500 m.

Exercise 2.4

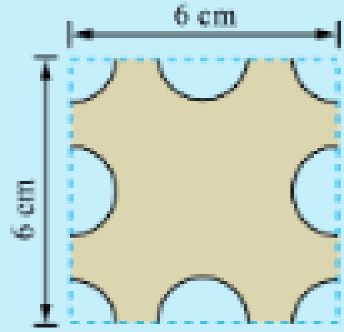
Answer the following questions $\left(\text{use } \pi = 3.14 \text{ or } \frac{22}{7}\right)$.

1. Find the circumference of a circle whose radius is:
   (a) 5 cm (b) 3.2 cm
2. The circumference of a circle is $\frac{45}{2}\pi$ units. Find its radius.
3. The circumference of a circle is 44 dm. Find its radius.
4. Calculate the area of a circle whose radius is:
   (a) 4.2 cm (b) 0.8 cm
5. Find the area enclosed between two concentric circles shown in the following figure:



6. The circumference of a circle is 176 cm. Find the diameter of a circle.
7. Find the diameter of a circle whose enclosed area is 2 464 cm^2.
8. Given that the area of an inscribed circle is 324π square units, calculate its radius.
9. Calculate the cost of making 1 metre – wide path round a circular pool of radius 10 metres, if the cost is 800 shillings per square metre.
10. A circular tin is placed in a square. If the side of the square is congruent to the diameter of the tin, calculate the percentage of the area of the square which remains uncovered.

11. Calculate the perimeter of the shaded region in the following figure if the semi circles and the quarter circles are of radius 1cm each.



Area of similar polygons

Activity 2.5: Finding the ratio of sides of regular polygons

In a group or individually, perform the following tasks:

1. On grid paper, draw right – angled triangle ABC with sides 9cm and 12cm.
2. On grid paper, draw right – angled triangle DEF with sides 6cm and 8cm.
3. How do you find the lengths of hypotenuse in task 1 and task 2?
4. Find the value of: $\frac{\overline{AB}}{\overline{DE}}$, $\frac{\overline{BC}}{\overline{EF}}$, and $\frac{\overline{CA}}{\overline{FD}}$.
5. Are the triangles similar? Explain.
6. Find the area of ΔABC and ΔDEF.
7. Find the ratio between the area of ΔABC and ΔDEF.
8. Compare the values of $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF}$, $\frac{\overline{AB}}{\overline{DE}}$, $\frac{\overline{BC}}{\overline{EF}}$, and $\frac{\overline{CA}}{\overline{FD}}$.
   Describe your results.
9. Share your findings with the rest of the class through presentantion followed by class discussion.

Let ΔABC and $\Delta A'B'C'$ be two similar triangles as shown in Figure 2.17.

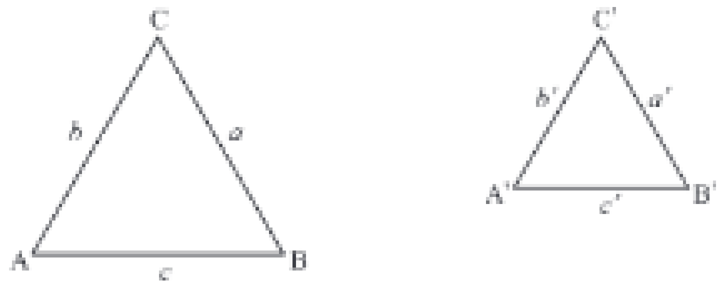


Figure 2.17: *Similar triangles*

Let $k = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ where k is the ratio of the lengths of the corresponding sides.

$$\text{Area of } \Delta ACB = \frac{1}{2}ab\sin\hat{C}$$

$$\text{Area of } \Delta A'C'B' = \frac{1}{2}a'b'\sin\hat{C}'$$

But, $\sin\hat{C} = \sin\hat{C}'$ because $\Delta ACB \sim \Delta A'C'B'$

Then, $$\frac{\text{Area of } \Delta ACB}{\text{Area of } \Delta A'C'B'} = \frac{\frac{1}{2}ab\sin\hat{C}}{\frac{1}{2}a'b'\sin\hat{C}'}$$

$$= \frac{ab}{a'b'} = \left(\frac{a}{a'}\right)\left(\frac{b}{b'}\right)$$

$$= k^2, \text{ where } \frac{a}{a'} = k \text{ and } \frac{b}{b'} = k$$

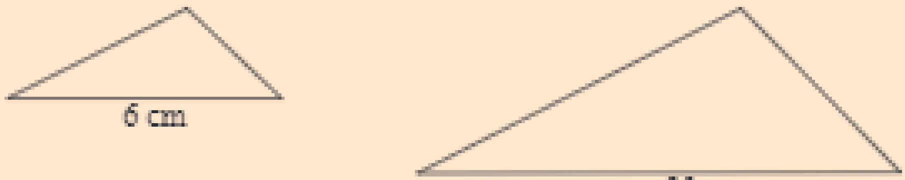
For any two similar polygons, one polygon can be divided into triangles which are similar to triangles of the other polygon. The ratio of their areas will be k^2.

Generally, if the ratio of the lengths of corresponding sides of two similar polygons is k, then the ratio of their areas is k^2.

FOR ONLINE USE ONLY
DO NOT DUPLICATE

Example 2.17

The following triangles are similar. A side of small triangle is 6 cm long and its corresponding side of the larger triangle is 20 cm. If the area of the small triangle is 90 cm², what is the area of the larger triangle?



Solution

Let the area of small and large triangles be A_1 and A_2 respectively.

Given $S_1 = 6$ cm and $S_2 = 20$ cm

Then, the ratio of their corresponding sides is $k = \frac{S_1}{S_2} = \frac{6\text{ cm}}{20\text{ cm}} = \frac{3}{10}$

Ratio of the areas $= k^2$

Thus, $\frac{A_1}{A_2} = k^2$

$$\frac{90}{A_2}\text{ cm}^2 = \left(\frac{3}{10}\right)^2$$

$$\frac{90}{A_2}\text{ cm}^2 = \frac{9}{100}$$

$$A_2 = \frac{100 \times 90\text{ cm}^2}{9}$$

$$A_2 = 1\,000\text{ cm}^2$$

Therefore, the area of the larger triangle is 1 000 cm².

Example 2.18

The ratio of the areas of two similar polygons is 50:72. If the length of a side of the smaller polygon is 12 cm, find the length of the corresponding side of the larger polygon.

Solution

Ratio of the areas = k^2

Then, $k^2 = \frac{50}{72} = \frac{25}{36}$

$$k = \sqrt{\frac{25}{36}} = \frac{5}{6}$$

Let the length of the larger polygon be x.

$$\frac{12}{x}\text{ cm} = \frac{5}{6}$$

$$5x = 72 \text{ cm}$$

$$x = 14.4 \text{ cm}$$

Therefore, the length of the corresponding side of the larger polygon is 14.4 cm.

Exercise 2.5

Answer the following questions:

1. Two triangles are similar. A side of one triangle is 2 units long and the corresponding side of the other triangle is 5 units long. What is the ratio of their areas?
2. Two triangles are similar. The sides of the first triangle are three times as long as the sides of the other. What is the ratio of their areas?
3. Two triangles are similar. The ratio of their areas is $\frac{25}{9}$. What is the ratio of their corresponding sides?
4. Two polygons are similar. Their areas are 16 cm² and 49 cm². A side of the first polygon is 28 cm long. What is the length of the corresponding side of the second polygon?

5. The sides of a triangle are 5 cm, 6 cm, and 7 cm. The longest side of a similar triangle is 21 cm. Find the length of the other sides of a similar triangle.

6. Two polygons are similar. A side of one is 5 m long. The corresponding side of the other is 1 m long. The area of the first is 100 m². What is the area of the second polygon in square metres?

7. It costs Tsh. 25 500 to plough a rectangular field 40 m long and 20 m wide. How much will it cost to plough a similar field 30 m wide?

8. Two triangles are similar. The ratio of their heights is $5:3$. If the area of the smaller triangle is 36 cm², what is the area of the larger triangle?

9. Two similar triangles have their corresponding bases in the ratio 1:5. If the area of the larger triangle is 50 square units, find the area of the smaller triangle.

10. Two triangles are similar. The ratio of their areas is $\frac{64}{49}$.
    (a) Find the ratio of their corresponding heights?
    (b) If the shorter height is 14 cm, find the corresponding height of the other.

11. Two regular heptagons have areas in the ratio 16:25.
    (a) Find the ratio of their sides.
    (b) If the smaller hectagon has side 6 cm, find the corresponding side of the larger hectagon.
    (c) If the larger hectagon has an area 100 cm², find the corresponding area of the smaller hectagon.

Chapter Summary

1. Area of $\Delta ABC = \frac{1}{2}bh$
   $= \frac{1}{2}ab\sin\hat{C} = \frac{1}{2}bc\sin\hat{A}$
   $= \frac{1}{2}ac\sin\hat{B}$, where a, b, and c are the sides of a triangle, h is its height and $\hat{A}$, $\hat{B}$, and $\hat{C}$ are angles of the triangle.
2. Area of the trapezium,
   $A = \frac{1}{2}(b_1 + b_2)h$, where b_1 and b_2 are the bases of the trapezium and h is its height.
3. Area of a parallelogram $A = bh$ where b is the base and h is the height of the parallelogram.
4. Area of a rhombus $A = bh$ or the area of rhombus is equal to half the product of the lengths of the diagonals. *i.e.*, $A = \frac{1}{2}l_1 \times l_2$.

where;

l_1 is the length of the first diagonal.

l_2 is the length of the second diagonal.

b is the length of the base.

h is the height of the rhombus.

5. Area of a rectangle is equal to length times width. $A = lw$ where l is the length and w is width.

6. Area of square is equal to side squared or half the product of the length of the diagonals.

   That is $A = l^2$ or $A = \frac{1}{2} d_1 \times d_2$, where l denotes the length of one side of a square and d_1 and d_2 denotes the lengths of the diagonals.

7. The length of a side of a regular polygon with n sides and radius r is given by $s = 2r \sin\left(\frac{180°}{n}\right)$ or $s = d \sin\left(\frac{180°}{n}\right)$ where d is diameter such that $d = 2r$.

8. The perimeter p of a regular polygon of n sides and radius r is given by:

   $p = 2nr \sin\left(\frac{180°}{n}\right)$

   $p = nd \sin\left(\frac{180°}{n}\right)$

   where $d = 2r$.

9. The area of a regular polygon of n sides inscribed in a circle of radius r is given by

   $A = \frac{1}{2} nr^2 \sin\left(\frac{360°}{n}\right)$.

10. The circumference of the circle of radius r is given by $c = 2\pi r$.

11. The enclosed area of a circle of radius r is given by $A = \pi r^2$.

12. If the ratio of the lengths of corresponding sides of two similar polygons is k, then the ratio of their areas is k^2.

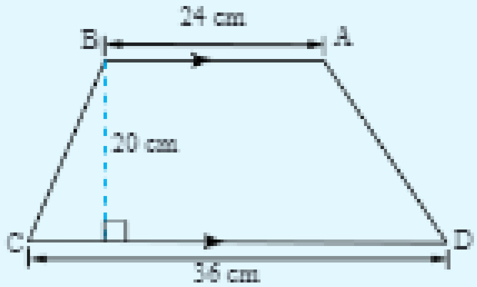
Revision exercise 2

Answer the following questions:

1. Find the area of the square whose perimeter is represented by $12x$ units.

2. If the circumference of a circle is 10π cm, find the enclosed area of the circle. Write the answer in terms of π.

3. Calculate the area of trapezium ABCD in the following:



4. Find the length of a side and perimeter of a regular nine – sided polygon inscribed in a circle of radius 10 cm.

5. Find the radius of a circle which inscribes a regular hexagon with perimeter 72 cm.

6. Find the area of a regular polygon having 36 sides inscribed in a circle of radius 20 cm.

7. The diagonals of a rhombus are 80 cm and 60 cm. Find the area of the rhombus.

8. The perimeter of a rhombus is 260 cm and one of its diagonal is 60 cm. Find the area of the rhombus and the length of the other diagonal.

9. Find the area and perimeter of a square whose diagonal is 10 cm.

10. Find the area of an equilateral triangle whose side is 10 cm.

11. Find the area of a regular hexagon inscribed in a circle of radius 8 cm.

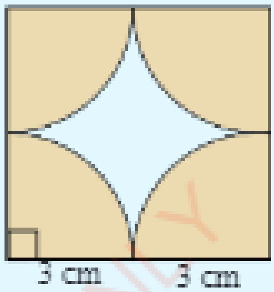
12. The circumference of a circle is 44 cm. Find its radius $\left(\text{Use } \pi = \frac{22}{7}\right)$.

13. A circular field has a circumference of 66 m. A plot in a shape of a square having its vertices on the surface is marked in the field. Calculate the area of the square.

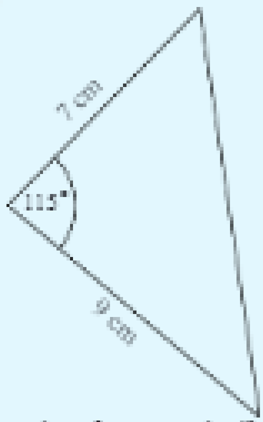
14. Two adjacent sides of a parallelogram are 4 cm and 3 cm. Find the ratio of its two altitudes.

15. The area of a rhombus is 112 cm^2 and the length of one diagonal is 7 cm. Find the length of the other diagonal.

16. A designer created the logo as shown in the figure. The logo consists of a square and four quarter – circles of equal size. Express in terms of π the area of the shaded region in square centimetres.



17. Find the area of the triangle shown in the following figure:



18. Two triangles are similar. A side of one is 12 units long and the corresponding side of the other is 30 units long. Find the ratio of their area.

Project 2

1. Show how to separate isosceles trapezium RSTU into four congruent trapezoidal regions.
2. Draw the parallelogram ABCD on a graph paper. Label the vertices on the interior of the angles with letters A, B, C, and D. Fold the parallelogram ABCD so that A lies on B and C lies on D forming a square.
   (a) Find the formula for the area of the square.
   (b) List the squares forming the parallelogram.
   (c) Deduce the formula for the area of the parallelogram.
   (d) Compare and contrast between the base and altitude of the parallelogram with the length and width of the square.
3. Attempt the following tasks:
   (a) Draw a triangle on graph paper so that one edge is along a horizontal line. Label the vertices on the interior of the angles of the triangle as A, B, and C.
   (b) Draw a line perpendicular to $\overline{AC}$ through B and mark it as F.
   (c) Draw a line perpendicular to $\overline{AC}$ through C and mark it as D.
   (d) Draw a line perpendicular to $\overline{AC}$ through A and mark it as E.
   (e) Draw a line parallel to $\overline{AC}$ through B.
   (f) Cut out a rectangle ACDE. Then, cut out triangle ABC. Place the two smaller pieces over triangle ABC completely to cover the triangle ABC.
   (g) Find the area of triangle BCD, ABE, and ABC.
   (h) Find the sum of areas of triangle BCD, ABE, and ABC.
   (i) Find the area of rectangle ACDE.
   (j) What do you observe about the two smaller triangles ABE, BCD, and triangle ABC?
   (k) What fraction of the rectangle ACDE is triangle ABC?
   (l) What is the conclusion you can make basing on the answers found in (h) and (i) above?
   (m) Make and label a scale drawing of your bedroom. Then, find its area in square centimetres.

Chapter Three

Three – dimensional figures

Introduction

In our real life situation, we usually meet and use things that are in three dimensions, for example; we live in a house, use a bar of soap, use cupboards just to mention a few. In this chapter, you will learn the meaning of three– dimensional figures, characteristics of three – dimensional figures, constructing and sketching three–dimensional figures. You will also identify properties of three– dimensional figures, calculate the angle between lines and planes, and angle between two intersecting planes. You will be able to derive formulae for calculating surface area of prisms, cylinders, pyramids, and cones. You will also be able to apply the formulae to calculate the surface area of spheres, derive formulae for calculating volume of prisms, cylinders, pyramids, and cones. The competencies developed will be applied in various activities such as: carpentry, construction of buildings, manufacturing industries, architecture, and other daily activities.

Three – dimensional figures

Activity 3.1: Classification of three – dimension figures

In a group or individually, perform this activity using the following steps:

1. Draw a cuboid, cylinder, cone, and rectangular pyramid.
2. Identify the number of sides, faces, edges, and vertices of the object listed in step 1.
3. Which object in step 1 has the largest number of faces?
4. Which object in step 1 has the fewest number of vertices?
5. What is the maximum number of sides of the objects?

A three – dimensional figure is a geometrical object which has three dimensions, namely length, width, and height. Three – dimensional figures occupy space and they are also called solid figures. Three – dimensional shapes have thickness or depth. Figure 3.1 shows some common three – dimensional figures. The flat or curved sides of solids are called faces or surfaces. The lines where two faces meet are called edges. The points where edges meet are called vertices.

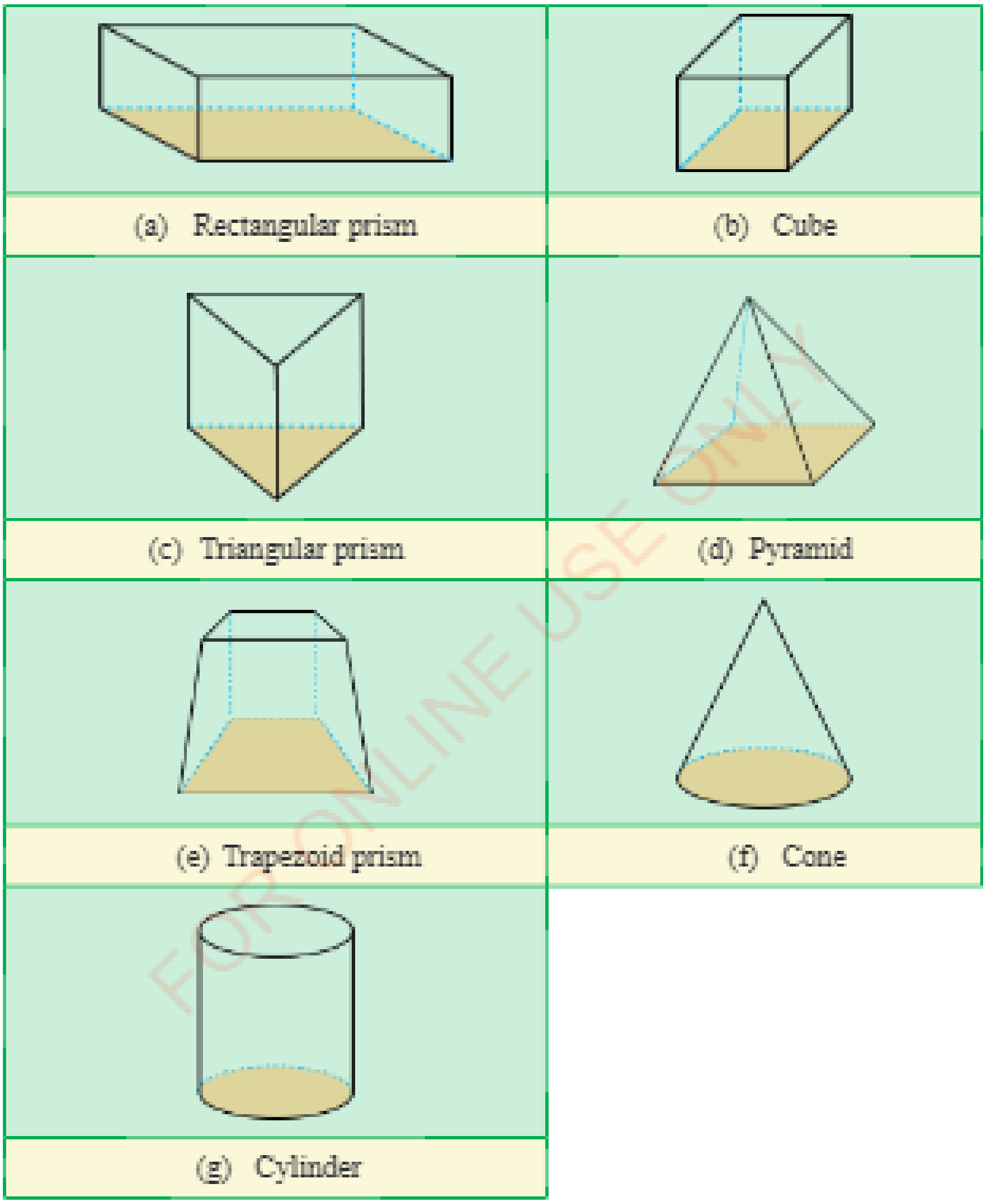


Figure 3.1: *Three – dimensional figures*

Cone

The three – dimensional shape in Figure 3.2 is a cone. The base or bottom of a cone is circular. The distance measured straight down the curved side is called slant height. The height to the vertex above the centre of the base is the height of the cone.

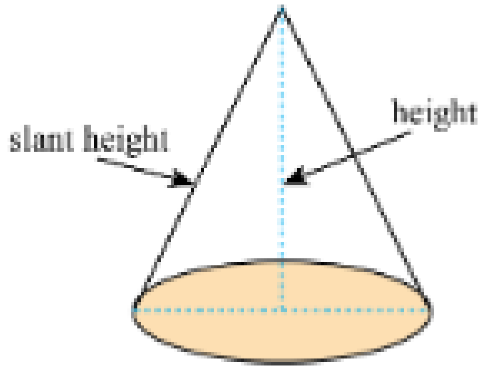


Figure 3.2: *The shape of a cone*

Note: Circular base looks like an oval because of presenting three – dimensional figure in two – dimensional plane.

Pyramid

The three – dimensional shape in Figure 3.3 is a pyramid. The distinctive feature of a pyramid is that all the faces, except one are triangles. The non – triangular side is called the base. The base can be any shape and a pyramid is named according to the shape of its base. The pyramid in Figure 3.3 is called a rectangular pyramid because its base is a rectangle.

A rectangular based pyramid has 5 faces, 8 edges, and 5 vertices. Other pyramids include a hexagonal, triangular, square pyramid and so on.

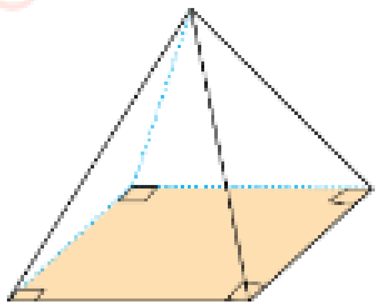


Figure 3.3: *Rectangular pyramid*

Prism

The three – dimensional shapes in Figure 3.4 are prisms. A prism is a geometrical figure whose two ends are similar, equal, and parallel and its sides are parallelograms. A prism is named according to the shape of the ends. The hexagonal prism has 8 faces, 18 edges, and 12 vertices. A square prism has square ends. The square prism has 6 faces, 12 edges, and 8 vertices. A triangular prism has triangular ends. The triangular prism has 5 faces, 9 edges, and 6 vertices.

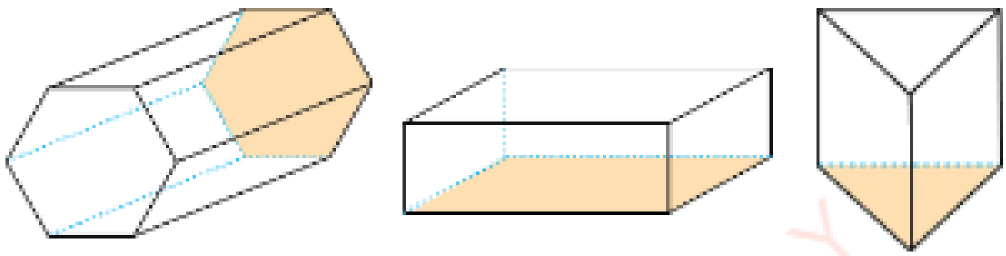


Figure 3.4: *Hexagonal, rectangular, and triangular prisms*

Cylinder

The three – dimensional shape in Figure 3.5 is a cylinder. The top and bottom surfaces of a cylinder are circular with the same diameter.
The distance measured straight down the curved surface is the height of the cylinder.

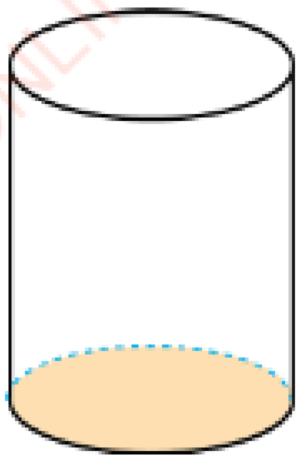
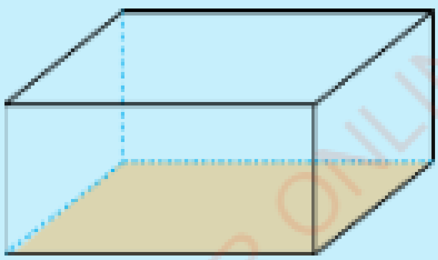


Figure 3.5: *A cylinder*

Exercise 3.1

Answer the following questions:

1. Describe the shapes of the plane faces of a cylinder.
2. Describe the shapes of the faces of a triangular prism.
3. How many faces, edges, and vertices does a triangular prism have?
4. A prism has five rectangular faces. Describe the shapes of the other two faces.
5. A square pyramid has four triangular faces. What is the total number of faces of the pyramid?
6. Study the following rectangular prism and answer the questions that follow.



   (a) How many faces does the prism has?
   (b) How many edges does the prism has?
   (c) How many vertices does the prism has?
7. Describe the shape of the faces of a cuboid.
8. Describe the shape of the faces of a cube.
9. Name two solids which have five faces.
10. Describe two geometrical objects which have no vertices.
11. Name the geometrical objects whose shapes of all their faces are described as follows:
   (a) Triangle
   (b) Rectangle
   (c) Square

Construction of three – dimensional figures

Activity 3.2: Construction of models of three – dimensional figures

In a group or individually, perform this activity using the following steps:

1. Take any two equal rectangular boxes.
2. Open the boxes into two different ways by cutting their edges without separating them.
3. Sketch the figures formed by the shapes in step 2.
4. Share your observations in step 3 with your neighbour.
5. Present your findings to your fellow students.

The cuboid is one of the most common solid figures. All solids have faces. Most solid figures have edges. Figure 3.6 shows the faces, edges, and vertices of a rectangular box.

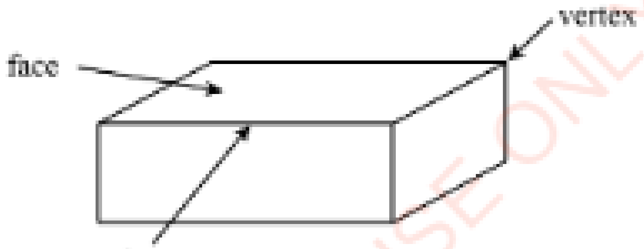
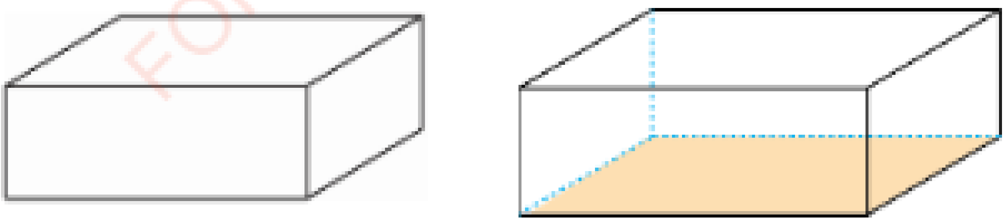


Figure 3.6: *Rectangular box*

A face can be flat (plane) or curved. An edge is a line where two faces meet and it is straight or curved. A vertex is a point or corner where three or more edges meet. The plural of vertex is vertices. Figure 3.7 shows three ways of drawing cuboids.

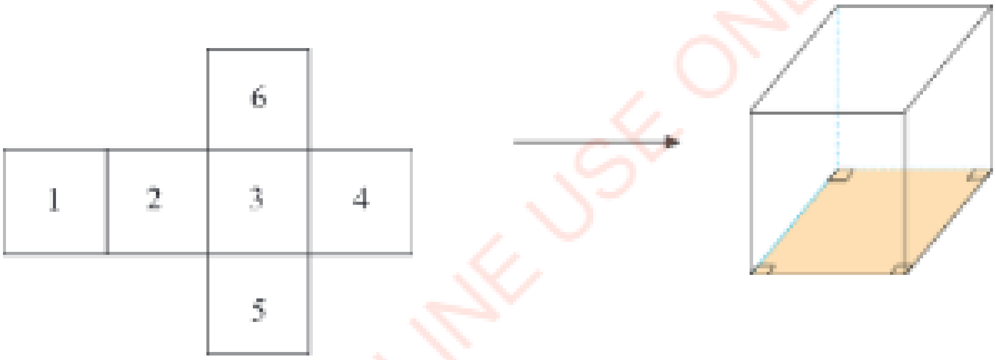


(a) Isometric projection (b) Skeleton view of a cuboid

Figure 3.7: *Isometric projection and skeleton view of a cuboid*

Figure 3.7 (c): *Net of a cuboid*

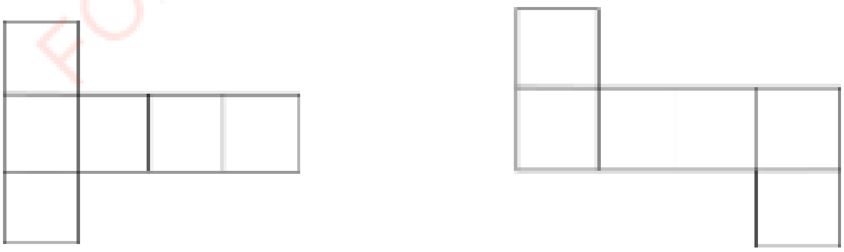
Figure 3.7(c) represents a shape of a closed cuboid when it is cut along the edge and opened out. This shape can be folded in order to make a cuboid. The shapes in this form are called nets. That is, a net is a pattern obtained when a three – dimensional figure is laid out flat showing all its faces. The squares numbered 1, 2, 3, and 4 in Figure 3.8(a) can be folded to form the sides of a cube. The square numbered 5 would form the bottom of the cube and square numbered 6 would form the top of the cube.



(a) Net of a cube (b) A closed cube

Figure 3.8: *Net and Skeleton view of a cuboid*

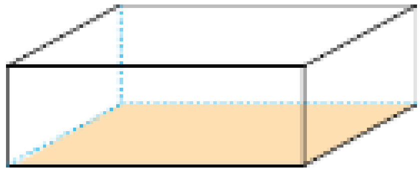
Other alternatives of representing nets of Figure 3.8 (b) are shown in Figure 3.9.



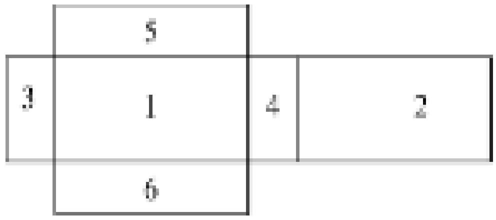
(a) (b)

Figure 3.9: *Nets of a cube*

Figure 3.10 (b) shows a net of a cuboid represented in Figure 3.10 (a). Rectangles 5 and 6 will form the two longer sides, rectangle 3 and 4 will form the ends or shorter sides. Rectangle 2 will be the top and rectangle 1 will be the bottom of a cuboid.



(a) Cuboid



(b) Net of cuboid

Figure 3.10: *A cuboid and its net*

When a triangular prism is laid out flat as shown in Figure 3.11(a), its net will be as shown in Figure 3.11(b). The triangles in the net form the ends, while the rectangles form the sides of a triangular prism.

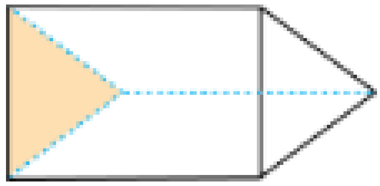


Figure 3.11 (a): *Triangular prism*

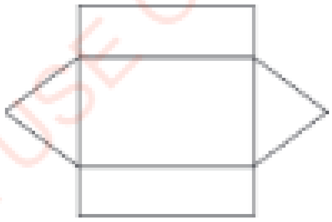


Figure 3.11 (b): *A net of triangular prism*

When a hexagonal prism is laid out flat as shown in Figure 3.12 (a), its net will be as shown in Figure 3.12 (b). The hexagons will form the ends while the rectangles will form the sides of a hexagonal prism.

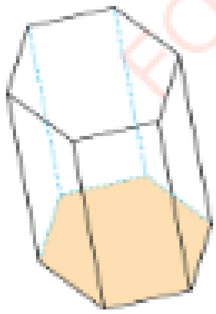


Figure 3.12(a): *Hexagonal prism*

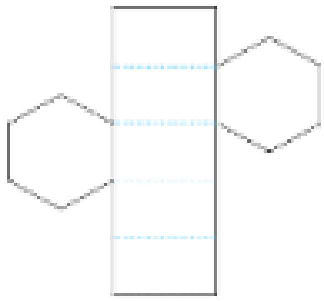


Figure 3.12(b): *A net of a hexagonal prism*

When a square pyramid is laid out flat as shown in Figure 3.13 (a), its net will be as shown in Figure 3.13 (b). The square in the net will form the base of the pyramid and all triangles will be its sides.

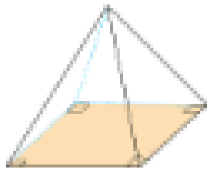


Figure 3.13 (a): *Square pyramid*

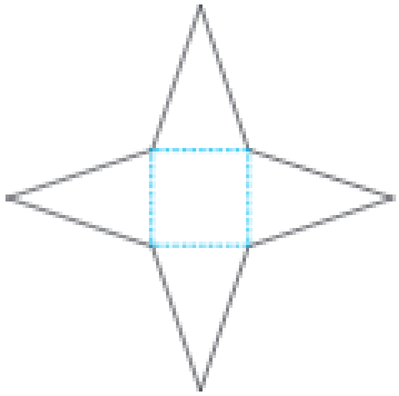


Figure 3.13 (b): *A net of a square pyramid*

Figure 3.14 (b) shows a net of a cylinder represented in Figure 3.14 (a). The net is made up of two circles and one rectangle.

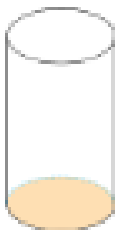


Figure 3.14 (a): *Cylinder*

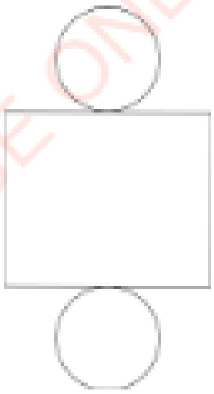


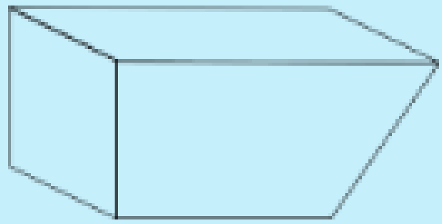
Figure 3.14 (b): *A net of a cylinder*

Exercise 3.2

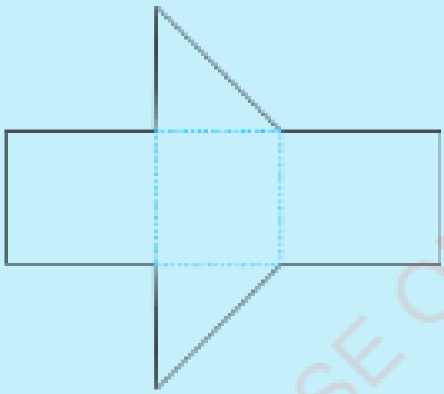
Answer the following questions:

1. Sketch two different nets of a triangular prism.
2. Draw a skeleton view of a cylinder and a triangular prism.
3. Sketch the net of a cone.
4. Sketch the net of a solid cylinder.
5. Sketch the net of a rectangular prism.
6. Sketch the net of a hexagonal prism.

7. Sketch the net for the following three – dimensional figure.



8. Draw a skeleton view of the geometrical figure formed by folding the following net.



Sketching three – dimensional figures

Activity 3.3: Sketching three – dimensional figures

In a group or individually, perform the following tasks:

1. Sketch a rectangular prism whose dimensions are 6 cm high, 8 cm long, and 10 cm wide.
2. Sketch a cube whose edge is 5 cm.
3. Sketch a triangular prism of height 4 cm, right angle bases, and sides of lengths 5 cm and 4 cm.
4. Share your work with your neighbours for more inputs.

When drawing a three – dimensional object, it is important to show that it is not a drawing of a flat object. Three – dimensional figures are usually presented on a two – dimensional plane as oblique drawings. These drawings are presented by projections known as oblique projections.

Three – dimensional figures are usually drawn on a two – dimensional plane by making oblique drawings under certain rules. The rules are as follows:

(a) Parallel lines of the object are drawn parallel.

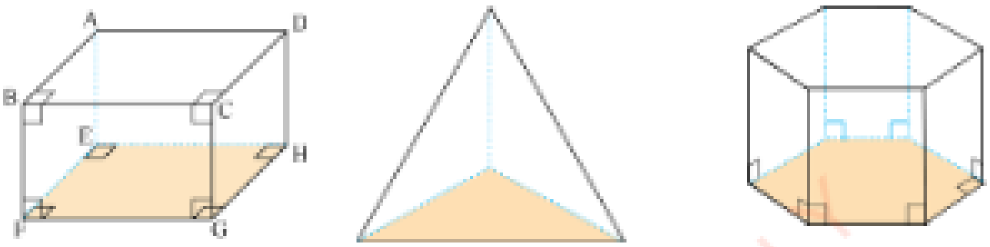
(b) Vertical lines of the object are drawn up and down the page.

(c) Hidden edges of the object are drawn dotted.

(d) Construction lines to guide the eye are drawn thinly.

(e) Right angles in the object are marked correctly in the drawings.

The oblique drawings of three – dimensional figures are shown in Figure 3.15.

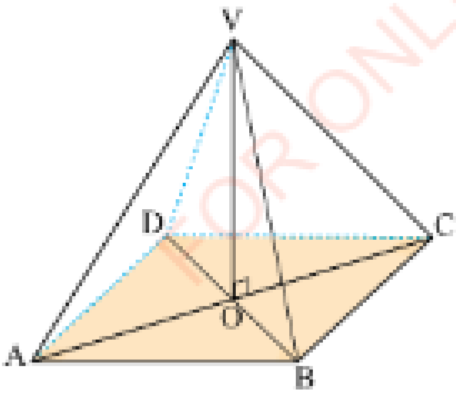


(a) Cube *(b) Right triangular pyramid* *(c) Right hexagonal prism*

Figure 3.15: *The oblique drawings of three – dimensional figures*

When drawing three – dimensional figures, the measures of angles and lengths may not represent their actual measures. For example, in Figure 3.15 (a), the actual measure of $\hat{FEH} = 90°$ but it looks greater than 90°. Also, $\overline{FG}$ is equal to $\overline{GH}$.

Lines and planes



Identification of lines and planes is important for describing and drawing three – dimensional figures. Consider the right – rectangular pyramid shown in Figure 3.16.

Figure 3.16: *A right rectangular pyramid*

A line is identified either by two points or by two intersecting planes. For example, points V and A determine a line $\overline{VA}$. Also, plane VAB intersect with plane ABCD to determine line $\overline{AB}$, which is one of the edges of the base. An example of this is the intersection of a wall and the floor of a room in a house. Note that, two distinct parallel lines or planes have no point of intersection. In general, three planes will intersect in a point. In Figure 3.16, planes VAB, VAD, and ABCD intersect at point A. An example of this is two intersecting walls and the floor of a room in a house.

There are lines which do not intersect. These lines are known as skew lines.

For example, in Figure 3.16 $\overline{VA}$ and $\overline{BC}$ are skew lines. These lines cannot lie in the same plane.

A plane is identified by any one of the following conditions as shown in Figure 3.16.

1. Three points not in the same straight line. For example, points V, C, and A;
2. Two parallel lines. For example, lines $\overline{AB}$ and $\overline{DC}$;
3. Two intersecting straight lines. For example, lines $\overline{VA}$ and $\overline{VC}$;
4. A line and a point not on the line. For example, line $\overline{BC}$ and a point V.

Consider a plane and a line segment as shown in Figure 3.17.

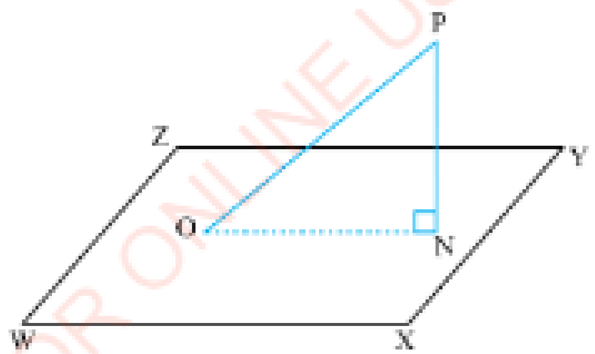


Figure 3.17: *A plane and line segment*

In Figure 3.17, if a line segment $\overline{PO}$ intersects a given plane at O and $\overline{PN}$ is perpendicular to the plane from P, then the line segment $\overline{ON}$ is called the projection of $\overline{PO}$ on the plane. Thus, the projection of $\overline{PO}$ on WXYZ is $\overline{ON}$, where $\overline{PN}$ is perpendicular to the plane WXYZ.

A line which is perpendicular to a plane is perpendicular to every line in the plane. Its projection on the plane is a point. Figure 3.18 shows that the projection of $\overline{PO}$ to the plane is the point O.

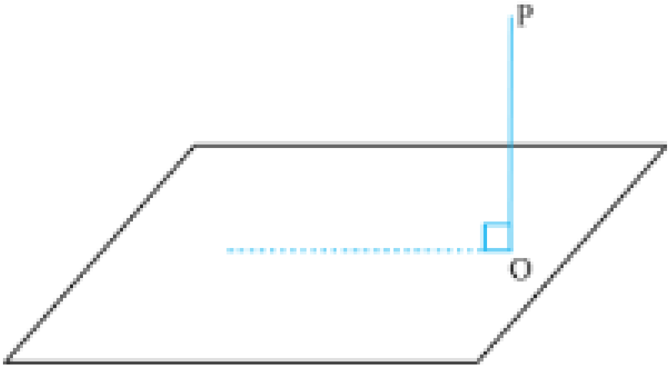


Figure 3.18: *The projection of* $\overline{PO}$ *on the plane at point O*

If a line is perpendicular to two non – parallel lines on the plane, then the line is perpendicular to the plane. Figure 3.19 shows $\overline{PO}$ is perpendicular to two lines in the plane, thus $\overline{PO}$ is perpendicular to the plane.

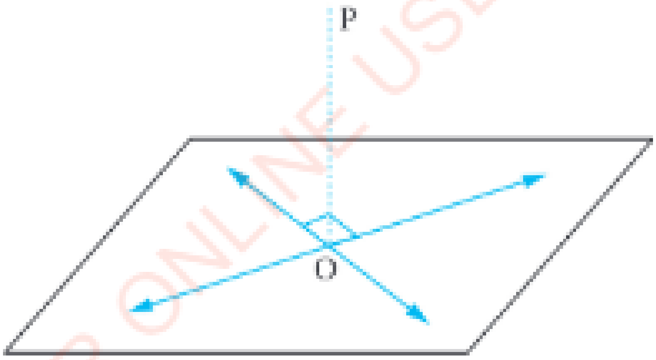


Figure 3.19: *Perpendicular line* $\overline{OP}$ *in the plane*

Angle between two lines

The angle between two skew lines is equal to the angle between lines which intersect and which are parallel to the skew lines. Consider two skew lines $\overline{HF}$ and $\overline{AD}$ of a cube shown in Figure 3.20.

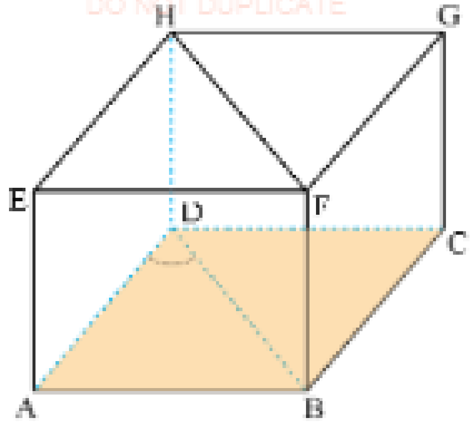


Figure 3.20: *The skew lines* $\overline{HF}$ *and* $\overline{AD}$ *of a cube*

The angle between the skew lines is found by translating $\overline{HF}$ to $\overline{BD}$ and the angle required is the angle between $\overline{AD}$ and $\overline{BD}$ which is $A\hat{D}B$. Note that $\overline{AD}$ can also be translated to $\overline{EH}$ to form $E\hat{H}F$ or translated to $\overline{GF}$ to form angle GFH.

Angle between a line and a plane

If a line $\overline{PO}$ intersects a given plane WXYZ at O and $\overline{PN}$ is the perpendicular from P to the plane, the angle between the line and the plane is $P\hat{O}N$ (as shown in Figure 3.21). It is the angle between the line and its projection on the plane.

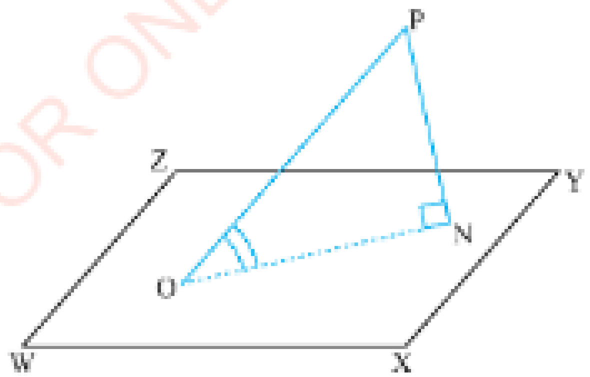


Figure 3.21: *Angle between the line and its projection on the plane*

Angle between two planes

Two planes which are not parallel intersect in a straight line. The angle between two planes is the angle between any pair of intersecting lines (one in each plane) each perpendicular to the line of intersection of the planes. Figure 3.22 shows an angle A between two planes.

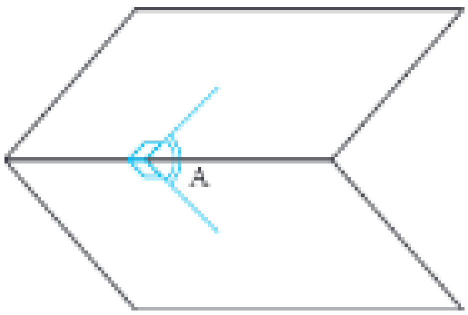


Figure 3.22: *Angle A between two planes*

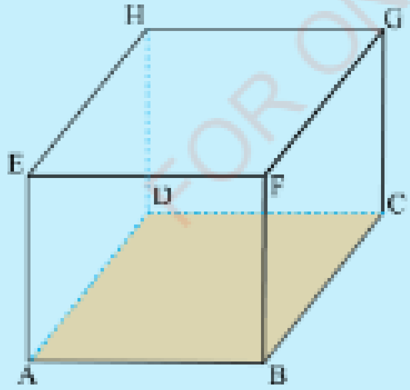
Exercise 3.3

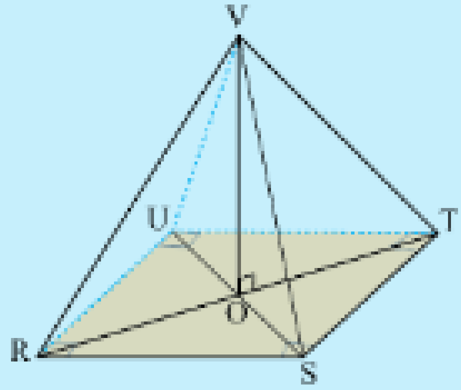
Answer the following questions:

1. In the given figures, which of the pairs of lines determine a plane?

(a) $\overline{GF}$ and $\overline{ED}$ (b) $\overline{HF}$ and $\overline{BG}$ (c) $\overline{FC}$ and $\overline{ED}$

(d) $\overline{BC}$ and $\overline{BH}$ (e) $\overline{RT}$ and $\overline{VO}$ (f) $\overline{VU}$ and $\overline{TS}$

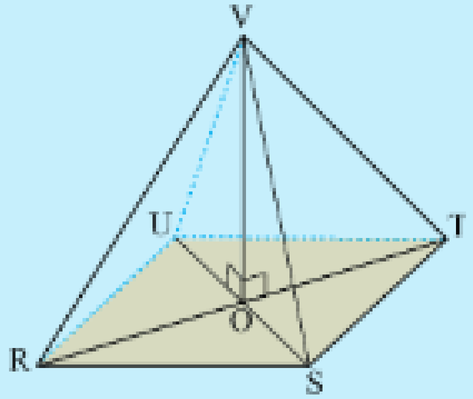




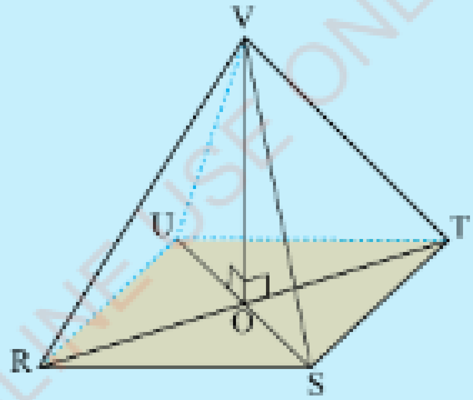
2. The following figure is a right pyramid RSTUV. State the projection of:

(a) $\overline{RV}$ on RSTU (b) $\overline{RV}$ on VSU (c) $\overline{SV}$ on VTR

In each case, name the angle between the line and the plane.



3. Study the given right square pyramid with base RSTU and then answer the questions that follow.

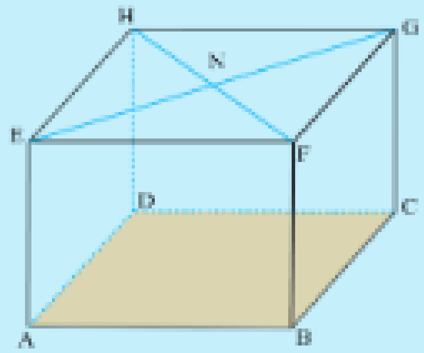


(a) Name the line of intersection of the planes RTV and USV.

(b) Name the angle between the planes RTV and USV at the base.

(c) Name the angle between the planes RSTU and RTV.

4. Study the following cube and answer the questions that follow.

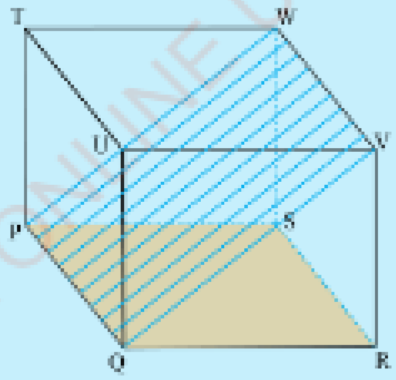


State the line of intersection of the following planes:

(a) ABCD and BCGF
(b) ABCD and BCHE
(c) ABCD and BDHF
(d) BDHF and ACGE
(e) ADHE and EFGH

5. Make an oblique drawing of the following three – dimensional objects:
(a) A right pentagonal pyramid
(b) A right square pyramid
(c) A right rectangular prism

6. Study the following cube and answer the questions that follow.



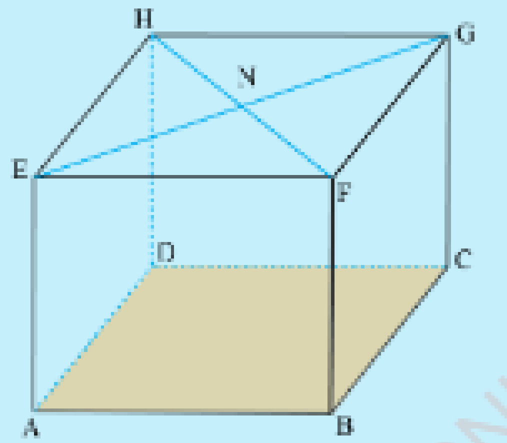
Name the angle between:

(a) planes PQRS and PQVW
(b) planes PQVW and RSWV
(c) planes QSWU and PSWT
(d) line $\overline{QV}$ and plane PQRS
(e) line $\overline{SU}$ and plane PQRS
(f) line $\overline{UV}$ and plane PQVW

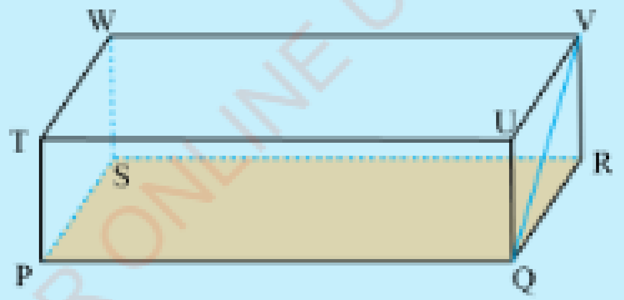
7. The following figure is a cube. If the length of its edges is 5 cm, state the distance between the given point and the plane:

(a) F and ABCD (b) F and CDHG

(c) N and CDHG



8. The following figure is a right rectangular prism. Identify and name the line segments which are perpendicular to $\overline{QV}$.



9. Use the figure in question 8 to state the angles formed by the following skew lines:

(a) $\overline{PQ}$ and $\overline{UV}$ (b) $\overline{PS}$ and $\overline{QV}$

(c) $\overline{TV}$ and $\overline{QR}$ (d) $\overline{SQ}$ and $\overline{WV}$

10. Square pyramid ABCDV has the base ABCD and the point V is vertically above the centre E of the base.
   (a) Draw an oblique projection of the pyramid.
   (b) State the angle between the planes AVC and BVD.
   (c) Are $\overline{BV}$ and $\overline{AD}$ skew lines? Justify your answer.
   (d) State the angle between skew lines $\overline{DV}$ and $\overline{AB}$.
   (e) What is the intersection of the planes VEC, VEB, and VBC?
   (f) What is the intersection of planes VAD and ABCD?

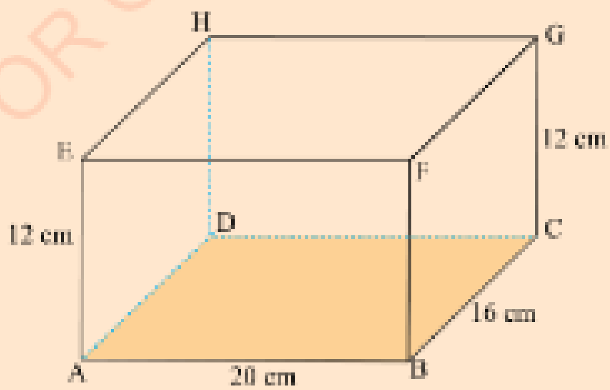
Calculations of angles and lengths of three – dimensional objects

The knowledge of trigonometry and the Pythagoras' theorem is used in calculations of lengths and angles when dealing with problems in three dimensions. It is essential to choose and draw suitable triangles in different planes, and find the sides and angles of these triangles as necessary.

Example 3.1

The following figure is a rectangular prism in which $\overline{AB}$ = 20 cm, $\overline{BC}$ = 16 cm, and $\overline{CG}$ = 12 cm. Calculate:

(a) the length of $\overline{AG}$.
(b) the angle $\overline{AG}$ makes with the plane ABCD.
(c) the angle that the plane HABG makes with plane ABCD.

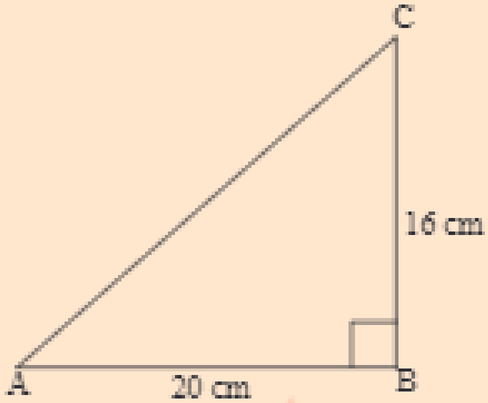


Solution

(a) Consider a right – angled triangle ACG as shown in the figure. In order to find $\overline{AG}$, first find $\overline{AC}$ from the right – angled triangle ABC.

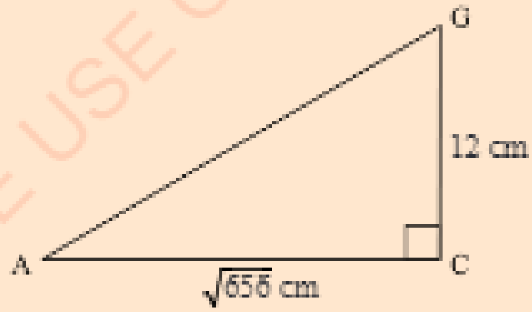
From Δ ABC, by using Pythagoras' theorem we have:

$$(\overline{AC})^2 = (\overline{AB})^2 + (\overline{BC})^2$$
$$= (20\text{ cm})^2 + (16\text{ cm})^2$$
$$= 400\text{ cm}^2 + 256\text{ cm}^2$$
$$= 656\text{ cm}^2$$
$$\overline{AC} = \sqrt{656}\text{ cm}$$



From Δ ACG, by using Pythagoras' theorem we have:

$$(\overline{AG})^2 = (\overline{AC})^2 + (\overline{CG})^2$$
$$= (\sqrt{656}\text{ cm})^2 + (12\text{ cm})^2$$
$$= 656\text{ cm}^2 + 144\text{ cm}^2$$
$$= \sqrt{800\text{ cm}^2}$$
$$\overline{AG} = 28.28\text{ cm}.$$



Therefore, the length of $\overline{AG}$ is 28.28 cm.

(b) The projection of $\overline{AG}$ on the plane ABCD is $\overline{AC}$. The angle between $\overline{AG}$ and the plane ABCD is $C\hat{A}G$. Calculation of $C\hat{A}G$ is done as follows:

Since $\overline{AG}$ = 28.28 cm, then

$\sin(C\hat{A}G) = \dfrac{\overline{CG}}{\overline{AG}}$, from ΔACG

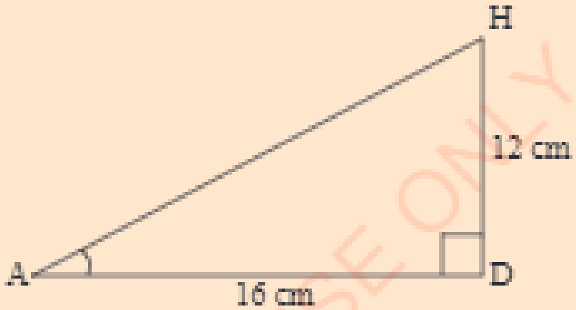
$$\sin(C\hat{A}G) = \frac{12 \text{ cm}}{28.28 \text{ cm}} = 0.4243$$

$$C\hat{A}G = \sin^{-1}(0.4243)$$

$$= 25°6'.$$

Therefore, the angle $\overline{AG}$ makes with the plane ABCD is 25°6′.

(c) The intersection of a line between a plane HABG with the plane ABCD is $\overline{AB}$. The angle which HABG makes with ABCD is $H\hat{A}D$ or $G\hat{B}C$. Consider the triangle ADH:



From Δ ADH, we have:

$$\tan(D\hat{A}H) = \frac{\overline{DH}}{\overline{AD}}$$

$$= \frac{12 \text{ cm}}{16 \text{ cm}} = 0.75$$

$$D\hat{A}H = \tan^{-1}(0.75)$$

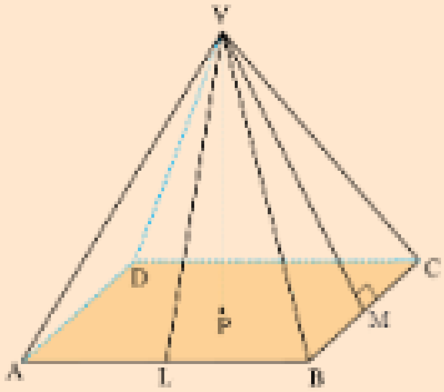
$$= 36°52'$$

Thus, $D\hat{A}H = 36°52'$.

Therefore, the angle that the plane HABG makes with plane ABCD is 36°52′.

Example 3.2

The following figures is a right angle rectangular pyramid of base ABCD, vertex V, and centre of base P.



If $\overline{AB} = \overline{CD} = 8$ cm, $\overline{AD} = \overline{BC} = 6$ cm, and $\overline{VA} = \overline{VB} = \overline{VC} = \overline{VD} = 13$ cm, calculate:

(a) the height $\overline{VP}$ of the pyramid.

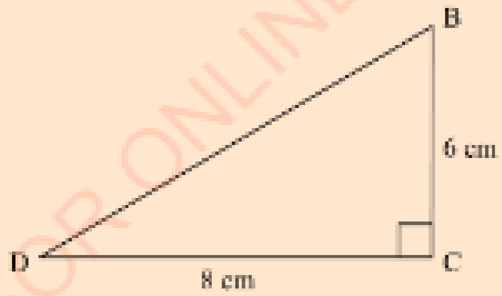
(b) the angle between $\overline{VB}$ and the base ABCD.

(c) the angle between triangle VBC and the base ABCD.

(d) the angle between triangle VCD and the base ABCD.

Solution

(a) Consider a right – angled triangle BCD in the pyramid (as shown in the figure).



From the figure, by using Pythagoras' theorem we have

$$(\overline{BD})^2 = (\overline{BC})^2 + (\overline{CD})^2$$
$$= (6 \text{ cm})^2 + (8 \text{ cm})^2$$
$$= 100 \text{ cm}^2$$
$$\overline{BD} = \sqrt{100 \text{ cm}^2}$$

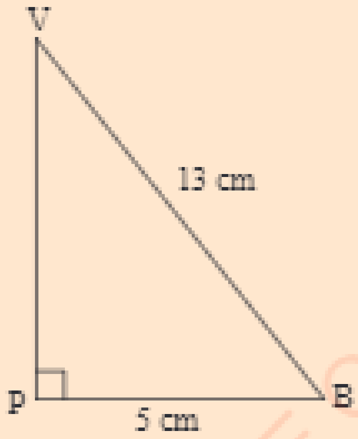
Hence, $\overline{BD} = 10$ cm

To calculate the length of $\overline{VP}$, consider the right – angled triangle VPB in the pyramid.

From the given figure, we have:

$$\overline{BP} = \frac{1}{2}\overline{BD}$$

$$= \frac{1}{2} \times 10 \text{ cm} = 5 \text{ cm}$$



Then by using pythagoras' theorem, we have:

$$(\overline{VP})^2 + (\overline{PB})^2 = (\overline{VB})^2$$

$$(\overline{VP})^2 = (13 \text{ cm})^2 - (5 \text{ cm})^2$$

$$= 169 \text{ cm}^2 - 25 \text{ cm}^2$$

$$= 144 \text{ cm}^2$$

$$\overline{VP} = \sqrt{144 \text{ cm}^2}$$

$$\overline{VP} = 12 \text{ cm}.$$

Therefore, the height $\overline{VP}$ of the pyramid is 12 cm.

(b) The angle between $\overline{VB}$ and the base ABCD is $V\hat{B}P$.

Since $\overline{VP} = 12$ cm and $\overline{BP} = 5$ cm, then the angle VBP is obtained as follows:

$$\tan(V\hat{B}P) = \frac{\overline{VP}}{\overline{PB}}$$

$$= \frac{12 \text{ cm}}{5 \text{ cm}} = 2.4$$

$$V\hat{B}P = \tan^{-1}(2.4)$$

$$= 67°23'$$

Thus, $V\hat{B}P = 67°23'$.

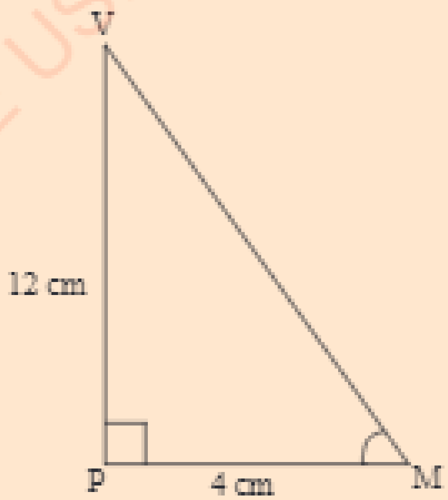
Therefore, the angle between $\overline{VB}$ and the base ABCD is $67°23'$.

(c) The angle between triangle VBC and the base ABCD is $V\hat{M}P$.

The lines $\overline{VP}$ and $\overline{PM}$ are both perpendicular to $\overline{BC}$ and M is the midpoint of $\overline{BC}$. Consider a right – angled triangle VMP in the pyramid.

From the figure we have:

$\overline{PM} = \frac{1}{2}\overline{DC} = 4$ cm



Thus, $\tan(V\hat{M}P) = \frac{\overline{VP}}{\overline{PM}}$

$$= \frac{12\text{cm}}{4\text{cm}} = 3$$

$$V\hat{M}P = \tan^{-1}(3)$$

Thus, $V\hat{M}P = 71°34'$.

Therefore, the angle between triangle VBC and the base ABCD is $71°34'$.

(d) The angle between triangle VCD and the base ABCD is $V\hat{N}P$. To find $V\hat{N}P$, consider a right – angled triangle VNP in the pyramid.

From triangle VNP we have:

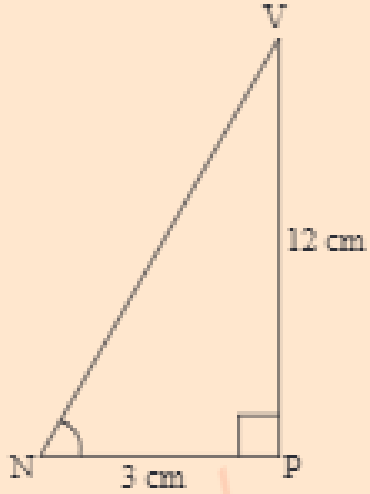
$$\tan(V\hat{N}P) = \frac{\overline{VP}}{\overline{PN}}$$

$$= \frac{12 \text{ cm}}{3 \text{ cm}}$$

$$= 4$$

$$V\hat{N}P = \tan^{-1}(4)$$

Thus, $V\hat{N}P = 75°58'$.



Therefore, the angle between triangle VCD and the base ABCD is 75°58′.

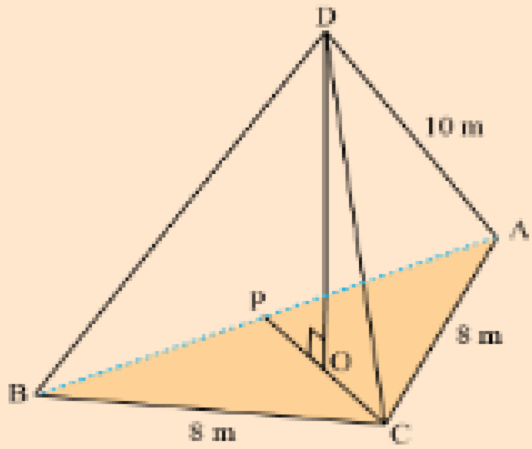
Example 3.3

The following figure ABCD is a right triangular pyramid with the baseABC which is an equilateral triangle. If $\overline{AB} = \overline{BC} = \overline{CA} = 8$ m and $\overline{DA} = \overline{DB} = \overline{DC} = 10$ m, calculate:

(a) the height $\overline{DO}$ of the pyramid.

(b) the angle which $\overline{BD}$ makes with the plane ABC.

(c) the angle between the planes DAB and ABC.



Solution

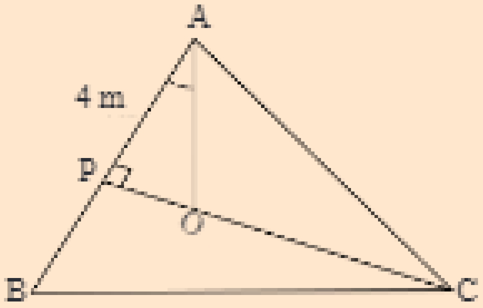
(a) The perpendicular $\overline{DO}$ from D to the plane ABC meets ABC at the centre of the triangle ABC. Since P is the midpoint of $\overline{AB}$, then,

$$\overline{AP} = \frac{1}{2}\overline{AB}$$

$$\overline{AP} = \frac{1}{2} \times 8\text{ m}$$

$$= 4\text{ m}$$

$$P\hat{A}O = \frac{1}{2}P\hat{A}C = \frac{1}{2} \times 60^\circ = 30^\circ$$



Consider the base ABC of the pyramid and the triangle APO, then

$$\cos P\hat{A}O = \frac{\overline{PA}}{\overline{AO}},$$

$$\overline{AO} = \frac{\overline{PA}}{\cos 30^\circ} = \frac{4\text{ m}}{0.866} = 4.62\text{ m}$$

Also,

$$\tan(P\hat{A}O) = \frac{\overline{PO}}{\overline{PA}}$$

$$\tan 30^\circ = \frac{\overline{PO}}{4\text{ m}}$$

$$\overline{PO} = 4\text{ m} \times 0.5774$$

$$= 2.31\text{ m}$$

From a right –angled triangle DAO, by using Pythagoras' theorem we have:

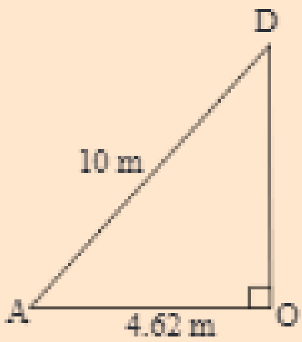
$$(\overline{DO})^2 + (\overline{OA})^2 = (\overline{DA})^2$$

$$(\overline{DO})^2 = (\overline{DA})^2 - (\overline{OA})^2$$

$$= (10\text{ m})^2 - (4.62\text{ m})^2$$

$$= 100\text{ m}^2 - 21.34\text{ m}^2$$

$$\overline{DO} = \sqrt{78.66\text{m}^2} = 8.87\text{m}$$



Therefore, the height $\overline{DO}$ of the pyramid is 8.87 m.

(b) The projection of $\overline{BD}$ on the plane ABC is $\overline{BO}$. The angle $\overline{BD}$ makes with the plane ABC is $D\hat{B}O$. Consider the right – angled triangle DBO.

We know that $\overline{AO} = \overline{BO} = 4.62$ m.

From ΔDBO, we have:

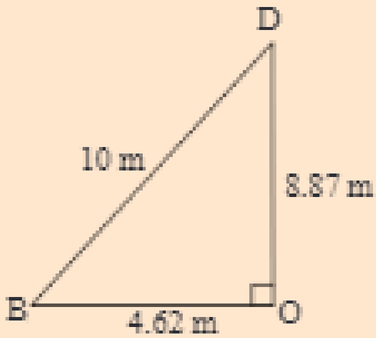
$$\sin(D\hat{B}O) = \frac{\overline{DO}}{\overline{DB}}$$

$$\sin(D\hat{B}O) = \frac{8.87 \text{ m}}{10 \text{ m}}$$

$$= 0.887$$

$$D\hat{B}O = \sin^{-1}(0.887)$$

$$D\hat{B}O = 62°30'.$$



Therefore, the angle which $\overline{BD}$ makes with the plane ABC is $62°30'$.

(c) The planes DAB and ABC intersect in $\overline{AB}$. The lines $\overline{OP}$ and $\overline{PD}$ are perpendicular to $\overline{AB}$. The angle between the planes DAB and ABC is $D\hat{P}O$. Consider the triangle DPO.

From triangle DPO, we have:

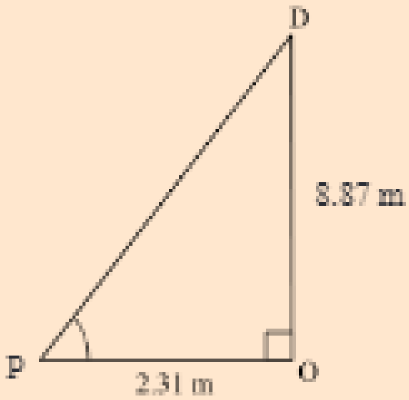
$$\tan(D\hat{P}O) = \frac{\overline{DO}}{\overline{OP}}$$

$$\tan(D\hat{P}O) = \frac{8.87 \text{ m}}{2.31 \text{ m}}$$

$$D\hat{P}O = \tan^{-1}(3.84)$$

Thus, $D\hat{P}O = 75°24'$.

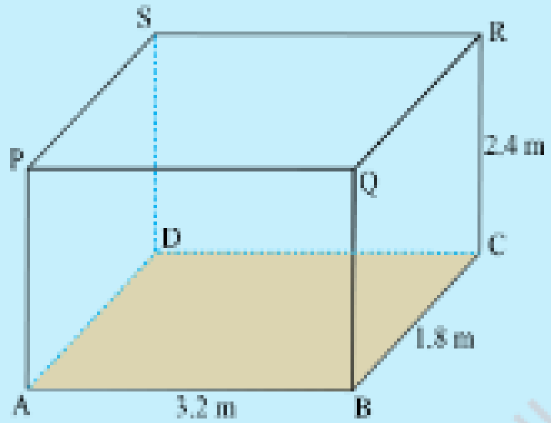
Therefore, the angle between the planes DAB and ABC is $75°24'$.



Exercise 3.4

Answer the following questions:

1. A box has a rectangular base ABCD and P, Q, R, S are vertically above, A, B, C, D, respectively as shown in the following figure:



If $\overline{AB} = 3.2$ m, $\overline{BC} = 1.8$ m and $\overline{CR} = 2.4$ m, find:

(a) the length of diagonal $\overline{BR}$.

(b) the length of the diagonal $\overline{AR}$.

(c) the angle between the diagonal $\overline{AR}$ and the base ABCD.

(d) the angle between the plane ADRQ and the base ABCD.

2. Given a pyramid APQRS on a square base PQRS of side 10 cm.

If $\overline{AP} = \overline{AQ} = \overline{AR} = \overline{AS} = 8$ cm, find:

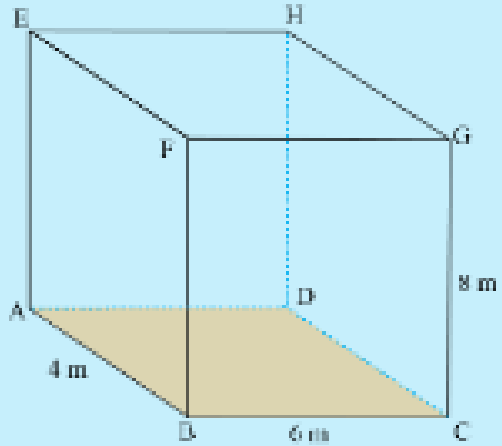
(a) the height of the pyramid.

(b) the angle between two opposite faces.

(c) the angle between a face and the base.

(d) the angle between the edges $\overline{SA}$ and $\overline{QA}$.

3. The following figure is a right prism.

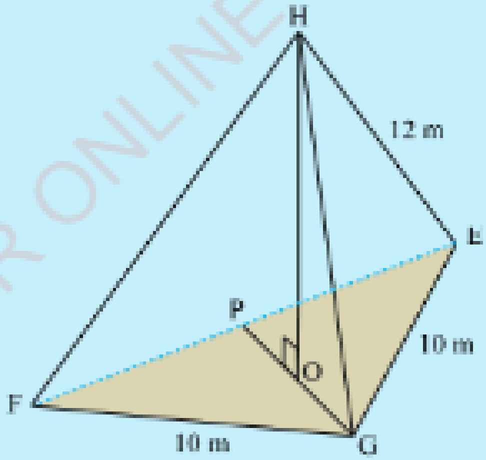


Find:

(a) the length of the diagonal $\overline{FD}$.

(b) the angle between the diagonal $\overline{BH}$ and plane ADHE.

4. The following figure EFGH is a right triangular pyramid with base EFG which is an equilateral triangle. If $\overline{EF} = \overline{FG} = \overline{GE} = 10$ m and $\overline{HE} = \overline{HF} = \overline{HG} = 12$ m.

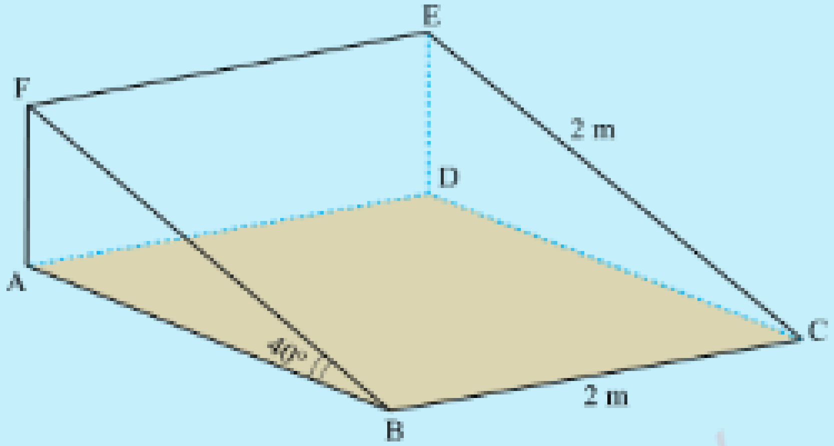


Calculate:

(a) the height $\overline{HO}$ of the pyramid.

(b) the angle between the planes HEF and EFG.

5. The following figure shows a drawing table with a square top BCEF of length 2 m inclined at an angle 40° to the base ABCD. Find the inclination of the diagonal $\overline{BE}$ to the horizontal.



6. The following figure shows a right pyramid OPQRS of height 10 cm on a horizontal rectangular base of dimensions 6 cm by 8 cm.

(a) Calculate the inclination of the following to the horizontal:

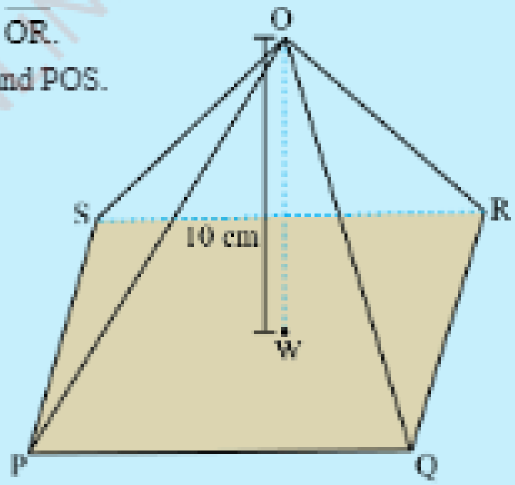
(i) the face OQR.

(ii) the edge $\overline{OQ}$.

(b) Calculate the angle between:

(i) the lines $\overline{SR}$ and $\overline{OR}$.

(ii) the planes ORQ and POS.



Activity 3.4: The concept of surface area of three – dimensional figures

In a group or individually, perform the following tasks:

1. Draw a net diagram of a closed cylinder with radius r and height h.
2. Find the surface area of the top and bottom part of the cylinder drawn in task 1 in terms of r and h.
3. Find the surface area of the folded net in terms of r and h.
4. Share your findings with your neighbour.

Right circular cone

A right circular cone is a cone whose vertex is vertically above the centre of the base of the cone. Consider a right circular cone of radius r, height h, and length of slant height l.

Let AB, BC, CD, DE etc., be approximately small line segments along the circular base forming small triangles VAB, VBC, VCD, VDE etc. as shown in Figure 3.23.

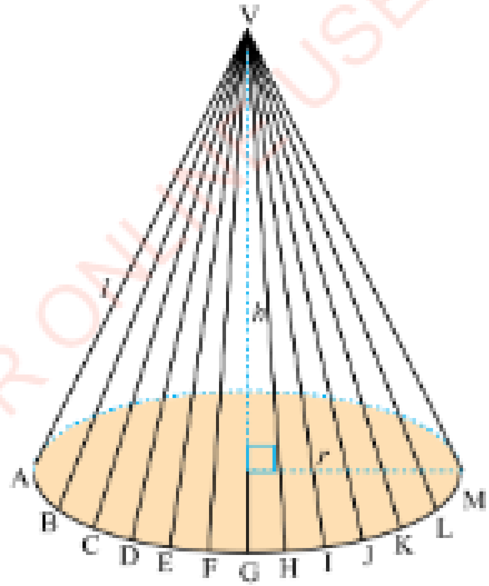


Figure 3.23: *A right circular cone*

Area of curved surface (lateral surface) of a right circular cone is the sum of areas of all small triangles VAB, VBC etc.

Area of curved surface of a right circular cone

$$=\frac{1}{2}(l\times\overline{AB})+\frac{1}{2}(l\times\overline{BC})+\cdots$$

$$=\frac{1}{2}l(\overline{AB}+\overline{BC}+\cdots)$$

The sum of the lengths of the line segments that is $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\cdots$ gives the circumference of the circular base of the cone. That is

$$\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\cdots=2\pi r.$$

Thus,

Area of curved surface of a right circular cone $=\frac{1}{2}l\times 2\pi r$

$=\pi rl$

Therefore, the area of the curved surface of the right circular cone is πrl.

Recall that the surface area of circular base is πr^2. Hence, the surface area of a right circular cone is given by:

The surface area of a right circular cone = the surface area of a circular base + area of curved surface.

The surface area of a right circular cone $=\pi r^2+\pi rl$

Therefore, the surface area of a right circular cone is $\pi r(r+l)$.

Example 3.5

Find the surface area of a right circular cone whose slant height is 10 cm and whose base radius is 8 cm (use $\pi = 3.14$).

Solution

The surface area of the right circular cone $=\pi r(r+l)$

$=3.14\times 8\text{ cm}\times(8\text{ cm}+10\text{ cm})$

$=452.16\text{ cm}^2$.

Therefore, the surface area of the right circular cone is 452.16 cm^2.

Right cylinder

Consider a right cylinder of radius r and height h, as shown in Figure 3.24 (a). The curved surface or lateral surface of the cylinder looks like that in Figure 3.24 (b).

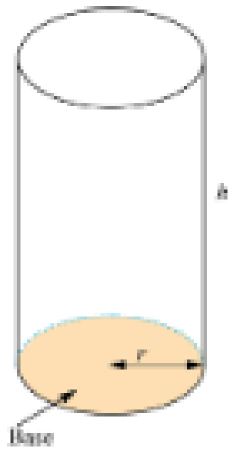


Figure 3.24 (a): *Right cylinder*

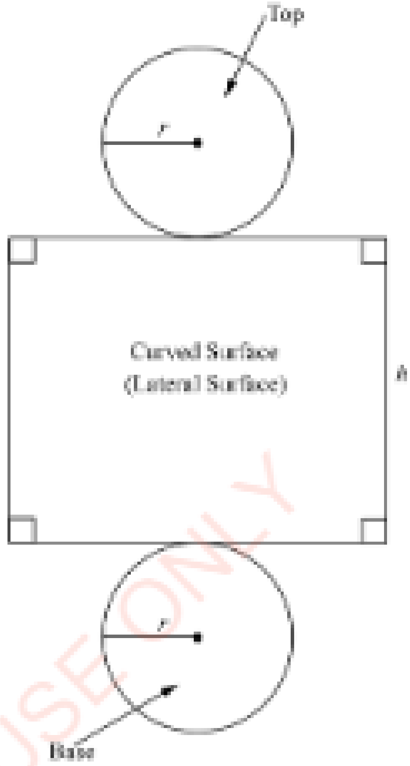


Figure 3.24 (b): *Lateral and circular bases of a right cylinder*

Area of the curved surface of the cylinder = Circumference of the base × height

$= 2\pi r \times h$

$= 2\pi rh$

Area of two circular bases of the cylinder = 2 × Area of a circular base

$= 2 \times \pi r^2$

$= 2\pi r^2$

Thus, the surface area of the cylinder

= Area of the curved surface + Area of the two circular bases

$= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

Therefore, the surface of a right cylinder is $2\pi r(h + r)$.

Example 3.5

Find the surface area of a right cylinder having base radius 10 cm and height 20 cm (Use $\pi = 3.14$).

Solution

Let, the radius $r = 10$ cm and the height $h = 20$ cm.

The surface area of the right cylinder $= 2\pi r(r + h)$

$= 2 \times 3.14 \times 10 \text{ cm} \times (10 \text{ cm} + 20 \text{ cm})$

$= 1\,884 \text{ cm}^2$

Therefore, the surface area of the right cylinder is $1\,884 \text{ cm}^2$.

Right pyramid

A right pyramid is a pyramid in which the slant edges joining the vertex to the corner of the base are equal. Consider a right rectangular pyramid as shown in Figure 3.25.

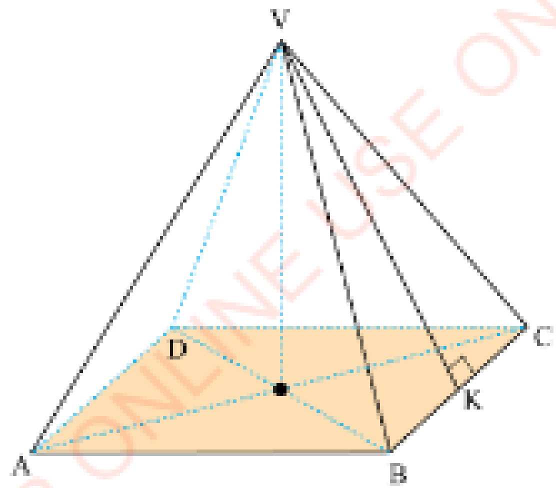


Figure 3.25: *Right rectangular pyramid*

In Figure 3.25, the lengths of $\overline{VA}$, $\overline{VB}$, $\overline{VC}$, and $\overline{VD}$ are equal and are called slant edges. The line $\overline{VK}$ is the line segment joining the vertex to the midpoint of a side of the base.

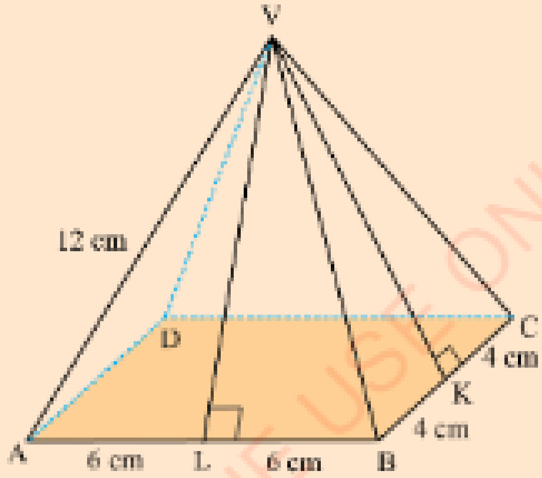
Area of one face, say VBC is:

$$\text{Area of VBC} = \frac{1}{2}\overline{BC} \times \overline{VK}.$$

The surface area of a rectangular pyramid = Area of lateral surface + Area of the base
= Area of ΔVAB + Area of ΔVBC + Area of ΔVCD + Area of ΔVAD + Area of ABCD
Generally, for any right pyramid:
the surface area = Area of lateral surface + Area of the base.

Example 3.6

A right rectangular pyramid is such that the rectangle is 12 cm by 8 cm and each slant edge is 12 cm as shown in the following figure. Find the surface area of the pyramid.



Solution

Consider a right – angled triangle VAL where $\overline{AL} = 6$ cm and $\overline{VA} = 12$ cm.

By Pythagoras' theorem, we have:

$$\overline{VL} = \sqrt{(\overline{VA})^2 - (\overline{AL})^2}$$

$$\overline{VL} = \sqrt{(12\text{ cm})^2 - (6\text{ cm})^2}$$

$$= \sqrt{108\text{ cm}^2}$$

$$= 6\sqrt{3}\text{ cm}$$

Also, for a right – angled triangle VKC, by Pythagoras' theorem we have:

$$(\overline{VK})^2 + (\overline{KC})^2 = (\overline{VC})^2$$

$$(\overline{VK})^2 = (\overline{VC})^2 - (\overline{KC})^2$$

$$\overline{VK} = \sqrt{(\overline{VC})^2 - (\overline{KC})^2}$$

$$\overline{VK} = \sqrt{(12\text{ cm})^2 - (4\text{ cm})^2}$$

$$= \sqrt{128\text{ cm}^2}$$

$$= 8\sqrt{2}\text{ cm}$$

Area of lateral surface = Area of ΔVAB + Area of ΔVCD + Area of ΔVBC + Area of ΔVAD

$$= 2 \times \text{Area of } \Delta VAB + 2 \times \text{Area of } \Delta VBC$$

$$= 2 \times \left(\frac{1}{2} \times \overline{AB} \times \overline{VL}\right) + 2 \times \left(\frac{1}{2} \times \overline{BC} \times \overline{VK}\right)$$

$$= 2\left(\frac{1}{2} \times 12\text{ cm} \times 6\sqrt{3}\text{ cm}\right) + 2\left(\frac{1}{2} \times 8\text{ cm} \times 8\sqrt{2}\text{ cm}\right)$$

$$= 72\sqrt{3}\text{ cm}^2 + 64\sqrt{2}\text{ cm}^2$$

$$= 215.2\text{ cm}^2$$

Area of the rectangular base $= \overline{AB} \times \overline{BC}$

$$= 12\text{ cm} \times 8\text{ cm}$$

$$= 96\text{ cm}^2$$

The surface area of the pyramid $= (215.2 + 96)\text{ cm}^2$

$$= 311.2\text{ cm}^2$$

Therefore, the surface area of the pyramid is 311.2 cm^2.

Right prism

A right prism is a prism in which each of the vertical edges are perpendicular to the plane of the base. An example of a right prism is shown in Figure 3.26. The lateral surface is made of faces EABF, FBCG, HDCG, and EADH. The bases are ABCD and EFGH.

Figure 3.26: *Right rectangular prism*

Area of lateral surface = $(\overline{AB} \times \overline{BF}) + (\overline{BC} \times \overline{CG}) + (\overline{DC} \times \overline{DH}) + (\overline{AD} \times \overline{DH})$

But, $\overline{AE} = \overline{BF} = \overline{CG} = \overline{DH}$.

Area of lateral surface = $(\overline{AB} + \overline{BC} + \overline{DC} + \overline{AD}) \times \overline{AE}$

= Perimeter of the base × height

Area of the bases = Area of ABCD + Area of EFGH

= 2 × Area of one base.

Generally, for any right prism, the surface area

= Area of lateral surface + Area of the bases.

Example 3.7

The height of a right prism is 4 cm and the perimeter of its base is 30 cm. Find the area of its lateral surface.

Solution

Area of lateral surface of a right prism = Perimeter of the base × height

= 30 cm × 4 cm

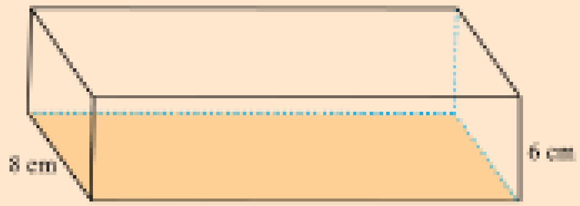
= 120 cm^2.

Therefore, area of lateral surface of the right prism is 120 cm^2.

Example 3.8

Find the surface area of a rectangular prism 12 cm long, 8 cm wide, and 6 cm high.

Solution



Area of lateral surface = Perimeter of the base × height

$= (12 \text{ cm} + 12 \text{ cm} + 8 \text{ cm} + 8 \text{ cm}) \times 6 \text{ cm}$

$= 240 \text{ cm}^2$

Area of the bases $= 2 \times 12 \text{ cm} \times 8 \text{ cm}$

$= 192 \text{ cm}^2$

The surface area of the rectangular prism = Area of lateral surface + Area of the bases

$= (240 + 192) \text{ cm}^2$

$= 432 \text{ cm}^2$

Therefore, the surface area of the rectangular prism is 432 cm^2.

Sphere

Consider a sphere of radius r as shown in Figure 3.27. The surface area of a sphere is given by $S = 4\pi r^2$.

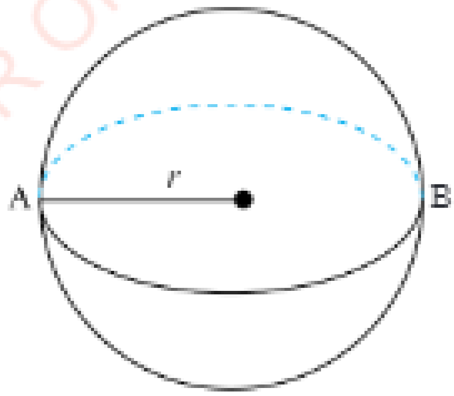


Figure 3.27: *Sphere of radius r*

Example 3.9

Find the surface area of a sphere of radius 5 cm (Use $\pi = 3.14$)

Solution

The surface area of a sphere $= 4\pi r^2$

$= 4 \times 3.14 \times 5 \text{ cm} \times 5 \text{ cm}$

$= 314 \text{ cm}^2$.

Therefore, the surface area of a sphere is 314 cm^2.

Exercise 3.5

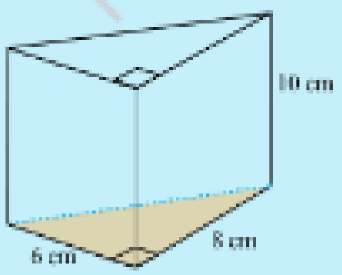
Answer the following questions:

(Use $\pi = 3.14$)

1. The altitude of a rectangular prism is 4 cm, the width and length of its base are 2 cm and 3 cm, respectively. Find the surface area of the prism.

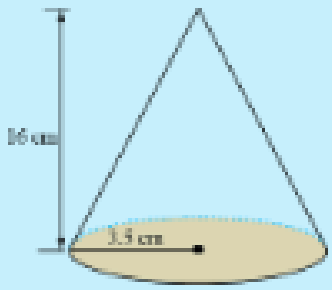
2. One side of a cube is 4 dm. Find:
   (a) the lateral surface area of the cube.
   (b) the total surface area of the cube.

3. The following figure is a right triangular prism whose base is a right – angled triangle. Calculate its surface area.



4. The radius of the base of a right circular cylinder is 7 dm and its height is 10 dm. Find its:
   (a) lateral surface area.
   (b) total surface area.

5. Calculate the lateral surface area of the following right circular cone:



6. The radius of the base of a right circular cone is 12 cm. If its lateral surface area is 72π cm^2, find its slant height.

7. Calculate the surface area of a regular tetrahedron of edge 6 units long.

8. The base of a right pyramid is a rectangle of dimensions 6 cm by 8 cm and the slant edges are each 9 cm long. Calculate its lateral surface area.

9. Find the surface area of a right circular cone whose height is 24 cm and slant height 25 cm.

10. Find the surface area of a right circular cylinder of diameter 8 cm and height 6 cm.

11. The surface area of a rectangular prism is 108 cm^2. If the base is 4 cm by 3 cm, find the height of the prism.

12. The altitude of a square pyramid is 5 units and a side of the base is 5 units long. Find the area of a horizontal cross – section at a distance 2 units above the base.

13. Find the surface area of a right circular cone of base radius 6 cm and height 8 cm.

14. One edge of the base of a square pyramid is 6 cm long. Each slant edge is 5 cm. Find the area of the lateral surface of the pyramid.

15. Calculate the surface area of a square pyramid having altitude 4 units and edge of the base 6 units.

16. Find the area of the curved surface of a right circular cone of base radius 3 cm and height 4 cm.

17. Find the surface area of a sphere with diameter 14 cm.

18. The radius of one sphere is twice the radius of a second sphere. What is the ratio of their surface areas?

19. If the earth is a sphere of radius 6 400 km, find its surface area.

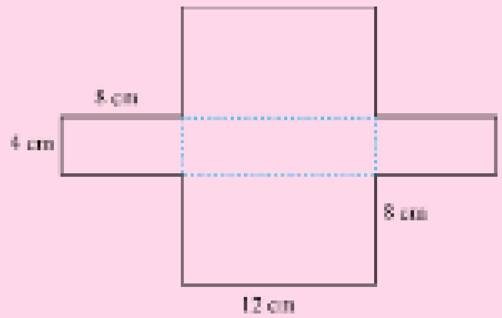
20. Find the radius of a sphere whose surface area is 108 cm^2.

Volume of three – dimensional figures

Activity 3.5: Estimation of volumes of different three – dimensional figures

Perform the following tasks individually or in your group:

1. Draw other alternative nets of the following figure on a manila sheet.



2. Cut out the nets drawn in task 1 and fold on the dashed line.
3. Tape the edges in task 2 together to form models of the solids with one face removed.
4. Construct a model of a rectangular pyramid whose base and height are the same as the prism in task 3.
5. Fill in the pyramid with rice, and pour the rice into the prism. Repeat until the prism is filled.
6. How many pyramids of rice did you take to fill the prism?
7. Compare the areas of the bases of the prism and pyramid.
8. Compare the height of the prism with that of the pyramid.
9. Estimate the volume of the prism and that of the pyramid.
10. Share your findings with your friends through discussion.

Volume of a right prism

Consider a right triangular prism shown in Figure 3.28.

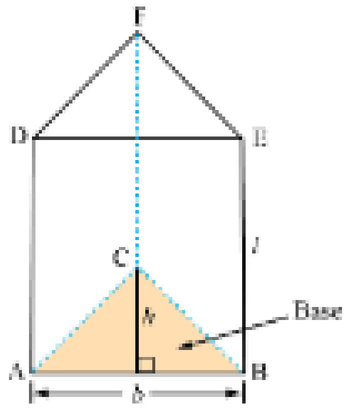


Figure 3.28: *A triangular prism*

If h is height of the prism and A is the area of triangle ABC, the volume of the prism is given by:

Volume of the prism = Area of ΔABC × length of the prism (l)

Generally, the volume of any right prism is equal to the product of the cross – sectional area and the length of the prism. That is,
Volume of any right prism = A × l, where A is the cross–sectional area of the base and l is the length of the prism. Thus,
Volume of any right prism $= \frac{1}{2}bh \times l$, where b is the length of the triangular base of the prism, h is the height of the triangular base, and l is the length of the prism.

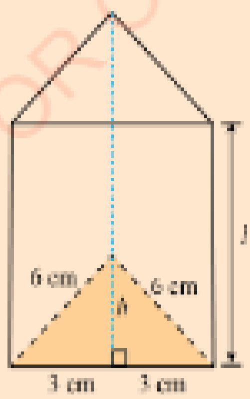
Similarly, for a rectangular prism, the volume is given by:
Volume of the rectangular prism = A × l, where A is the area of the rectangular base of the prism and l is the length of the prism.

Example 3.10

The volume of a triangular prism is 256 cm^3. If its base is an equilateral triangle with sides of length 6 cm, calculate the height and length of the prism.

Solution

Consider the following sketch of a triangular prism:



From the figure, calculate the height h, by using Pythagoras' theorem.

We have:

$h^2 + (3\text{ cm})^2 = (6\text{ cm})^2$

$h^2 = (36 - 9)\text{ cm}^2$

$h^2 = 27\text{ cm}^2$

$h = 3\sqrt{3}\text{ cm}$

$h = 5.196\text{ cm}$

Volume of the triangular prism $= \frac{1}{2}bh \times l$

$256\text{ cm}^3 = \frac{1}{2} \times 6\text{ cm} \times 5.196\text{ cm} \times l$

$31.176l = 512\text{ cm}$

$l = 16.42\text{ cm}$

Therefore, the height and length of the triangular prism are 5.196 cm and 16.42 cm respectively.

Example 3.11

Calculate the volume of a rectangular prism whose base is 8 cm by 5 cm and whose height is 10 cm.

Solution

Volume of the rectangular prism = Area of the base × height

$= (8\text{ cm} \times 5\text{ cm}) \times 10\text{ cm}$

$= 400\text{ cm}^3$

Therefore, the volume of the rectangular prism is 400 cm^3.

Volume of a right circular cylinder

Consider a right circular cylinder with radius r and height h as shown in Figure 3.29.

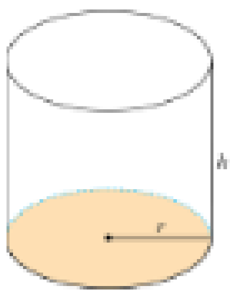


Figure 3.29: *A right circular cylinder*

The volume of a right circular cylinder is equal to the product of the area of the base and the height. If V is the volume, A is the area of the base and h is the height, then

$$\begin{aligned} V &= A \times h \\ &= \pi r^2 \times h \\ &= \pi r^2 h \end{aligned}$$

Therefore, the volume of the right circular cylinder is $\pi r^2 h$.

Example 3.12

Calculate the radius of a right circular cylinder of volume 1 570 m^3 and height 20 m (Use $\pi = 3.14$).

Solution

Volume of a right circular cylinder, $V = \pi r^2 h$ where height $h = 20$ m

Thus, 1 570 $m^3 = 3.14 \times r^2 \times 20$ m

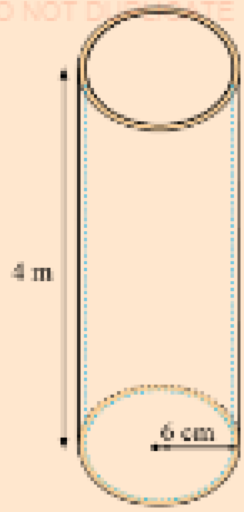
$$r^2 = \frac{1\,570 \text{ m}^3}{62.8 \text{ m}} = 25 \text{ m}^2$$

$$r = 5 \text{ m}$$

Therefore, the radius of the right circular cylinder is 5 m.

Example 3.13

A pipe made of metal 1 cm thick, has an external radius of 6 cm as shown in the figure. Find the volume of metal used in making a 4 m height of the pipe (use $\pi = 3.14$).



Solution

Let R represents the external radius, r represents the inner radius, and h represents the length of the pipe.

Volume of a metal used to make the pipe equals to the difference of the volume of the big pipe and that of the smaller pipe.

Volume of a metal used to make the pipe $= \text{Area of cross}-\text{section} \times \text{height}$

$= \pi(R^2 - r^2)h$

$= \pi(R - r)(R + r)h$

Given R = 6 cm, $r = 5$ cm, $h = 4$ m = 400 cm, then we have that

$$\text{Volume of metal, V} = 3.14(6\text{ cm} - 5\text{ cm})(6\text{ cm} + 5\text{ cm}) \times 400\text{ cm}$$

$$= 3.14 \times 1 \times 11 \times 400\text{ cm}^3$$

$$= 13\ 816\text{ cm}^2$$

Therefore, the volume of metal used is 13 816 cm^3.

Exercise 3.6

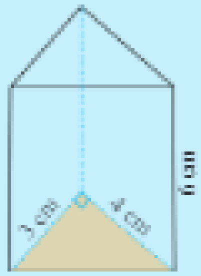
Answer the following questions:

(use $\pi = 3.14$)

1. Calculate the volume of the following right prisms.

(a)

(b)



2. Find the volume of a wire of length 1 km with circular cross–section of radius 0.5 cm. Give your answer in cubic metres.
3. Eight litres of water are poured into a cylindrical jug of inside diameter 20 cm. Calculate the height of the water level.
4. The cross – section of a right prism 18 dm long is an isosceles triangle with sides of dimensions 5 dm, 5 dm, and 8 dm. Calculate its volume.
5. The volume of a right triangular prism is 1 200 cm^3. If the magnitude of its base area is three times its eight, find the height of the prism.
6. A garden roller is 28 cm in diameter and 30 cm wide. Calculate its volume.
7. Find the volume of a right circular cylinder of diameter 80 mm and height 150 mm.
8. A cylindrical pipe of outer diameter 10 cm is 2 cm thick. What is the diameter of the hole? Find the volume of material in such a pipe of length 30 cm.
9. A lateral edge of a right prism is 4 cm and its base is an equilateral triangle with perimeter 30 cm. Calculate its volume.
10. Find the volume of a right prism whose base is a regular hexagon with a side of the base 4 cm long and its height is 10 cm.
11. The length of the diagonal of the base of a cube is 2.3 cm. Find the volume of the cube.
12. A water pipe made of material 1 cm thick has an external diameter of 8 cm. Find the volume of material of the pipe 100 m long.
13. The radius of the base of a right circular cylinder is 2.5 dm and its altitude is 5 dm. Find the volume of the cylinder.
14. The outside dimensions of a closed metal rectangular box are 16 cm, 28 cm, and 40 cm. If the metal is 2 cm thick, what are the inside measurements of the box? Find the volume of space inside the box.
15. The altitude of a right rectangular prism is 4 dm, and its base is 5 dm long and 3 dm wide. Find the volume of the prism.

Volume of pyramid

Consider a triangular prism shown in Figure 3.30 which is cut into three triangular pyramids AEFG, ACFG, and ADCF as shown in Figure 3.31.

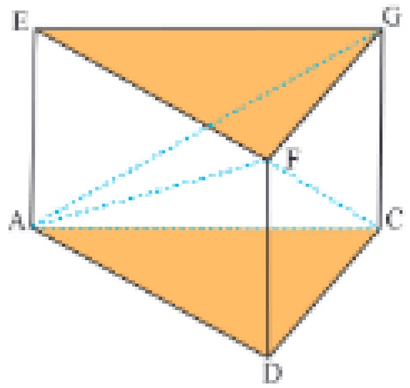


Figure 3.30: *Triangular prism*

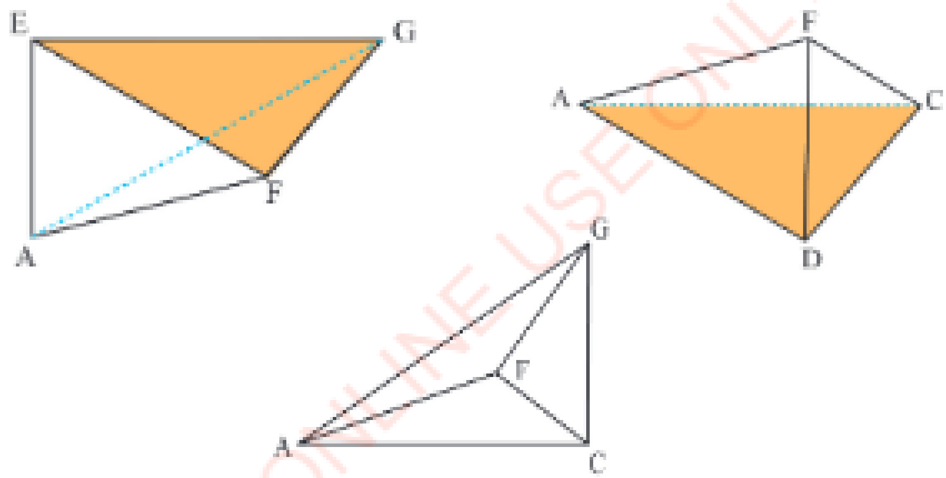


Figure 3.31: *Three triangular pyramids from the triangular prism*

The pyramids AEFG and ACFG with bases AEF and ACF, respectively have a common vertex F.

$\Delta AEF \cong \Delta ACF$ (by SSS theorem).

Therefore, the pyramids have the same base area. Since they have the same vertex F, they have the same altitude.

Volume of Pyramid AEFG = Volume of Pyramid ACFG.

Similarly, pyramids ADCF and ACFG with the bases DCF and FCG, respectively have a common vertex A.

Thus, $\Delta DCF \sim \Delta FCG$ (By SSS theorem).

Therefore, the pyramids have the same base area. Since the pyramids have the same vertex A, then they have the same altitude.

Thus, Volume of the pyramid ADCF = Volume of the pyramid ACFG.

Therefore, it follows that the three pyramids have the same volume.
But, Volume of prism = Base area × height.

Therefore, volume of each pyramid = $\frac{1}{3}$(Base area × height).

Generally, the volume of a pyramid is one third the product of its altitude and its base area.

Example 3.14

Calculate the volume of a square pyramid whose altitude is 10 cm and a base of length 6 cm.

Solution

Volume of the square pyramid = $\frac{1}{3}$(base area × height)

$V = \frac{1}{3}Bh$ where B is the base area and h is the attitude.

$= \frac{1}{3}(6 \text{ cm} \times 6 \text{ cm}) \times 10 \text{ cm}$

$= 120 \text{ cm}^3$

Therefore, the volume of the square pyramid is 120 cm^3.

Volume of a cone

Consider a cone of radius r and altitude h as shown in Figure 3.32.

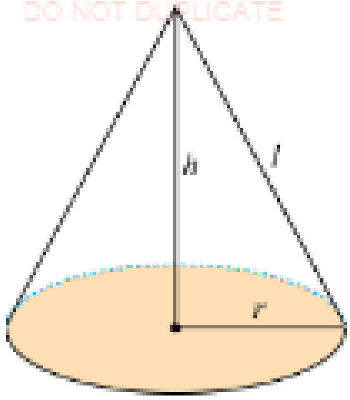


Figure 3.32: *A cone*

A base of a circular cone can be considered to be a regular polygon with an infinite number of sides. The volume of a cone can therefore be found using the method applied for pyramids. That is, Volume of a cone $\frac{1}{3}$(base area × height)

Thus, volume of a circular cone is one third the product of its altitude and its base area.

If the base is of radius r, the base area is πr^2.

Therefore, the volume of a cone is given by:

Volume of a cone = $\frac{1}{3}\pi r^2 h$, where h is the height of the cone and r is the radius.

Example 3.15

Calculate the volume of a cone of base radius 10 cm and altitude 12 cm (use $\pi = 3.14$).

Solution

Let, the radius r = 10 cm and the altitude h = 12 cm.

$$\text{Volume of the cone} = \frac{1}{3}(\text{Base area} \times \text{height})$$

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}(3.14 \times (10\text{ cm})^2) \times 12\text{ cm}$$

$$= 1\,256\text{ cm}^3$$

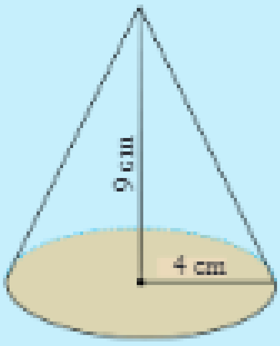
Therefore, the volume of the cone is 1 256 cm^3.

Exercise 3.7

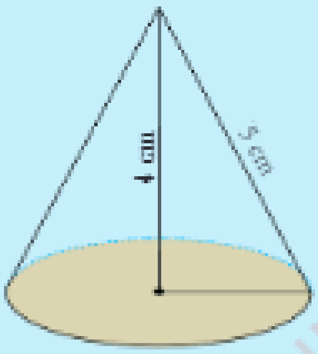
Answer the following questions:
(use $\pi = 3.14$)

1. Calculate the volume of each of the following right circular cones:

   (a)

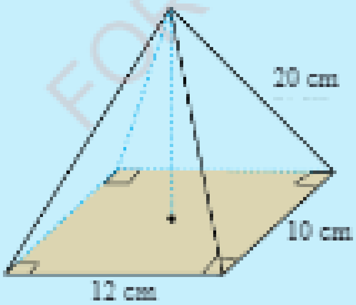


   (b)

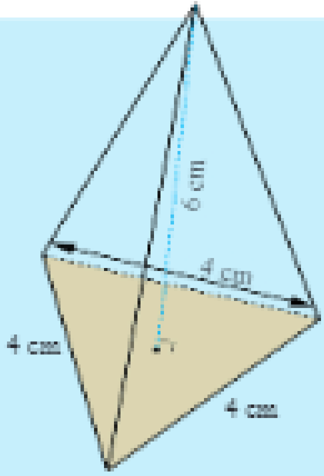


2. Calculate the volume of each of the following right pyramids:

   (a)



   (b)



3. Show that the volume of a square pyramid whose lateral faces are equilateral triangles of side x is $\frac{x^3\sqrt{2}}{6}$ cubic units.

4. Find the volume of a square pyramid of height 24 cm and slant edges 25 cm each.

5. Calculate the volume of a right pyramid if the area of the regular hexagonal base is 90 cm^2, and its altitude is 12 cm.

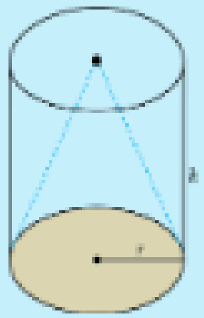
6. The base of a right rectangular pyramid is 8 cm, and each slant edge is 9 cm long. Find the volume of the pyramid.

7. The volume of a cone is 120 cm^3 and its altitude is 5 cm. Find its base area.

8. The slant height of a cone is 20 cm and the radius of its base is 12 cm. Find its volume in terms of π.

9. The radius of the base of a right circular cone is 6 cm, and its volume is 720π cm^3. Find its:
   (a) altitude.
   (b) slant height.

10. A cone is contained in a cylinder so that their base and height are the same as shown in the following figure. Calculate the volume of space lying inside the cylinder but outside the cone.



Chapter summary

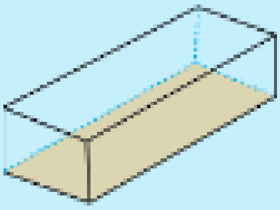
1. A three – dimensional figure is a geometric figure which has three – dimensions, namely length, width, and height. For example, cubes, prisms, cylinders, pyramids, and cones.
2. Rules to follow when sketching a net diagram of various three – dimensional figures:
   (a) Parallel lines of the object are drawn parallel.
   (b) Vertical lines of the object are drawn up and down the page.
   (c) Hidden edges of the object are drawn dotted.
   (d) Construction lines to guide the eye are drawn thinly.
   (e) Right angles in the object are marked correctly in the drawings.
3. Rules to follow when finding angles between two planes, between a line and a plane and between two skew lines are:
   (a) Angle between two planes intersecting point must be considered.
   (b) The angle between the line and its projection on the plane, must be considered when finding an angle between a line and a plane.
   (c) Angle between two skew lines, is equal to the angle between parallel lines which intersect with skew lines.
4. The formulae for finding the surface areas of three – dimensional objects are:
   (a) Right circular cone:
   $A = \pi r(r + l)$
   (b) Right cylinder:
   $A = 2\pi r(r + h)$
   (c) The surface area of right pyramid is equal to Lateral surface area + The base areas
   (d) The surface area of right prism is equal to the base area + Lateral surface area
   (e) The surface area of a sphere:
   $A = 4\pi r^2$
5. The formulae for finding volumes of three – dimensional objects are:
   (a) Volume of a cylinder:
   $V = \pi r^2 h$
   (b) Volume of a pyramid:
   $V = \frac{1}{3}(\text{Base area} \times \text{Height})$
   (c) Volume of a cone:
   $V = \frac{1}{3}\pi r^2 h$
   (d) Volume of any right prism:
   $V = Al$

Revision exercise 3 (a)

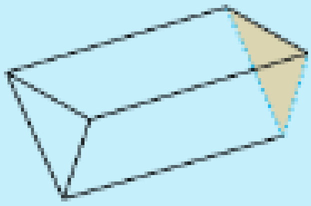
Answer the following questions:

1. Draw a net for each of the following geometrical figures:

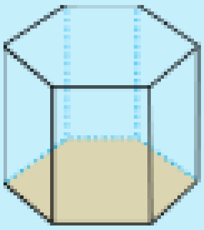
(a)



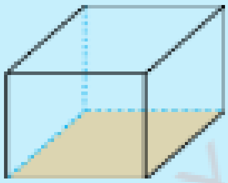
(b)



(c)

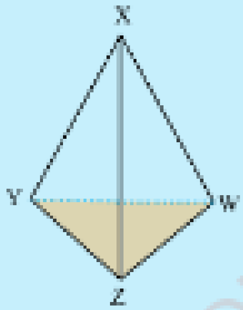


(d)

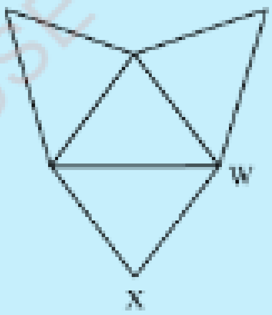


2. For each of the following prisms, copy its net and label the remaining vertices:

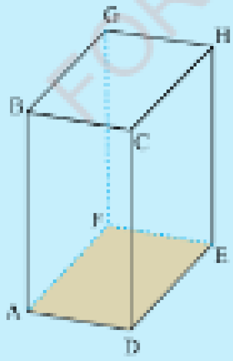
(a)



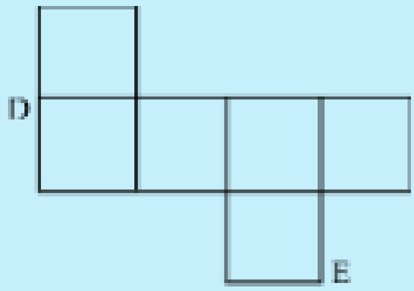
(b)



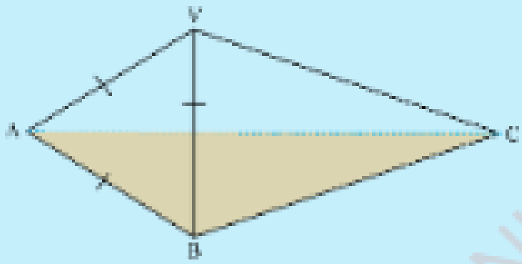
(c)



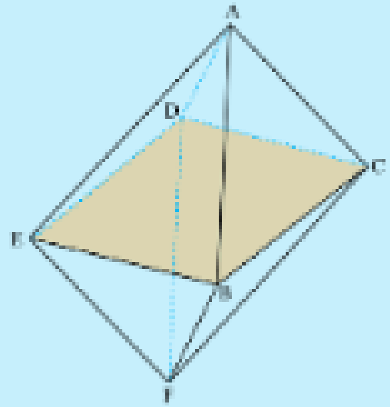
(d)



3. A right square pyramid ABCDV has a base ABCD of length 8 cm and height 6 cm vertically above the centre E of the base.
   (a) Draw an oblique projection of the pyramid.
   (b) Calculate the length BV.
   (c) Find the angle between the planes AVC and BVD.
   (d) Is it true that CV and AB are skew lines?
   (e) Calculate the angle between the planes BCV and ABCD.
4. In the following pyramid, the face VAB is an equilateral triangle with sides of length 6 cm, $\overline{CA}$ = 10 cm, $\overline{CB}$ = 10 cm and $\overline{CV}$ = 8 cm.



   (a) State which angles formed by adjacent edges are right angles.
   (b) Copy the figure and mark:
      (i) the angle between the planes VAB and CAB by x.
      (ii) the angle between planes VBC and VAC by y.
5. ABCD is the base and EFGH is the top face of a cube. $\overline{AG}$ is a diagonal of a cube and $\overline{AB}$ = 10 cm.
   (a) Calculate $\overline{AC}$ and $\overline{AG}$.
   (b) Find the angle between $\overline{AG}$ and the base ABCD.
6. The following figure shows a regular octahedron with 5 cm long for each edge.

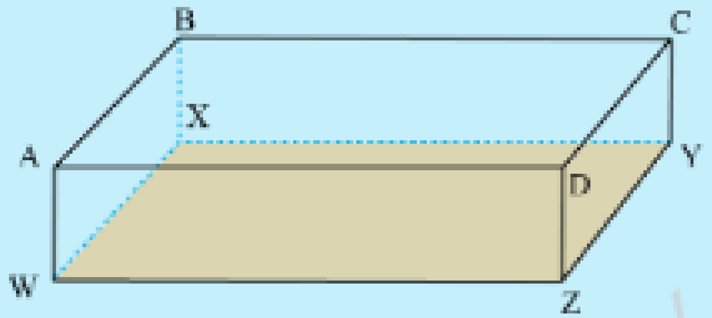


(a) Name two squares which are congruent to BCDE.

(b) Find the volume of the octahedron.

(Hint: Volume of pyramid $= \frac{1}{3}$ (Area of the base) × (Perpendicular height)).

7. A school hall is 28 m long, 21 m wide, and 12 m high as shown in the following figure.



Find the angles between:

(a) $\overline{WC}$ and the plane CDZY.

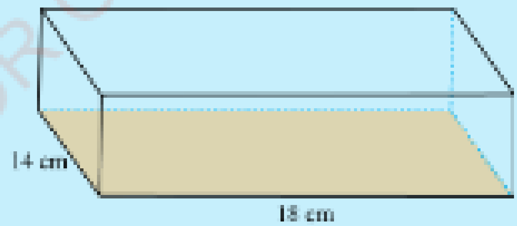
(b) the planes WBCZ and XBCY.

8. The base of a right pyramid is a rectangle of dimensions 12 cm by 8 cm and the slant edges are each 15 cm long. Calculate its:

(a) curved surface area.

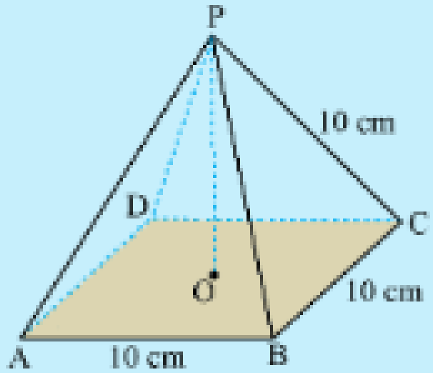
(b) surface area.

9. The following figure represents the right rectangular prism.



If the surface area of the prism is 1 144 cm^2, then find its height.

10. The base of the pyramid PABCD is a square whose side has length 10 cm. If each of the other faces is an equilateral triangle of side 10 cm as shown in the following figure:



Find:

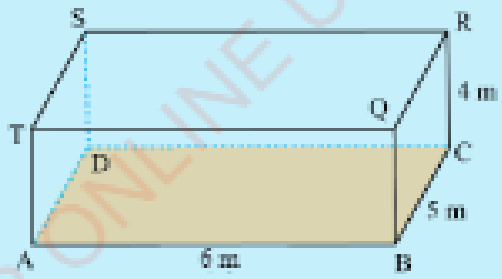
(a) the projection of $\overline{AP}$ to the base.

(b) the height of the pyramid $\overline{PO}$.

(c) the angle with which each face makes with the base.

(d) the angle with which the edge $\overline{PC}$ makes with the base.

11. The given figure represents a room of dimensions 6 m by 5 m by 4 m.



(a) Calculate the diagonal $\overline{AR}$.

(b) Find the angle $\overline{AR}$ makes with the floor.

(c) Find the angle with which the plane SABR makes with the floor.

12. The face of a desk is a rectangle ABCD, where $\overline{AB}$ = 2 cm and $\overline{BC}$ =1 cm, and slopes at 30° to the horizontal. Calculate the angle which the diagonal $\overline{AC}$ makes with the horizontal.

13. A right pyramid ABCDV has a square horizontal base ABCD of side of length 8 cm. If V is a vertex and $\overline{VA} = \overline{VB} = \overline{VC} = \overline{VD} = 12$ cm, calculate:
    (a) the perpendicular height of the pyramid.
    (b) the angle which $\overline{VA}$ makes with the horizontal.
    (c) the angle with which the plane VAB makes with the horizontal.
    (d) the angle between $\overline{VA}$ and $\overline{VC}$.

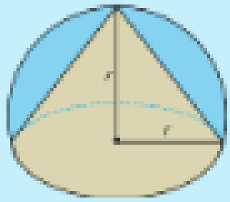
Revision Exercise 3 (b)

Answer the following questions:

(use $\pi = 3.14$)

1. Find the base radius of a right circular cone having height of 5 cm and a volume of 2 310 cm^3.
2. The base of a right pyramid is a square with a side of length 4 cm and the height of the pyramid is 6 cm. Find:
   (a) the volume of the pyramid.
   (b) the lateral surface area of the pyramid.
3. The base radius of a right circular cone is 6 cm and its height is 9 cm. Find in terms of π:
   (a) the volume of the cone.
   (b) the area of the curved surface.
4. The area of a regular pentagon inscribed in a circle of radius r is 15 cm^2. Calculate the diameter of the circle.
5. The lengths of two sides of a triangle are 10 cm and 12 cm. If the area of a triangle is 58 cm^2, find the value of the included angle.
6. A rectangular prism is 8 cm long, 7 cm wide, and 5 cm high. Calculate its:
   (a) volume.
   (b) surface area.
7. The area of a spherical surface is 616 cm^2. Find its radius in terms of π.
8. The radius of the base of a right circular cone is 10 cm, and its volume is 720π cm^3. Calculate its height.
9. Find the surface area of a sphere of diameter 18 cm.
10. A hole of radius 2 cm is drilled through the centre of a right circular cylinder whose base radius is 3 cm and altitude 4 cm. Find the surface area of the resulting figure.
11. A triangular prism is such that the sides of the base are 6 cm, 8 cm, and 10 cm. If the height of the prism is 20 cm, calculate its:
    (a) surface area.
    (b) volume.
12. Calculate the lateral surface area of a triangular prism if the sides of the base are 5 cm, 5 cm, and 8 cm, and its height is 8 cm.

13. Find a surface area of a rectangular prism 20 cm long, 12 cm wide, and 8 cm high.
14. Calculate the volume of a right regular pentagonal prism of radius 10 cm and height 10 cm.
15. A right circular cylinder 2 dm high holds 30 litres of water. Find its internal radius.
16. In a hemispherical solid, a conical part is removed as shown in following figure. Find the volume and surface area of the resulting solid figure.

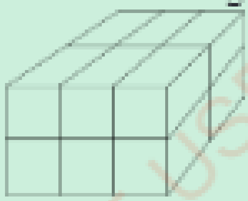


Project 3

1. Write down the methods for finding the surface areas of prisms, cylinders, pyramids, and cones. Hence, compare the methods.
2. Materials: cubes/blocks.

   Step 1: Make a prism like the one shown in the following figure.

   Step 2: Make at least three different rectangular prisms.



(a) Assume that the edge of each cube represents 1 unit then, the area of each surface of each cube is 1 square unit, and the volume of each cube is 1 cubic unit. Copy and complete the following table.

Prism	Area of base	Height	Volume
1st	6	2	12
2nd			
3rd			
4th			

(b) Find the area and volume of the prism.

(c) Describe how the areas of the base and the height of a prism are related to its volume.

3. Coastal Soap Company have hired you to design a new soap container. The company wants the container to be a right cylinder that will hold 150 cubic centimetres using as little material as possible. How would you design the container that will meet the needs of the company?

Chapter Four

Probability

Introduction

In everyday life, we are always faced with situations involving uncertainties. For example, when two equally strong football teams play a match, it is difficult to predict the outcome of the game. One team may win while the other loose the game or the match may end in a draw. Similarly, when a fair die is rolled once, it is not easy to tell in advance which of the numbers 1 to 6 will show up. In predicting such uncertainties, the theory of probability is applied. In this chapter, you will learn the meaning of probability, how to apply the knowledge of probability to determine the occurrence of an event in real life situations, to determine the probability of an event through experiments, interpret experimental results in relation to the real life situations. Morever, you will learn to calculate the probability of combined events using a formula, perform experiments of two combined events, and draw tree diagrams of combined events. The competencies developed will help you to predict the occurrence of some events in different fields such as science, technology, commerce, economics, social sciences, sports, and other daily life activities.

Meaning of probability

Activity 4.1: Defining probability

In a group or individually perform this activity using the following steps:

1. Mention any four coins which are in use in our country.
2. Identify the number of faces of each coin mentioned in step 1.
3. Identify the features found in each face of the coin mentioned in step 1.
4. Choose one coin and toss it 20 times. Record your observations or outcomes.
5. Using the result of step 4, estimate the probability of occurrence of each outcome.
6. Use step 5 to deduce the meaning of probability.
6. Share your findings with your neighbours for discussion.

The concept of probability occurs in different life experiences as illustrated by some practical experiments shown in Figure 4.1.

Figure 4.1: *Playing cards, dice, and coins*

Probability occurs when conducting practical experiments such as tossing a coin, rolling a die, playing cards, and many other experiments resulting into more than one outcome.

Probability in everyday life

In predicting uncertainties, the theory of probability is often applied. Probability is a branch of Mathematics which deals with the analysis of random phenomena. It provides quantitative occurrences of situations or events. In other words, it is a measure of chances.

Activity 4.2: Determining the measure of chances of an event

In a group or individually, perform this activity using the following steps:

1. Using manila sheet, draw a number line as shown in Figure 4.2.

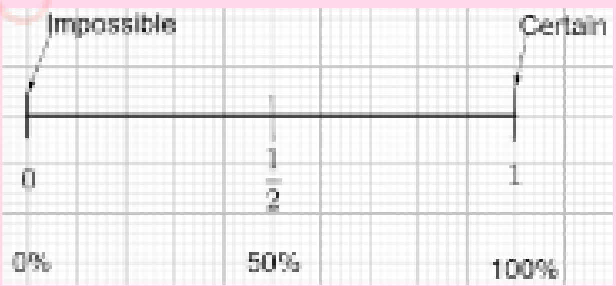


Figure 4.2: *A number line from 0 to 1*

2. Locate the estimate of occurrence for each of the following situations or events on the number line in step 1:
   (a) You will have Basic Mathematics homework tonight.
   (b) A baby to be born today will be a girl.
   (c) The local meteorologist predicts a 40% chance that it will rain tomorrow.
   (d) It will rain in your town in December.
   (e) A student chosen by chance from your class is wearing jean trousers.
   (f) A student will score 100% in final Basic Mathematics examination.
   (g) Your grandparents will live 200 years.
   (h) You will live forever.
3. Share your findings in step 2 with your neighbours for more inputs.

The sample space and the possible outcomes of an event

Suppose that a fair coin is tossed once. The expected results from this experiment are the head (H) or tail (T) showing up. The results of the experiment are called outcomes. A set of all expected outcomes from the experiment are called sample space, which is denoted by a letter S.

A specified outcome is called an event, and is denoted by a letter E. For example, in this experiment of tossing a fair coin once, the outcome that a head shows up is an event. It is a subset of the sample space. Thus, $S = \{H, T\}$ and $E = \{H\}$.

In an experiment, the event that may not occur is denoted by E'. For example, in tossing a fair coin once, if the head (H) shows up then an event set is written as $E = \{H\}$. The event that the head does not show up is written as $E' = \{T\}$.

Activity 4.3: Finding the sample space of the experiment

Perform each of the following experiment in a group or individually and write the respective sample space:

1. Roll a six sided fair die once.
2. Play cards.
3. Share your findings with your neighbours for more inputs.

Example 4.1

If a fair die is rolled once, find:

(a) the sample space of an experiment.

(b) the set of event for the occurence of an even numbers.

(c) the set of event that an even number does not occur.

Solution

(a) The sample space for the experiment is, S = {1, 2, 3, 4, 5, 6}.

(b) The set of event for the occurence of an even number is, E = {2, 4, 6}.

(c) The set of event that an even number does not occur is, E' = {1, 3, 5}.

Example 4.2

In an experiment of selecting an even number less than 20. Find an event set for the multiple of 3.

Solution

Let E denotes an event of multiple of 3.

The sample space for the experiment is, S = {2, 4, 6, 8, 10, 12, 14, 16, 18}.

Therefore, the event set for the multiple of 3 is, E = {6, 12, 18}.

Example 4.3

Find an event of not selecting an even number from a set of counting numbers less than 9.

Solution

Let E' denotes an event of not selecting an even number.

The sample space for the experiment is, S = {1, 2, 3, 4, 5, 6, 7, 8}

Thus, the event set for even numbers is, E = {2, 4, 6, 8}.

Therefore, the event of not selecting an even number is, E' = {1, 3, 5, 7}.

Exercise 4.1

Answer the following questions:

1. Write a sample space for each of the following experiments:
   (a) A fair die is rolled once and the face showing up is read.
   (b) Two teams play a football match.
   (c) A friend is asked for the month of his birthday.
   (d) Anna asked Juma the day of the week he visited his grandparents.
   (e) A card is drawn from a box containing five cards bearing the numerals 2, 4, 6, 8, and 10.

2. Write the set of specified events in each of the following experiments:
   (a) When a fair die is rolled, the number obtained is greater or equal to 5.
   (b) Odd numbers between 0 and 10 whose product is 3.
   (c) A prime number between 20 and 40 is chosen.
   (d) Cards with odd numbers chosen from a pack of cards bearing the numbers 1 up to 20.
   (e) Names of days of the week which begin with consonant letter.

3. Write in set notation the elements of each of the following events:
   (a) A fair die is rolled and the number showing up is not less than 3.
   (b) Names of months which do not begin with letter "J".
   (c) Odd numbers chosen between 4 and 15.
   (d) The complement of set E where the set of sample space is S = {a, b, c, d, e} and the set of event is E = {b, d, e}.
   (e) The complement of event set E when the possibility set is S = {Jane, Issa, Bakari} and E = {Jane, Issa , Bakari}.

Probability of an event

The probability of an event is determined from the following experiment.

Consider an experiment of tossing a fair coin 10 000 times and the results recorded as shown in Table 4.1.

Table 4.1: An experimental results of tossing a fair coin

Experiment	Number of tosses	Number of heads	Number of tails	Ratio of heads occurrences	Ratio of tails occurrences
First 100	100	54	46	0.54	0.46
First 500	500	254	246	0.508	0.492
First 1 000	1 000	501	499	0.501	0.499
First 5 000	5 000	2 516	2 484	0.5032	0.4968
All 10 000	10 000	4 979	5 021	0.4979	0.5021

Observations of column 5 and 6 in Table 4.1 reveal that, the ratio of the number of heads and the total number of tosses as well as that of the tails comes closer to 0.5 000 as the number of tosses increases. This ratio is called the **observed relative frequency of the event**. Using this experiments, it can be deduced that the probability of an event, P(E) is;

$$P(E) = \frac{\text{Number of outcomes in the event}}{\text{Number of outcomes in the sample space}}$$

or $$P(E) = \frac{n(E)}{n(S)},$$

where P(E) is the probability of an event, n(E) is the number of outcomes in the event and n(S) is the number of outcomes in the sample space. The probability found using experiments such as of Table 4.1 is called experimental probability.

Activity 4.4: Finding the relative frequency of an event

In a group or individually perform this activity using the following steps:

1. Toss a fair coin 100 times.
2. Record the results for the first 20, 40, 60, 80, and all 100 tosses in a form similar to Table 4.1.
3. Calculate:
   (a) the ratio of the number of heads occurrences to the total number of tosses.
   (b) the ratio of the number of tails occurrences and the total number of tosses
4. Compare your results with those obtained in Table 4.1.
5. Present your findings to the class for discussions.

Example 4.4

A drawing pin was tossed 1 000 times, where as the pin fell flat in 563 tosses. Find the probability that when such a pin is tossed, it will fall flat.

Solution

Let E represents the number of tosses where the pin fell flat.

Given n(E) = 563, n(S) = 1 000

$$\text{Probability of an event} = \frac{\text{Number of pins fell flat}}{\text{Total number of tosses}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{563}{1\,000} = 0.563$$

Therefore, the probability that the pin will fall flat is 0.563.

Example 4.5

A certain factory manufactured torch bulbs. If 5% of the manufactured torch bulbs were defective. What is the probability that a bulb from the factory will be non–defective?

Solution

Let E be 5%, the percentage of defective torch bulb.

E′ be 100% – 5% = 95%

the percentage of non defective torch bulb.

Thus, $n(E') = 95$

$n(S) = 100$

$$\text{Hence, } P(E') = \frac{n(E')}{n(S)}$$

$$= \frac{95}{100}$$

$$= 0.95$$

Therefore, the probability of a bulb being non–defective is 0.95.

In general, the formula for probability of an event is defined under conditions that every expected outcome has an equal chance of occurring as others.

Phrases such as selection at random and fair die or fair coin are used to imply that the choice is impartial, that is, each event have equal chance of occuring.

Steps for calculating the probability of an event:

1. Identify the elements of all the outcomes, that is, the sample space.
2. Define the elements of the event, that is, identify the set of events.
3. Calculate the probability of an event using the formula,

$$P(E) = \frac{n(E)}{n(S)}.$$

Example 4.6

A fair coin is tossed once. Find the probability that a head will occur.

Solution

Let E denotes the event that a head will occur. Hence, the sample space and event are S = {H, T} and E = {H} respectively.

Thus, n(S) = 2 and n(E) = 1.

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{1}{2}$$

Therefore, the probability that a head will occur is $\frac{1}{2}$.

Example 4.7

A piece of chalk is picked at random from a box containing 5 identical pieces, two of which are red and three are white. Find the probability that the piece of chalk picked is red.

Solution

Distinguish the red and white pieces of chalks by the letters r_1, r_2 and w_1, w_2, w_3 respectively.

Then, the sample space, $S = \{r_1, r_2, w_1, w_2, w_3\}$, thus n(S) = 5.

The event, $E = \{r_1, r_2\}$, thus n(E) = 2

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{2}{5}$$

Therefore, the probability that a piece of chalk picked is red is $\frac{2}{5}$.

Example 4.8

Find the probability that a king appears in drawing a single card from an ordinary deck of 52 cards.

Solution

Let E denotes the event that a king appears.

$n(S) = 52$, $n(E) = 4$ since there are 4 kings in one ordinary deck of 52 cards.

$$P(E) = \frac{n(E)}{n(S)}$$
$$= \frac{4}{52}$$
$$= \frac{1}{13}$$

Therefore, the probability that a king appears is $\frac{1}{13}$.

Remember that the event that E does not happen is given by the complement of E, which is E'. From set notation, the set of all outcomes is S and are related to those in E and E' by the following relation:

$$n(S) = n(E) + n(E')$$

$$\text{But, } P(E) = \frac{n(E)}{n(S)} \quad (1)$$

$$P(E') = \frac{n(E')}{n(S)} \quad (2)$$

Then adding equations (1) and (2) gives:

$$P(E) + P(E') = \frac{n(E)}{n(S)} + \frac{n(E')}{n(S)}$$
$$= \frac{n(E) + n(E')}{n(S)}$$
$$= \frac{n(S)}{n(S)}, \text{ where } n(E) + n(E') = n(S)$$
$$= 1.$$

Hence, $P(E) + P(E') = 1$ which can also be written as $P(E') = 1 - P(E)$ or $P(E) = 1 - P(E')$; where P(E) is the probability of an event E to occur and P(E′) is the probability of an event E not to occur.

Note: The probability of an event is 1 when there is complete certainty that it will occur. The complete certainty is also known as a sure event. The probability of an event is 0 when the event is impossible to occur. Generally, for any event E, its respective probability lies between 0 and 1 inclusively; that is, $0 \le P(E) \le 1$ and can be expressed as fraction, decimal or percentage. The closer the probability of an event to 0, the less likely the event is to occur. The closer the probability of an event to 1, the more likely the event to occur. In set notation, an impossible event is denoted by ∅ or { } (empty set).

Example 4.9

Find the probability of not getting a prime number when a fair die is rolled once.

Solution

The sample space, S = {1, 2, 3, 4, 5, 6}, thus n(S) = 6

Let E denotes the event of getting a prime number.

Then, E = {2, 3, 5}

$E' = \{1, 4, 6\}$

$$P(E') = \frac{n(E')}{n(S)}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

Therefore, the probability of not getting a prime number is $\frac{1}{2}$.

Example 4.10

The probability of selecting a red ball from a box containing red and green balls is $\frac{1}{4}$. Find the probability of not selecting a red ball.

Solution

Let E be event of selecting a red ball, then E' is the event of not selecting a red ball.

$$\text{From } P(E) + P(E') = 1$$

$$\text{Then, } P(E') = 1 - P(E), \text{ but } P(E) = \frac{1}{4}$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

Therefore, the probability of not selecting a red ball is $\frac{3}{4}$.

Example 4.11

When rolling a fair die, what is the probability of getting a number greater or equal to 1?

Solution

Let E denotes the event of getting a number greater than or equal to 1.

The sample space, $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$

Then, $E = \{1, 2, 3, 4, 5, 6\}$, $n(E) = 6$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{6}{6}$$

$$= 1$$

Therefore, the probability of getting a number greater than or equal to 1 is 1.

Activity 4.5: Finding the probability of an event

In a group or individually perform the following tasks:

1. Toss a 500 shillings Tanzanian coin on a flat hard surface and note whether it shows a buffalo or a head. Find the probability that a head shows up by making:
   (a) 10 trials of the experiment.
   (b) 20 trials of the experiment.
2. Write your name on one side of the box with six sides. Drop it 15 times on the floor. Record the results and find the probability of seeing your name on the top side of your box.
3. Open any page of a book and count the number of *a*'s and the total number of letters in one line of the page. Find the probability of getting a letter '*a*' in that particular line.
4. Investigate the birthday of each student in your class and give the experimental probability that any student picked out from your class has a birthday in March.
5. Share your findings with your neighbours through discussion, for more inputs.

Exercise 4.2

Answer the following questions:

1. A fair die was rolled 100 times, the six numbers and their frequency of occurrence are summarized in the following table:

Number	Frequency
1	14
2	17
3	20
4	18
5	15
6	16

Find the probability for each of the following events:
   (a) 5 appears
   (b) 3 appears
   (c) an even number appears
   (d) a prime number appears.

2. A class has 18 boys and 22 girls. If a student is selected at random from the class, find the probability that the selected student is a girl.
3. A survey conducted at a certain maternity ward showed that 60% of the children born were females. Find the probability that a child born in that ward is a male.

4. In 5 pages of a certain book, the number of letter *e* and the number of all letters were 155 and 1 455 respectively. Find the probability of a letter *e* occurring in a collection of all letters in those 5 pages of the book.

5. A box contains 300 bolts and 700 nuts. If one item is picked at random from the box, find the probability that it is a nut.

6. A survey of patients' blood groups in a certain hospital showed that 25 out of 100 patients had group O positive. Find the probability that the blood group of a patient in that hospital is group O positive.

7. Find the probability that a month selected at random from the twelve months of the year will have:
(a) 31 days (b) 30 days.

8. The winning tickets for selling 20 000 lottery tickets are five. If a ticket is picked at random, what is the probability that it is a winning ticket?

9. If a fair die is rolled, find the probability that a number divisible by 3 appears.

10. Six identical cards bearing the names Maria, Ali, Juma, Idi, Elizabeth, and Lea are put in a box and one card is drawn. What is the probability that the card chosen has:
(a) a girl's name?
(b) a boy's name?
(c) Lea's name?

11. A card is selected at random from 50 cards numbered 1 to 50. Find the probability that the number on the card is:
(a) divisible by 5
(b) a prime number
(c) a two digit number ending with the digit 2.

12. A box contains 6 volleyballs and 8 footballs of the same size. If one ball is picked at random, what is the probability that it will be a football?

13. One hundred packets of tea leaves were weighed and the following results were obtained.

Mass in grams	Number of packets
490 – 494	18
495 – 500	50
501 – 504	30
505 – 510	2

Find the probability that a packet of tea leaves picked at random will weigh:
(a) more than 500 grams.
(b) less than 495 grams.

14. Find the probability that a year selected at random from the years; 1949, 1950, 1951, 1952, 1953, and 1954, will be a leap year.

15. If the probability that it rains in Dar es Salaam on 1st April in any year is 0.4, what is the probability that it will not rain in Dar es Salaam on 1st April next year?

16. There are 40 boys in Juma's class. 22 of them are chosen at random to play a football game. Find the probability that Juma will not be chosen.
17. Find the probability of picking a white bead from a box which contains 144 white beads.
18. If Sara is pregnant, what is the probability that her baby will be born a girl?
19. The probability of winning a game is 0.459. Find the probability of not winning the game.
20. Find the probability that a number selected at random from the numbers; –4, –2, 0, 2, 4, and 16 will be a solution set of the equation $x^2 - 2x - 8 = 0$.

Combined events using tree diagrams, tables, and formulae

Activity 4.6: Finding the probability of the two combined events

1. In a group or individually, place two fair coins on a plate and toss them into your desk.
2. Count the number of heads.
3. Keep track of each trial like the one shown in Table 4.2.

Table 4.2: Experimental trials of tossing a coin twice

Trials	HH	TH	HT	TT
Number of tails				
Number of heads				

4. Find the experimental probability that two heads occur.
5. Share your findings with your neighbours through a discussion.

Combined event are more than one event which occur together in one experiment. Consider the experiment of tossing two fair coins at the same time. Let the event of interest be E = {Obtaining two heads}.

In this case, there are two simple events which are:

E_1 = {Obtaining a head on the first coin}   and

E_2 = {Obtaining a head on the second coin}

Two or more simple events represented by a single event are called combined events. In this case, the event E is a combination of the simple events E_1 and E_2. Finding the probability of combined events, requires the use of tree diagrams, tables, or formulae.

(a) Tree diagram of combined events

A tree diagram is one of the ways of organizing outcomes in order to identify the possible combinations of the outcomes. The possibility set can be found using a tree diagram as shown in Figure 4.3.

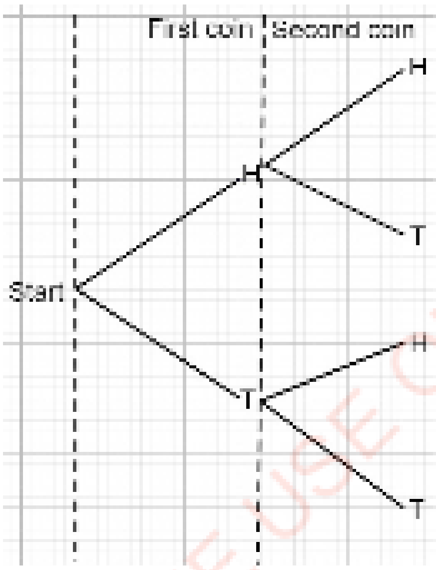


Figure 4.3: *Experimental results of tossing a fair coin twice by using the tree diagram*

From Figure 4.3 there are four possible outcomes. Hence, the sample space for the experiment is, S = {HH, HT, TH, TT}.

(b) A table for combined events

A sample space can be found without using the tree diagram. For example, in the preceding experiment, one could obtain the results of the experiment using the technique shown in Table 4.3.

Table 4.3: Tossing two coins whose sides are Head (H) and Tail (T)

		2nd coin	
1st coin		H	T
	H	HH	HT
	T	TH	TT

From Table 4.3, the sample space, S = {HH, HT, TH, TT}.

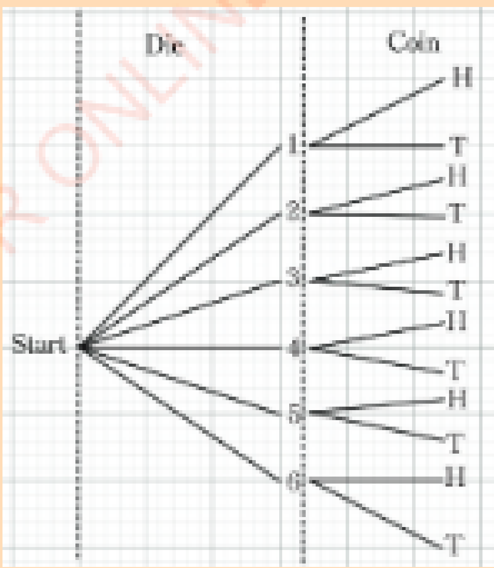
A table can be used in finding the possible outcome of combined events. This means that both coins can show heads or one can show a head while the other shows a tail or both coins can show tails.

Example 4.12

A fair die is rolled and a fair coin is tossed simultaneously. Find the sample space and hence determine the probability that a head and a number less than 3 will occur.

Solution

Start with rolling a fair die. From one point, draw six line segments indicating the possible outcomes (the six numbers) on the die as shown in the given figure. Then to each of these line segments, draw two branches to indicate the possible outcomes of the head and tail of the fair coin.



From the diagram, there are twelve possible outcomes.

Let E denotes an event of a head and a number less than 3 to occur.

The sample space,

$S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$, $n(s) = 12$

The event, $E = \{1H, 2H\}$, $n(E) = 2$

Thus, $P(E) = \frac{n(E)}{n(S)}$

$= \frac{2}{12}$

$= \frac{1}{6}$

Therefore, the sample space, $S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$ and the probability that a head and a number less than 3 to occur is $\frac{1}{6}$.

Example 4.13

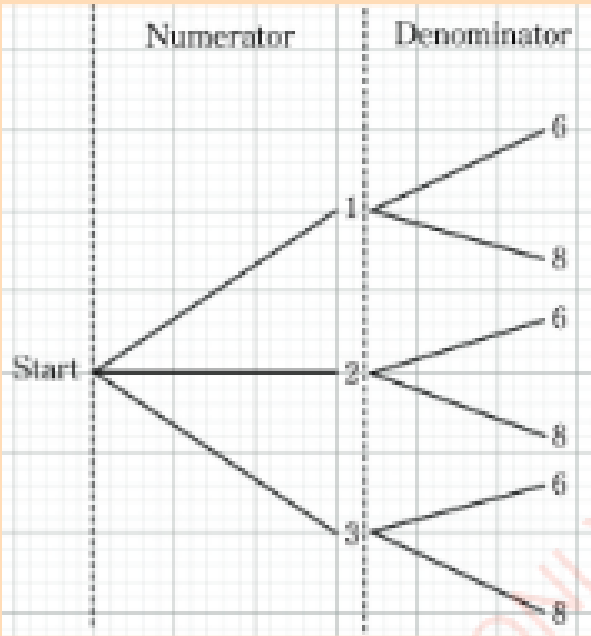
A fraction is written by selecting the numerator from the digits 1, 2, 3 and the denominator from the digits 6, 8.

(a) Find the sample space of this experiment.

(b) Find the probability that the fraction written is less than $\frac{1}{2}$.

Solution

(a) Draw a tree diagram as shown in the following figure:



Therefore, the sample space, $S = \left\{\frac{1}{6}, \frac{1}{8}, \frac{2}{6}, \frac{2}{8}, \frac{3}{6}, \frac{3}{8}\right\}$.

(b) Let E denotes an event that a fraction written is less than $\frac{1}{2}$,

thus, $E = \left\{\frac{1}{6}, \frac{1}{8}, \frac{2}{6}, \frac{2}{8}, \frac{3}{8}\right\}$,

$n(E) = 5$ and $n(S) = 6$

Thus, $P(E) = \frac{n(E)}{n(S)}$

$= \frac{5}{6}$

Therefore, the probability that the fraction written is less than $\frac{1}{2}$ is $\frac{5}{6}$.

Alternatively, the outcome can be found as shown in the following table:

Numerator / Denominator	1	2	3
6	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$
8	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$

(a) Therefore, the sample space, $S = \left\{\frac{1}{6}, \frac{1}{8}, \frac{2}{6}, \frac{2}{8}, \frac{3}{6}, \frac{3}{8}\right\}$.

(b) The event, $E = \left\{\frac{1}{6}, \frac{1}{8}, \frac{2}{6}, \frac{2}{8}, \frac{3}{8}\right\}$, $n(E) = 5$ and $n(S) = 6$

Thus, $P(E) = \frac{n(E)}{n(S)}$

$= \frac{5}{6}$

Therefore, the probability that the fraction written is less than $\frac{1}{2}$ is $\frac{5}{6}$.

Example 4.14

In a family of 2 children, what is the probability that:

(a) all are girls?

(b) one is a boy and the other is a girl?

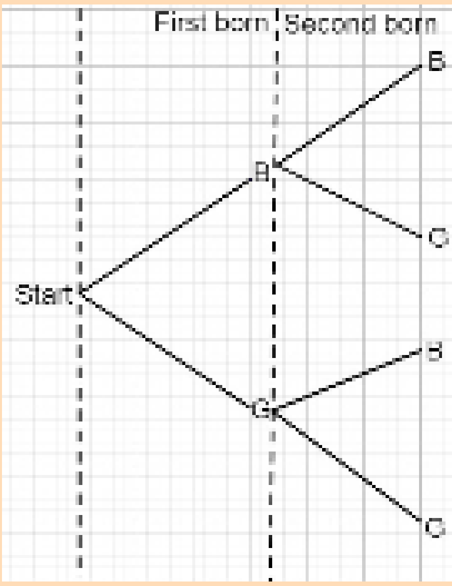
(c) at least two are boys?

Solution

Let B denotes an event that a family has a boy child.

Let G denotes an event that a family has a girl child.

Draw a tree diagram as shown in the following figure.



Then, the sample space is, $S = \{BB, BG, GB, GG\}$, thus, $n(S) = 4$.

(a) $E = \{GG\}$, $n(E) = 1$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{1}{4}$$

Therefore, the probability that all children are girls is $\frac{1}{4}$.

(b) $E = \{BG, GB\}$, $n(E) = 2$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

Therefore, the probability that one child is a boy and the other is a girl is $\frac{1}{2}$.

(c) $E = \{BB\}$, $n(E) = 1$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{1}{4}$$

Therefore, the probability that at least two children are boys is $\frac{1}{4}$.

(c) The probability of combined events using a formula

If A and B are any combined events from a sample space S, then;

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Exampl 4.15

If $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{6}$, find $P(A \cup B)$

Solution

From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{3+4-2}{12}$$

$$= \frac{5}{12}$$

Therefore, $P(A \cup B) = \frac{5}{12}$.

Example 4.16

A student picked a pair of socks out of his drawer without looking in it. If he has 3 pairs of black socks, 4 pairs of brown socks, 3 pairs of black sport socks, and 5 pairs of white sport socks. Find the probability that the student will pick a pair of black socks or a pair of sport socks.

Solution

The events are inclusive because a pair of socks can be both a sport and black.

P (black socks) = $\frac{6}{15}$, P (Sport socks) = $\frac{8}{15}$, P (black and sport socks) = $\frac{3}{15}$.

P (black or sport socks) = P (black socks) + P (Sport socks) – P (black and sport socks)

$$= \frac{6}{15} + \frac{8}{15} - \frac{3}{15}$$

$$= \frac{11}{15}$$

Therefore, the probability that a student will pick a pair of black socks or a pair of sport socks is $\frac{11}{15}$.

(d) **Mutually exclusive events**

Events are mutually exclusive if they cannot occur together. For example, two students entering the same race, the event that one student wins and the other wins too cannot happen together.

For two mutual exclusive events A and B, we have $P(A \cup B) = P(A) + P(B)$.

For mutual exclusive events, $P(A \cap B) = 0$ since the events cannot occur together.

Example 4.17

The probability that team A will win the football league is $\frac{1}{5}$, and the probability that team B will win is $\frac{1}{7}$. What is the probability that a team will win the football league?

Solution

P (team A to win) = $\frac{1}{5}$ P (team B to win) = $\frac{1}{7}$

P (a team to win) = P (team A to win or team B to win)

= P (team A to win) + P (team B to win)

$= \frac{1}{5} + \frac{1}{7}$

$= \frac{7+5}{35}$

$= \frac{12}{35}$

Therefore, the probability that a team will win the football league is $\frac{12}{35}$.

Example 4.18

There are two prizes. The probabilities that Asha will win the first or second prize are $\frac{1}{10}$ and $\frac{1}{8}$ respectively. Find the probability that she will win one of the prizes.

Solution

P(Asha winning the first prize) = $\frac{1}{10}$

P(Asha winning the second prize) = $\frac{1}{8}$

P(Asha winning one of the prizes) = P(Asha winning the first or second prize)

= P(Asha winning the first prize) + P(Asha winning the second prize)

$= \frac{1}{10} + \frac{1}{8}$

$= \frac{4+5}{40}$

$= \frac{9}{40}$

Therefore, the probability that Asha will win one of the prizes is $\frac{9}{40}$.

(e) Independent events

Two events are said to be independent if the occurrence of one does not affect the occurrence of the other. For example, when a fair coin is tossed twice, the outcome of the first toss does not affect the outcome of the second toss.

When two events A and B are independent, the probability of both to occur is $P(A \text{ and } B) = P(A) \times P(B)$, or $P(A \cap B) = P(A) \times P(B)$.

Example 4.19

Give that $P(A) = 0.6$ and $P(B) = 0.4$, find $P(A \cap B)$ if A and B are independent events.

Solution

Given $P(A) = 0.6,\ P(B) = 0.4$

Then $P(A \cap B) = P(A) \times P(B)$

$= 0.6 \times 0.4$

$= 0.24$

Therefore, $P(A \cap B) = 0.24$.

Example 4.20

A fair coin is tossed and a fair six sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.

Solution

P (Landing on the head side) = P(H) = $\frac{1}{2}$

P (rolling a 3 on the die) = P(3) = $\frac{1}{6}$

$$P(H \text{ and } 3) = P(H) \times P(3)$$

$$= \frac{1}{2} \times \frac{1}{6}$$

$$= \frac{1}{12}$$

Therefore, the probability of landing on the head side of the coin and rolling a 3 on the die is $\frac{1}{12}$.

Exercise 4.3

Answer the following questions:

1. Two fair coins are tossed simultaneously.
   Find the probability that:
   (a) two heads appear.
   (b) one head appears.
   (c) two tails appear.

2. If a two digit number is written choosing the ten's digit from the set {1, 2, 3} and the one's digit from the set {5, 6}, find the probability that a number greater than 20 will be written.

3. A pair of fair dice is rolled. Find the probability that the sum of the two numbers showed up is:
   (a) at least 8
   (b) at most 10
   (c) exactly 7
   (d) exactly 1

4. A regular tetrahedron having its faces marked 1, 2, 3, and 4 is rolled and a fair die is rolled. The number facing down on the tetrahedron and the number showing up on the die are read. Find the probability that the sum of the numbers involved is:
   (a) 8
   (b) 9
   (c) 10
   (d) 11

5. A fair die and a fair coin are tossed simultaneously. Find the probability that a number greater than 4 and a tail will appear.

6. In a family with 2 children, what is the probability that:
   (a) both are boys?
   (b) at most two girls?

7. Maria has two blouses. One is green and the other is yellow. She also has three skirts, which are blue, white, and black. What is the probability that she will put on a yellow blouse and a blue skirt?

8. In a certain village there are two horses. One is brown and the other is white. Three boys, David, Issa, and Stephen want to ride the horses. What is the probability that a boy will ride a white horse?

9. A Furniture Company has four designs for chairs and three designs for tables. By using a tree diagram find the number of different pairs of table and chair designs the company can provide.

10. There are 8 girls and 12 boys in the class. Five of the girls play sports and 3 do not play sports. Eight of the boys play sports and 4 do not play sports. If a student is selected a random, what is the probability that the student is a boy or plays sports?

    A six sided fair die is rolled. Let A be an event in which the number that shows up is odd and B be an event that the number is greater than 3. Find the probability that the number shows up is odd or greater than 3.

12. Jane and Hassan are competing for a managerial post at a certain company. Their probabilities of getting the post are 0.5 and 0.4 respectively. Find the probability that the company will get a manager?

13. Given that $P(A) = 0.3$, and $P(B) = 0.45$, find $P(A \cup B)$ if;
    (a) A and B are mutually exclusive events
    (b) A and B are mutually inclusive where $P(A \cap B) = 0.2$

14. Two fair dice, one colored white and another colored red, are thrown. Find the probability that;
    (a) the score on the red die is 2 and white die is 5.
    (b) the score on the white die is 1 and the red die is even.

Chapter summary

1. Probability deals with measurement of uncertainties.
2. All expected outcomes in a given experiment form the sample space, which is denoted by a letter "S".
3. The results of the experiments are called events which is usually denoted by the letter E.
4. The total number of outcomes in the experiment is the number of sample space denoted as n(S) and the number of events is denoted by n(E).
5. For any event E, its respective probability lies between 0 and 1 inclusively that is, $0 \leq P(E) \leq 1$.
6. Probability of an event denoted by P(E) is the ratio of number of outcomes in the event and the number of outcomes in the sample space.
   That is, $P(E) = \frac{n(E)}{n(S)}$.
7. Probability of an event not to occur is denoted by $P(E')$.
   $P(E) = 0$, when an event is impossible to occur or cannot happen.
   $P(E) = 1$, when there is complete certainty that the event will occur.
   Hence, $P(E) + P(E') = 1$
   which can also be written as:
   $P(E) = 1 - P(E')$ or
   $P(E') = 1 - P(E)$.
8. If A and B are two mutual exclusive events then,
   $P(A \cup B) = P(A) + P(B)$.
9. If A and B are two combined events then,
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
10. If A and B are two independent events then,
    $P(A \cap B) = P(A) \times P(B)$.

Revision exercise 4

Answer the following questions:

1. A survey about the birth months of students in a certain school was carried out and the observations was recorded as shown in the following table.

Months	Number of students
January	40
February	52
March	60
April	10
May	28
June	9
July	31
August	20
September	15
October	45
November	33
December	57

Calculate the probability that a student selected at random was born in:
(a) January
(b) a month beginning with a letter J.
(c) one of the months of the year except June.

2. A pair of regular tetrahedron with their faces marked 1, 4, 9, and 16 are tossed, and the numbers on the face down are read. Use a tree diagram to find;
(a) the possibility set,
(b) the probability that the sum of two numbers involved is at least:
(i) 7 (ii) 20
(iii) 32 (iv) 1

3. A bag contains 10 red balls, 5 blue balls and 7 green balls.
Find the probability of selecting at random:
(a) a red ball
(b) a green ball
(c) a blue or a red ball
(d) a red or a green ball

4. An integer is chosen at random from 2, 3, 4, 5, ..., 10.
What is the probability that:
(a) it is a factor of 18?
(b) it is divisible by 3?
(c) it is either less than 3 or greater than 7?

5. A letter is chosen from the word "random". What is the probability that it is:
(a) an n?
(b) a vowel?
(c) an x?

6. In a class of 40 students, 24 take Chemistry, and 8 take Biology. Find the probability that a student selected at random will take either Chemistry or Biology.

7. What does it mean when we say that the probability is:
(a) Zero? (b) One?

8. A bicycle tyre manufacturing factory kept a record of the distance at which a particular kind of bicycle tyre needed to be replaced. The following table shows the results from 800 such tyres.

Distance (km)	Frequency
Less than 501	125
501 to 1000	257
1001 to 1500	328
More than 1501	90

What is the probability that if you buy a tyre of this kind:

(a) it will need to be replaced before it has covered 500 km?

(b) it will last for more than 1000 km?

(c) it will not last for more than 1000 km?

9. There is a 60% chance that tomorrow will rain. If it does, there is an equal likely chance that Michael will read, watch TV or play table tennis. What is the probability that it will rain and Michael will read?

10. Two pairs of shoes are in a bag. If two shoes are taken at random from the bag, what is the probability of:

(a) drawing a matched pair?

(b) drawing a shoe for the left and right feet?

(c) getting shoes of the same feet?

(d) getting two shoes of left feet?

11. Let A and B be independent events with $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$, find;

(a) $P(A \cap B)$ (b) $P(A \cup B)$

12. In a group of students, the probability that a student is chosen at random walking to school is 0.35 and the probability that a student has blonde hair is 0.2. If the probability that a student's walks to school or has blonde hair is 0.45. Find the probability that a student has blonde hair and walks to school.

13. A bag contains 6 green balls and 9 blue balls. Two balls are drawn one at a time and without replacement. Find the probability that:

(a) both balls are of the same colour

(b) the balls are of different colours.

(c) at least one of the balls is blue.

Project 4

1. The experiment of rolling a fair die 24 times was done and the data were recorded accordingly:

   (a) Roll a fair die 24 times. Record your data as shown in the following sample graph.

   (b) How many times do you think each number will occur in 24 rolls? Explain with reasons.

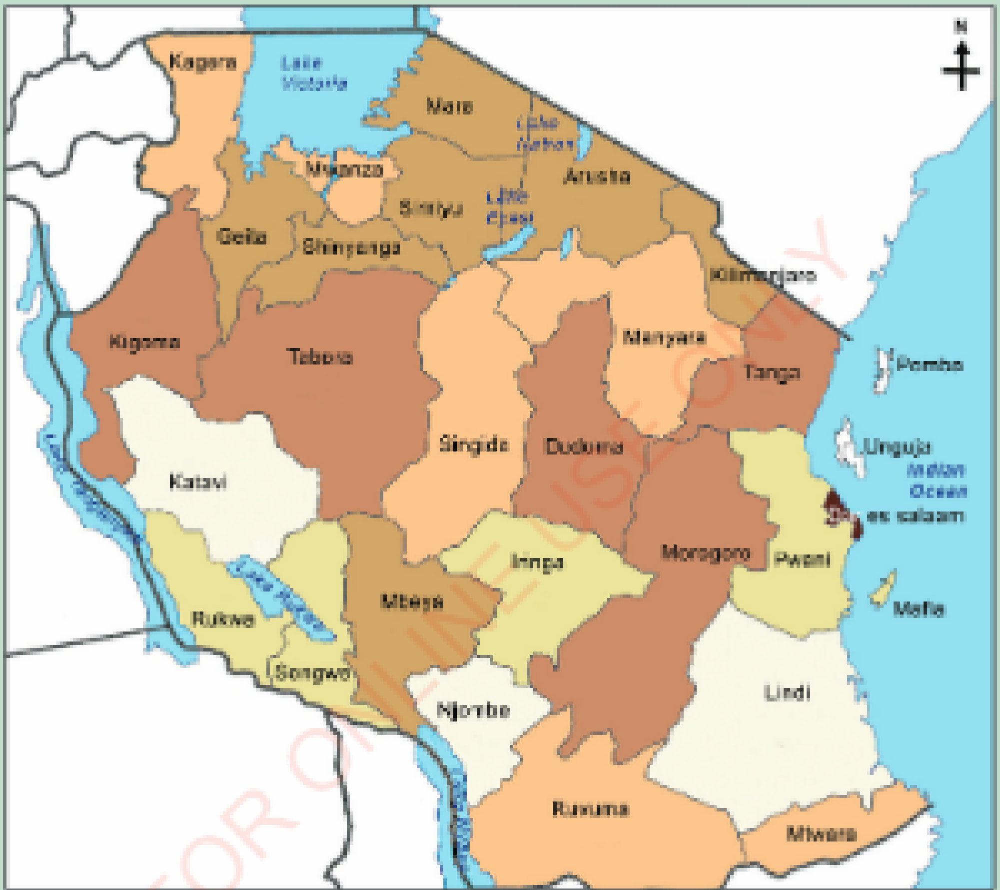
   (c) Compare your graph with the graph of the other groups in your class.

   (d) Find the ratio of number of times each number occurs and the total number of rolls.

2. Suppose you and your friends would like to have a vacation tour in the regions of Tanzania mainland as shown in the following map of Tanzania. You can decide where to visit and where to start.

   (a) Which region would you have the highest probability to visit? Why?

   (b) Explain why the method you have used in (a) in order to arrive at the answer? What is the usefulness of a map given in arriving at your answer?



Chapter Five

Trigonometry

Introduction

Trigonometry is the study of relationships between sides, lengths, and angles of a triangle. Generally, trigonometry has great practical importance to builders, architects, surveyors, engineers, and to many other people of different fields. In this chapter, you will learn how to determine the sine, cosine, and tangent of an angle, apply trigonometric ratios to solve problems in daily life, interpret the graphs of sine and cosine functions, derive and apply the sine and cosine rules in solving problems on triangles and apply the compound angle formulae in solving trigonometric problems. The competencies of trigonometry can be applied in various real life situations such as in solving problems related to astronomy, navigation, and building construction.

Trigonometrical ratios in a unit circle

Activity 5.1: Introducing the relationship of sides, lengths, and angles of a triangle

In a group or individually, perform the following tasks:

1. Using scale, draw three right – angled triangles PQR, ABC, and MNT such that $\overline{PQ} = 4\text{ cm}$, $\overline{QR} = 3\text{ cm}$, $\overline{PR} = 5\text{ cm}$, $\overline{AB} = 8\text{ cm}$, $\overline{BC} = 15\text{ cm}$, $\overline{AC} = 17\text{ cm}$, $\overline{MN} = 6\text{ cm}$, $\overline{NT} = 8\text{ cm}$, $\overline{MT} = 10\text{ cm}$.

2. Calculate the ratios of $\frac{\overline{PQ}}{\overline{PR}}$, $\frac{\overline{AB}}{\overline{AC}}$, and $\frac{\overline{MN}}{\overline{MT}}$.

3. Calculate the ratios of $\frac{\overline{QR}}{\overline{PR}}$, $\frac{\overline{BC}}{\overline{AC}}$, and $\frac{\overline{NT}}{\overline{MT}}$.

4. Calculate the ratios of $\frac{\overline{PQ}}{\overline{QR}}$, $\frac{\overline{AB}}{\overline{BC}}$, and $\frac{\overline{MN}}{\overline{NT}}$.
5. Use scale drawing methods to find $P\hat{R}Q$, $A\hat{C}B$, and $M\hat{T}N$.
6. Compare the angles obtained in task 5 and the ratios in task 2.
7. Use scale drawing method to find $B\hat{A}C$, $Q\hat{P}R$, and $N\hat{M}T$.
8. Compare the angles obtained in task 7 and the ratios of task 2.
9. Comment on the results in tasks 6 and 8.
10. Share your findings with your neighbours for more inputs.

The three trigonometrical ratios of sine, cosine, and tangent have been defined using the sides of a right–angled triangle ABC as follows:

If A is an acute angle as shown in Figure 5.1 then:

$$\sin A = \frac{\text{length of opposite side}}{\text{length of hypotenuse side}} = \frac{a}{c}$$

$$\cos A = \frac{\text{length of adjacent side}}{\text{length of hypotenuse side}} = \frac{b}{c}$$

$$\tan A = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{a}{b}$$

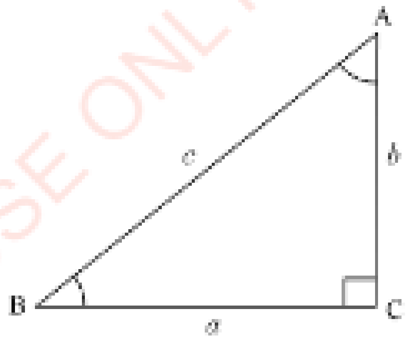


Figure 5.1: *Sides of a right – angled triangle*

From Figure 5.1: $\overline{BC}$ is called the opposite side to angle A whose length is denoted by *a*.

$\overline{CA}$ is called the adjacent side to angle A whose length is denoted by *b*.

$\overline{BA}$ is called the hypotenuse side to angle A whose length is denoted by *c*.

Consider a circle of unit radius subdivided into four congruent sectors by the coordinate axes whose origin is at the centre of the circle as shown in Figure 5.2. Let θ be any acute angle $(0° < \theta < 90°)$ and P be the point on the circle with coordinates (x, y), where $\overline{OP}$ is the radius of the unit circle.

The trigonometrical ratios in this circle can be obtained by using the sides of a right – angled triangle OPQ as follows:

$$\sin\theta = \frac{\overline{QP}}{\overline{OP}} = \frac{y}{1} = y.$$

$$\cos\theta = \frac{\overline{OQ}}{\overline{OP}} = \frac{x}{1} = x.$$

$$\tan\theta = \frac{\overline{QP}}{\overline{OQ}} = \frac{y}{x}.$$

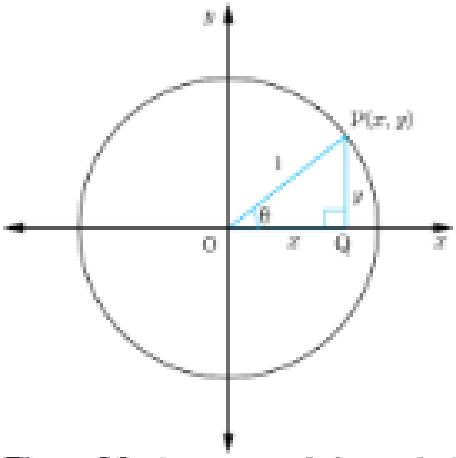


Figure 5.2: *An acute angle in a unit circle*

All the trigonometrical ratios in Figure 5.2 are positive as per corresponding axes of x and y.

In Figure 5.3, θ is an obtuse angle $(90° < \theta < 180°)$. The trigonometrical ratios of θ are the same as the trigonometrical ratios of $180° - \theta$.

The trigonometrical ratios in this circle can be obtained by using the sides of a right – angled triangle OQP as follows:

$$\sin\theta = \sin(180° - \theta) = \frac{\overline{QP}}{\overline{OP}} = \frac{y}{1} = y.$$

$$\cos\theta = -\cos(180° - \theta) = \frac{\overline{OQ}}{\overline{OP}} = \frac{-x}{1} = -x.$$

$$\tan\theta = -\tan(180° - \theta) = \frac{\overline{QP}}{\overline{OQ}} = \frac{y}{-x} = -\frac{y}{x}.$$

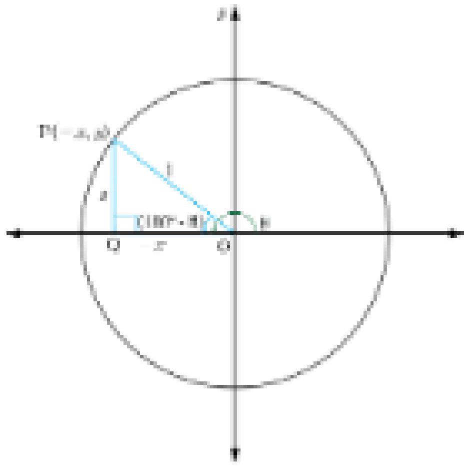


Figure 5.3: *An obtuse angle in a unit circle*

In Figure 5.3, the trigonometrical ratios of sine is positive while that of cosine and tangent are negative as per corresponding axes of x and y.

In Figure 5.4, θ is a reflex angle $(180° < \theta < 270°)$.

The trigonometrical ratios of θ are the same as the trigonometrical ratios of θ – 180°.

The trigonometrical ratios in this circle can be obtained by using the sides of a right – angled triangle OQP as follows:

$$\sin\,\theta = -\sin\,(\theta - 180°) = \frac{\overline{QP}}{\overline{OP}} = \frac{-y}{1} = -y.$$

$$\cos\,\theta = -\cos\,(\theta - 180°) = \frac{\overline{OQ}}{\overline{OP}} = \frac{-x}{1} = -x.$$

$$\tan\,\theta = \tan\,(\theta - 180°) = \frac{\overline{QP}}{\overline{OQ}} = \frac{-y}{-x} = \frac{y}{x}.$$

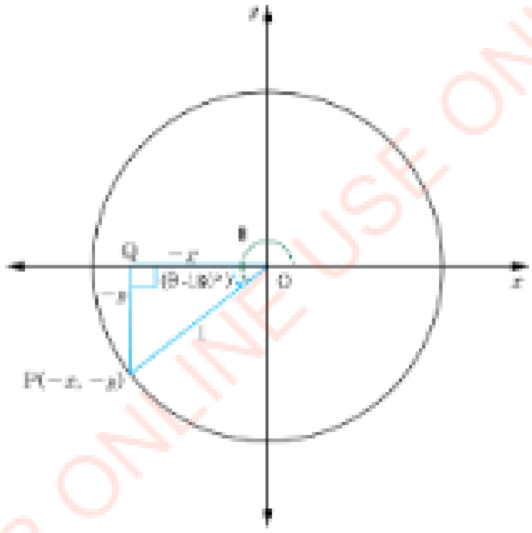


Figure 5.4: *Reflex angle in a unit circle*

In Figure 5.4, the trigonometrical ratios of sine and cosine are negative while for the tangent is positive as per corresponding axes of x and y.

From Figure 5.5, θ is a reflex angle $(270° < \theta < 360°)$. The trigonometrical ratios of θ are the same as the trigonometrical ratios of $360° - \theta$.

The trigonometrical ratios in this circle can be obtained by using the sides of a right – angled triangle OQP as follows:

$$\sin\theta = -\sin(360° - \theta) = \frac{\overline{QP}}{\overline{OP}} = \frac{-y}{1} = -y.$$

$$\cos\theta = \cos(360° - \theta) = \frac{\overline{OQ}}{\overline{OP}} = \frac{x}{1} = x.$$

$$\tan\theta = -\tan(360° - \theta) = \frac{\overline{QP}}{\overline{OQ}} = -\frac{y}{x}.$$

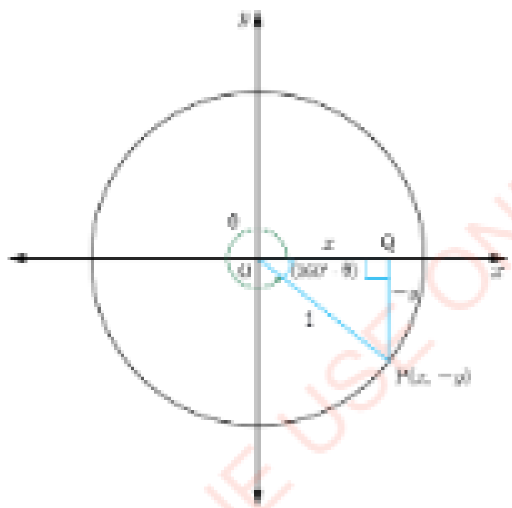


Figure 5.5: *A reflex angle in a unit circle*

In Figure 5.5, the trigonometrical ratios of sine and tangent are negative while for the cosine is positive as per corresponding axes of x and y.

Signs of the trigonometrical ratios

We have seen that the trigonometrical ratios are positive or negative depending on the size of the angle and the quadrant in which the angle is found.

The results obtained are illustrated in Figure 5.6. These results will be helpful in determining whether sine, cosine, and tangent of an angle is positive or negative.

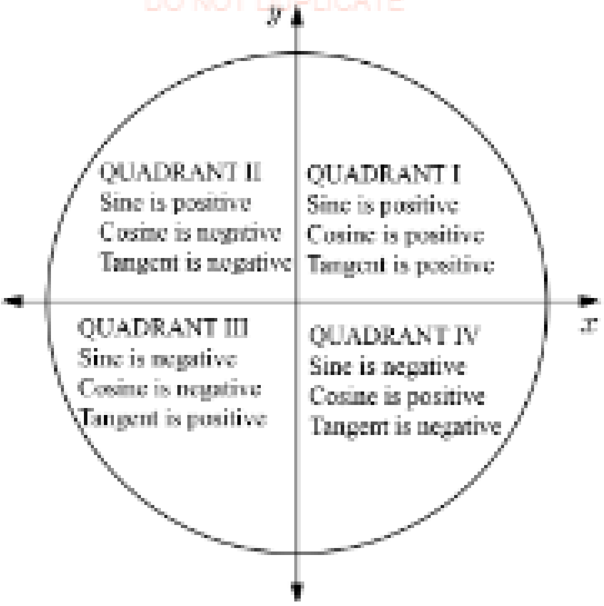


Figure 5.6: *Signs of trigonometric ratios*

The results can best be remembered using the letters ASTC (All, Sine, Tangent, Cosine) or CTSA (Cosine, Tangent, Sine, All) if read clockwise direction beginning with the fourth quadrant.

Example 5.1

Write the signs of each of the following trigonometrical ratios:

(a) $\sin 170^\circ$ (b) $\cos 240^\circ$ (c) $\tan 310^\circ$ (d) $\sin 300^\circ$.

Solution

(a) 170° is in the second quadrant hence, $\sin 170^\circ$ is positive.

(b) 240° is in the third quadrant hence, $\cos 240^\circ$ is negative.

(c) 310° is in the fourth quadrant hence, $\tan 310^\circ$ is negative.

(d) 300° is in the fourth quadrant hence, $\sin 300^\circ$ is negative.

Example 5.2

Express each of the following in terms of the sine, cosine or tangent of an acute angle:

(a) $\cos 165°$ (b) $\sin 317°$ (c) $\tan 95°$ (d) $\tan 258°$

Solution

(a) 165° is in the second quadrant

$\cos 165° = -\cos(180° - 165°) = -\cos 15°$.

(b) 317° is in the fourth quadrant

$\sin 317° = -\sin(360° - 317°) = -\sin 43°$.

(c) 95° is in the second quadrant

$\tan 95° = -\tan(180° - 95°) = -\tan 85°$.

(d) 258° is in the third quadrant

$\tan 258° = \tan(258° - 180°) = \tan 78°$.

Approximate values of sine and cosine of angles

Figure 5.7 represents a unit circle shown on a graph paper. Angles 0°, 10°, 20°, . . ., 360° marked on the circumference correspond to the angles at the centre O. Approximating values of sine and cosine of the respective angles are found by reading the coordinates of the points on the unit circle. For example, in finding the value of $\sin 20°$ we read the y – coordinate at the point as 0.34 and the value of $\cos 20°$ we read the x – coordinate at the point as 0.94.

Table 5.1 has been partly completed by entering approximate values of sine and cosine of an angle using Figure 5.7.

Activity 5.2: Approximating values of sine and cosine of angles using a unit circle

In a group or individually, perform the following tasks using Figure 5.7:

1. Copy and fill in the missing values in Table 5.1.

Table 5.1: Approximate values of sine and cosine of angles between 0° and 360°

Angle	Sine	cosine	Angle	Sine	cosine
20°	0.34	0.94	190°	−0.17	
30°			220°		
70°			290°		
120°		−0.5	310°		0.64
140°	0.64		330°		

2. Compare the values in task 1 with the values given in the tables of trigonometrical ratios.
3. What did you observe from this activity?
4. Share your findings with your friends for more inputs.

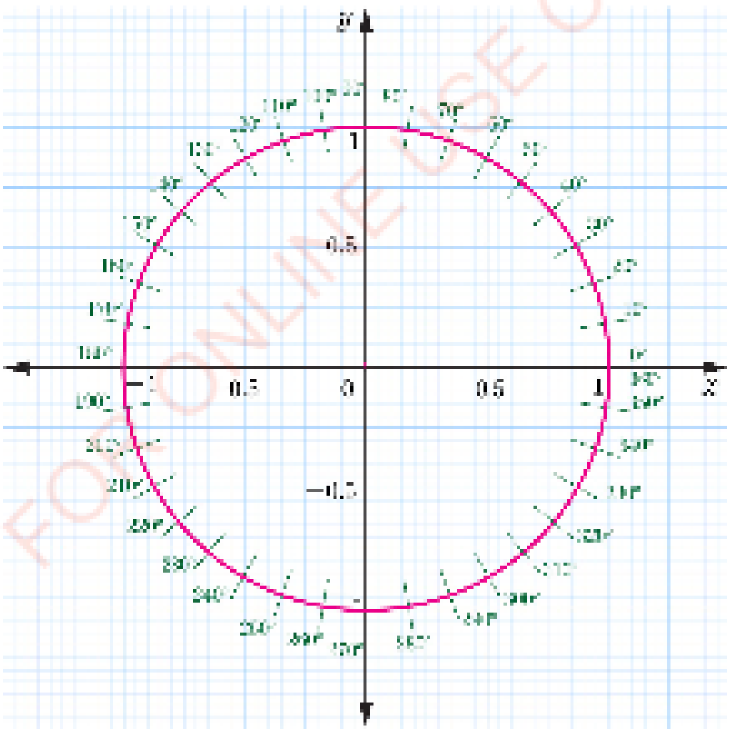


Figure 5.7: *Measurements of trigonometrical ratios of angles in a unit circle*

Figure 5.8 represents a unit circle shown on a graph paper. The line segments corresponding to central angles $0°, 10°, 20°, \ldots, 360°$ meet tangents to the circle at points A and A_1. The tangent of any angle can be found by using the coordinates at the point as follows:

$$\tan\theta = \frac{y-\text{coordinate value}}{x-\text{coordinate value}}.$$

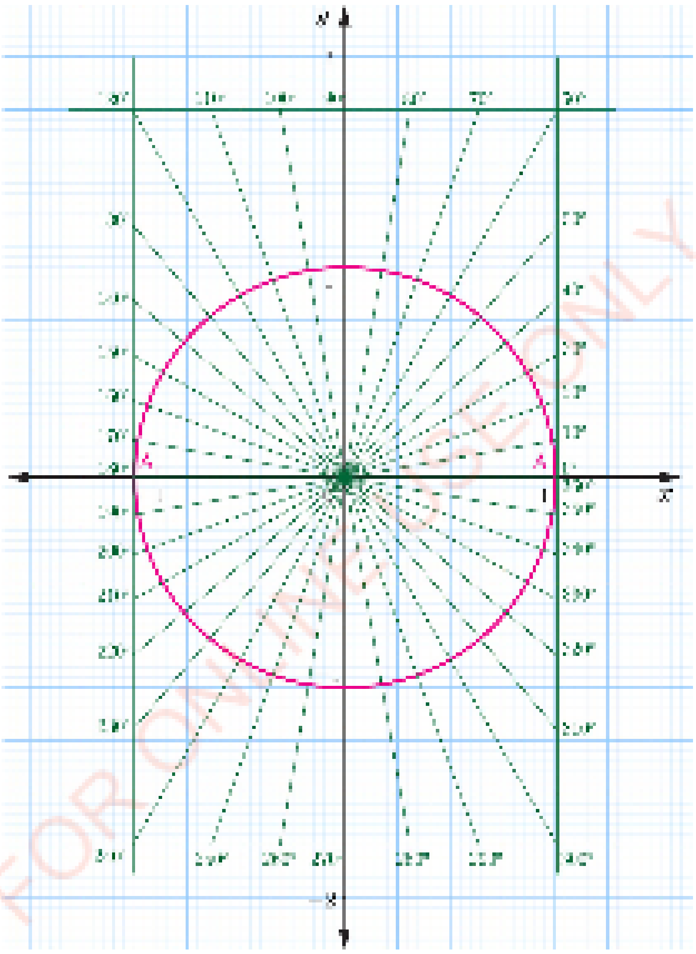


Figure 5.8: *Reading angles measured in a unit circle*

For example: $\tan 40° = \frac{0.84}{1} = 0.84.$

$$\tan 130° = \frac{1.2}{-1} = -1.2.$$

$$\tan 210° = \frac{-0.58}{-1} = 0.58.$$

Activity 5.3: Approximating values of tangent of an angle using a unit circle

In a group or individually, perform the following tasks using Figure 5.8:

1. Copy and fill the missing values in Table 5.2.

Table 5.2: Approximate values of tangent of angles between 0° and 360°

Angle	20°	60°	140°	210°	240°	310°	330°
Tangent	0.36		−0.84			−1.19	

2. Compare the values in task 1 with the values given in the tables of trigonometrical ratios.
3. What did you observe from this activity?
4. Share your findings with your neighbours for more inputs.

The use of coordinate of a point in determining trigonometrical ratios

Trigonometrical ratios and their respective signs can be determined by using coordinates of points in each quadrant. Figure 5.9 represents a circle with centre O and radius *r*, subdivided into four quadrants by the coordinate axes.

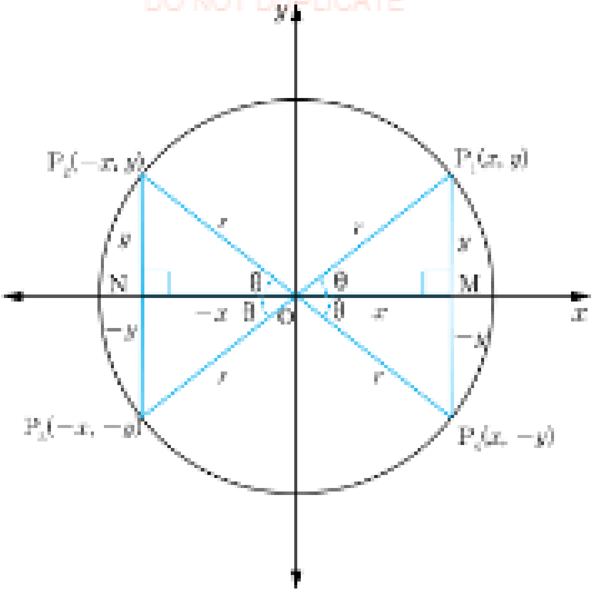


Figure 5.9: *Signs of trigonometrical ratios in a unit circle*

The x – coordinate is positive in the first and fourth quadrants but negative in the second and third quadrants. The y – coordinate is positive in the first and second quadrants but negative in the third and fourth quadrants.

Note that, r is the radius and θ is the angle of the triangle as shown in Figure 5.9. The trigonometrical ratios of θ from the first quadrant are as follows:

$$\sin\theta = \frac{y-\text{coordinate}}{\text{radius}} = \frac{y}{r}.$$

$$\cos\theta = \frac{x-\text{coordinate}}{\text{radius}} = \frac{x}{r}.$$

$$\tan\theta = \frac{y-\text{coordinate}}{x-\text{coordinate}} = \frac{y}{x}.$$

Therefore, we are able to find the trigonometrical ratios of angles using the coordinates of the terminal points of each side.

Let θ be any angle and P be a point with coordinate (–4, 3) which is the terminal point of $\overline{OP}$ as shown in Figure 5.10.

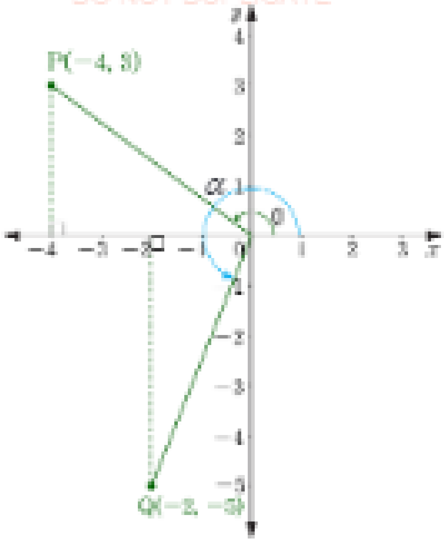


Figure 5.10: *Measurements of obtuse and reflex angles in xy – plane*

By Pythagoras' theorem: $r = \overline{OP} = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

Therefore, $\sin\ \theta = \sin\ (180° - \theta) = \frac{3}{5}$.

$\cos\ \theta = -\cos\ (180° - \theta) = -\frac{4}{5}$.

$\tan\ \theta = -\tan\ (180° - \theta) = -\frac{3}{4}$.

Consider another point $Q(-2, -5)$ being the terminal point of $\overline{OQ}$ making a reflex angle α (See Figure 5.10).

Hence, $r = \overline{OQ} = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

Therefore, $\sin\ \alpha = -\sin\ (\alpha - 180°) = \frac{-5}{\sqrt{29}} = -\frac{5\sqrt{29}}{29}$.

$\cos\ \alpha = -\cos\ (\alpha - 180°) = \frac{-2}{\sqrt{29}} = -\frac{2\sqrt{29}}{29}$.

$\tan\ \alpha = \tan\ (\alpha - 180°) = \frac{-5}{-2} = \frac{5}{2}$.

When such ratios are calculated out for each quadrant the signs of sine, cosine, and tangent will be as shown in Figure 5.6.

Example 5.3

The terminal side of line OQ has the coordinates $(-5,-12)$ and is inclined at the angle θ with the x – axis. Find sin θ, cos θ, and tan θ.

Solution

The hypotenuse side, $\overline{OQ} = \sqrt{(-5)^2 + (-12)^2}$

$$= \sqrt{169}$$

$$= 13$$

Therefore, $\sin\theta = -\frac{12}{13}$.

$$\cos\theta = -\frac{5}{13}.$$

$$\tan\theta = \frac{-12}{-5} = \frac{12}{5}.$$

Example 5.4

The line segment OQ is inclined at an angle θ with the x – axis. If $\overline{OQ} = 2\sqrt{6}$ and $Q = (a,-4)$, find sin θ, cos θ, and tan θ.

Solution

$$\overline{OQ} = \sqrt{a^2 + (-4)^2}$$

$\overline{OQ} = \sqrt{a^2 + 16}$, but $\overline{OQ} = 2\sqrt{6}$

$$2\sqrt{6} = \sqrt{a^2 + 16}$$

$$a^2 + 16 = 24$$

$$a^2 = 8$$

$$a = \pm 2\sqrt{2}$$

Therefore, $\sin\theta = -\frac{4}{2\sqrt{6}} = -\frac{\sqrt{6}}{3}$.

Note: Coordinate Q may be located to the third and fourth quadrants.

$$\cos\theta = \pm\frac{2\sqrt{2}}{2\sqrt{6}} = \pm\frac{\sqrt{3}}{3}.$$

$$\tan\theta = \pm\frac{-4}{2\sqrt{2}} = \mp\sqrt{2}.$$

Exercise 5.1

Answer the following questions:

1. Write the signs of each of the following trigonometric ratios:
   (a) $\cos 160^\circ$
   (b) $\sin 310^\circ$
   (c) $\tan 75^\circ$
   (d) $\sin 220^\circ$
   (e) $\cos 355^\circ$
   (f) $\tan 190^\circ$

2. Express each of the following in terms of sine, cosine or tangent of an acute angle:
   (a) $\cos 308^\circ$
   (b) $\sin 217^\circ$
   (c) $\tan 175^\circ$
   (d) $\sin 333^\circ$
   (e) $\cos 268^\circ$
   (f) $\tan 103^\circ$

3. Express each of the following in terms of $\sin 50^\circ$:
   (a) $\sin 130^\circ$ (b) $\sin 230^\circ$
   (c) $\sin 310^\circ$

4. Express each of the following in terms of $\cos 20^\circ$:
   (a) $\cos 160^\circ$
   (b) $\cos 200^\circ$
   (c) $\cos 340^\circ$

5. Express each of the following in terms of $\tan 40^\circ$:
   (a) $\tan 140^\circ$
   (b) $\tan 220^\circ$
   (c) $\tan 320^\circ$

6. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ if θ is the angle made by the positive x – axis and the line from the origin to each of the following points:
   (a) $(2, 6)$
   (b) $(-12, 5)$
   (c) $(-4, -3)$
   (d) $\left(4, -2\sqrt{2}\right)$

Relationship between trigonometrical ratios

Consider ΔABC as shown in Figure 5.11. Angles A and C are complementary, that is:

$$\widehat{A}+\widehat{C}=90^\circ$$

$$\widehat{C}=90^\circ-\widehat{A}$$

But, $\sin A=\frac{a}{b}$ and $\cos C=\frac{a}{b}$

Thus, $\sin A = \cos C = \cos\left(90^\circ - A\right)$

Hence, $\sin A = \cos\left(90^\circ - A\right)$.

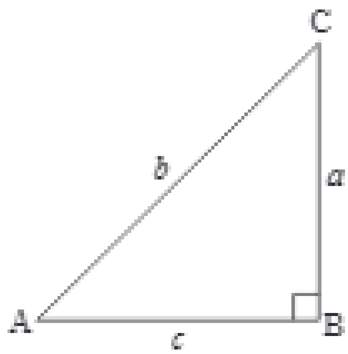


Figure 5.11: *Relationship between trigonometrical ratios in the right – angled triangle*

The sine of an angle is equal to the cosine of its complement and vice versa.

From Figure 5.11,

$$\frac{\sin A}{\cos A}=\frac{a}{b}\div\frac{c}{b}=\frac{a}{c}$$

But, $\tan A=\frac{a}{c}$

Hence, $\frac{\sin A}{\cos A}=\tan A$.

Also, $\sin^2 A=\frac{a^2}{b^2}$ and $\cos^2 A=\frac{c^2}{b^2}$

$$\sin^2 A+\cos^2 A=\frac{a^2}{b^2}+\frac{c^2}{b^2}=\frac{a^2+c^2}{b^2}$$

But, $a^2+c^2=b^2$ (Pythagoras' theorem)

Hence, $\sin^2 A+\cos^2 A=\frac{b^2}{b^2}=1$

Therefore, $\sin^2 A+\cos^2 A=1$.

Generally, for any angle θ the corresponding trigonometrical identity is $\sin^2\theta+\cos^2\theta=1$, which can also be written as $\cos^2\theta=1-\sin^2\theta$ or $\sin^2\theta=1-\cos^2\theta$.

Example 5.5

Given that $\sin\theta = \frac{4}{9}$, find $\cos\theta$ and $\tan\theta$ for $0° \le \theta \le 90°$.

Solution

From $\sin^2\theta + \cos^2\theta = 1$

$$\cos^2\theta = 1 - \sin^2\theta,$$

But, $\sin^2\theta = \frac{16}{81}$.

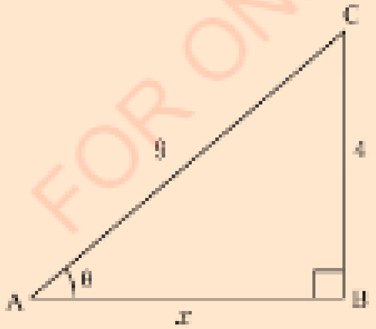
$$\cos^2\theta = 1 - \frac{16}{81} = \frac{65}{81}$$

Hence, $\cos\theta = \sqrt{\frac{65}{81}} = \frac{\sqrt{65}}{9}$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4}{9} \div \frac{\sqrt{65}}{9} = \frac{4\sqrt{65}}{65}.$$

Therefore, $\cos\theta = \frac{\sqrt{65}}{9}$ and $\tan\theta = \frac{4\sqrt{65}}{65}$.

Alternatively: Sketch a right – angled triangle such that the side opposite to θ has 4 units and the hypotenuse side has 9 units as shown in the following figure.



Using Pythagoras' theorem, the other side of the triangle is given by:

$$\left(\overline{AB}\right)^2 = \left(\overline{AC}\right)^2 - \left(\overline{BC}\right)^2$$

$$x^2 = 9^2 - 4^2$$
$$= 81 - 16 = 65$$

Hence, $x = \sqrt{65}$

$$\cos\theta = \frac{\sqrt{65}}{9}$$

$$\tan\theta = \frac{4\sqrt{65}}{65}$$

Therefore, $\cos\theta = \frac{\sqrt{65}}{9}$ and

$\tan\theta = \frac{4\sqrt{65}}{65}$.

Example 5.6

Given that α and β are complementary angles and $\sin\alpha = \frac{5}{13}$, find $\tan\beta$.

Solution

Since, α and β are complementary angles then,

$\alpha + \beta = 90^\circ$

$\alpha = 90^\circ - \beta$

Hence, $\sin\alpha = \cos(90^\circ - \alpha) = \cos\beta = \frac{5}{13}$.

But, $\sin^2\beta = 1 - \cos^2\beta$

$= 1 - \frac{25}{169} = \frac{144}{169}$

Hence, $\sin\beta = \frac{12}{13}$.

$\tan\beta = \frac{\sin\beta}{\cos\beta}$

$= \frac{12}{13} \div \frac{5}{13} = \frac{12}{5}$.

Therefore, $\tan\beta = \frac{12}{5}$.

Example 5.7

If $\sin\theta = 0.9397$ and $\cos\theta = 0.3420$, find without using trigonometrical tables or calculators the value of tan θ.

Solution

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$= \frac{0.9397}{0.3420}$

$= 2.748$

Therefore, $\tan\theta = 2.748$.

Exercise 5.2

Without using trigonometrical tables and calculators answer the following questions:

1. Given that $\cos\theta = \frac{4}{5}$, find the value of $\sin\theta$.
2. If $\tan\alpha = \frac{\sqrt{2}}{5}$, find the value of $\sin(90° - \alpha)$.
3. If α and β are complementary angles and $\tan\alpha = \frac{\sqrt{6}}{3}$, find the value of $\tan\beta$.
4. If $\sin A = 0.9733$ and $\cos A = 0.2250$, find the value of $\tan A$.
5. Given that $\cos\theta = 0.9272$ and $\tan\theta = 0.4040$. Find the value of $\sin\theta$.
6. If $\tan\theta = \frac{\sqrt{3}}{4}$, find the value of $\sin\theta$.
7. Given that $\tan\theta = 0.75$, find the value of $\cos\theta$.
8. If $\cos x = \frac{p}{q}$ and $\tan x = \frac{r}{p}$, find the value of $\sin x$.
9. If $\tan\beta = \frac{r}{\sqrt{s}}$ and $\tan\alpha = \frac{\sqrt{s}}{r}$, show that $\cos\beta = \sqrt{\frac{s}{r^2 + s}}$, where α and β are complementary angles.
10. What will be the sine and cosine of an acute angle whose tangent is $2\frac{2}{5}$?
11. If α and β are complementary angles and $\sin\alpha = \frac{\sqrt{3}}{2}$, find the value of $\sin\beta$.
12. If $\sin\alpha = \frac{5}{8}$, find the value of $\cos\alpha$.
13. Given that $\cos\lambda = \frac{6}{7}$, find the value of $\tan\lambda$.
14. Given that $\sin x = -\frac{3}{8}$, find the value of $\cos x$.
15. If $\cos\alpha = -\frac{1}{3}$, find the value of $\tan\alpha$.
16. If $\sin\theta = 0.9848$, find the value of $\cos\theta$.
17. Find the value of $\tan A$ if $\sin A = 0.8192$.
18. If $\cos P = 0.3746$, find the value of $\sin P$.
19. If $\cos Q = 0.9063$, find the value of $\tan Q$.
20. Given that $\cos\theta = 0.3090$, find the value of $\sin\theta$ and $\tan\theta$.

Positive and negative angles

Angles may be positive or negative depending on the direction in which the angle is measured. An angle may be measured in a clockwise or anticlockwise direction as shown in Figure 5.12.

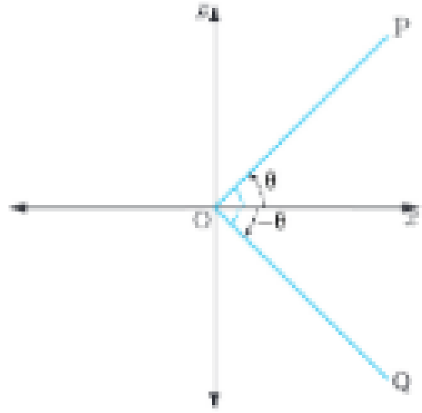


Figure 5.12: *Measurements of angles in a clockwise and anticlockwise directions*

Angles measured in the clockwise direction from the positive x – axis are negative. Angles measured in the anticlockwise direction from the positive x – axis are positive.

Figures 5.13 and 5.14 illustrate how positive and negative angles can be located in the four quadrants. The corresponding positive and negative angles whose trigonometrical ratios are the same can easily be found.

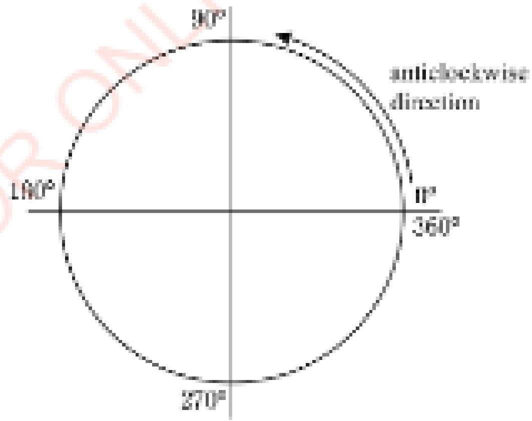


Figure 5.13: *Location of positive angles in the first quadrant*

If θ is positive, the negative angle corresponding to θ is $(-360° + \theta)$. If θ is negative, the positive angle corresponding to θ is $(360° + \theta)$.

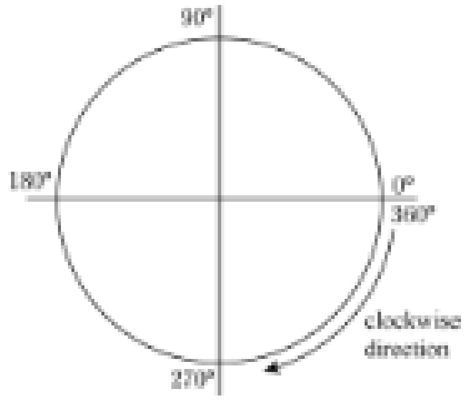


Figure 5.14: *Location of negative angles in the fourth quadrant*

Example 5.8

Find a positive or negative angle corresponding to each of the following angles:

(a) 273° (b) –210° (c) 304° (d) –115°

Solution

(a) The negative angle corresponding to 273° is $(-360° + 273°) = -87°$.
(b) The positive angle corresponding to –210° is $(360° - 210°) = 150°$.
(c) The negative angle corresponding to 304° is $(-360° + 304°) = -56°$.
(d) The positive angle corresponding to –115° is $(360° - 115°) = 245°$.

Example 5.9

Find the sine, cosine, and tangent of each of the following angles:

(a) 144° (b) –231° (c) –70° (d) 310°

Solution

(a) $\sin 144° = \sin(180° - 144°) = \sin 36° = 0.5878.$

$\cos 144° = -\cos(180° - 144°) = -\cos 36° = -0.8090.$

$\tan 144° = -\tan(180° - 144°) = -\tan 36° = -0.7265.$

(b) $\sin(-231°) = \sin(360° - 231°) = \sin 129° = \sin(180° - 129°) = \sin 51° = 0.7771.$

$\cos(-231°) = -\cos 51° = -0.6293.$

$\tan(-231°) = -\tan 51° = -1.2349.$

(c) $\sin(-70°) = \sin(360° - 70°) = \sin 290° = -\sin(360° - 290°) = -\sin 70° = -0.9397.$

$\cos(-70°) = \cos 70° = 0.3420.$

$\tan(-70°) = -\tan 70° = -2.7475.$

(d) $\sin(310°) = -\sin(360° - 310°) = -\sin 50° = -0.7660.$

$\cos(310°) = \cos 50° = 0.6428.$

$\tan(310°) = -\tan 50° = -1.1918.$

Trigonometrical ratios of special angles

Referring to Figures 5.7 and 5.8 the trigonometrical ratios of angles 0°, 90°, 180°, 270°, and 360° are summarized in Table 5.3.

Table 5.3: Trigonometrical ratios of angles 0°, 90°, 180° 270°, and 360°

Angle	0°	90°	180°	270°	360°
Sine	0	1	0	–1	0
Cosine	1	0	–1	0	1
Tangent	0	∞	0	∞	0

The trigonometrical ratios of other special angles such as 30°, 45°, and 60° can be examined.

Figure 5.15 represents an equilateral triangle with length of 2 units on each side.

Line AX is perpendicular to $\overline{BC}$.

Since ΔABX is congruent to ΔACX, then:

$\overline{BX} = \overline{XC} = 1$ unit

Therefore, using Pythagoras' theorem the length $\overline{AX}$ can be obtained as follows:

$(\overline{AX})^2 = (\overline{AB})^2 - (\overline{BX})^2 = (\overline{AC})^2 - (\overline{CX})^2 = 2^2 - 1^2 = 3$

Thus, $\overline{AX} = \sqrt{3}$ units.

ΔABX is a right – angled where $A\hat{B}C = 60°$ and $B\hat{A}X = 30°$.

From Figure 5.15 the following trigonometrical ratios can be obtained:

$\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$, $\sin 30° = \frac{1}{2}$,

$\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{\sqrt{3}}{3}$.

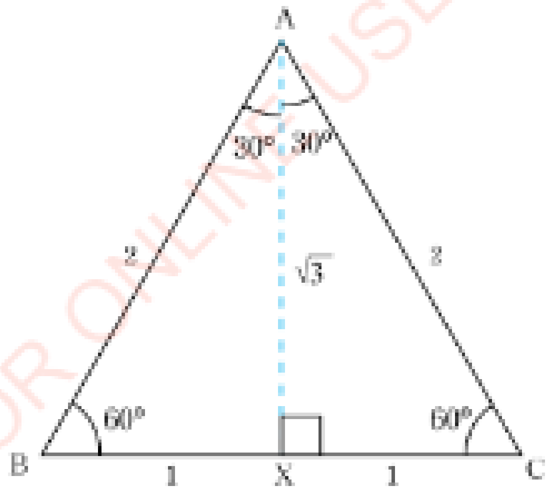


Figure 5.15: *Equilateral triangle ABC*

The trigonometrical ratios of 45° can be obtained as follows:

Consider an isosceles right – angled triangle ABC shown in Figure 5.16 in which:

$\overline{AC} = \overline{BC} = 1$ unit

Then, by using Pythagoras' theorem;

$$\left(\overline{AC}\right)^2 + \left(\overline{CB}\right)^2 = \left(\overline{BA}\right)^2$$

$$1^2 + 1^2 = \left(\overline{BA}\right)^2$$

$$\overline{BA} = \sqrt{1+1} = \sqrt{2}$$

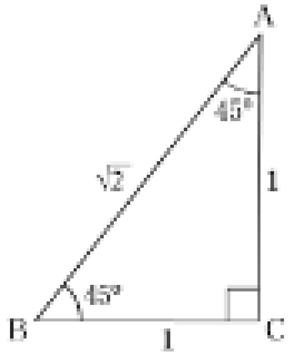


Figure 5.16: *An isosceles right – angled triangle*

The trigonometrical ratios of 45° are:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \ \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \text{ and } \tan 45° = 1.$$

The results of special acute angles are summarized in Table 5.4 as follows:

Table 5.4: Trigonometric ratio for special acute angles

Angle	30°	45°	60°
Sine	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
Cosine	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
Tangent	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

By using ratios of the special angles, the equivalent trigonometrical ratios for both positive and negative angles in each quadrant can be determined without using trigonometrical tables.

Example 5.10

Find the sine, cosine, and tangent of each of the following angles without using trigonometrical tables and calculators:

(a) $-135°$ (b) $120°$ (c) $330°$

Solution

(a) $\sin(-135°) = -\sin 45° = -\frac{\sqrt{2}}{2}$.

$\cos(-135°) = -\cos 45° = -\frac{\sqrt{2}}{2}$.

$\tan(-135°) = \tan 45° = 1$.

(b) $\sin(120°) = \sin 60° = \frac{\sqrt{3}}{2}$.

$\cos(120°) = -\cos 60° = -\frac{1}{2}$.

$\tan(120°) = -\tan 60° = -\sqrt{3}$.

(c) $\sin(330°) = -\sin 30° = -\frac{1}{2}$.

$\cos(330°) = \cos 30° = \frac{\sqrt{3}}{2}$.

$\tan(330°) = -\tan 30° = -\frac{\sqrt{3}}{3}$.

Example 5.11

Find the value of θ if $\cos\theta = -\frac{1}{2}$ and $0° \le \theta \le 360°$.

Solution

The value of cos θ is negative in the second and third quadrants.

Hence, $-\cos(180° - \theta) = \cos(180° + \theta) = -\frac{1}{2} = -\cos 60°$.

Thus, $\theta = 180° - 60° = 120°$ or $\theta = 180° + 60° = 240°$.

Therefore, the values of θ are $120°$ and $240°$.

Example 5.12

Find the values of each of the following without using trigonometrical tables and calculators:

(a) $\frac{\tan 30^\circ \sin 60^\circ}{\cos 45^\circ}$ (b) $\frac{\tan 60^\circ \sin 30^\circ}{\sin 45^\circ}$ (c) $\frac{\sin(-150)^\circ \cos 315^\circ}{\tan 300^\circ}$

Solution

(a) $\frac{\tan 30^\circ \sin 60^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{3}}{3}\times\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \frac{3}{6}\times\frac{2}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$

(b) $\frac{\tan 60^\circ \sin 30^\circ}{\sin 45^\circ} = \frac{\sqrt{3}\times\frac{1}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{6}}{2}.$

(c) $\frac{\sin(-150^\circ)\cos 315^\circ}{\tan 300^\circ} = \frac{-\frac{1}{2}\times\frac{\sqrt{2}}{2}}{-\sqrt{3}} = \frac{-\sqrt{2}}{-4\sqrt{3}} = \frac{\sqrt{6}}{12}.$

Exercise 5.3

Answer the following questions:

1. Find the sine, cosine, and tangent of each of the following:
   (a) 162° (b) 250°
   (c) 318° (d) –72°
   (e) –157° (f) –245°

2. Find the angles θ between 0° and 360° for each of the following equations:
   (a) $\sin\theta = -0.4365$
   (b) $\cos\theta = -0.8766$
   (c) $\tan\theta = 0.4321$

3. Find the angles θ between –360° and 0° for each of the following equations:
   (a) $\sin\theta = -0.4365$
   (b) $\tan\theta = 0.7856$
   (c) $\tan\theta = -0.3412$

4. Find the value of y if $2\cos y \tan 30^\circ = 1$ and $-360^\circ \le y \le 360^\circ$.

In question 5 to 8, find the value for each of the following without using trigonometrical tables or calculators:

5. $\sin 60° \cos 60°$
6. $\cos 45° \tan 30°$
7. $\dfrac{\sin 45° \tan 60°}{\cos 30°}$
8. $\dfrac{\tan(-30°)\cos 60°}{\sin(-45°)}$

9. Find positive or negative angles corresponding to each of the following:
   (a) $-333°$
   (b) $192°$
   (c) $-204°$
   (d) $265°$

10. If $\sin\theta = \sin 160°$ and θ is an acute angle, find θ.

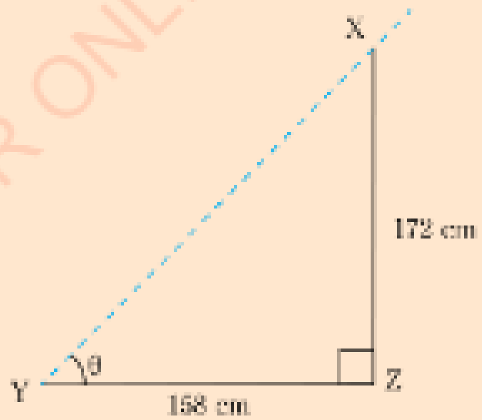
Applications of trigonometrical ratios

Trigonometrical ratios can be used in solving real life problems. For example, in surveying, trigonometrical ratios can be used to determine the angle of elevation and depression of buildings, trees, mountains, and many others.

Example 5.13

A man who is 172 cm tall, observes that the length of his shadow is 158 cm. Find the angle of elevation of the sun.

Solution



From the figure, $\overline{XZ}$ represents the height of a man, $\overline{YZ}$ represents the length of his shadow and θ is the angle of elevation of the sun.

$$\tan\theta = \frac{\overline{XZ}}{\overline{YZ}}$$

$$\tan\theta = \frac{172\text{ cm}}{158\text{ cm}}$$

$$= 1.089$$

$$\theta = 47^\circ\ 26'$$

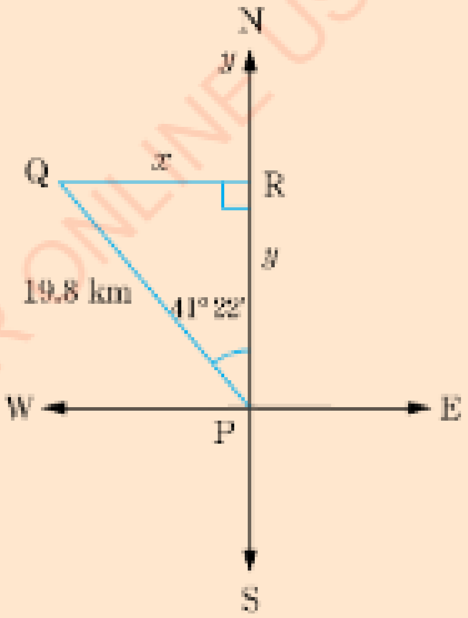
Therefore, the angle of elevation of the sun is $47^\circ\ 26'$.

Example 5.14

Mr. Petro starts from point P by travelling 19.8 km in a direction N 41° 22′W. How far has he travelled West and North respectively?

Solution

Present the information as shown in the following figure.



Let x and y be the distances in km due west and north of P respectively as shown in the figure.

Using a right – angled triangle PRQ:

$\sin 41° 22' = \frac{x}{19.8 \text{km}}$

$x = 19.8 \text{ km} \times \sin 41° 22'$

$= 13.09 \text{ km}$

$\cos 41° 22' = \frac{y}{19.8 \text{km}}$

$y = 19.8 \text{ km} \times \cos 41°22'$

$= 14.86 \text{ km}$

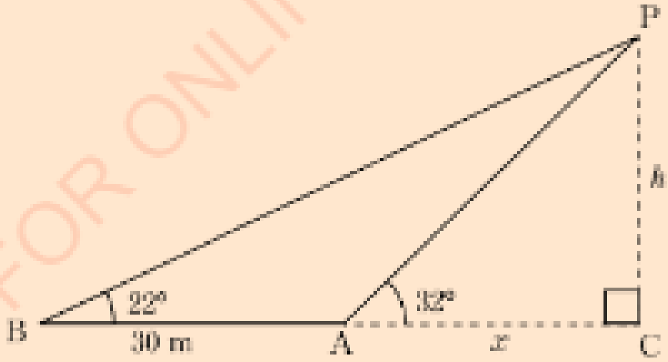
Therefore, Mr. Petro travelled 13.09 km west of P and 14.86 km north of P.

Example 5.15

From a certain point A, Fatuma observes the angle of elevation of the top of a church tower to be 32°. Moving 30m further away to a point B on the same horizontal level as the bottom of the tower C, she observes the angle of elevation to be 22°. Find the distance $\overline{AC}$ and the height of the tower.

Solution

Present the information as shown in the following figure.



Let h be the height of the tower and x be the distance $\overline{AC}$ as shown in the figure.

Then: $\tan 32° = \frac{h}{x}$ and $\tan 22° = \frac{h}{x+30}$

$h = x \tan 32°$ (1)

$h = (x + 30) \tan 22°$

$h = x \tan 22° + 30 \tan 22°$ (2)

Eliminating h from equations (1) and (2), gives:

$x \tan 32° = x \tan 22° + 30 \tan 22°$

$x (\tan 32° - \tan 22°) = 30 \tan 22°$

$$x = \frac{30 \tan 22°}{\tan 32° - \tan 22°}$$

$$= \frac{30 \times 0.4040}{0.6249 - 0.4040}$$

$$= \frac{12.12}{0.2209}$$

$= 54.88$

Substituting this value of x in equation (1), gives:

$h = 54.88 \times \tan 32°$

$= 54.88 \times 0.6249$

$= 34.29$

Therefore, distance $\overline{AC}$ is 54.88 m and the height of the tower is 34.29 m.

Exercise 5.4

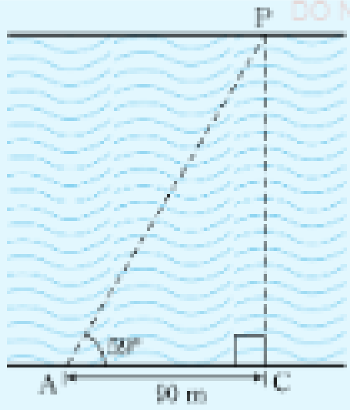
Answer the following questions:

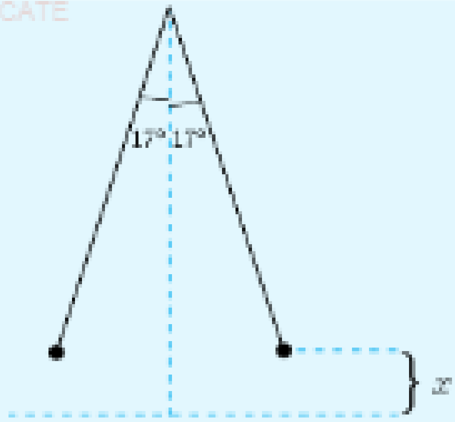
1. At a point 182 m from the foot of a tower on a level road, the angle of elevation of the top of the tower is 36°44′. Find the height of the tower.

2. A ladder rests against the top of the wall and makes an angle of 62° with the ground. If the foot of the ladder is 8.3m from the wall, find the height of the wall.

3. From the top of a cliff 35 m high, the angles of depression of two boats lying in a line due east of the cliff are 27° and 23°. Find the distance between the boats.

4. The points A and C are on the same bank of the river (see the sketched figure). If $\overline{AC}$ measures 90 m and $P\hat{A}C = 59°$, find the width of the river.





5. The lengths of a rectangular garden are 163 m long and 109 m wide. Find the angles made by a canal and the edges if the canal cuts the garden diagonally.

6. A ship starts from a point P and travels 22 km in a direction N32° 41′E. How far North and East of P is the ship?

7. A ladder 18 m long rests against a vertical wall. Find the inclination of the ladder to the horizontal if the foot of the ladder is 8.5m from the wall.

8. A pendulum 28 cm long swings on either side of the vertical through an angle of 17° (see the sketched figure). What height does the pendulum bob rise?

9. A ladder lies along a vertical wall. If the ladder makes angles of 75°10′ and 60° 27′ to the horizontal with 3 m between the two angles. Find:
   (a) the length of the ladder.
   (b) the distance from the foot of the ladder to the wall.

10. An electric pole casts a shadow 5.5 m long. If the angle of elevation of the sun is 55°, find the height of the pole.

11. A rope 15 m long is stretched out from the top of a flag – post 10 m high to a point on a level ground.
   (a) What angle does it make with the ground?
   (b) How far is this point from the foot of the flag post?

12. The angle of depression of a boat from the cliff 25 m high is 12°. Find the distance of the boat from the bottom of the cliff.

Activity 5.4: Determine the value of sine, cosine, and tangent of angles in a right – angled triangle

In a group or individually, perform the following tasks:

1. Draw a set of axes in your exercise book, each axis should be 8 cm (8 units in the first quadrant).
   (a) Plot the point (4, 5) and label it as A.
   (b) Join A to the origin O.
   (c) Draw a perpendicular line down from A to a point B on the x – axis.
   (d) You should now have a right – angled triangle. Label the angle between the x – axis and the hypotenuse as α and fill in the lengths of the sides of the triangle. (You will need to calculate the length of hypotenuse).
   (e) Now determine the values of the following using your diagram:
   (i) sin α (ii) cos α (iii) tan α
   (f) Instead of using the numbers, use the letters x, y, and r, to compute the expressions of the following:
   (i) sin α (ii) cos α (iii) tan α
2. Share your findings with your friends for more inputs.

The determination of trigonometrical ratios for both positive and negative angles has been done earlier. The relationship between an angle and its trigonometrical ratio defines a function.

For example, if sin θ = y, then, the ordered pairs (θ, y) define a cordinate on the sine function.

Similarly, the ordered pairs (θ, x) define a cordinate on the cosine function and $\left(\theta, \frac{y}{x}\right)$ define a cordinate on the tangent function.

Examples of such ordered pairs are:

$(45°, \sin 45°) = (45°, 0.71)$

$(120°, \cos 120°) = (120°, -0.5)$

$(-70°, \tan(-70)°) = (-70°, -2.75)$

Sine and cosine functions

Sine function is a function of the form $f(\theta)=\sin\theta$ and cosine function is the function of the form $f(\theta)=\cos\theta$.

For example: Tables 5.5 and 5.6 show a set of ordered pairs for sine and cosine functions respectively for angles between $-720°$ and $720°$.

Table 5.5: Ordered pairs of sine for angles between $-720°$ and $720°$

θ	–720°	–630°	–540°	–450°	–300°	–270°	–180°	–90°	0°
sin θ	0.00	1.00	0.00	–1.00	0.00	1.00	0.00	–1.00	0.00
θ	90°	180°	270°	300°	450°	540°	630°	720°	
sin θ	1.00	0.00	–1.00	0.00	1.00	0.00	–1.00	0.00	

Table 5.6: Ordered pairs of cosine for angles between $-720°$ and $720°$

θ	–720°	–630°	–540°	–450°	–300°	–270°	–180°	–90°	0°
cos θ	1.00	0.00	–1.00	0.00	1.00	0.00	–1.00	0.00	1.00
θ	90°	180°	270°	300°	450°	540°	630°	720°	
cos θ	0.00	–1.00	0.00	1.00	0.00	–1.00	0.00	1.00	

These ordered pairs can be plotted to obtain the graphs of sin θ and cos θ as shown in Figure 5.17.

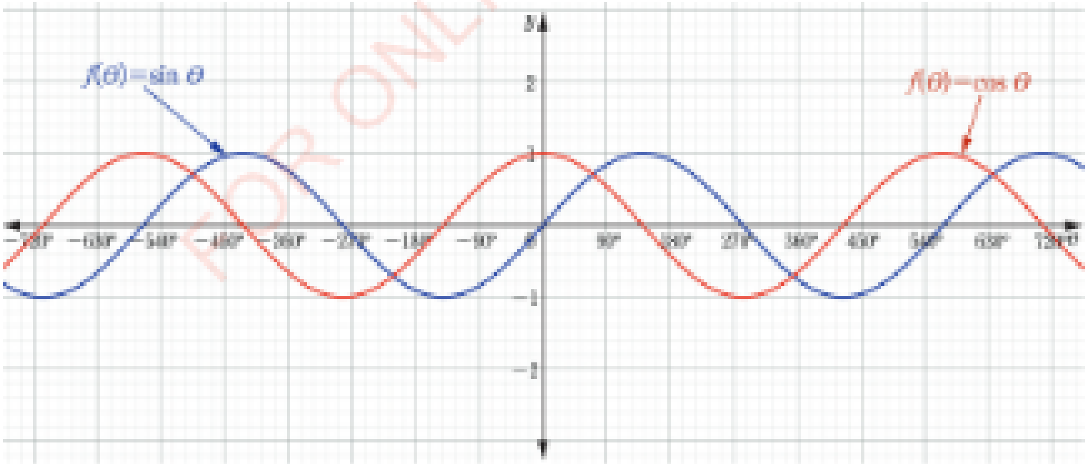


Figure 5.17: *Sine and cosine graphs of some special angles*

The graphs of sin θ and cos θ in Figure 5.17 shows that the values of the functions repeat at the intervals of 360°.

For example, when θ = –270°, sin θ = 1. This value occurs again when θ = 90°. Similarly, when θ = –180°, cos θ = –1 and this value occurs again when θ = 180° after a difference of 360°.

This means that $\sin\theta = \sin(\theta + 360°) = \sin(\theta + 2\times 360°)$ and so on.

Also, $\cos\theta = \cos(\theta + 360°) = \cos(\theta + 2\times 360°)$ and so on.

Generally, $\sin\theta = \sin(\theta + 360° n)$ and $\cos\theta = \cos(\theta + 360° n)$ where n is an integer.

Each of these functions is called a periodic function and the interval 360° is called the period.

The values of sine and cosine can be found from graphs of trigonometrical functions by drawing a horizontal line equal to, say, the value as shown in Figure 5.18.

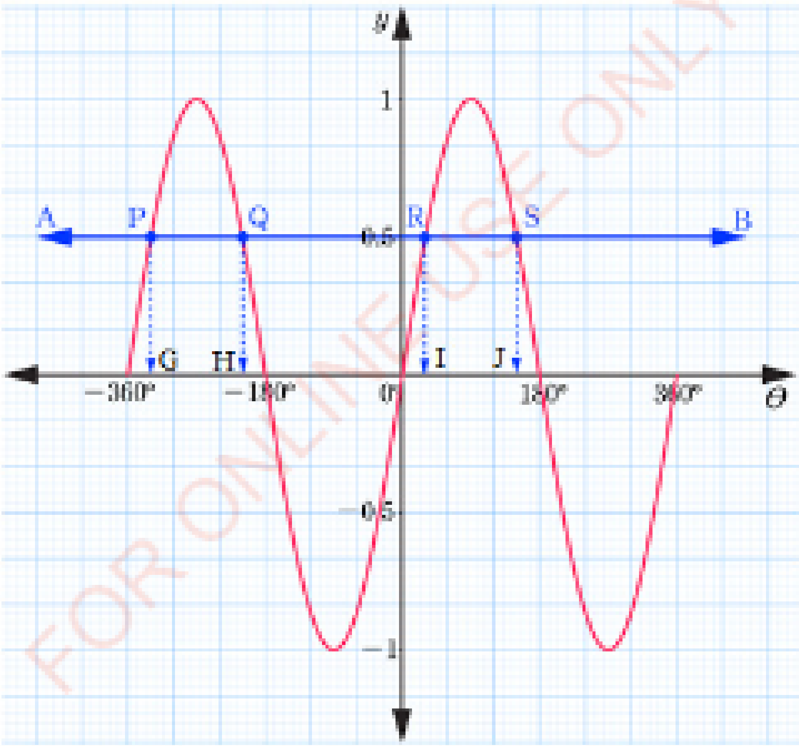


Figure 5.18: *Sine graph showing values of some special angles*

Line AB corresponds to y = sin θ = 0.5. The line intersect the graph at points P, Q, R, and S. Angles corresponding to these points are obtained vertically down on the x – axis at points G, H, I, and J respectively.

Example 5.16

Using trigonometrical graphs in the interval $-360° \leq \theta \leq 360°$, find the values of θ such that:

(a) $\sin\theta = 0.4$ (b) $\cos\theta = 0.9$

Solution

Therefore, from Figures 5.17 and 5.18 the required angles are:

(a) $-336°, -204°, 24°, 156°$.

(b) $-334°, -26°, 26°, 334°$.

Example 5.17

Using the graph of sin θ in the interval $-360° \leq \theta \leq 360°$ find the values of θ if $4\sin\theta = -1.8$.

Solution

$4\sin\theta = -1.8$

$\sin\theta = -0.45$

Therefore, from Figure 5.18 the required angles are $-153°, -27°, 207°$ and $333°$.

Example 5.18

Using the trigonometrical graphs, find the sine and cosine of each of the following angles:

(a) $-300°$ (b) $155°$ (c) $-40°$ (d) $249°$

Solution

From Figures 5.17 and 5.18 the required values are:

(a) $\sin(-300°) = 0.87$
$\cos(-300°) = 0.5$

(b) $\sin 155° = 0.42$
$\cos 155° = -0.91$

(c) $\sin(-40°) = -0.64$
$\cos(-40°) = 0.77$

(d) $\sin 249° = -0.93$
$\cos 249° = -0.36$

Exercise 5.5

Answer the following questions:

1. Use the graphs of Figure 5.17 and Figure 5.18 to read approximate values of θ such that:

   (a) $\sin\theta = 0.6$ (b) $\cos\theta = 0.2$

2. Use the graphs to find approximate values for each of the following:

   (a) $\sin(-245°)$ (b) $\cos(-190°)$

   (c) $\cos 320°$ (d) $\sin 130°$

3. Use the graph of $\cos\theta$ to find the values of θ if $5\cos\theta = -1.5$ and $0° \leq \theta \leq 360°$.

The sine rule

Activity 5.5: Explaining the concept of the sine rule

In a group or individually, perform the following tasks:

1. Draw a triangle ABC. By using a protractor, measure $A\hat{B}C$, $B\hat{C}A$, and $B\hat{A}C$.
2. Using a ruler measure $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$. Then, find the value of $\frac{\overline{BC}}{\sin A}$, $\frac{\overline{AC}}{\sin B}$, and $\frac{\overline{AB}}{\sin C}$.
3. What conclusion can you make from task 2?
4. Give the meaning of the sine rule.
5. Share your findings with your friends for more inputs.

Consider ΔABC in which A, B, and C are the angles and letters a, b, and c denotes the corresponding sides of the triangle as shown in Figure 5.19.

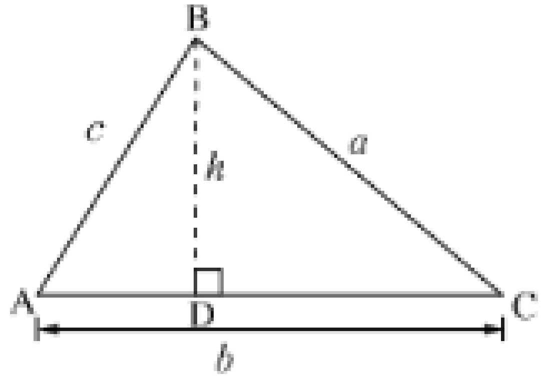


Figure 5.19: *Illustration for the sine rule using the sides* $\overline{AB}$ *and* $\overline{BC}$

By drawing a line perpendicular to side $\overline{AC}$ from angle B, ΔABD and ΔCBD are formed. The two triangles share the same height h.

Using a right – angled ΔABD, the following is deduced:

$$\sin A = \frac{h}{c}, \text{ thus, } h = c\sin A \qquad (1)$$

Using, right – angled ΔCBD, the following is deduced:

$$\sin C = \frac{h}{a}, \text{ thus, } h = a\sin C \qquad (2)$$

Eliminating h from equations (1) and (2) gives:

$$c \sin A = a \sin C \qquad (3)$$

Dividing by $\sin A \sin C$ on both sides of equation (3) results to;

$$\frac{c\sin A}{\sin A \sin C} = \frac{a\sin C}{\sin A \sin C}$$

$$\text{Hence, } \frac{c}{\sin C} = \frac{a}{\sin A} \qquad (4)$$

Suppose ΔABC is divided by drawing a perpendicular line from angle A to the side $\overline{BC}$ as shown in Figure 5.20.

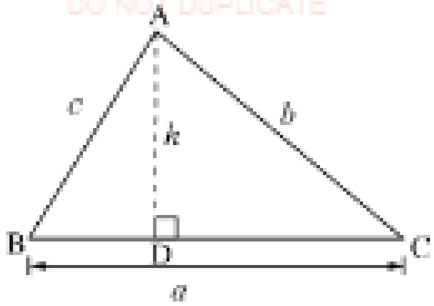


Figure 5.20: *Illustration for the sine rule using the sides* $\overline{AB}$ *and* $\overline{AC}$

Using a right – angled ΔBAD the following is obtained:

$$\sin B = \frac{h}{c}, \text{ thus, } h = c\sin B \qquad (5)$$

Using a right – angled ΔCAD, the following is obtained:

$$\sin C = \frac{h}{b}, \text{ thus, } h = b\sin C \qquad (6)$$

Eliminating h from equations (5) and (6) gives:

$$c\sin B = b\sin C \qquad (7)$$

Dividing by $\sin B \sin C$ on both sides of equation (7) results to:

$$\frac{c\sin B}{\sin B\sin C} = \frac{b\sin C}{\sin B\sin C}.$$

Hence, $$\frac{c}{\sin C} = \frac{b}{\sin B} \qquad (8)$$

Using equations (4) and (8) results to:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (9)$$

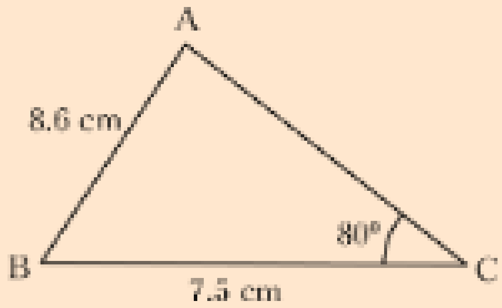
The relations in equation (9) is called the sine rule.
The sine rule states that if a, b, and c are the lengths of the sides of a ΔABC, then,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The sine rule is used when both two angles and one side are given, or two sides and one angle are given.

Example 5.19

Find the length of the unknown side and angles in the following triangle ABC:



Solution

The following unknowns have to be found: $\overline{AC}$, $\hat{A}$, and $\hat{B}$.

Using the sine rule,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, where $a = 7.5$ cm, $c = 8.6$ cm, and $\hat{C} = 80°$

$$\frac{7.5 \text{ cm}}{\sin A} = \frac{8.6 \text{ cm}}{\sin 80°}$$

$$\sin A = \frac{7.5 \text{ cm} \times \sin 80°}{8.6 \text{ cm}} = 0.8588$$

Hence, $\hat{A} = 59° \, 11'$

But, $\hat{A} + \hat{B} + \hat{C} = 180°$

$$\hat{B} = 180° - (80° + 59° \, 11')$$

$$= 180° - 139° \, 11'$$

Hence, $\hat{B} = 40° \, 49'$

$\frac{b}{\sin 40° \, 49'} = \frac{8.6 \text{ cm}}{\sin 80°}$, where $b = \overline{AC}$

$$b = \frac{8.6 \text{ cm} \times \sin 40° \, 49'}{\sin 80°}$$

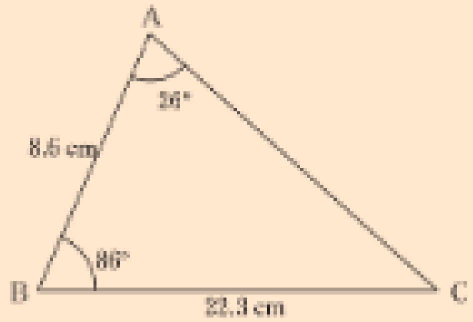
$$= 5.7 \text{ cm}$$

Therefore, $\overline{AC} = 5.7$ cm, $B\hat{A}C = 59° 11'$, and $C\hat{B}A = 40° \, 49'$.

Example 5.20

In the following triangle, find:

(a) $B\hat{C}A$ (b) $\overline{AC}$



Solution

(a) $B\hat{C}A + C\hat{A}B + A\hat{B}C = 180°$ (sum of angles in a triangle)

$B\hat{C}A = 180° - (26° + 86°)$

$= 68°$

Therefore, $B\hat{C}A = 68°$.

(b) Using the sine rule, $\frac{b}{\sin B} = \frac{a}{\sin A}$ where $a = 22.3$ cm, $\hat{A} = 26°$, and $\hat{B} = 86°$

Then, $\frac{b}{\sin 86°} = \frac{22.3 \text{ cm}}{\sin 26°}$

$$b = \frac{22.3 \text{ cm} \times \sin 86°}{\sin 26°}$$

$= 50.7$ cm

Therefore, $\overline{AC} = 50.7$ cm.

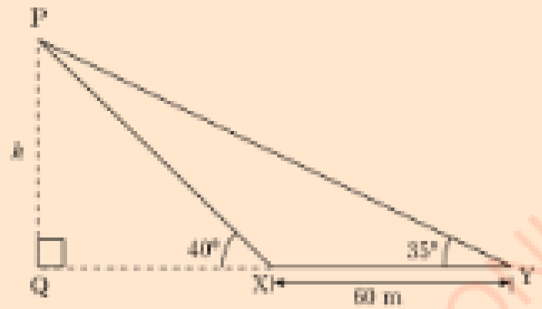
Example 5.21

Points X and Y lie East of a stationary kite P and are 60 m apart. The angles of elevation of the kite from X and Y are 40° and 35° respectively. Present this information in a triangle, then find:

(a) $\overline{PY}$ (b) the height of a kite.

Solution

The information given is presented as shown in the following figure:



(a) $P\hat{X}Y = 180° - 40° = 140°$ (Supplementary angles)

Hence, $X\hat{P}Y = 180° - (140° + 35°)$

$= 180° - 175° = 5°$

In ΔPXY,

$$\frac{\overline{PY}}{\sin 140°} = \frac{\overline{XY}}{\sin 5°}$$

$$\overline{PY} = \frac{\overline{XY} \sin 140°}{\sin 5°}$$

$$= \frac{60 \text{ m} \sin 140°}{\sin 5°}$$

$$= 442.5 \text{ m}$$

Therefore, $\overline{PY} = 442.5$ m.

(b) Using a right–angled triangle PQY where $\hat{Y} = 35°$, $\overline{PY} = 442.5\text{m}$

$$\frac{\overline{PQ}}{\sin 35°} = \frac{442.5\text{m}}{\sin 90°}$$

Then, $$\overline{PQ} = \frac{442.5\text{m} \times \sin 35°}{\sin 90°}$$
$$= 253.8\text{m}$$

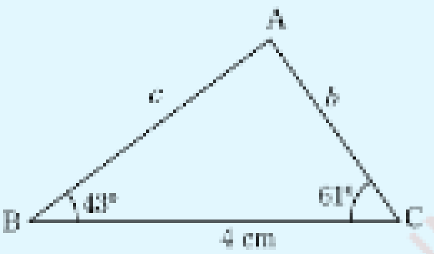
Therefore, the height of the kite is 253.8 m.

Exercise 5.6

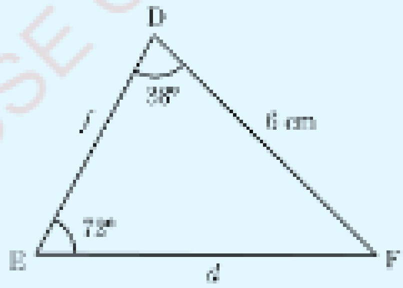
Answer the following questions:

1. Find the unknown sides and angles for each of the following triangles:

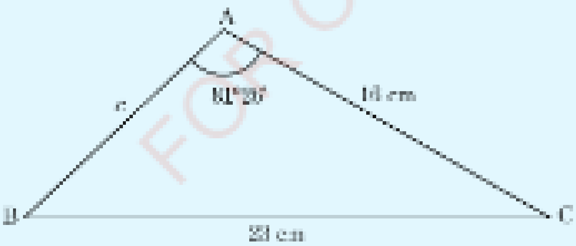
(a)



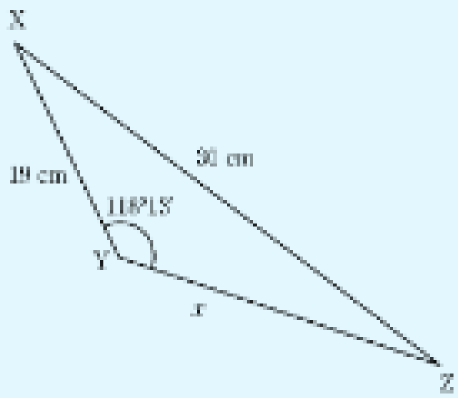
(b)



(c)

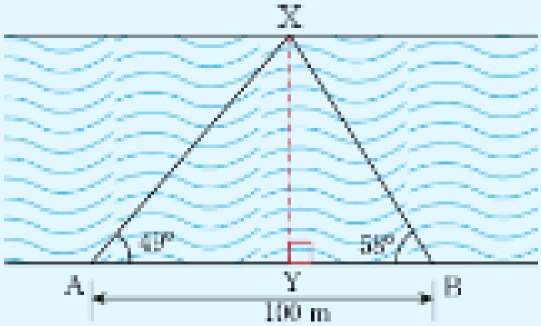


(d)



2. Juma notices that the angle of elevation of the top of a coconut tree is 32°. Walking 11 metres in a direction towards the tree he observes that the angle of elevation is 45°. Find the height of the tree.

3. Anna wishes to find the width $\overline{XY}$ of the Kilombero river. She measures a distance $\overline{AB}$ = 100 m along the bank of the river. She observes that a point X on the other bank of the river is such that $X\hat{A}B = 49°$ and $X\hat{B}A = 58°$.

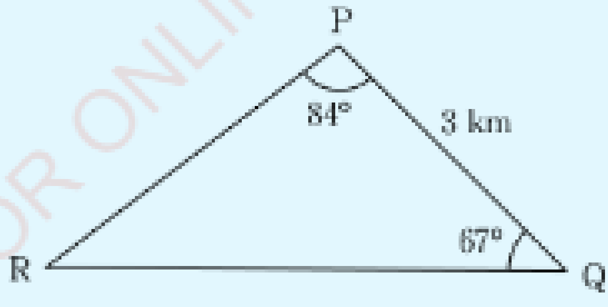


Find the width of the river.

4. Two boats A and B are due south of a cliff, the two boats are 890 m apart. The angles of elevation at the top of the cliff at the two points A and B are 26°14′ and 17°56′ respectively. Find the height of the cliff.

5. A boat at point R is observed from two points P and Q which are 3 km apart (see the figure). If $R\hat{P}Q = 84°$ and $P\hat{Q}R = 67°$ find:

(a) $\overline{PR}$ (b) $\overline{QR}$



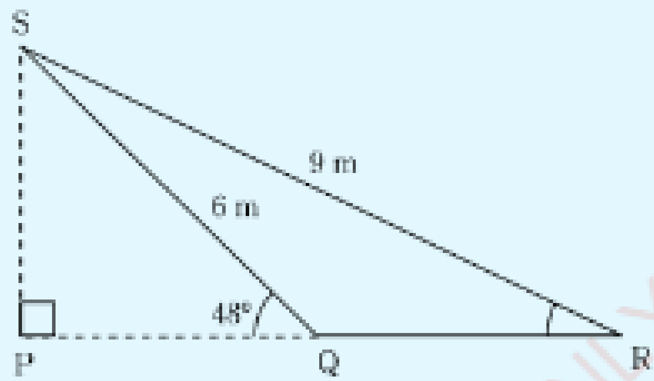
5. A ship sails 302 km from port O to port P on a bearing $N35°W$ and then moves to port Q on a bearing $W50°S$. If $\overline{OQ}$ = 400 km, find:

(a) the distance PQ.

(b) the bearing of O from Q.

6. A railway signal $\overline{SP}$ is supported by two chains $\overline{SQ}$ and $\overline{SR}$ of length 6 m and 9 m respectively as shown in the following figure. If the angle of elevation of the top of the signal from point Q is 48°, find:

   (a) $S\hat{R}Q$.
   (b) the distance QR.
   (c) the height of the signal.



The cosine rule

Activity 5.6: Explaining the concept of the cosine rule

In a group or individually, perform the following tasks through the following steps:

1. On the graph paper, draw xy – axis on the first quadrant, draw a convenient triangle ABC with vertices A(0, 0), B(c, 0), and $C(b\cos A,\ b\sin A)$.
2. Using the distance formula, find the distance CB.
3. Repeat steps 1 and 2 when the altitude is drawn from point A and from point C.
4. Give the meaning of the cosine rule.
5. Share your findings with your friends and discuss.

Consider a triangle ABC in which the side $\overline{BD}$ is perpendicular to the line drawn from point B to the side $\overline{AC}$.

Let, the side $\overline{BD}$ be h, $\overline{AD} = x$, and $\overline{DC} = b - x$ as shown in Figure 5.21.

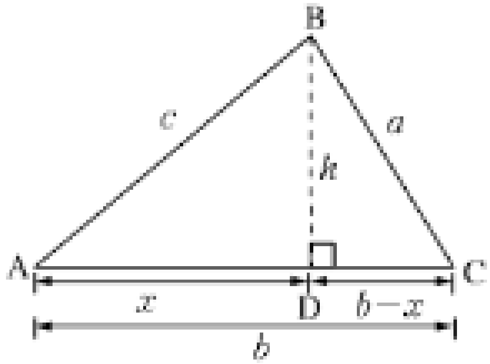


Figure 5.21: *Illustration for the cosine rule*

By drawing a line perpendicular to side $\overline{AC}$ from angle B, the triangle ABD and CBD are formed. From a right – angled triangle ABD, using the Pythagoras' theorem gives:

$h^2 + x^2 = c^2$

Thus, $h^2 = c^2 - x^2$ (1)

Also from a right – angled triangle CBD, using the Pythagoras' theorem gives:

$h^2 + (b - x)^2 = a^2$

Thus, $h^2 = a^2 - (b - x)^2$ (2)

Eliminating h from equations (1) and (2) gives:

$c^2 - x^2 = a^2 - (b - x)^2$

$c^2 - x^2 = a^2 - b^2 + 2bx - x^2$

Thus, $c^2 = a^2 - b^2 + 2bx$ (3)

From a right – angled triangle ABD,

$x = c\cos A$ (4)

On substituting equation (4) into equation (3) gives:

$c^2 = a^2 - b^2 + 2bc\cos A$ (5)

Rearranging equation (5) gives:

$a^2 = b^2 + c^2 - 2bc\cos A$ (6)

Also, when the altitude is drawn from point A or from point C, the similar expression for the other side of the triangle ABC can be obtained as:

$b^2 = a^2 + c^2 - 2ac\cos B$ (7)

$c^2 = a^2 + b^2 - 2ab\cos C$ (8)

Hence, equations (6), (7), and (8) are the formulae for the cosine rule.
Therefore, the cosine rule states that the square of the length of any side of a triangle equals the sum of the squares of the length of the other sides minus twice their product multiplied by the cosine of their included angle.
The cosine rule is used when:

(a) two sides and an included angle are given.
(b) three sides are given.

Example 5.22

Find the unknown side and angles in ΔABC given that $a = 3$ cm, $c = 4$ cm, and $\hat{B} = 30°$.

Solution

In the given figure, A, B, and C are angles of a triangle and a, b, and c are the sides.

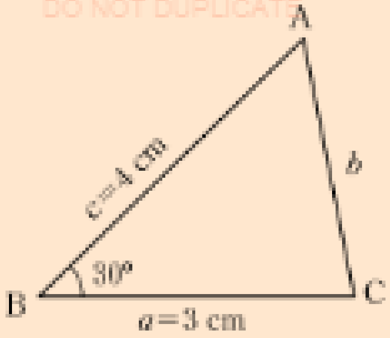


Figure 5.22: *A triangle ABC with two sides and included angle*

By using the cosine rule:

$$b^2 = c^2 + a^2 - 2ac\cos B$$

$$= 16\,\text{cm}^2 + 9\,\text{cm}^2 - 2 \times 3\,\text{cm} \times 4\,\text{cm} \cos 30^\circ$$

$$= (16 + 9 - 24 \times 0.866)\,\text{cm}^2$$

$$= 4.216\ \text{cm}^2$$

$$b = 2.05\ \text{cm}.$$

To find angle A the cosine rule is used again as follows:

$a^2 = c^2 + b^2 - 2bc\cos A$ where A is an angle between the sides b and c.

$$\cos A = \frac{c^2 + b^2 - a^2}{2bc} = \frac{(4\,\text{cm})^2 + (2.05\,\text{cm})^2 - (3\,\text{cm})^2}{2 \times 2.05\,\text{cm} \times 4\,\text{cm}}$$

$$= \frac{11.2025\,\text{cm}^2}{16.4\,\text{cm}^2} = 0.6831$$

$$A = 46^\circ 55'$$

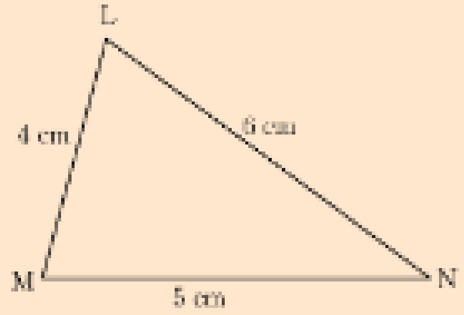
$$C = 180^\circ - (46^\circ 55' + 30^\circ)$$

$$= 180^\circ - (76^\circ 55') = 103^\circ 5'$$

Therefore, $b = 2.05$ cm, $A = 46^\circ 55'$, and $C = 103^\circ 5'$.

Example 5.23

Find the unknown angles in the following ΔLMN:



Solution

Let, M, N, and L represent the angles in the ΔLMN.

By using the cosine rule;

$l^2 = m^2 + n^2 - 2mn \cos L$

$\cos L = \frac{m^2 + n^2 - l^2}{2mn}$

$= \frac{(6\,\text{cm})^2 + (4\,\text{cm})^2 - (5\,\text{cm})^2}{2 \times 6\,\text{cm} \times 4\,\text{cm}}$

$= 0.5625$

$L = 55° 46'$

$m^2 = n^2 + l^2 - 2nl \cos M$

$\cos M = \frac{n^2 + l^2 - m^2}{2nl}$

$= \frac{(4\,\text{cm})^2 + (5\,\text{cm})^2 - (6\,\text{cm})^2}{2 \times 4\,\text{cm} \times 5\,\text{cm}}$

$= 0.125$

$M = 82°49'$

$N = 180° - (M + N)$

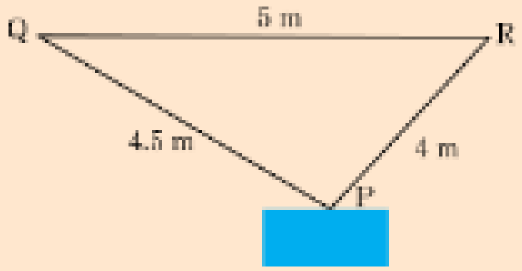
$= 180° - (82°49' + 55° 46')$

$= 41°25'$

Therefore, the unknown angles are $55° 46'$, $82° 49'$, and $41° 25'$.

Example 5.24

An object is hung from a horizontal beam by two chains fastened at points Q and R, 5 m apart. If the chains are 4.5 m and 4 m long, find the angles made by the chains with the beam.



Solution

In the figure, P, Q, and R are angles of a triangle PQR.

Let, $p = 5$ m, $r = 4.5$ m, and $q = 4$ m

Using the cosine rule:

$$q^2 = p^2 + r^2 - 2pr\cos Q$$

$$\cos Q = \frac{p^2 + r^2 - q^2}{2pr}$$

$$= \frac{(5\text{m})^2 + (4.5\text{m})^2 - (4\text{m})^2}{2\times 5\text{m}\times 4.5\text{m}}$$

$$= 0.65$$

$$Q = 49°28'$$

Similarly $r^2 = p^2 + q^2 - 2pq\cos R$

$$\cos R = \frac{p^2 + q^2 - r^2}{2pq}$$

$$= \frac{25\text{m}^2 + 16\text{m}^2 - 20.25\text{m}^2}{2\times 5\text{m}\times 4\text{m}}$$

$$= \frac{20.75\text{m}^2}{40\text{m}^2}$$

$$= 0.5188$$

$$R = 58°45'$$

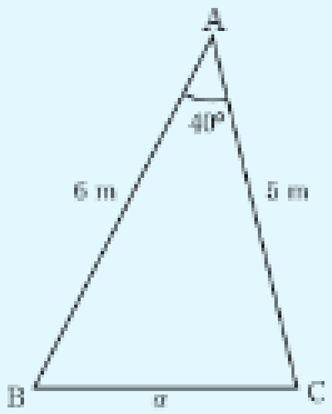
Therefore, the angles which the chains make with the beam are $49°28'$ and $58°45'$.

Exercise 5.7

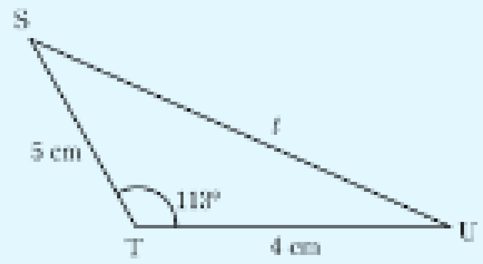
Answer the following questions:

1. Find the unknown side and angles in each of the following triangles:

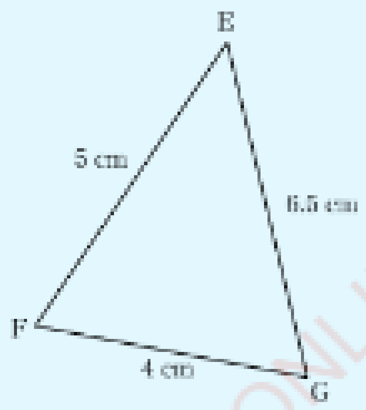
(a)



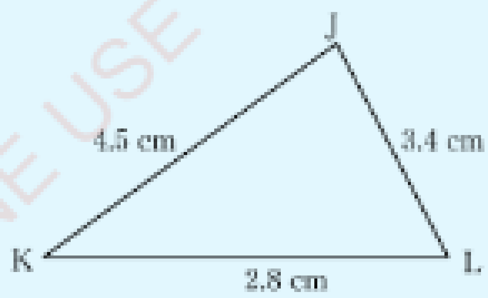
(b)



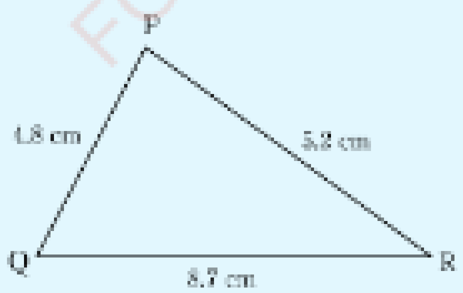
(c)



(d)



(e)



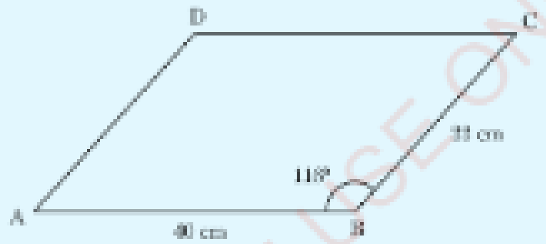
2. Given ΔUVW with sides $u = 11$ cm, $v = 14$cm, and $w = 21$ cm. If $\hat{W} = 67°$, find the measures of $\hat{V}$ and $\hat{U}$.

3. In ΔABC, $a = 14$ cm, $b = 4$ cm, $c = 12$cm, find the middle – sized angle.

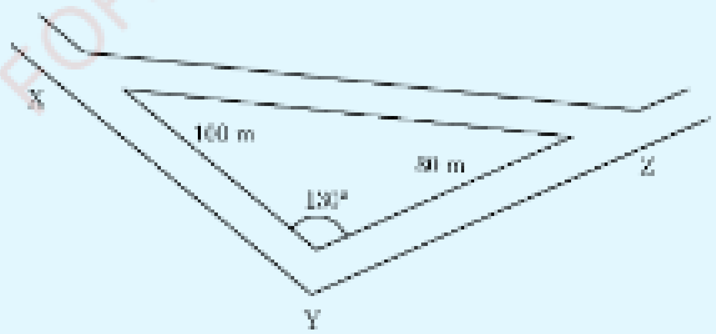
4. If EFGH is a parallelogram whose sides are 12 cm and 16 cm, find the length of the diagonal EG given that $\hat{F} = 119°$.

5. The diagonals PR and QS of the parallelogram PQRS are 24 cm and 16 cm respectively. If the acute angle formed by the diagonals is $38° 27'$, find the longer side of the parallelogram.

6. Given a parallelogram ABCD with adjacent sides measure 40 cm and 22 cm. If the largest angle of the parallelogram measures 116°, find the length of the larger diagonal.



7. A road is bent at point Y so that $X\hat{Y}Z = 130°$ and makes a short road $\overline{XZ}$. If $\overline{XY}$ is 100 m and $\overline{YZ}$ is 80 m, what a distance is saved by walking along the short cut $\overline{XZ}$?



8. The ports E and F on a straight coast line are such that F is 53 km East of E. A ship starting from port E sails 40 km to a point G in a direction N 65°30′E. Find:
   (a) the distance of the ship from F.
   (b) the ship's bearing from port F.
9. The hands of a clock are 4 cm and 5 cm long. Find the distance between the tips of the hands at 5.00 p.m.
10. A rhombus PQRS has a side of length 6 cm and with the longest diagonal of 7 cm. If one of its smallest angles is 37°, find the sum of its largest angles.

Compound angles

Activity 5.7: Comparing values of sine and cosine of compound angles

Perform the following tasks in a group or individually:

1. Check whether the following values of angles are the same.
   (a) $\sin(45° + 30°) = \sin 45° + \sin 30°$
   (b) $\sin(90° - 60°) = \sin 90° - \sin 60°$
   (c) $\cos(45° + 30°) = \cos 45° + \cos 30°$
   (d) $\cos(90° - 60°) = \cos 90° - \sin 60°$
2. What did you observe?
3. Share your findings with your neighbours for more inputs.

Other useful trigonometrical relations can be determined by considering the sine and cosine of the sum or difference of any two angles.

Let us investigate whether $\sin(60° + 30°) = \sin 60° + \sin 30°$.

$\sin(60° + 30°) = \sin 90° = 1$.

$\sin 30° = \frac{1}{2}$, and $\sin 60° = \frac{\sqrt{3}}{2}$.

Therefore, $\sin(60° + 30°) \neq \sin 60° + \sin 30°$ since, $1 \neq \frac{\sqrt{3}}{2} + \frac{1}{2}$.

Similarly, we can investigate whether

$\cos(180° - 90°) = \cos 180° - \cos 90°$.

$\cos(180° - 90°) = \cos 90° = 0$, $\cos 180° = -1$ and $\cos 90° = 0$

Therefore, $\cos(180° - 90°) \neq \cos 180° - \cos 90°$ since $0 \neq -1$.

From these examples, it can be deduced that:

(a) $\sin(A + B) \neq \sin A + \sin B$

(b) $\sin(A - B) \neq \sin A - \sin B$

(c) $\cos(A + B) \neq \cos A + \cos B$

(d) $\cos(A - B) \neq \cos A - \cos B$

The cosine of the sum and difference of any two angles

The sum and difference identities for the sine and cosine functions are identities involving angles A and B. These identities are as follows:

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Let us derive the relation for $\cos(A + B)$. From the fact that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, $\cos(A + B)$ can be derived as follows:

$$\begin{aligned}\cos(A + B) &= \cos(A - (-B)) \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \sin B\end{aligned}$$

Hence, $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

This relation is true for $\cos(A - B)$ and $\cos(A + B)$ for angles A and B, will also be the same for functions of numbers.

Therefore, $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Example 5.25

(a) Find the value of cos 135° from cos (90° + 45°) without using trigonometrical tables or calculators.

(b) Simplify cos(90° + θ).

Solution

(a) $\cos 135^\circ = \cos\left(90^\circ + 45^\circ\right)$

$$= \cos 90^\circ \cos 45^\circ - \sin 90^\circ \sin 45^\circ$$

$$= \left(0\times\frac{\sqrt{2}}{2}\right)-\left(1\times\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{2}}{2}$$

Therefore, $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

(b) $\cos\left(90^\circ + \theta\right)$

$$= \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$$

$$= (0)\cos\theta - (1)\sin\theta$$

$$= 0 - \sin\theta$$

$$= -\sin\theta$$

Therefore, $\cos(90^\circ + \theta) = -\sin\theta$.

Example 5.26

Find the value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$ without using trigonometrical tables or calculators.

Solution

$$\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$=\left(\frac{1}{2}\times\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Therefore, $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{\sqrt{2}+\sqrt{6}}{4}$.

Exercise 5.8

Answer the following questions:

1. Verify that $\cos(180°-90°) = \cos 180° \cos 90° + \sin 180° \sin 90°$.
2. Verify that $\cos(90°-60°) = \cos 90° \cos 60° + \sin 90° \sin 60°$.
3. Find the value of $\cos 15°$ by using $\cos(45°-30°)$.
4. Verify that for any angle C, $\cos(90°-C) = \sin C$.
5. Verify that $\cos\left(\pi-\frac{2}{3}\pi\right) = \cos\pi\cos\frac{2}{3}\pi+\sin\pi\sin\frac{2\pi}{3}$.
6. Verify that $\cos(\pi-\pi) = \cos\pi\cos\pi+\sin\pi\sin\pi$.
7. Verify that the following relations are true for all values of t:
   (a) $\cos(\pi-t) = -\cos t$.
   (b) $\cos(2\pi-t) = \cos t$.
   (c) $\cos\left(\frac{\pi}{2}-t\right) = \sin t$.
   (d) $\cos(\pi-t) = \sin t$.
8. Verify that $\cos(30°+60°) = \cos 30° \cos 60° - \sin 30° \sin 60°$.
9. Find the value of $\cos 105°$ from $\cos(45°+60°)$.
10. Find the value of $\cos 255°$ from $\cos(45°+210°)$.
11. Verify that $\cos(180°+45°) = \cos 180° \cos 45° - \sin 180° \sin 45°$.

12. Verify that the following relations are true for all values of t.

(a) $\cos(\pi + t) = -\cos t.$

(b) $\cos(2\pi + t) = \cos t.$

(c) $\cos\left(\frac{\pi}{2} + t\right) = -\sin t.$

(d) $\cos\left(\frac{3}{2}\pi + t\right) = -\sin t.$

13. Verify using the formula for $\cos(t + s)$ that $\cos(t + 2\pi) = \cos t.$ What does this show about the period of the cosine function?

14. Simplify the following expressions:

(a) $\frac{\cos 2A}{\cos A + \sin A}$

(b) $\cos\left(\frac{\pi}{2} - \beta\right)$

(c) $\cos\left(\theta - \frac{\pi}{2}\right)$

(d) $\frac{\cos 2A}{\cos A - \sin A}$

The sine of the sum and difference of any two angles

If C is any acute angle, then

$$\begin{aligned}\cos(90° - C) &= \cos 90° \cos C + \sin 90° \sin C \\ &= 0 \times \cos C + 1 \times \sin C \\ &= \sin C.\end{aligned}$$

Hence, $\cos(90° - C) = \sin C.$

Let, C be $(90° - A)$, then $90° - C = 90° - (90° - A) = A$

This means that $\cos(90° - C) = \sin C$ and $\cos A = \sin(90° - A).$

Now, let C be another name for $(A + B)$. The formula for the sine of the sum of the two angles A and B can be found as follows:

$$\begin{aligned}\sin(A + B) &= \cos(90° - (A + B)) \\ &= \cos[(90° - A) - B] \\ &= \cos(90° - A)\cos B + \sin(90° - A)\sin B \\ &= \sin A \cos B + \cos A \sin B\end{aligned}$$

Hence, $\sin(A + B) = \sin A \cos B + \cos A \sin B.$

The formula for the sine of the difference of any two angles A and B can also be found by considering $\sin[A + (-B)]$ using the derived formula for the sum.

Note that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, results to:

$\sin[A + (-B)] = \sin A \cos(-B) + \cos A \sin(-B)$

$\sin(A - B) = \sin A \cos B + \cos A(-\sin B)$

Hence, $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

The relations for $\sin(A + B)$ and $\sin(A - B)$ for angles A and B will also be the same for function of numbers.

Therefore, $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and

$\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Example 5.27

(a) Verify that $\sin(90° + 30°) = \sin 90° \cos 30° + \cos 90° \sin 30°$.

(b) Find the value of $\sin 75°$ without using trigonometrical tables or calculators.

Solution

(a) LHS: $\sin(90° + 30°) = \sin 120°$

$$= \sin(180° - 120°)$$

$$= \sin 60° = \frac{\sqrt{3}}{2}.$$

RHS:

$$\sin 90° \cos 30° + \cos 90° \sin 30° = \left(1 \times \frac{\sqrt{3}}{2}\right) + \left(0 \times \frac{1}{2}\right) = \frac{\sqrt{3}}{2}.$$

Therefore, $\cos(90° + 30°) = \sin 90° \cos 30° + \cos 90° \sin 30°$.

(b) $\sin 75° = \sin(45° + 30°)$

$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2} \times \frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Therefore, $\sin 75° = \frac{\sqrt{6} + \sqrt{2}}{4}$.

Example 5.28

(a) Verify that $\sin\left(\frac{2}{3}\pi - \frac{1}{6}\pi\right) = \sin\frac{2\pi}{3}\cos\frac{\pi}{6} - \cos\frac{2}{3}\pi\sin\frac{\pi}{6}$.

(b) Simplify (i) $\sin 2A$ (ii) $\sin\left(\theta + \frac{\pi}{3}\right)$

Solution

(a) LHS: $\sin\left(\frac{2}{3}\pi - \frac{\pi}{6}\right) = \sin\frac{3}{6}\pi = \sin\frac{\pi}{2} = 1$

RHS: $\sin\frac{2}{3}\pi\cos\frac{\pi}{6} - \cos\frac{2}{3}\pi\sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} \times \frac{1}{2}\right)$

$$= \frac{3}{4} + \frac{1}{4} = 1$$

Therefore, $\sin\left(\frac{2}{3}\pi - \frac{\pi}{6}\right) = \sin\frac{2\pi}{3}\cos\frac{\pi}{6} - \cos\frac{2}{3}\pi\sin\frac{\pi}{6}$.

(b) (i) $\sin 2A = \sin(A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2\sin A \cos A$

Therefore, $\sin 2A = 2\sin A \cos A$.

(ii) $\sin\left(\theta + \frac{\pi}{3}\right) = \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}$

$= \sin\theta \times \frac{1}{2} + \cos\theta \times \frac{\sqrt{3}}{2}$

$= \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$

$= \frac{1}{2}\left(\sin\theta + \sqrt{3}\cos\theta\right)$

Therefore, $\sin\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}\left(\sin\theta + \sqrt{3}\cos\theta\right)$.

Exercise 5.9

Answer the following questions:

1. Verify that $\sin(30° + 60°) = \sin 30° \cos 60° + \cos 30° \sin 60°$.
2. Verify that $\sin\left(\frac{2}{3}\pi + \frac{5}{3}\pi\right) = \sin\frac{2}{3}\pi \cos\frac{5}{3}\pi + \cos\frac{2}{3}\pi \sin\frac{5}{3}\pi$.
3. Use $\sin[s + (-t)]$ to find a compound formula for $\sin(s - t)$.
4. By using the formula for $\sin(A - B)$, show that $\sin(90° - C) = \cos C$.
5. Find the value of $\sin 15°$ from $\sin(315° - 300°)$.
6. Verify that $\sin\left(\frac{2}{3}\pi - \frac{7}{6}\pi\right) = \sin\frac{2}{3}\pi \cos\frac{7}{6}\pi - \cos\frac{2}{3}\pi \sin\frac{7}{6}\pi$.
7. By letting $A = B$ in the expression for $\sin(A + B)$, find the formula for $\sin 2A$.
8. Verify that $\sin 90° = 2 \sin 45° \cos 45°$.
9. Find the value of $\sin 90°$ using the fact that $\sin 90° = 2 \sin 45° \cos 45°$.
10. Simplify $\sin(t + \pi)$.
11. Find the value of $\sin 225°$ from $\sin(180° + 45°)$.
12. Find the value of $\sin\frac{5}{12}\pi$ from $\sin\left(\frac{2}{3}\pi - \frac{\pi}{4}\right)$.
13. Verify that $\sin(180° + 45°) = \sin 180° \cos 45° + \cos 180° \sin 45°$.
14. Simplify each of the following expressions:
    (a) $\sin(290° - A)$
    (b) $\sin\left(B - \frac{2\pi}{3}\right)$
    (c) $\sin\left(\frac{\pi}{6} - K\right)$.

The tangent of the sum and difference of any two angles

Activity 5.8: Deriving the compound angle formula for tangent

In a group or individually, perform the following tasks:

1. Find the ratio between the compound angle formula for $\sin(A+B)$ and $\cos(A+B)$.
2. Use the ratio obtained in task 1 to obtain the compound angle formula for $\tan(A+B)$.
3. Repeat task 1 and 2 for the compound angle formula for $\tan(A-B)$.
4. Give a conjecture from the formula obtained in task 2 and task 3.
5. Share your findings with your neighbours for more inputs.

From activity 5.8, the relation for tan (A + B) and tan (A – B) for acute angles A and B are also the same for a function of numbers.

To find the value of $\tan(A+B)$, one needs to consider the trigonometric ratio relations for tan θ, sin θ, and cos θ where θ is an acute angle.

Generally, it is known that: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Then, suppose that $\theta = A + B$.

Thus, $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$.

$$\tan(A+B) = \frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B}.$$

Then divide all the terms of $\tan(A+B)$ by $\cos A\cos B$ to obtain:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Conversely, $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$.

Example 5.29

Find the value of $\tan 105°$ from $\tan(45° + 60°)$.

Solution

$$\tan(45° + 60°) = \frac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{1+2\sqrt{3}+3}{1-3}$$

$$= \frac{4+2\sqrt{3}}{-2}$$

Therefore, $\tan 105° = -2-\sqrt{3}$.

Example 5.30

(a) Simplify $\tan\left(\frac{\pi}{4} + \theta\right)$.

(b) Verify that $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}}$.

Solution

(a) $$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} \times \tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta}$$

Therefore, $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$.

(b) LHS: $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$.

RHS: $$\frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}} = \frac{\sqrt{3} - \frac{\sqrt{3}}{3}}{1+\sqrt{3}\times\frac{\sqrt{3}}{3}} = \frac{3\sqrt{3}-\sqrt{3}}{6} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}.$$

Therefore, $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}}$.

Exercise 5.10

Answer the following questions:

1. Find the value of $\tan 75°$ from $\tan(45°+30°)$.

2. Verify that $\tan 225° = \frac{\tan 180° + \tan 45°}{1-\tan 180° \times \tan 45°}$.

3. By substituting $B = A$ in $\tan(A+B)$, find the formula for $\tan 2A$.

4. Simplify $\tan\left(\theta - \frac{\pi}{3}\right)$.

5. Find the value of $\tan B$ if $\tan(A+B) = 2$ and $\tan A = \frac{1}{2}$.

6. If $\tan 105° = -2-\sqrt{3}$, find the value of $\tan 165°$ from $\tan(105° + 60°)$.

7. Find the value of $\frac{\tan 40° + \tan 20°}{1-\tan 40° \times \tan 20°}$ without using trigonometrical tables or calculators.

8. Verify that $\tan 120° = \frac{2\tan 60°}{1-\tan^2 60°}$.

9. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{4}$, then find:

   (a) $\tan(A+B)$.   (b) $\tan(A-B)$.

10. Simplify $\tan(A+B)$ if $\tan A = \frac{1}{4}$ and $\tan B = \alpha$

11. Show that $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

Chapter summary

1. Trigonometrical ratios in a unit circle for an angle A:

   (a) $\sin A = \dfrac{\text{length of an opposite side}}{\text{length of hypotenuse side}}$.

   (b) $\cos A = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse side}}$.

   (c) $\tan A = \dfrac{\text{length of an opposite side}}{\text{length of adjacent side}}$.

2. Relationship between trigonometrical ratios for an angle α:

   (a) $\sin\alpha = \cos(90° - \alpha)$

   (b) $\cos\alpha = \sin(90° - \alpha)$

   (c) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

   (d) $\sin^2\alpha + \cos^2\alpha = 1$ where α is an acute angle.

3. Trigonometrical ratios for special angles 0°, 30°, 45°, 60°, 90°, 180°, 270°, and 360°.

Angle	0°	30°	45°	60°	90°	180°	270°	360°
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

4. Even and odd functions:

   (a) $\sin(-\alpha) = -\sin\alpha$

   (b) $\cos(-\alpha) = \cos\alpha$

   (c) $\tan(-\alpha) = -\tan\alpha$

5. The sine rule is given as: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

6. The cosine rule is given as:

   (a) $a^2 = b^2 + c^2 - 2bc\cos A$

   (b) $b^2 = a^2 + c^2 - 2ac\cos B$

   (c) $c^2 = a^2 + b^2 - 2ab\cos C$

7. Cosine of the sum and difference of any two angles:
   $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

8. Sine of the sum and difference of any two angles:
   $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

9. Tangent of the sum and difference of any two angles:
   $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Revision exercise 5

Answer the following questions:

1. Evaluate the following without using trigonometrical tables or calculators:
   (a) $\frac{\tan 60° \sin 30°}{\sin 45°}$
   (b) $\frac{\sin 30° \cos 60°}{\tan 30°}$
   (c) $\frac{\sin 45° \tan 60°}{\cos 30°}$
   (d) $2 \cos 135° - \sin 30°$
   (e) $\tan(-300°) - \tan 120°$
   (f) $\sin 225° + \cos 45°$

2. If α and β are complementary angles and $\sin \alpha = \frac{\sqrt{3}}{5}$, find the value of:
   (a) $\cos \alpha$ (b) $\tan \beta$

3. Using the trigonometrical tables, find the value of each of the following:
   (a) $\sin 192°$
   (b) $\cos 224°$
   (c) $\tan 321°$
   (d) $\sin(-15°)$
   (e) $\cos(-129°)$
   (f) $\tan(-310°)$.

4. Find the angles between 0° and 360° which satisfy each of the following:
   (a) $\sin \theta = -0.2468$
   (b) $\cos \theta = 0.3579$
   (c) $\tan \alpha = -2.356$.

5. Find the angles between − 360° and 360° which satisfy each of the following:
   (a) $\sin \theta = 0.1234$
   (b) $\cos \theta = -0.5678$
   (c) $\tan \theta = 0.3546$.

6. If P is the point (−3, 8), find the sine, cosine, and tangent of the obtuse angle between $\overline{OP}$ and the x − axis, where O is the point at (0, 0).

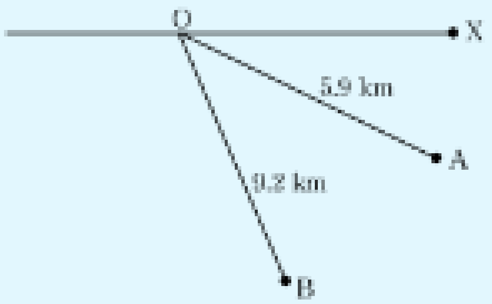
7. Express each of the following in terms of the sine, cosine or tangent of an acute angle:
   (a) $\sin 238°$
   (b) $\cos(-263°)$
   (c) $\tan(-36°)$.

In questions 8 to 11, find the unknown angles and sides in each of the given ΔABC such that:

8. $a = 9.1$ cm, $c = 7.8$ cm, $C = 29°27'$
9. $b = 14$ cm, $B = 41°37'$, $C = 84°23'$
10. $a = 16$ cm, $b = 18$ cm, $C = 110°$
11. $a = 19$ cm, $b = 21$ cm, $c = 36$ cm

12. The length of the shortest side of a triangle is 5.4 cm. Two of the angles are 31° and 40°. Find the length of the longest side.

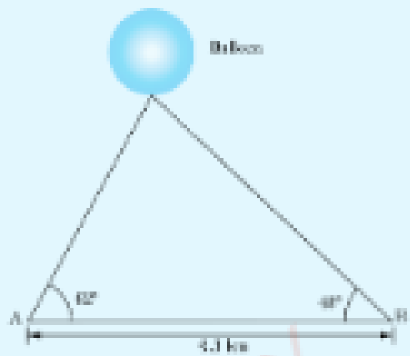
13. Two islands A and B are 5.9 km and 9.2 km respectively from a point O of a straight coast line OX. If $X\hat{O}A = 29°$ and $X\hat{O}B = 46°$, find the distance between the islands.



14. From a certain point X, Hamisi observes the angle of elevation of the top of a tall building to be 40°. Moving 50 m further away to a point Y on a level road, he notices the angle of elevation to be 29°. Find:
    (a) the distance of Y from the bottom of the building.
    (b) the height of the building.

15. Calculate the angle between the diagonals of a parallelogram whose sides are 5.1 cm and 2.5 cm enclosing an angle of 70°.

16. The angles of elevation of a balloon from two points A and B which are 0.3 km apart are 62° and 48° respectively. If the balloon is vertically above the line AB, find its distance above the line.



17. In triangle ABC, $\overline{AB} = 11$ cm, $\overline{AC} = 12$ cm, and $\overline{BC} = 15$ cm. Find the value of the largest angle of triangle ABC.

18. Khadija travels from village A to village B, 8.6 km away in a direction N 30° E and then 5.6 km to village C in a direction S 28° E. Find:
    (a) the distance AC.
    (b) the bearing of C from A.
    (c) the bearing of A from C.

19. Find the value of each of the following:

(a) $\sin\left(-\frac{\pi}{3}\right)$

(b) $\tan\left(\frac{3\pi}{2}\right)$

(c) $\cos\left(\frac{3\pi}{4}\right)$

(d) $\tan\left(-\frac{3\pi}{4}\right)$

20. (a) Simplify the following:

(i) $\tan\left(\theta-\frac{2\pi}{3}\right)$

(ii) $\tan(A-B)$ if $\tan B=\frac{1}{8}$

(b) If $\sin A=\frac{1}{12}$ and $\cos B=\frac{4}{5}$, evaluate:

(i) $\tan(A+B)$

(ii) $\sin(A-B)$

(iii) $\cos(A-B)$.

Project 5

Make a visual display that shows the vertical drop and angle of elevation for several different roller coasters. In your presentation, include the following:

1. Scale drawing of the right triangles.
2. The vertical drop, horizontal change, length of track, and angle of elevation for each roller coaster.
3. An explanation of how you used trigonometry to find measures in your display.

Chapter Six

Vectors

Introduction

Various daily activities require the use of quantities which have both magnitude (size) and direction. These quantities are generally called vectors. In this chapter, you will learn the meaning of a vector quantity, concept of displacement and position vectors, resolve vectors into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components, magnitude and direction of a vector. You will also learn the sum and difference of vectors, multiplication of vectors by a scalar, and solve real life problems on velocities, displacements, acceleration, and forces. The competencies developed will help you to solve problems related to magnitude and direction in military, games and aircraft, navigations, tourism, sports, designing, and other daily life activities.

Displacement and position vectors

Displacement vector

Activity 6.1: Defining the displacement vector

In a group or individually, perform this activity using the following steps:

1. On the graph paper, locate a point A.
2. Locate another point B which is 4 cm in the direction of North East from point A.
3. Join the points A and B by a straight line showing an arrow to the end at point B.
4. What did you observe in step 3? Give the name of the vector obtained in step 3.
5. In your own words, define the term displacement vector.

The distance moved by an object from point A to point B in the direction of B is called the displacement. The displacement from point A to point B is denoted by $\overrightarrow{AB}$. Figure 6.1 shows the displacement from the point A to point B.

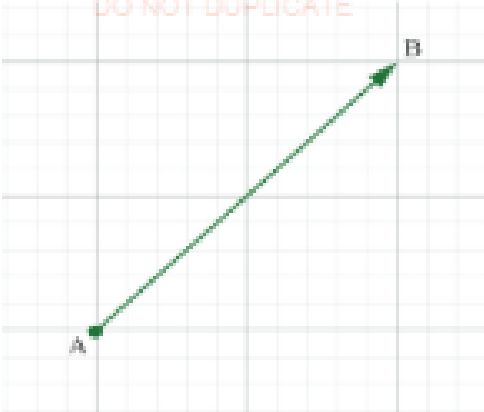


Figure 6.1: *The displacement from point A to point B*

A vector is defined as a physical quantity which has both magnitude and direction. In Figure 6.1, the displacement $\overrightarrow{AB}$ is a vector because it has magnitude as well as direction. Some examples of vector quantities are displacement, velocity, acceleration, force, momentum, electric field, and magnetic field. Quantities which have magnitude only are called scalars. For example, distance, speed, pressure, time, and temperature are scalar quantities.

Vectors are named by either two capital letters with an arrow above, like $\overrightarrow{OA}$ or by a single capital or small letter in bold case, like ***a***. Sometimes as a single small letter with a bar below like $\underline{a}$. In Figure 6.2, $\overrightarrow{OA}$ is the displacement from the origin O to the point A.

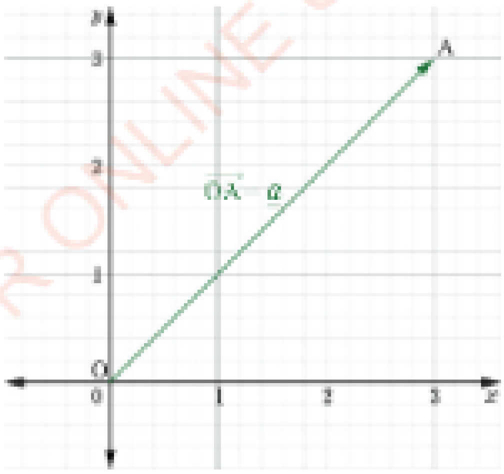


Figure 6.2: *Displacement vector* $\overrightarrow{OA}$ *from point O to point A on the positive* xy *– axes*

When the notation $\overrightarrow{OA}$ is used to describe a vector, the arrow indicates the direction of the vector. That is, vector $\overrightarrow{OA}$ has initial point at O and end point at A, whereas vector $\overrightarrow{AO}$ has its initial point at A and end point at O.

Equivalent vectors

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if they have the same magnitude and direction. In Figure 6.3, the vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$ are equivalent because they have the same magnitude and direction.

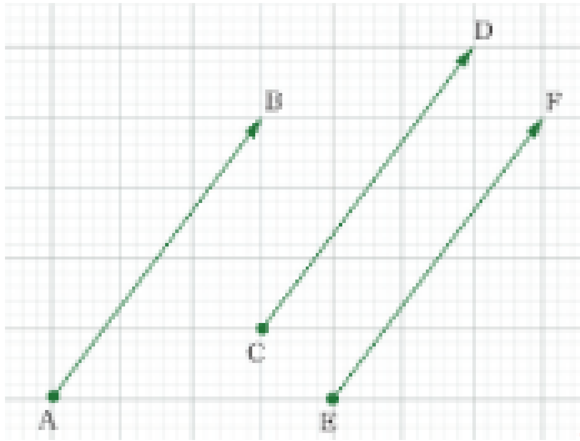


Figure 6.3: *Equivalent vectors*

The notation ≡ means equivalent. Therefore, $\overrightarrow{AB} \equiv \overrightarrow{CD} \equiv \overrightarrow{EF}$ as shown in Figure 6.3.

Position vectors

Activity 6.2: Explaining the concept of position vector

In a group or individually, perform this activity using the following steps:

1. Draw an *xy* – plane and mark scales from –8 to 8 on both axes.
2. Locate points A(2, 5), B(–3, 7), and C(–4, –1) on the *xy* – plane.
3. Draw vectors $\overrightarrow{OB}$, $\overrightarrow{OA}$, and $\overrightarrow{OC}$ where O is the origin.
4. What did you observe in step 3?
5. Using your own words, explain the meaning of position vector.
6. Share your findings with your neighbours and discuss.

In the *xy* – plane all vectors having the same origin and different end points are called position vectors. Position vectors are named using the coordinates of their end points.

In Figure 6.4, $\overrightarrow{OA} = (2, 3)$, $\overrightarrow{OB} = (-1, 2)$, $\overrightarrow{OC} = (-1, -2)$, $\overrightarrow{OD} = (3, -1)$, and

$\overrightarrow{OP} = (x, y)$ are position vectors of points, A, B, C, D, and P respectively.

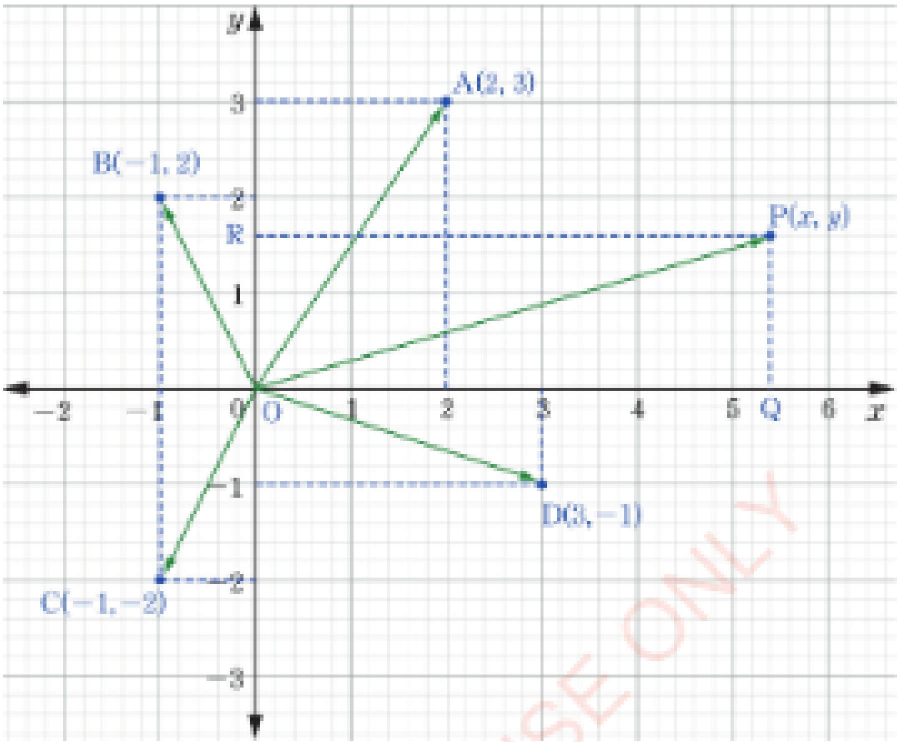


Figure 6.4: *Position vectors in the xy – plane*

Components of position vectors

With reference to Figure 6.4, any position vector $\overrightarrow{OP} = (x, y)$ can be resolved into two components; namely, a horizontal component and a vertical component. For example, the components of $\overrightarrow{OP} = (x, y)$ are $\overrightarrow{OQ} = (x, 0)$ and $\overrightarrow{OR} = (0, y)$. Thus, $\overrightarrow{OQ} = (x, 0)$ is the horizontal component and $\overrightarrow{OR} = (0, y)$ is the vertical component.

Example 6.1

Write the position vectors and their components for each of the following points:

(a) $A(1, -1)$ (b) $B(-5, 5)$ (c) $C(-4, -3)$ (d) $D(u, v)$ where u and v are any real numbers.

Solution

(a) $\overrightarrow{OA} = (1, -1)$,
Horizontal component is $(1, 0)$,
Vertical component is $(0, -1)$.

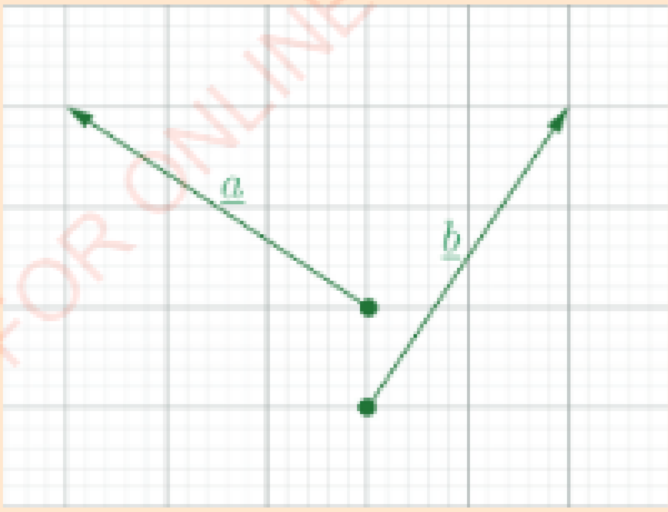
(b) $\overrightarrow{OB} = (-5, 5)$,
Horizontal component is $(-5, 0)$,
Vertical component is $(0, 5)$.

(c) $\overrightarrow{OC} = (-4, -3)$,
Horizontal component is $(-4, 0)$,
Vertical component is $(0, -3)$.

(d) $\overrightarrow{OD} = (u, v)$,
Horizontal component is $(u, 0)$,
Vertical component is $(0, v)$.

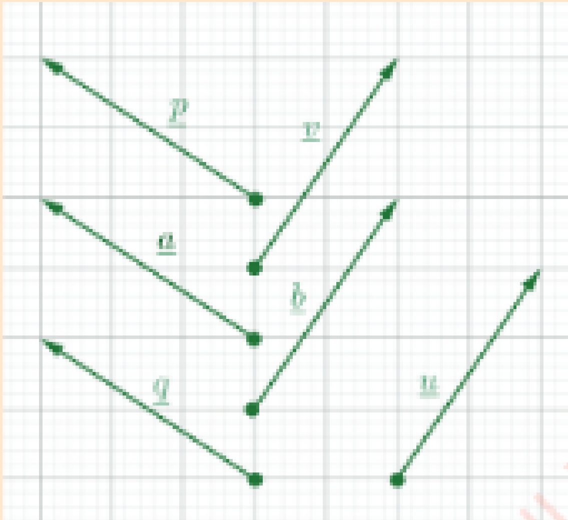
Example 6.2

For each of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the following figure, draw a pair of equivalent vectors.



Solution

In the following figure, the vectors $\underline{p}$ and $\underline{q}$ are equivalent to $\underline{a}$, while the vectors $\underline{u}$ and $\underline{v}$ are equivalent to $\underline{b}$.



The unit vectors, $\underline{i}$ and $\underline{j}$

In the xy – plane, the position vector with unit length in the positive x – axis is named $\underline{i}$ and the position vector with unit length in the positive y – axis is named $\underline{j}$. That is, both $\underline{i}$ and $\underline{j}$ are unit vectors. Figure 6.5 shows the unit vectors in the positive xy – plane.

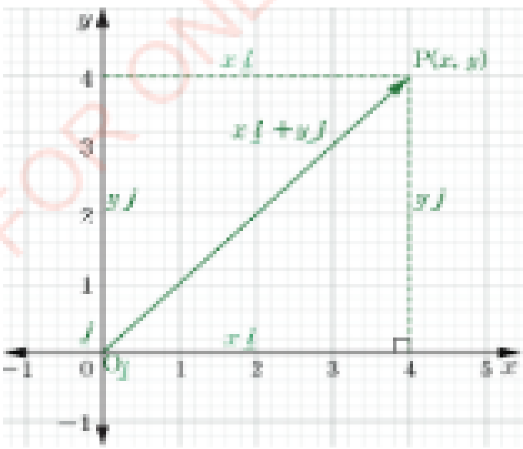


Figure 6.5: *Unit vectors $\underline{i}$ and $\underline{j}$ in the positive xy – plane.*

In terms of position vectors, these vectors are written as $\underset{\sim}{i} = (1, 0)$ and $\underset{\sim}{j} = (0, 1)$.

Now, consider the position vector $\overrightarrow{OP} = (x, y)$ in Figure 6.5. The vector $\overrightarrow{OP}$ can be resolved into $\overrightarrow{OP} = (x, 0) + (0, y)$ and factorized into $\overrightarrow{OP} = x(1, 0) + y(0, 1)$. But, $\underset{\sim}{i} = (1, 0)$ and $\underset{\sim}{j} = (0, 1)$. Hence, $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

Generally, any position vector $\overrightarrow{OP} = (x, y)$ can be written as an algebraic sum of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

Example 6.3

Write each of the following in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$:

(a) $\underset{\sim}{a} = (-3, -4)$ (b) $\underset{\sim}{b} = (-5, 5)$ (c) $\underset{\sim}{c} = (-1, 6)$ (d) $\underset{\sim}{d} = (u, v)$

where u and v are any real numbers.

Solution

(a) $\underset{\sim}{a} = (-3, -4)$

The horizontal component of $\underset{\sim}{a}$ is $(-3, 0)$.

The vertical component of $\underset{\sim}{a}$ is $(0, -4)$.

Thus,

$\underset{\sim}{a} = (-3, 0) + (0, -4)$

$= -3(1, 0) + -4(0, 1)$

$= -3\underset{\sim}{i} - 4\underset{\sim}{j}$

Therefore, $\underset{\sim}{a} = -3\underset{\sim}{i} - 4\underset{\sim}{j}$.

(b) $\underset{\sim}{b} = (-5, 5) = (-5, 0) + (0, 5)$

$= -5(1, 0) + 5(0, 1)$

$= -5\underset{\sim}{i} + 5\underset{\sim}{j}$

Therefore, $\underset{\sim}{b} = -5\underset{\sim}{i} + 5\underset{\sim}{j}$.

(c) $\underset{\sim}{c} = (-1, 6)$

$= -1(1, 0) + 6(0, 1)$

$= -\underset{\sim}{i} + 6\underset{\sim}{j}$

Therefore, $\underset{\sim}{c} = -\underset{\sim}{i} + 6\underset{\sim}{j}$.

(d) $\underset{\sim}{d} = (u, v)$

$= (u, 0) + (0, v)$

$= u(1, 0) + v(0, 1)$

$= u\underset{\sim}{i} + v\underset{\sim}{j}$

Therefore, $\underset{\sim}{d} = u\underset{\sim}{i} + v\underset{\sim}{j}$.

Example 6.4

Write each of the following vectors as position vectors:

(a) $\underline{s} = -8\underline{i}$ (b) $\underline{u} = 7\underline{j}$ (c) $\underline{v} = \frac{2\underline{i}}{3} + \frac{2\underline{j}}{5}$

(d) $\underline{w} = w_1\underline{i} + w_2\underline{j}$ where w_1 and w_2 are real numbers.

Solution

(a) $\underline{s} = -8\underline{i}$

$= -8\underline{i} + 0\underline{j}$

$= -8(1, 0) + 0(0, 1)$

$= (-8, 0) + (0, 0)$

$= (-8, 0)$

Therefore, $\underline{s} = (-8, 0)$.

(b) $\underline{u} = 7\underline{j}$

$= 0\underline{i} + 7\underline{j}$

$= 0(1, 0) + 7(0, 1)$

$= (0, 0) + (0, 7)$

$= (0, 7)$

Therefore, $\underline{u} = (0, 7)$.

(c) $\underline{v} = \frac{2\underline{i}}{3} + \frac{2\underline{j}}{5}$

$= \frac{2}{3}(1, 0) + \frac{2}{5}(0, 1)$

$= \left(\frac{2}{3}, 0\right) + \left(0, \frac{2}{5}\right)$

$= \left(\frac{2}{3}, \frac{2}{5}\right)$

Therefore, $\underline{v} = \left(\frac{2}{3}, \frac{2}{5}\right)$.

(d) $\underline{w} = w_1\underline{i} + w_2\underline{j}$

$= w_1(1, 0) + w_2(0, 1)$

$= (w_1, 0) + (0, w_2)$

$= (w_1, w_2)$

Therefore, $\underline{w} = (w_1, w_2)$.

Addition and subtraction of position vectors

If $\mathbf{U} = (u_1, u_2)$ and $\mathbf{V} = (v_1, v_2)$ where u_1, u_2, v_1, and v_2 are any real numbers, then, $\mathbf{U} + \mathbf{V} = (u_1 + v_1, u_2 + v_2)$ and $\mathbf{U} - \mathbf{V} = (u_1 - v_1, u_2 - v_2)$.

Similarly, if $\mathbf{U} = u_1\underline{i} + u_2\underline{j}$ and $\mathbf{V} = v_1\underline{i} + v_2\underline{j}$, then

$$\mathbf{U} + \mathbf{V} = (u_1\underline{i} + u_2\underline{j}) + (v_1\underline{i} + v_2\underline{j}) = (u_1 + v_1)\underline{i} + (u_2 + v_2)\underline{j} \text{ and}$$

$$\mathbf{U} - \mathbf{V} = (u_1\underline{i} + u_2\underline{j}) - (v_1\underline{i} + v_2\underline{j}) = (u_1 - v_1)\underline{i} + (u_2 - v_2)\underline{j}.$$

Example 6.5

Given $\mathbf{U} = (3,2)$ and $\mathbf{V} = (1,3)$, find $\mathbf{U}+\mathbf{V}$ and $\mathbf{U}-\mathbf{V}$ in terms of:

(a) Position vectors.

(b) Unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

Solution

(a) Given $\mathbf{U} = (3,2)$ and $\mathbf{V} = (1,3)$, then

$$\mathbf{U}+\mathbf{V} = (3,2)+(1,3)$$
$$= (3+1, 2+3)$$
$$= (4,5)$$

Therefore, $\mathbf{U}+\mathbf{V} = (4,5)$.

Similarly, $\mathbf{U}-\mathbf{V} = (3,2)-(1,3)$
$$= (3-1,\ 2-3)$$
$$= (2,-1)$$

Therefore, $\mathbf{U}-\mathbf{V} = (2,-1)$.

(b) $\mathbf{U} = (3,2) = 3\underset{\sim}{i}+2\underset{\sim}{j}$ and
$\mathbf{V} = (1,3) = \underset{\sim}{i}+3\underset{\sim}{j}$

$$\mathbf{U}+\mathbf{V} = (3\underset{\sim}{i}+2\underset{\sim}{j})+(\underset{\sim}{i}+3\underset{\sim}{j})$$
$$= (3\underset{\sim}{i}+\underset{\sim}{i})+(2\underset{\sim}{j}+3\underset{\sim}{j})$$
$$= 4\underset{\sim}{i}+5\underset{\sim}{j}$$

Therefore, $\mathbf{U}+\mathbf{V} = 4\underset{\sim}{i}+5\underset{\sim}{j}$.

Similarly,
$$\mathbf{U}-\mathbf{V} = (3\underset{\sim}{i}+2\underset{\sim}{j})-(\underset{\sim}{i}+3\underset{\sim}{j})$$
$$= (3\underset{\sim}{i}-\underset{\sim}{i})+(2\underset{\sim}{j}-3\underset{\sim}{j})$$
$$= 2\underset{\sim}{i}-\underset{\sim}{j}$$

Therefore, $\mathbf{U}-\mathbf{V} = 2\underset{\sim}{i}-\underset{\sim}{j}$.

Example 6.6

If $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$, show that the coordinates of a point P are (x, y), where x and y are real numbers.

Solution

Given $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then we have,

$$\overrightarrow{OP} = x(1,\ 0) + y(0,\ 1)$$
$$= (x,\ 0) + (0,\ y)$$
$$= (x,\ y)$$

Therefore, the coordinates of a point P are (x, y).

Exercise 6.1

Answer the following questions:

1. Draw two displacement vectors which are equivalent to $\underset{\sim}{a} = (3, 4)$ such that one has its origin at $(3, 0)$ and the other at $(0, 4)$.

2. Write each of the following in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$:
   (a) $\underset{\sim}{p} = (-3, 6)$
   (b) $\underset{\sim}{q} = (5, -2)$
   (c) $\underset{\sim}{r} = (-4, 2)$
   (d) $\underset{\sim}{s} = (-7, -3)$
   (e) $\underset{\sim}{u} = (0, -9)$

3. Write each of the following as a position vector:
   (a) $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$
   (b) $\underset{\sim}{b} = 6\underset{\sim}{i} - 7\underset{\sim}{j}$
   (c) $\underset{\sim}{c} = \underset{\sim}{i} + 4\underset{\sim}{j}$
   (d) $\underset{\sim}{d} = 2\underset{\sim}{i}$
   (e) $\underset{\sim}{e} = -3\underset{\sim}{j}$

4. Two forces of magnitudes 3 N and 4N, are acting in the positive x – axis and negative y – axis, respectively at the same point O $(0, 0)$.
   (a) Write each of the forces as a position vector.
   (b) Write the two forces as a single vector in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

5. Write the position vector of each of the following points:
   (a) $A(3, 4)$
   (b) $B(5, 3)$
   (c) $C(7, 8)$
   (d) $D(u_1, u_2)$

   where u_1 and u_2 are real numbers.

6. Given $\underset{\sim}{u} = -6\underset{\sim}{i} + 12\underset{\sim}{j}$, $\underset{\sim}{v} = -12\underset{\sim}{i} - 18\underset{\sim}{j}$, find $\mathbf{U} + \mathbf{V}$ and $\mathbf{U} - \mathbf{V}$ in terms of:
   (a) position vectors.
   (b) unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

7. If the components of the vector $\underset{\sim}{r}$ are $(x, 0)$ and $(0, y)$, express $\underset{\sim}{r}$ as a single vector in terms of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

8. Draw a position vector which is equivalent to $\overrightarrow{AB}$ joining A(3, 0) and B(0, 3).

9. Draw the position vectors **P** and **–P** on the same diagram if $\mathbf{P} = (-4, 3)$.

10. An aeroplane flies 200 km due East from town A to town B. It makes another flight 300 km from town B to town C due South.

    (a) Write the final destination of the aeroplane as a position vector with reference from town A.

    (b) Write the position vector of the aeroplane at town C in terms of the unit vectors $\underline{i}$ and $\underline{j}$, taking town A as the origin.

    (c) Find the vector which will represent the direct journey of the aeroplane from town C to town A in terms of $\underline{i}$ and $\underline{j}$.

Magnitude and direction of a vector

Activity 6.3: The concept of magnitude and direction of a vector

In a group or individually perform this activity using the following steps:

1. On a graph paper, draw x and y – axes.
2. Locate a point $P(x, y)$ in the first quadrant.
3. Join the origin and point $P(x, y)$ by a straight line with an arrow in the direction of P.
4. What is the name of the resulting vector?
5. Find the distance d of the vector $\overrightarrow{OP}$ in terms of x and y.
6. State the physical meaning of the distance obtained in step 5.
7. Share your findings with your friends through discussion.

Magnitude of a vector

The term magnitude of a vector is used to define the size of a vector. Magnitude is sometimes called modulus. The magnitude or modulus of a vector is a scalar quantity. The magnitude of a vector is calculated by using the distance formula which is based on the Pythagoras' theorem. Consider the right – angled triangle ABC as shown in Figure 6.6.

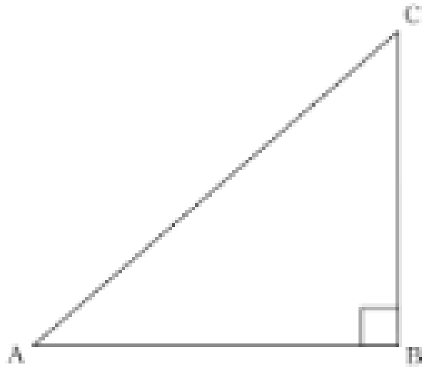


Figure 6.6: *Right angled – triangle ABC*

Using Figure 6.6 and Pythagoras' theorem, $\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2$. Taking the square root on both sides we obtain $\overline{AC} = \sqrt{\left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2}$.

Now, consider the position vector $\underset{\sim}{r} = (x, y)$ as shown in Figure 6.7.

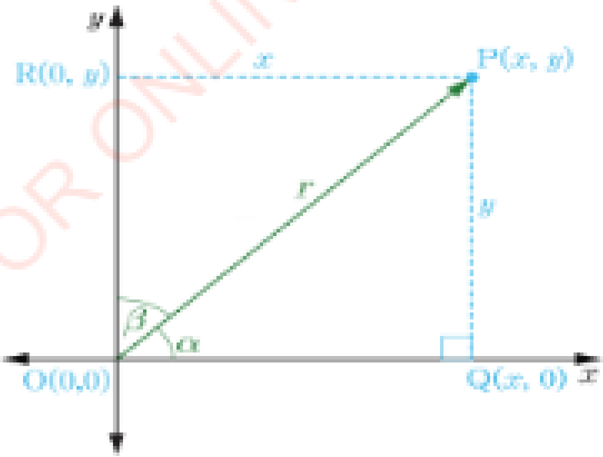


Figure 6.7: *The position vector* $\underset{\sim}{r}$

From the position vector $\overrightarrow{OP} = \underset{\sim}{r} = (x, y)$ draw $\overrightarrow{PQ}$ perpendicular to the x – axis and $\overrightarrow{PR}$ perpendicular to the y – axis. The coordinates of Q are $(x, 0)$ and that of R are $(0, y)$. The ΔOPQ is a right – angled triangle at Q, thus by the Pythagoras' theorem we get

$$\left(\overline{OP}\right)^2 = \left(\overline{OQ}\right)^2 + \left(\overline{QP}\right)^2$$

But, $\overline{OQ} = x$ and $\overline{QP} = \overline{OR} = y$. Substitute x and y for $\overline{OQ}$ and $\overline{QP}$ respectively to get

$$\left(\overline{OP}\right)^2 = (x-0)^2 + (y-0)^2$$
$$= x^2 + y^2$$
$$\overrightarrow{OP} = \sqrt{x^2 + y^2}\text{, but } \overrightarrow{OP} = \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

Since $\overrightarrow{OP}$ is a position vector, the magnitude of $\overrightarrow{OP}$ is denoted by $\overline{OP}$ or $|\overrightarrow{OP}|$. Therefore, $r = \sqrt{x^2 + y^2}$, where r is the magnitude of the vector $\underset{\sim}{r}$.

This formula is used to calculate the magnitude of any position vector. In Figure 6.7, the position vector of $P(x, y)$ is defined by $\overrightarrow{OP}$. The position vector of point P can also be denoted by $\underset{\sim}{r}$ as shown in Figure 6.7.

In Figure 6.8, the position vector $\underset{\sim}{r}$ has its initial point at $P(x_1, y_1)$ and the end point at $Q(x_2, y_2)$. Thus, the vector $\overrightarrow{PQ}$ is given by:

$$\overrightarrow{PQ} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}$$

Therefore, $\overrightarrow{PQ} = \underset{\sim}{r} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}$.

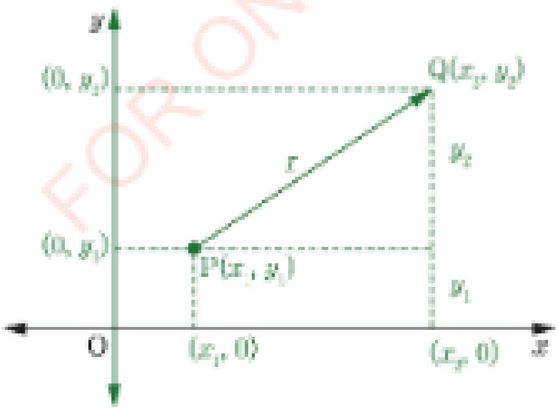


Figure 6.8: *The position vector* $\overrightarrow{PQ}$

From Figure 6.8, the magnitude of the vector $\overrightarrow{PQ}$ where the initial point is $P(x_1, y_1)$ and the end point is $Q(x_2, y_2)$ is given by the distance formula:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, $|\underset{\sim}{r}| = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

A vector whose magnitude is one is called a unit vector. If $\underset{\sim}{u}$ is a vector, the unit vector in the direction of $\underset{\sim}{u}$ is given by $\frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$ and denoted by $\hat{\underset{\sim}{u}}$ (notation for unit vector), that is $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$.

Example 6.7

(a) Calculate the magnitude of $\overrightarrow{OV} = (-3, 4)$.

(b) If $\underset{\sim}{r} = k\underset{\sim}{i} + 7\underset{\sim}{j}$, find the values of k when $|\underset{\sim}{r}| = 25$.

Solution

(a) Let, $(x, y) = (-3, 4)$

Given $\overrightarrow{OV} = (-3, 4)$, then the magnitude of $\overrightarrow{OV}$ is given by:

$$|\overrightarrow{OV}| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

Therefore, the magnitude of $\overrightarrow{OV}$ is 5 units.

(b) Given $\underset{\sim}{r} = k\underset{\sim}{i} + 7\underset{\sim}{j}$ and $|\underset{\sim}{r}| = 25$

we have that: $|\underset{\sim}{r}| = \sqrt{k^2 + 7^2}$

$$25 = \sqrt{k^2 + 49}$$

Square both sides to get:

$$625 = k^2 + 49$$
$$k^2 = 625 - 49$$
$$= 576$$
$$k = \pm\sqrt{576}$$
$$= \pm 24$$

Therefore, $k = 24$ or $k = -24$.

Example 6.8

Find the magnitude of the vector $\overrightarrow{PQ}$ whose initial point is P(1, 1) and its end point is $Q(5, 3)$.

Solution

Let, $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (5, 3)$

The magnitude of the vector $\overrightarrow{PQ}$ is given by

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substitute the values of the vector x_1, y_1, x_2, and y_2 in the distance formula.

That is,

$$\left|\overrightarrow{PQ}\right| = \sqrt{(5-1)^2 + (3-1)^2}$$

$$= \sqrt{4^2 + 2^2}$$

$$= \sqrt{20}$$

$$\approx 4.5$$

Therefore, the magnitude of the vector $\overrightarrow{PQ}$ is about 4.5 units.

Example 6.9

Find the unit vector in the direction of $\underline{u} = (12, 5)$, hence verify that its magnitude is one.

Solution

Let, $(x, y) = (12, 5)$

Then we have,

$$|\underline{u}| = \sqrt{x^2 + y^2}$$

$$|\underline{u}| = \sqrt{12^2 + 5^2}$$

$$= \sqrt{144 + 25}$$

$$= \sqrt{169}$$

$$= 13.$$

Let $\hat{\underline{u}}$ be the unit vector in the direction of $\underline{u}$, then.

$$\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|}$$

$$= \frac{1}{13}(12, 5)$$

$$= \left(\frac{12}{13}, \frac{5}{13}\right)$$

$$= \frac{12}{13}\underline{i} + \frac{5}{13}\underline{j}$$

Therefore, $\hat{\underline{u}} = \left(\frac{12}{13}, \frac{5}{13}\right)$ or $\hat{\underline{u}} = \frac{12}{13}\underline{i} + \frac{5}{13}\underline{j}$.

The magnitude of $\hat{\underline{u}}$ is given by:

$$|\hat{\underline{u}}| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2}$$

$$= \sqrt{\left(\frac{144}{169}\right) + \left(\frac{25}{169}\right)}$$

$$= \sqrt{\left(\frac{169}{169}\right)} = 1$$

Therefore, the magnitude of a unit vector $\hat{\underline{u}}$ is one.

Direction of a vector

The direction of a vector is the measure of the angle it makes with the horizontal line. The direction of a vector can be obtained by using either bearings or direction cosines.

Bearing

The bearing of the point is the angle measured in degrees in a clockwise direction from due North. Figure 6.9 shows the bearings of points P and Q.

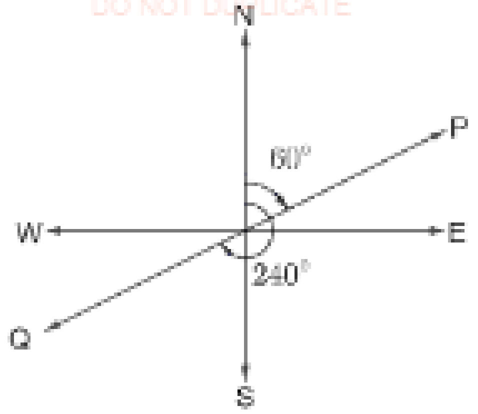


Figure 6.9: *The bearing of points P and Q*

The bearing of the point P is 060° and the bearing of point Q is 240°.

Note: The bearing is represented by three full digits (plus any decimal places).

The vector $\underset{\sim}{v}$ and the angle θ makes with the horizontal line can also be obtained by using formulae. Figure 6.10 shows a vector $\underset{\sim}{v}$ whose initial point is at $P(x_1, y_1)$ and end point is at $Q(x_2, y_2)$. The direction of the vector $\underset{\sim}{v}$ is the measure of angle θ.

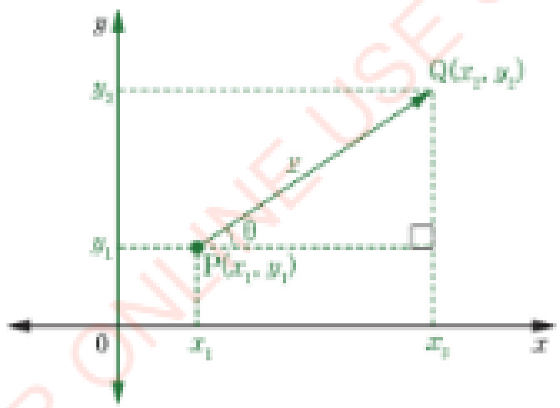


Figure 6.10: *Direction of a vector*

In Figure 6.10 the vector $\underset{\sim}{v}$ is given by: $\underset{\sim}{v} = \overrightarrow{PQ}$

Therefore, $\underset{\sim}{v} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}$.

The measure of the angle θ is given by:

$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) is the initial point and (x_2, y_2) is the end point.

If the initial point of $\underset{\sim}{v}$ is $(0,0)$ and the end point is (x, y), then the measure of the angle θ is $\tan\theta = \frac{y}{x}$.

Example 6.10

Find the bearing of the position vector $\underset{\sim}{v} = (6, 8)$.

Solution

Let, $(x, y) = (6, 8)$

The magnitude of $\underset{\sim}{v}$ is given by:

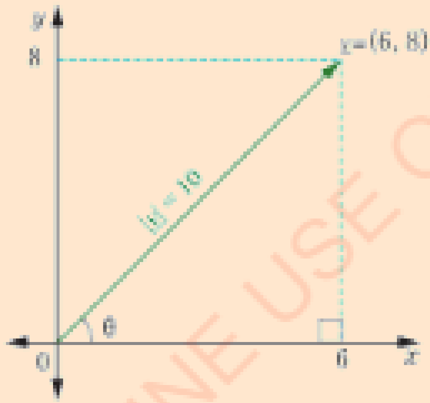
$$|\underset{\sim}{v}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{6^2 + 8^2}$$

$$= \sqrt{100}$$

$$= 10$$

Hence, $|\underset{\sim}{v}| = 10$.



Using trigonometric ratios we have:

$$\cos\theta = \frac{6}{10}$$

$$\Rightarrow \theta = \cos^{-1} 0.6$$

$$\theta = 53.1°.$$

Since the bearing of the vector is measured in a clockwise direction from due North (N), then the bearing of $\underset{\sim}{v} = 090° - 053.1° = 036.9°$.

Therefore, the bearing of the position vector $\underset{\sim}{v}$ is 036.9°.

Example 6.11

Find the direction of the vector $\overrightarrow{PQ}$ whose initial point P is at (2, 3) and end point Q is at $(5,\ 8)$.

Solution

Let, $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 8)$

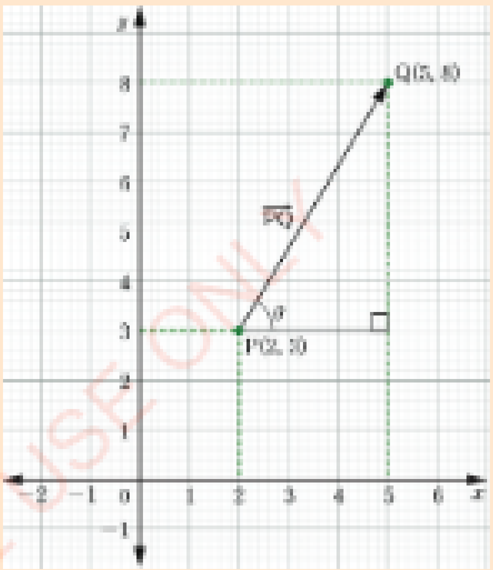
Then the vector $\overrightarrow{PQ}$ is given by:

$\overrightarrow{PQ} = (x_2 - x_1)\,\underset{\sim}{i} + (y_2 - y_1)\,\underset{\sim}{j}$

$\overrightarrow{PQ} = (5-2)\underset{\sim}{i} + (8-3)\underset{\sim}{j}$

$= 3\underset{\sim}{i} + 5\underset{\sim}{j}$

Hence, $\overrightarrow{PQ} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$.



Thus, $\tan\theta = \frac{y}{x}$ where $x = 3$ and $y = 5$.

$\Rightarrow \tan\theta = \frac{5}{3}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{5}{3}\right)$

$= \tan^{-1}(1.667)$

$\theta = 59°$

Therefore, the direction of the vector $\overrightarrow{PQ}$ is 59°.

Direction cosines

Figure 6.11 shows the vector $\overline{OP} = (x, y)$ which makes angles α and β with the positive x and y – axes, respectively, the cosines of α and β are called the direction cosines of $\overrightarrow{OP}$.

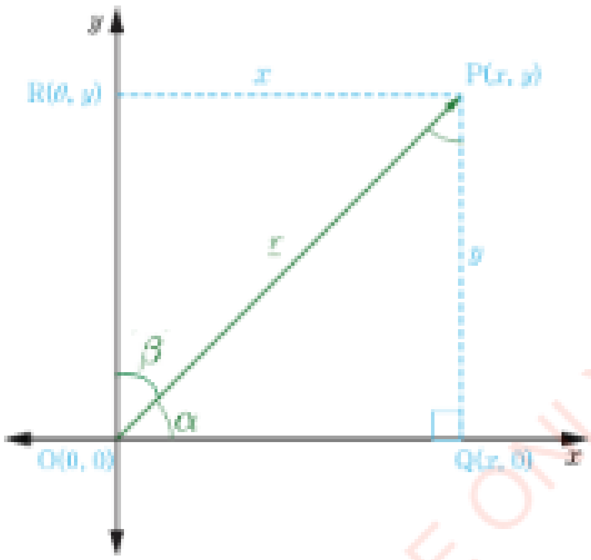


Figure 6.11: *The position vector*

Figure 6.11 shows that the coordinates of Q are $(x, 0)$ and that of R are $(0, y)$. The ΔOQP is right – angled at Q and $P\hat{O}R = O\hat{P}Q$ because they are alternate interior angles.

Thus, $O\hat{P}Q = \beta$, $\overline{OP} = |\overrightarrow{OP}|$, $\overline{OQ} = x$, and $\overline{QP} = \overline{OR} = y$.

Therefore, $\cos\alpha = \frac{x}{|\overline{OP}|}$ and $\cos\beta = \frac{y}{|\overline{OP}|}$ or $\cos\alpha = \frac{x}{|\underline{r}|}$ and $\cos\beta = \frac{y}{|\underline{r}|}$ are direction cosines of $\overrightarrow{OP}$.

Example 6.12

Given $\underset{\sim}{a} = 6\underset{\sim}{i} + 8\underset{\sim}{j}$, find the direction cosines of $\underset{\sim}{a}$ and the angle that $\underset{\sim}{a}$ makes with the positive x – axis.

Solution

Let, $(x, y) = (6, 8)$

Then, the magnitude of a vector $\underset{\sim}{a}$, is given by

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$$

$$|\underset{\sim}{a}| = \sqrt{6^2 + 8^2}$$

$$= \sqrt{36 + 64}$$

$$= \sqrt{100}$$

$$= 10$$

Hence, $|\underset{\sim}{a}| = 10$.

Suppose $\underset{\sim}{a}$ makes angles α with the positive x – axis and β with the positive y – axis, then, direction cosines are given by:

$$\cos\alpha = \frac{x}{|\underset{\sim}{a}|} = \frac{6}{10} = \frac{3}{5}$$

and $$\cos\beta = \frac{y}{|\underset{\sim}{a}|} = \frac{8}{10} = \frac{4}{5}$$

Therefore, the direction cosines of $\underset{\sim}{a}$ are $\frac{3}{5}$ and $\frac{4}{5}$.

Since $\underset{\sim}{a}$ makes angle α with the positive x – axis then;

$$\tan\alpha = \frac{y}{x} = \frac{8}{6} = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}(1.333) = 53.1°$$

Alternatively, the angle α can be obtained from

$$\cos\alpha = \frac{x}{|\underset{\sim}{a}|} = \frac{6}{10} = 0.6$$

$$\alpha = \cos^{-1}(0.6) = 53.1°$$

Therefore, the vector $\underset{\sim}{a}$ makes an angle of $53.1°$ with the positive x – axis.

Example 6.13

Find a position vector in the form of $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ whose direction cosines are $-\frac{5}{13}$ and $-\frac{12}{13}$.

Solution

For any vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ the direction cosine are

$\frac{x}{|\underset{\sim}{r}|}$ and $\frac{y}{|\underset{\sim}{r}|}$. Thus $\frac{x}{|\underset{\sim}{r}|} = -\frac{5}{13}$ and $\frac{y}{|\underset{\sim}{r}|} = -\frac{12}{13}$.

On comparing we get $x = -5$ and $y = -12$. Hence, $\underset{\sim}{r} = -5\underset{\sim}{i} - 12\underset{\sim}{j}$.

Therefore, a position vector is $\underset{\sim}{r} = -5\underset{\sim}{i} - 12\underset{\sim}{j}$.

Note: Given a set of direction cosine, one can find the corresponding vector.

Exercise 6.2

Answer the following questions:

1. (a) Define the magnitude of a vector.

   (b) Calculate the magnitude and bearing of $\underline{a} = -12\underline{i} - 5\underline{j}$.

2. Calculate the magnitude of $\underline{a} = -8\underline{i} + 6\underline{j}$.

3. Calculate the direction cosines of $\underline{c} = 3\underline{i} + 4\underline{j}$, hence show that the sum of the squares of the direction cosines is one.

4. Find the unit vector in the direction of $\underline{a} = 8\underline{i} - 6\underline{j}$.

5. Find:

   (a) $|\underline{i} + \underline{j}|$ (b) $|\underline{i} - \underline{j}|$

   (c) $|3\underline{i}|$ (d) $|\underline{i} + 2\underline{j}|$

   (e) $|14\underline{j}|$

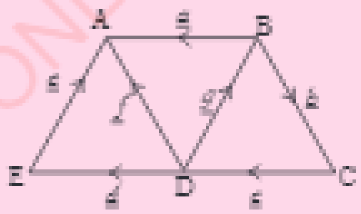
6. Find the direction cosines and bearing of each of the following vectors:

   (a) $\underline{a} = \underline{i} + \underline{j}$ (b) $\underline{b} = \underline{i} - \underline{j}$

Sum and difference of vectors

Activity 6.4: Finding the resultant of the vectors

In a group or individually, study the following figure and perform the tasks that follow:



1. Use the figure to identify the vectors represented by:

   (a) $\underline{g} + \underline{b}$ (b) $\underline{d} + \underline{e}$ (c) $\underline{b} + \underline{c} + \underline{g}$

2. What do you observe in tasks 1 (b) and 1 (c)?
3. Use task 1 to deduce the meaning of a resultant of a vector.
4. Share your observations with other students in the class.

Addition of vectors

Resultant vector is the sum of two or more vectors. The resultant vector has its own magnitude and direction. There are three laws used in adding vectors namely; the triangular law, the parallelogram law, and the polygon law of vectors.

The triangle law of vector addition

The process of vectors addition involves joining two vectors such that the initial point of the second vector is joined to the end point of the first vector. The resultant is obtained by completing the triangle with the vector whose initial point is the initial point of the first vector and its end point coincides with the end point of the second vector. For example, if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in Figure 6.12(a) are vectors, the resultant vector $\underset{\sim}{a}+\underset{\sim}{b}$ is obtained as shown in Figure 6.12(b).

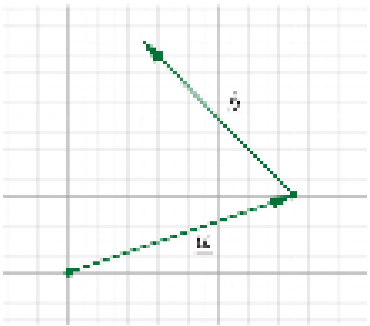


Figure 6.12(a): *Vectors in space*

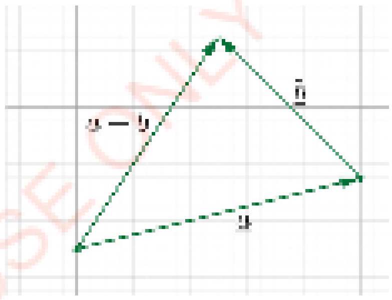


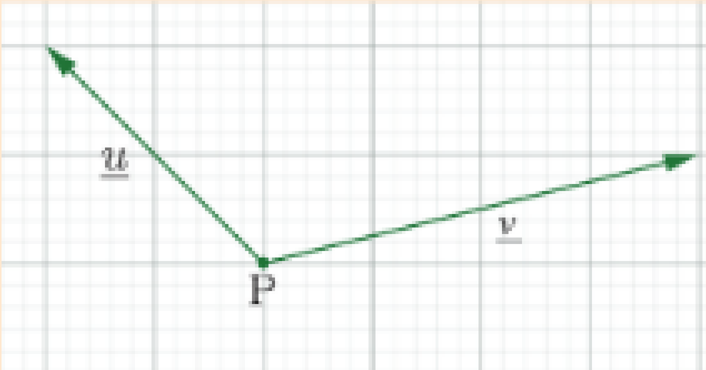
Figure 6.12(b): *Resultant of vectors by triangular law*

The parallelogram law of vector addition

When two vectors are sharing a common initial point P, then the resultant vector is obtained by completing the parallelogram. The two vectors form the sides of the parallelogram and the resultant is the diagonal with its initial point at P and its end point at R as illustrated in the given example.

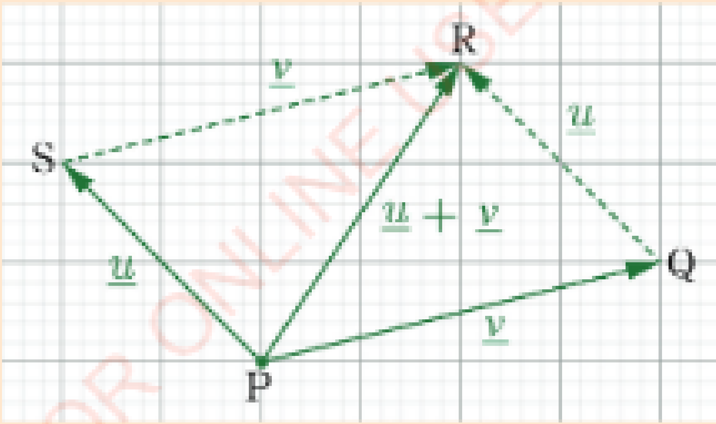
Example 6.14

Find the resultant of vectors $\underline{u}$ and $\underline{v}$ given in the following figure.



Solution

Complete the parallelogram by joining the end point of $\underline{u}$ at S by an equivalent vector of $\underline{v}$ to R and the end point of $\underline{v}$ at Q by an equivalent vector of $\underline{u}$ to R using dotted lines to obtain the parallelogram PQRS.



Therefore, from the figure the resultant vector is $\overrightarrow{PR} = \underline{u} + \underline{v} = \underline{v} + \underline{u}$.

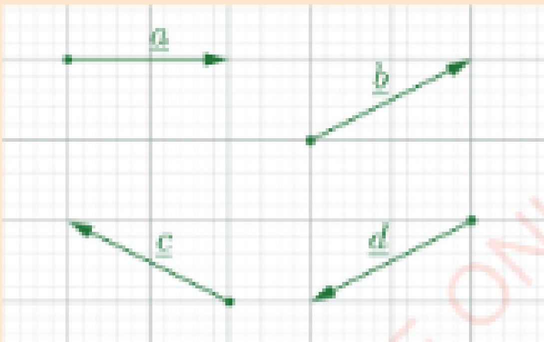
Note: In this type of vector addition the commutative property hold.

The polygon law of vector addition

When adding more than two vectors, the resultant of the vectors is obtained by joining the end point to the initial point of the vectors one after the other. The resultant of the vectors is the vector joining the initial point of the first vector to the end point of the last vector. The resulting figure is usually a polygon.

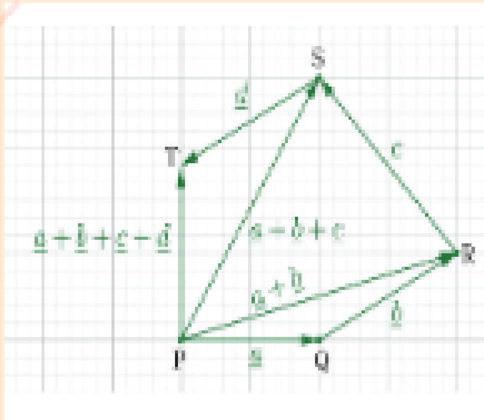
Example 6.15

Find the resultant of the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, and $\underset{\sim}{d}$ given in the following figure.



Solution

Start with $\underset{\sim}{a}$ and let its initial point be P and end point be Q.



Join $\underset{\sim}{b}$ to $\underset{\sim}{a}$ at Q to obtain $\overrightarrow{PR} = \underset{\sim}{a} + \underset{\sim}{b}$.

Join $\underset{\sim}{c}$ to $\underset{\sim}{a} + \underset{\sim}{b}$ at R to obtain $\overrightarrow{PS}$

$= \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$.

Join $\underset{\sim}{d}$ to $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ at S to obtain $\overrightarrow{PT}$

$= \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}$.

Therefore, the resultant is $\overrightarrow{PT}$

$= \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}$.

The resulting figure is a polygon PQRST as shown in the figure.

Opposite vectors

Consider the vectors $\overrightarrow{OP}$ and $\overrightarrow{PO}$ as shown in Figure 6.13.

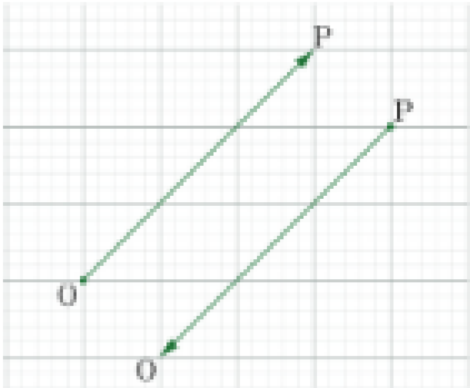
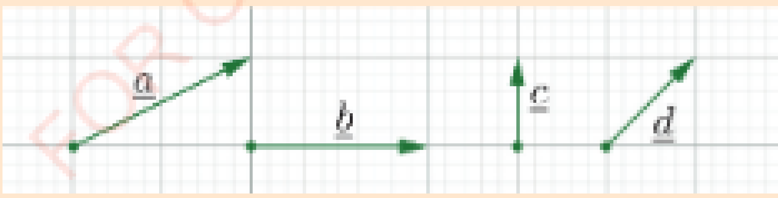


Figure 6.13: *Opposite vectors*

In Figure 6.13, $\overrightarrow{OP}$ and $\overrightarrow{PO}$ have equal magnitudes but opposite in direction. Thus, the vectors $\overrightarrow{OP}$ and $\overrightarrow{PO}$ are called opposite vectors. When $\overrightarrow{OP}$ is added to $\overrightarrow{PO}$ the resultant is a zero vector. That is, $\overrightarrow{OP} + \overrightarrow{PO} = \underset{\sim}{0}$. Thus, $\overrightarrow{OP} = -\overrightarrow{PO}$. The negative sign indicates that $\overrightarrow{OP}$ is in opposite direction to $\overrightarrow{PO}$.

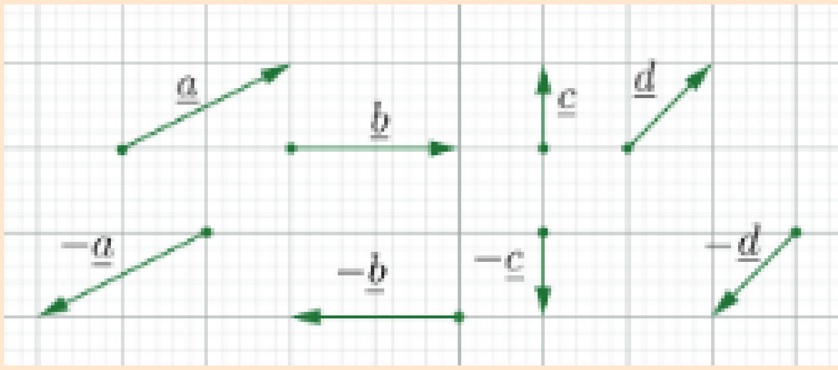
Example 6.16

Find and draw vectors which are opposite in direction to the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, and $\underset{\sim}{d}$ shown in the following figure.



Solution

The opposite vectors of $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ are $-\underline{a}$, $-\underline{b}$, $-\underline{c}$, and $-\underline{d}$ as shown in the following figure.



Subtraction of vectors

The process of subtracting one vector from another gives the same result as adding one vector to the opposite of the other vector. Thus, the difference of two vectors also gives a resultant vector. Two laws can be used when subtracting vectors, namely; the triangular law and the parallelogram law of vector subtraction.

Triangular law of vector subtraction

Given two vectors $\underline{a}$ and $\underline{b}$, the vector $\underline{b}$ can be subtracted from $\underline{a}$ using the triangular law as shown in Figure 6.14.

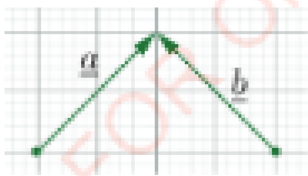


Figure 6.14 (a): *Vectors in space*

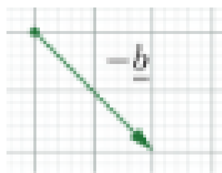


Figure 6.14 (b): *Opposite vector*

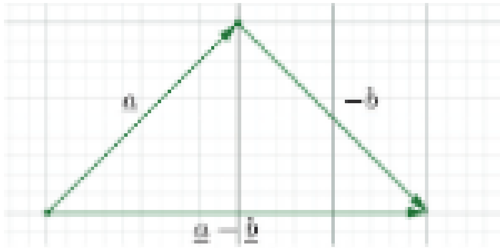


Figure 6.14 (c): *Resultant of vectors by triangular law*

Resultant of two or more forces

The resultant force of two or more forces is a force that has the same effect as two or more forces acting together.

When two or more forces are acting on a body, the total of all the forces which causes the resulting effect is the resultant force. The resultant force is obtained by taking the sum of all the forces. The resultant force can be geometrically obtained by resolving the given forces. If the two forces act at the right angle, then the Pythagoras' theorem can be applied to obtain the resultant force.

If several forces act on a particle, the resultant force can be obtained by finding the components of each force in two perpendicular directions. The algebraic sum of these components gives the magnitude and direction of a resultant of the several forces.

Suppose the force F acts on a particle at an angle θ to the x–axis, then F cos θ and F sin θ are the components along the x–axis and y–axis respectively.

That is $\text{F} = \text{F} \cos\theta \underset{\sim}{i} + \text{F} \sin\theta \underset{\sim}{j}$.

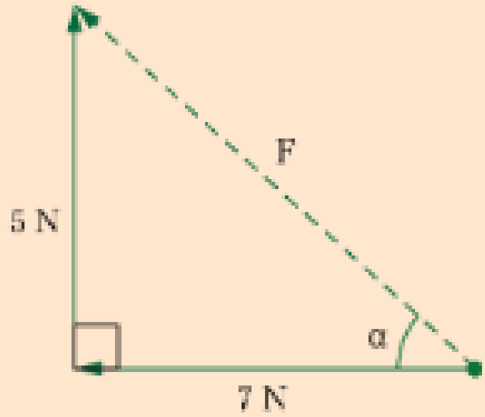
Example 6.17

The forces of magnitudes 5 N and 7 N act on a particle at an angle of 90° between them. Find the magnitude and direction of their resultant.

Solution

The forces are constructed as shown in the figure.

In this case, the resultant force can be obtained using the Pythagoras' theorem. That is,



$$F^2 = (7\,N)^2 + (5\,\text{N})^2$$

$$= (49 + 25)\ \text{N}^2$$

$$= \sqrt{74\,\text{N}^2}$$

$$F = \sqrt{74}\ \text{N}$$

Hence, $F = 8.6\text{N}$.

Also,

$$\tan\alpha = \frac{5\text{N}}{7\text{N}}$$

$$\tan\alpha = 0.7143$$

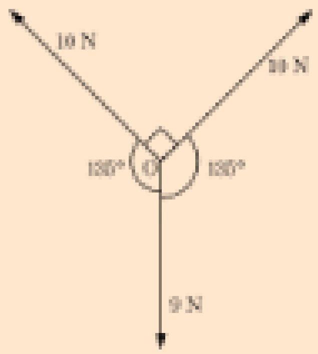
$$\alpha = \tan^{-1}(0.7143)$$

Hence, $\alpha = 35.5°$.

Therefore, the resultant is a force of magnitude 8.6 N at an angle of 35.5° to the 7N force.

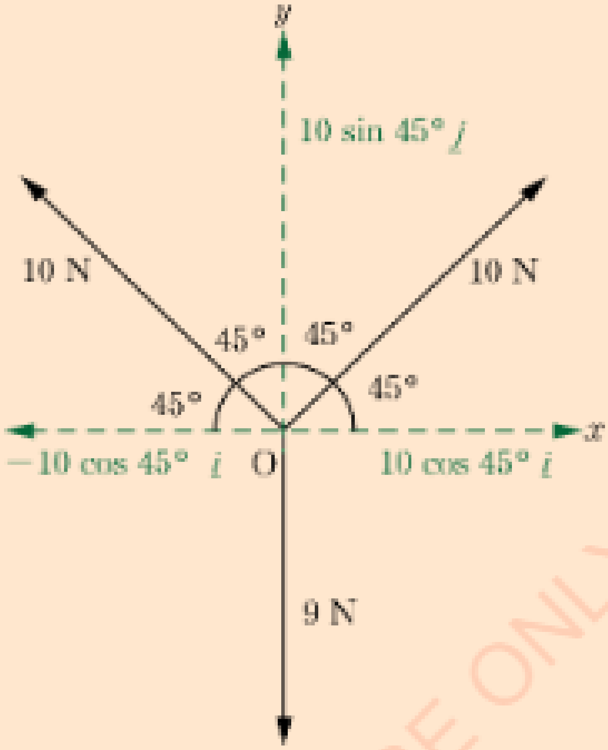
Example 6.18

Find the magnitude of the resultant force of a system of three forces 9 N, 10 N, and 10 N acting at point O as shown in the following figure:



Solution

The following figure shows the components of the forces along the x and y–axes.



The resultant force $\vec{R}$ is given by:

$$\vec{R} = \vec{R}_x + \vec{R}_y$$

$$= (10\cos 45°\underline{i} - 10\cos 45°\underline{i}) + (10\sin 45°\,\underline{j} + 9\sin 270°\,\underline{j}) \quad \text{or}$$

$$= (10\cos 45°\underline{i} - 10\cos 45°\underline{i}) + (10\sin 45°\,\underline{j} - 9\sin 90°\,\underline{j})$$

$$= \left(10\,\frac{\sqrt{2}}{2} - 10\,\frac{\sqrt{2}}{2}\right)\underline{i} + \left(10\,\frac{\sqrt{2}}{2} - 9\right)\underline{j}$$

$$= 0 + (5\sqrt{2} - 9)\,\underline{j}$$

$$= (5\sqrt{2} - 9)\,\underline{j}$$

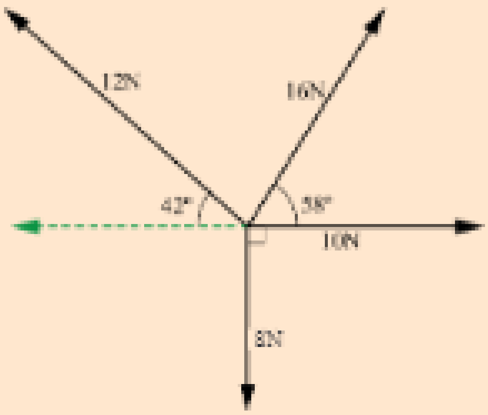
$$= -1.93\,\underline{j}$$

Thus, $\left|\vec{R}\right| = \left|-1.93\,\underline{j}\right| = 1.93$

Therefore, the magnitude of the resultant force is 1.93N.

Example 6.19

Find the magnitude and direction of the resultant force as shown in the following figure:



Solution

The 8N force will not contribute anything to the horizontal components and the 10N force will not contribute to the vertical component.

Resolving horizontally gives:

$$\vec{R}_x = (10 + 16\cos 58° - 12\cos 42°)\underset{\sim}{i}$$
$$= (10 + 16 \times 0.5299 - 12 \times 0.7431)\underset{\sim}{i}$$
$$= 9.5612\underset{\sim}{i}.$$

Resolving vertically gives:

$$\vec{R}_y = (16\sin 58° + 12\sin 42° + 8\sin 270°)\underset{\sim}{j}$$
$$= (16 \times 0.8480 + 12 \times 0.6691 + 8 \times -1)\underset{\sim}{j}$$
$$= 13.5972\underset{\sim}{j}.$$

The resultant force $\overline{R}$ is given by

$$\vec{R} = \vec{R}_x + \vec{R}_y$$
$$= 9.5612\underset{\sim}{i} + 13.5972\underset{\sim}{j}.$$

Thus, $|\vec{R}| = \sqrt{(9.5612)^2 + (13.5972)^2}$

$$= \sqrt{276.3}$$
$$= 16.6$$

$$\tan\theta = \frac{13.5972}{9.5612}$$
$$= 1.422$$
$$\theta = \tan^{-1}(1.422)$$
$$= 54.9°$$

Therefore, the magnitude of the resultant force is 16.6 N and makes an angle of 54.9° with the 10 N force.

Exercise 6.3

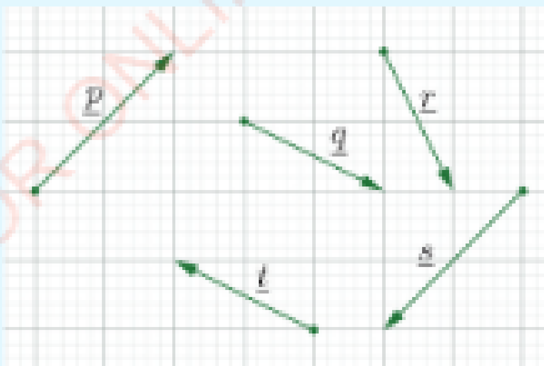
Answer the following questions:

1. Find the magnitude of the resultant force of a system of two forces each of magnitudes 7 making an angle of 120° to each other.

2. Find the resultant of two vectors of magnitudes 4 and 7 at angles of 30° and 60°, respectively with the horizontal.

3. Find the resultant of three vectors of magnitudes 3, 4, and 5 along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, and diagonal $\overrightarrow{CA}$ of a rectangle ABCD.

4. Given vectors $\underline{p}$, $\underline{q}$, $\underline{r}$, $\underline{s}$, and $\underline{t}$ as indicated in the following figure:



Find:

(a) $\underline{p}+\underline{q}$ (b) $\underline{p}+\underline{q}+\underline{r}$

(c) $\underline{p}+\underline{q}+\underline{r}+\underline{s}$ (d) $\underline{p}+\underline{q}+\underline{r}+\underline{s}+\underline{t}$

5. Use the vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$, $\underset{\sim}{s}$, and $\underset{\sim}{t}$ in question 4 to find:

(a) $\underset{\sim}{p}-\underset{\sim}{q}$ (b) $\underset{\sim}{r}-\underset{\sim}{s}$

(c) $\underset{\sim}{p}-\underset{\sim}{q}-\underset{\sim}{r}-\underset{\sim}{s}$ (d) $(\underset{\sim}{p}-\underset{\sim}{q})-(\underset{\sim}{r}-\underset{\sim}{s}-\underset{\sim}{t})$.

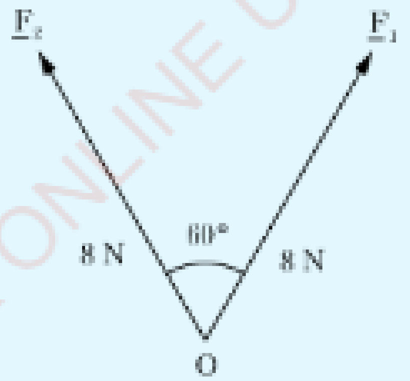
6. Find the resultant of three vectors of magnitude 6 units each along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ of an equilateral triangle ABC.

7. Three vectors with the same magnitude act along $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$, where $A\hat{O}B$ and $C\hat{O}B$ each measures to 60°. Find the resultant of the three vectors.

8. Three forces of magnitudes 10 N, 20 N, and 30 N make angles of 30°, 120°, and 210°, respectively with the positive x – axis. By scale drawing, find the magnitude and direction of the resultant force.

9. By scale drawing, find $\underset{\sim}{a}+\underset{\sim}{b}$ given that $|\underset{\sim}{a}|=10$ in the direction N45°E and $|\underset{\sim}{b}|=5$ in the direction N45°W.

10. Find the resultant of the system of forces $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ acting at point O as shown in the following figure.



11. Find the vector opposite to the resultant of the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$, where $\underset{\sim}{a}=(1,2)$, $\underset{\sim}{b}=(2,3)$ and $\underset{\sim}{c}=(3,4)$.

12. Two forces acting on a particle have magnitudes 8N and 3N. The angle between their directions is 60°. Find:
    (a) the resultant force acting on the particle.
    (b) the magnitude of the resultant force acting on the particle.
    (c) the direction of the resultant force.

13. Given that $\underset{\sim}{a} = 4i + 6j$ and $\underset{\sim}{b} = -2i - 3j$ then find $\underset{\sim}{a} - \underset{\sim}{b}$ by using a parallelogram law of vector subtraction.

Multiplication of vectors by a scalar

Activity 6.5: Multiplication of a vector by a scalar

In a group or individually, perfom the following tasks:

1. Draw a vector $\overrightarrow{OP} = (2, 2)$ in the first quadrant on the xy – plane.
2. Increase the magnitude (modulus) of $\overrightarrow{OP}$ three times to obtain $\overrightarrow{OP}'$.
3. Find the coordinates of P′.
4. Determine the relationship between the two position vectors.
5. Share your findings with your neighbours through discussion.

Consider a vector $\underset{\sim}{u}$ of magnitude 5 units making an angle of 45° with the positive x – axis. If the magnitude of $\underset{\sim}{u}$ is doubled, a vector $\underset{\sim}{v}$ of magnitude 10 units in the direction of $\underset{\sim}{u}$ is obtained as shown in Figure 6.15. Hence, $\underset{\sim}{v} = 2\underset{\sim}{u}$. The vector $\underset{\sim}{v}$ is called a scalar multiple of vector $\underset{\sim}{u}$.

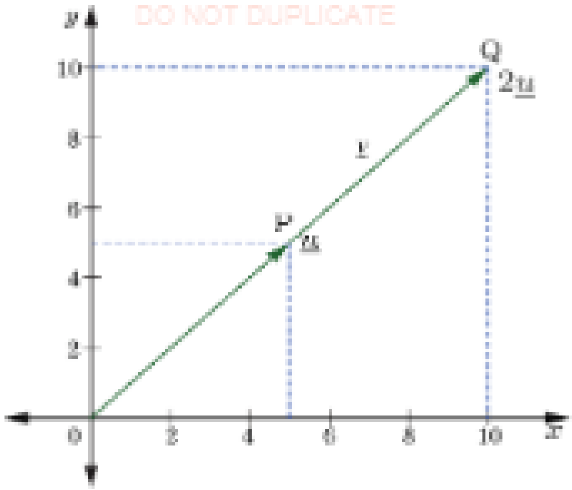


Figure 6.15: *Multiplication of a vector by a scalar*

Generally, if the vector $\underline{u} = (u_1, u_2)$ where u_1 and u_2 are real numbers and t is any non–zero real number, then $t\underline{u} = t(u_1, u_2) = (tu_1, tu_2)$. The vector $t\underline{u} = (tu_1, tu_2)$ is a scalar multiple of the vector $\underline{u} = (u_1, u_2)$ as shown in Figure 6.16.

Similarly, if $\underline{u} = u_1\underline{i} + u_2\underline{j}$, then $t\underline{u} = tu_1\underline{i} + tu_2\underline{j}$

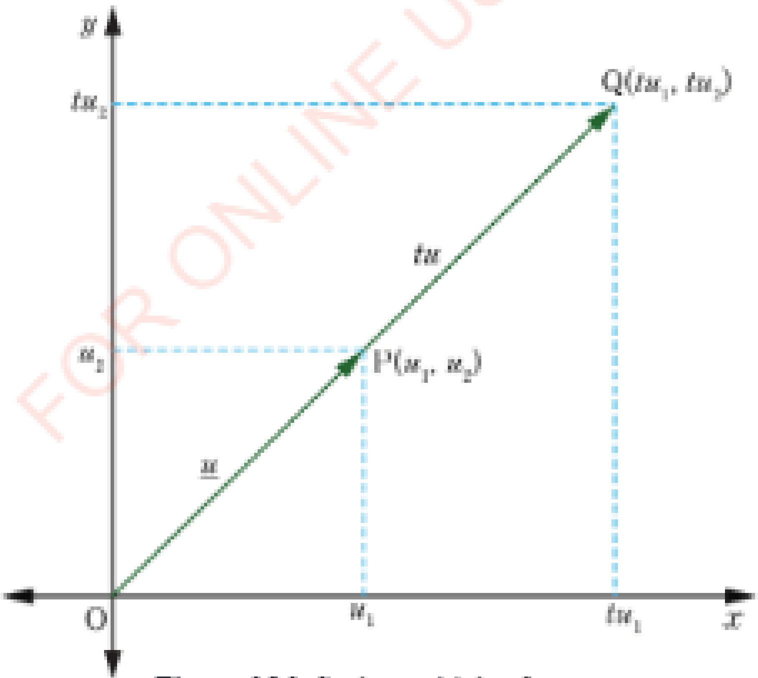


Figure 6.16: *Scalar multiple of vector*

Example 6.20

If $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 5\underset{\sim}{i} + 4\underset{\sim}{j}$, find $-5\underset{\sim}{a} + 3\underset{\sim}{b}$.

Solution

$$
\begin{aligned}
-5\underset{\sim}{a} + 3\underset{\sim}{b} &= -5(3\underset{\sim}{i} + 3\underset{\sim}{j}) + 3(5\underset{\sim}{i} + 4\underset{\sim}{j}) \\
&= -15\underset{\sim}{i} - 15\underset{\sim}{j} + 15\underset{\sim}{i} + 12\underset{\sim}{j} \\
&= (-15 + 15)\underset{\sim}{i} + (-15 + 12)\underset{\sim}{j} \\
&= -3\underset{\sim}{j}
\end{aligned}
$$

Therefore, $-5\underset{\sim}{a} + 3\underset{\sim}{b} = -3\underset{\sim}{j}$.

Example 6.21

Given $\underset{\sim}{p} = (8,\ 6)$ and $\underset{\sim}{q} = (7,\ 9)$, find $9\underset{\sim}{p} - 8\underset{\sim}{q}$.

Solution

$$
\begin{aligned}
9\underset{\sim}{p} - 8\underset{\sim}{q} &= 9(8,\ 6) - 8(7,\ 9) \\
&= (72,\ 54) - (56,\ 72) \\
&= (16,\ -18)
\end{aligned}
$$

Therefore, $9\underset{\sim}{p} - 8\underset{\sim}{q} = (16,\ -18)$.

Exercise 6.4

Answer the following questions:

1. Given $\underset{\sim}{a} = (1,\ 2)$, $\underset{\sim}{b} = (-2,\ -1)$, and $\underset{\sim}{c} = (3,\ 7)$, find $2\underset{\sim}{a} + 3\underset{\sim}{b} + 4\underset{\sim}{c}$.
2. Draw on the same diagram the vectors $\underset{\sim}{u}$, $3\underset{\sim}{u}$, $4\underset{\sim}{u}$, $-\underset{\sim}{u}$, and $-4\underset{\sim}{u}$ given that $\underset{\sim}{u} = (-1,\ 1)$.
3. Find $3\underset{\sim}{a} - 5\underset{\sim}{b} - 6\underset{\sim}{c}$ given that $\underset{\sim}{a} = (3,\ 2)$, $\underset{\sim}{b} = (-4,\ -2)$, and $\underset{\sim}{c} = (4,\ 6)$.

4. If t is a non–zero real number, find each of the following vectors:
   (a) $t\underset{\sim}{a}$, where $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$, $t = -10$
   (b) $t\underset{\sim}{u}$, where $\underset{\sim}{u} = 6\underset{\sim}{i} + 7\underset{\sim}{j}$, $t = 5$
   (c) $t\underset{\sim}{v}$ where $\underset{\sim}{v} = v_1\underset{\sim}{i} + v_2\underset{\sim}{j}$, t, v_1, and v_2 are any real numbers.

5. Draw both the vector and its scalar multiple on the same pair of axes, given that:
   (a) $\underset{\sim}{u} = (3, 4)$ and $t = 3$ (b) $\underset{\sim}{v} = (-3, 4)$ and $t = -3$

6. Given $\underset{\sim}{p} = (12, 5)$ and $\underset{\sim}{q} = (-24, -15)$ find:
   (a) $|5\underset{\sim}{p} - 3\underset{\sim}{q}|$ (b) $2\underset{\sim}{p} + 5\underset{\sim}{q}$
   (c) $\dfrac{2\underset{\sim}{p} + 5\underset{\sim}{q}}{|2\underset{\sim}{p} + 5\underset{\sim}{q}|}$ (d) $\dfrac{5\underset{\sim}{p} - 3\underset{\sim}{q}}{|5\underset{\sim}{p} - 3\underset{\sim}{q}|}$

7. Write $\underset{\sim}{g} = 3\underset{\sim}{i} - 9\underset{\sim}{j}$ as a scalar multiple of $\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

8. Simplify $\underset{\sim}{a} - 2\underset{\sim}{b} + \underset{\sim}{c}$ given that $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$, and $\underset{\sim}{c} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$.

9. Given $\underset{\sim}{p} = (9, 7)$ and $\underset{\sim}{q} = (7, 5)$ find:
   (a) $3(\underset{\sim}{p} - \underset{\sim}{q})$ (b) $3(\underset{\sim}{p} + \underset{\sim}{q})$

10. Given $\underset{\sim}{u} = (1, 2)$, $\underset{\sim}{v} = (3, 4)$ and $t = 8$, find:
   (a) $t\underset{\sim}{u} + t\underset{\sim}{v}$ (b) $3t\underset{\sim}{u} + 2\underset{\sim}{v}$
   (c) $t(\underset{\sim}{u} - \underset{\sim}{v})$ (d) $t^2\underset{\sim}{u} + 5t\underset{\sim}{v}$

Application of vectors

The knowledge of vectors is useful in solving some real practical problems, including situations involving forces or velocities. For example, the forces acting on a boat crossing from a point on one bank of a river to a point on the other bank. The boats motor generates a force in one direction and the current of the river generates a force in another direction. Both forces are vectors. In this case, both the magnitude and direction of each force must be considered in order to know the direction of the boat.

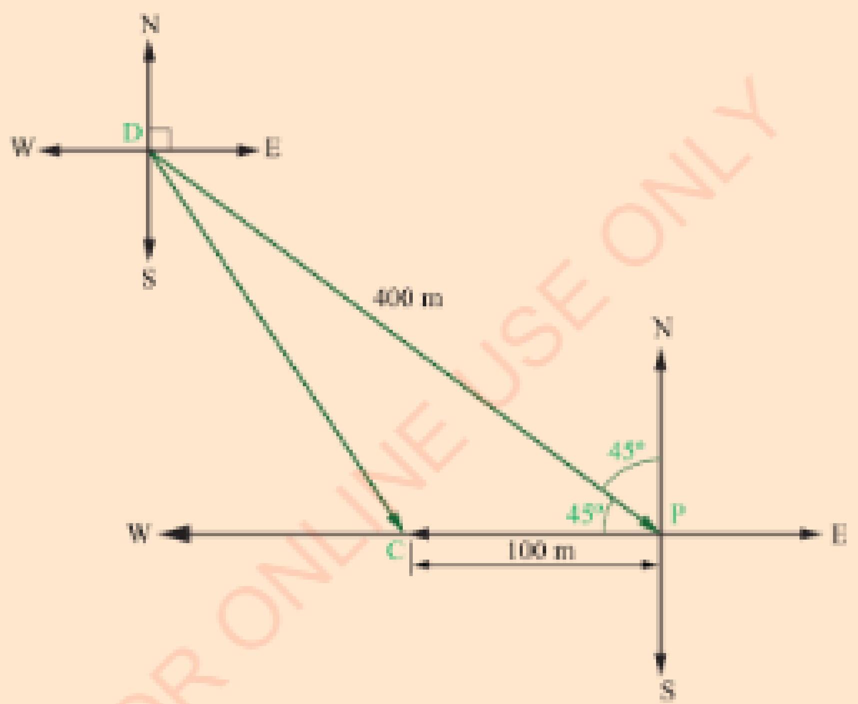
Example 6.22

A student walks 400 metres in the direction S45°E from the dormitory to the parade ground and then he walks 100 metres due west to his classroom. Find the student's displacement and direction from the dormitory to the classroom.

Solution

Let D, P, and C represent the dormitory, the parade ground, and the classroom respectively.

The following figure describes the displacements which join the dormitory D, parade ground P, and classroom C.



From the figure, the resultant $\overline{DC}$ is obtained by using the cosine rule that is

$$(\overline{DC})^2 = (\overline{DP})^2 + (\overline{PC})^2 - 2 \times \overline{DP} \times \overline{PC} \cos\theta$$
$$= (400\text{ m})^2 + (100\text{ m})^2 - 2 \times 400\text{ m} \times 100\text{ m} \times \cos 45°$$
$$= 160\,000\text{ m}^2 + 10\,000\text{ m}^2 - 80\,000\text{ m}^2 \times 0.7071$$
$$= 113\,432\text{ m}^2$$

Thus, $\overline{DC} = \sqrt{113\ 432\ m^2} = 336.8$ m.

Let, $C\hat{D}P = \alpha$

Then, by using the sine rule, we have:

$$\frac{\sin \alpha}{100\ m} = \frac{\sin 45^\circ}{336.8\ m}$$

$$\sin \alpha = \frac{0.7071 \times 100}{336.8}$$

$$= 0.2099$$

$$\alpha = \sin^{-1}(0.2099)$$

$$\alpha = 12^\circ$$

Bearing $= S(45^\circ - 12^\circ)E = S33^\circ E$.

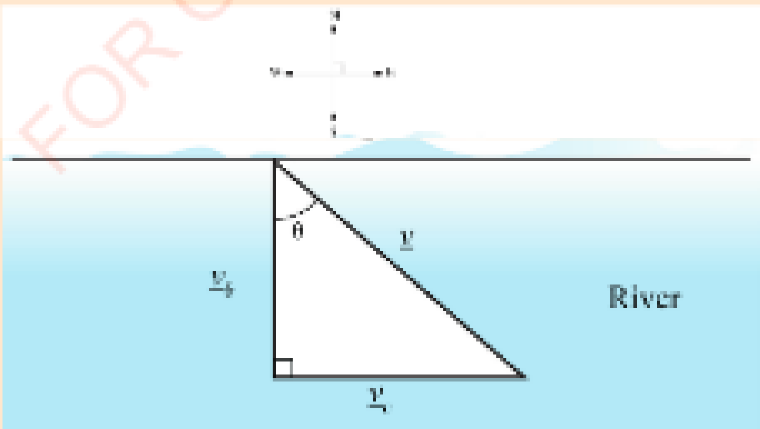
Therefore, the student's displacement from the dormitory to the classroom is 336.8 metres at a bearing of S33°E.

Example 6.23

A boat crosses a river at a velocity of 20 km/h southwards. The river has a current of 5 km/h due east. Calculate the resultant velocity of the boat.

Solution

Let the velocity of the boat be $\underline{v}_b$, the velocity of the current be $\underline{v}_c$ and the resultant velocity of the boat be $\underline{v}$. Consider the following figure:



The figure shows the velocity diagram which satisfies the triangle law of vector addition. The equation $\underline{v} = \underline{v}_b + \underline{v}_c$ is obtained, where $\underline{v}_b = -20\underline{j}$ and $\underline{v}_c = 5\underline{i}$. The unit vectors $\underline{i}$ and $\underline{j}$ represent velocities of 1 km/h in the East and North directions respectively.

Hence, $\underline{v} = \underline{v}_b + \underline{v}_c$

$= -20\underline{j} + 5\underline{i}$

$= 5\underline{i} - 20\underline{j}$

and $|\underline{v}| = \sqrt{(5)^2 + (-20)^2}$

$= \sqrt{25 + 400}$

$= \sqrt{425} = 5\sqrt{17}$

$= 20.6$

Let θ be the angle between $\underline{v}$ and $\underline{v}_b$, then

$$\tan\theta = \frac{|\underline{v}_c|}{|\underline{v}_b|}$$

$$\tan\theta = \frac{5}{20}$$

$$= 0.25$$

$$\theta = \tan^{-1}(0.25)$$

$$\theta = 14°2'$$

Hence, $\theta = 14°2'$.

Therefore, the resultant velocity of the boat is 20.6 km/h at a bearing of S14°2′E.

Example 6.24

Three forces $\underline{F}_1 = (3, 4)$, $\underline{F}_2 = (5, -2)$, and $\underline{F}_3 = (4, 3)$ measured in Newtons act at a point O(0, 0).

(a) Determine the magnitude and direction of their resultant.

(b) Find the magnitude and direction of the opposite of the resultant force.

Solution

(a) Let $\underline{R}$ be the resultant force.

$$\begin{aligned}\underline{R} &= \underline{F}_1 + \underline{F}_2 + \underline{F}_3\\ &= (3,\ 4) + (5,\ -2) + (4,\ 3)\\ &= (12,\ 5)\end{aligned}$$

$$\begin{aligned}|\underline{R}| &= \sqrt{12^2 + 5^2}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

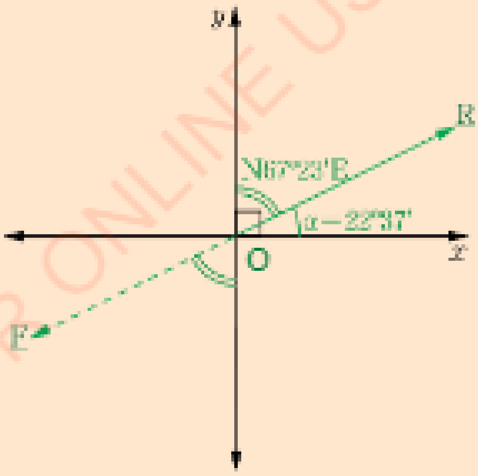
Hence, $|\underline{R}| = 13$.

Let α be the angle the resultant force makes with the horizontal axis.

$$\begin{aligned}\tan\alpha &= \frac{y}{x}\\ &= \frac{5}{12}\\ &= 0.4167\\ \alpha &= \tan^{-1}(0.4167)\end{aligned}$$

Hence, $\alpha = 22°37'$.

Now, consider the following vector diagram:



From the figure, we observe that the bearing of $\underline{R}$ is N 67° 23′ E.

Therefore, the magnitude of the resultant force is 13 Newton at a bearing of N67°23′E .

(b) Let $\underline{F}$ be the opposite of $\underline{R}$, then we have

$$\underline{F} = -\underline{R} = -(12, 5)$$

$$= (-12, -5)$$

$$|\underline{F}| = \sqrt{(-12)^2 + (-5)^2}$$

$$= \sqrt{169}$$

$$= 13$$

Therefore, the magnitude of the opposite of the resultant force is 13 Newton at a bearing of S67°23′W.

Exercise 6.5

Answer the following questions:

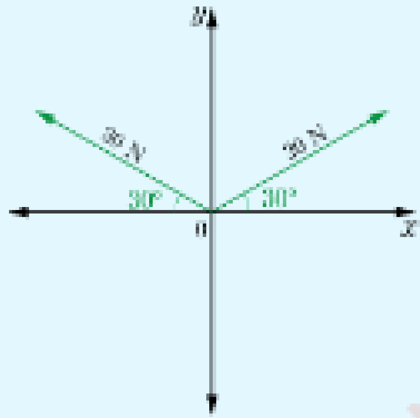
1. Three forces $\underline{F}_1 = (1, 2)$, $\underline{F}_2 = (-2, 0)$, and $\underline{F}_3 = (2, -1)$ measured in Newtons act at the origin. Calculate the magnitude and direction of the force opposite to the resultant force.

2. An aeroplane flies from Dar es Salaam to Kilimanjaro International Airport 400 km due north of Dar es Salaam at 800 km/h. A wind of current 20 km/h blows westwards.
   (a) Calculate the magnitude and direction of the resultant velocity of the aeroplane.
   (b) Which direction must the pilot take in order to land at Kilimanjaro International Airport?
   (c) How long does it take to cover the trip?

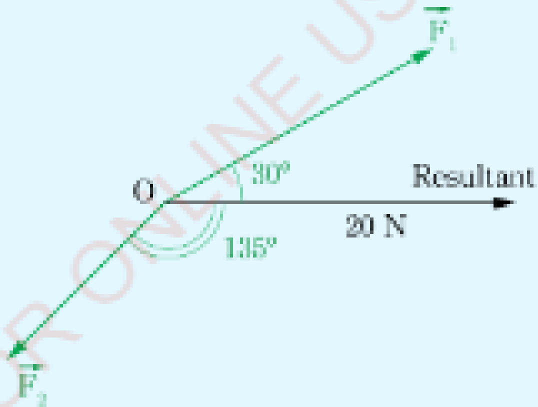
3. Two tracks are pulling on a track stuck in the mud. The first track is pulling with a force of 80 Newton at an angle of 51° from the horizontal while the second truck is pulling with a force of 75 Newton at an angle of 39° from the horizontal. Find the magnitude and direction of the resultant force.

4. A boat moves at a velocity of 10 km/h upstream against a down – stream current of 10 km/h. Calculate the velocity of the boat:
   (a) in still water.
   (b) when moving downstream.

5. Two forces of magnitudes 20 N and 30 N have direction as shown in the figure. Determine the resultant of components in x – axis and y – axis.



6. Two forces acting at a point 0 make angles of 30° and 135° with their resultant of magnitude 20 N as shown in the figure. Find the magnitude of the two forces.



7. An aeroplane leaves the airport on the bearing of 45° travelling at 640 km/h. The wind is blowing at a bearing of 135° at a speed of 64 km/h. Find the actual velocity of the aeroplane.

8. Three forces of magnitudes 6 N, 2 N, and 3 N act at the same point in the North, South, and West directions respectively. Find the magnitude and direction of the resultant force.

Chapter summary

1. Displacement is the change in position of an object. It has both magnitude and direction.
2. Equivalent vectors have the same magnitude and in the same direction.
3. Position vectors are vectors having the same origin and different end points.
4. The unit vectors are vectors whose magnitude is unit.
5. Magnitude of a vector is given as $|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$, for $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$.
6. The unit vector $\hat{\underset{\sim}{u}}$ in the direction of $\underset{\sim}{u}$ is given as $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$.
7. Bearing is the angle measured in degrees in clockwise direction from due north.
8. The direction of a vector is the measure of the angle it makes with the horizontal line.
9. For any vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ the direction cosines are given as:
   (a) $\cos\alpha = \frac{x}{|\underset{\sim}{r}|}$
   (b) $\cos\beta = \frac{y}{|\underset{\sim}{r}|}$
10. Opposite vectors are vectors with the same magnitude but in the opposite direction.
11. The resultant force of two or more forces is a force that has the same effect as two or more forces acting together.

Revision exercise 6

Answer the following questions:

In question 1 to 5 use the vectors:

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$, and $\underset{\sim}{c} = 4\underset{\sim}{j}$ to find:

1. $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{b} - \underset{\sim}{c}$.
2. $5(2\underset{\sim}{a} - 3\underset{\sim}{b} + 2\underset{\sim}{c}) - 2(-4\underset{\sim}{a} + 3\underset{\sim}{b} - 8\underset{\sim}{c})$.
3. $3\underset{\sim}{a} - 2\underset{\sim}{b} + 4\underset{\sim}{c}$.
4. $|\underset{\sim}{b} - \underset{\sim}{c}|$ and $|\underset{\sim}{b}| - |\underset{\sim}{c}|$
5. $|\underset{\sim}{a} + \underset{\sim}{b}|$ and $|\underset{\sim}{a}| + |\underset{\sim}{b}|$.
6. $\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ and $\left|\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}\right|$. What can you say about $\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$?
7. Two forces of magnitudes 5 N and 6 N act on a par0ticle. They act at right angles. Find the magnitude and direction of their resultant.
8. Determine a force $\underset{\sim}{p}$ such that $\underset{\sim}{p}$, $\underset{\sim}{q} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$, and $\underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j}$ have resultant zero.
9. A person walks 600 m at a bearing of 045°, 500 m at a bearing of 090°, 300 m at a bearing of 135°, and 400 m at a bearing of 225°. Find the resultant displacement made by the person.

10. The forces $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, and $\overrightarrow{OS}$ act at the same point O in the same plane such that $P\hat{O}Q = 45°$, $Q\hat{O}R = 135°$ and $R\hat{O}S = 90°$. Find the resultant if the magnitude of each of the four forces is 50 N and that $\overrightarrow{OP}$ acts northwards.

11. Given the vectors $\underset{\sim}{a} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$, and $\underset{\sim}{c} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$, find:
    (a) $|2\underset{\sim}{a} + \underset{\sim}{b} - 3\underset{\sim}{c}|$
    (b) Scalars t and k if $t\underset{\sim}{a} + k\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$.

12. A boat moves at a velocity of 15 km/h upstream against a downstream current of 12 km/h. Calculate the velocity of the boat:
    (a) In still water
    (b) when moving down stream.

13. In order to get across a river to a dock on time, a boat would have to row 65° upstream at a velocity of 80 km/h if no current was present. If the current is 25 km/h, what velocity and direction must the boat travelled?

Project 6

1. Draw a regular octagon ABCDEFGH, where $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AH} = \underset{\sim}{b}$
2. Use the figure in 1 above to express the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
   (a) $\overrightarrow{FE}$ (b) $\overrightarrow{ED}$ (c) $\overrightarrow{BG}$ (d) $\overrightarrow{HD}$
3. Draw the right – angled triangle PQR, the point S divides $\overrightarrow{PR} = \underset{\sim}{a}$ in the ratio 1:3, T divides $\overrightarrow{RQ}$ in the ratio 3:2, and $\overrightarrow{PQ} = \underset{\sim}{b}$.
4. A swimmer wishes to cross a river to an exactly opposite point on the far bank. If she can swim 5 km, what angle to the bank she must make? Use a scale drawing and verify the result by calculation.

Chapter Seven

Matrices and transformations

Introduction

A matrix represents the way of writing information in rectangular array of numbers, symbols, and expressions. The information is represented in rows and columns to simplify understanding and interpreting of the quantities they represent. In this chapter, you will learn basic operations on matrices such as addition, subtraction, and multiplication, calculating determinant of matrices, finding the inverse of matrices and applying matrices in solving simultaneous equations and transformations. The competencies developed will help you in different fields such as business, engineering, demography, airplanes during taking off and landing, and many others.

Operations on matrices

Activity 7.1: Determining the number of rows and columns in a matrix

In a group or individually perform the following tasks:

At a certain Secondary School, every class has four streams: Form One class; stream A has 35 students, stream B has 30 students, stream C has 29 students, and stream D has 34 students. Form Two class; stream A has 28 students, stream B has 25 students, stream C has 18 students, and stream D has 10 students. Form Three class; stream A has 20 students, stream B has 19 students, stream C has 17 students, and stream D has 8 students. Form Four class; stream A has 12 students, stream B has 15 students, stream C has 19 students, and stream D has 22 students.

1. Arrange the streams in rows and classes in columns.
2. How many rows and columns are formed in task 1?
3. What is your observation in the rows and columns formed?
4. Identify the number of elements in each row and in each column.
5. Share your findings with your neighbours through discussions.

Note: The rectangular array of rows (m) and columns (n) with the order $m \times n$ is enclosed by [ ] or ( ).

Types of matrices

There are various types of matrices; this chapter concentrates on the following:

(a) **Row matrix**: This is a matrix having only one row. For example, $A = (8 \quad 6)$ is a row matrix of the order 1×2.

(b) **Column matrix**: This is a matrix having only one column. For example, $B = \begin{pmatrix} 4 \\ 16 \end{pmatrix}$ is a column matrix of the order 2×1.

(c) **Scalar matrix**: This is a square matrix whose elements in the main diagonal are the same and the rest of the elements are zero. For example, $M = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$ is a scalar matrix of the order 2×2.

(d) **Null matrix**: This is a matrix with all zero elements. Thus, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a null matrix or a zero matrix of the order 2×2.

(e) **Square matrix**: This is a matrix having the same number of rows and columns.

For example, $P = \begin{pmatrix} 4 & 6 \\ 12 & 6 \end{pmatrix}$ is a square matrix having 2 rows and 2 columns .

(f) **Diagonal matrix**: This is the matrix with zeros elements except in the main diagonal. For example, $p = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ is the diagonal matrix of the order 2×2.

(g) **Unit matrix or identity matrix**. This is a diagonal or square matrix in which the main diagonal has all elements equal to 1. An identity matrix is usually denoted by the symbol I. Thus, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix of the order 2×2.

Addition and subtraction of matrices

Two matrices may be added or subtracted provided they are of the same order. It is done by adding or subtracting corresponding elements in each of the given matrices. The results give another matrix which has the same order as the original matrices.

Example 7.1

Given the matrices $A = \begin{pmatrix} 8 & 5 \\ 6 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 7 \\ 8 & 2 \end{pmatrix}$.

Find: (a) $A + B$ (b) $B - A$

Solution

(a) $A + B = \begin{pmatrix} 8 & 5 \\ 6 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 7 \\ 8 & 2 \end{pmatrix} = \begin{pmatrix} 8+3 & 5+7 \\ 6+8 & 4+2 \end{pmatrix} = \begin{pmatrix} 11 & 12 \\ 14 & 6 \end{pmatrix}$.

(b) $B - A = \begin{pmatrix} 3 & 7 \\ 8 & 2 \end{pmatrix} - \begin{pmatrix} 8 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 3-8 & 7-5 \\ 8-6 & 2-4 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$.

Example 7.2

Find the matrix C, when $C = A - B$ given that $A = \begin{pmatrix} 6 & 2 \\ 1 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}$.

Solution

$$C = A - B = \begin{pmatrix} 6 & 2 \\ 1 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 6-4 & 2-3 \\ 1-7 & 8-5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix}$$

Therefore, $C = \begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix}$.

Exercise 7.1

Answer the following questions:

1. If $P = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix}$ then evaluate:

   (a) $P + Q$ (b) $Q - P$

2. If the matrix $R = \begin{pmatrix} 9 & 4 \\ 7 & 5 \end{pmatrix}$ and the matrix $S = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}$, find the matrix T such that $R + T = S$.

3. If $A = \begin{pmatrix} 8 & 6 \\ 6 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 10 & 4 \\ 6 & 4 \end{pmatrix}$ find:

   (a) $A + B$ (b) $A - B$

4. Find the values of L and M in the following:

   $$\begin{pmatrix} L & 4 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ M & 1 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ -6 & 6 \end{pmatrix}.$$

5. If the matrices $A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}$, then calculate the matrix C such that $C = A + B$.

6. Given the following matrices:

   (a) $M = \begin{pmatrix} 16 & 2 \\ 10 & 8 \end{pmatrix}$, $N = \begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}$. Find $M - N$.

   (b) If $R = \begin{pmatrix} 8 & 6 \\ 7 & 5 \end{pmatrix}$, $S = \begin{pmatrix} 3 & 2 \\ 6 & 5 \end{pmatrix}$, then find $R - S$.

7. If $A = \begin{pmatrix} 4 & 6 \\ 4 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}$, then find:

   (a) $A + B$ (b) $A - B$

8. Given that $P = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ -3 & 4 \end{pmatrix}$, find the values of a and b if $P = Q$.

9. If $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, find each of the following:

   (a) $M + Q$ (b) $M - Q$

10. Find the matrix P in the equation $P + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & 5 \end{pmatrix}$.

11. Find the values of w, x, y, and z in the following:

   $\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 6 \\ 7 & -8 \end{pmatrix}$.

Scalar multiplication of matrices

A matrix may be multiplied by a real number, where by each entry in a matrix is multiplied by a scalar k. If M is any matrix and k is any scalar (real number), then the product kM is the matrix obtained my multiplying each entry of the matrix M by a scalar k. The matrix kM is said to be a scalar multiple of M.

For example, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and k be any constant real number, then $kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$.

Example 7.3

Given $A = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$, find 3A.

Solution

$$3A = 3\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 3\times2 & 3\times1 \\ 3\times6 & 3\times4 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 18 & 12 \end{pmatrix}$$

Therefore, $3A = \begin{pmatrix} 6 & 3 \\ 18 & 12 \end{pmatrix}$.

Example 7.4

If $W = \begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix}$, find: (a) 2W (b) 4W (c) –5W

Solution

(a) $2W = 2\begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 10 & 18 \end{pmatrix}$ (b) $4W = 4\begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 28 & 32 \\ 20 & 36 \end{pmatrix}$

(c) $-5W = -5\begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} -35 & -40 \\ -25 & -45 \end{pmatrix}$.

Multiplication of matrices

Two matrices can only be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. The multiplication is done by multiplying the elements in a row by the elements in a column, and the order of the resulting matrix will have the number of rows of the first matrix and number of columns of the second matrix. For example, if matrix P is of order $m \times u$ and matrix Q is of order $v \times n$, then PQ is possible if and only if $u = v$ and the resulting matrix will be of the order $m \times n$.

When matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is multiplied by another matrix $B = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$

the product matrix is $AB = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} e & g \\ f & h \end{pmatrix}$, where a, b, c, d, e, f, g, and h are any real numbers.

In this case, A is called the pre–multiplier matrix because it is multiplied to the left of B and B is called the post–multiplier matrix because it is multiplied to the right of A. Briefly pre–means done before and post–means done after.

Consider the 2×2 matrices where by $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} e & g \\ f & h \end{pmatrix}$.

The product $AB = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} e & g \\ f & h \end{pmatrix}$ is obtained as follows:

Step 1: The first row of matrix A is multiplied by the first column of matrix B. That is;

$$\begin{pmatrix} a & c \\ & \end{pmatrix} \times \begin{pmatrix} e \\ f \end{pmatrix} = (ae + cf).$$

Elements of the first row of matrix A are multiplied by the corresponding elements of the first column of matrix B to get the element in first row and first column of matrix AB.

Step 2: The first row of matrix A is multiplied by the second column of matrix B. That is;

$$\begin{pmatrix} a & c \\ & \end{pmatrix} \times \begin{pmatrix} g \\ h \end{pmatrix} = (ag + ch).$$

Elements of the first row of matrix A are multiplied by the corresponding elements of the second column of matrix B to get the element in the first row and second column of matrix AB.

Step 3: The second row of matrix A is multiplied by the first column of matrix B. That is;

$$\begin{pmatrix} & \\ b & d \end{pmatrix} \times \begin{pmatrix} e \\ f \end{pmatrix} = (be + df).$$

Elements of the second row of matrix A are multiplied by the corresponding elements of the first column of matrix B to get the element in the second row and first column of matrix AB.

Step 4: The second row of matrix A is multiplied by the second column of matrix B to get the element in the second row and second column of matrix AB. That is;

$$\begin{pmatrix} & \\ b & d \end{pmatrix} \times \begin{pmatrix} g \\ h \end{pmatrix} = (bg + dh).$$

Elements of the second row of matrix A are multiplied by the corresponding elements of the second column of matrix B to get the element in the second row and second column of matrix AB.

Thus, we have;

$$A \times B = \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{Pre-multiplier} \times \underbrace{\begin{pmatrix} e & g \\ f & h \end{pmatrix}}_{Post-multiplier} = \underbrace{\begin{pmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{pmatrix}}_{Product}$$

Note: Matrix multiplication is possible when the number of columns in the pre–multiplier matrix is equal to the number of rows in the post–multiplier matrix. Also, the number of rows of the pre–multiplier matrix becomes the number of rows in the product matrix. The number of columns in the post–multiplier matrix becomes the number of columns in the product matrix.

Generally, if the pre–multiplier matrix has order $m \times p$ and the post – multiplier matrix has order $p \times n$ then the product matrix will have order $m \times n$.

Example 7.5

Given $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$, then find AB.

Solution

$$A \times B = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2\times5+5\times3 & 2\times2+5\times6 \\ 4\times5+3\times3 & 4\times2+3\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 10+15 & 4+30 \\ 20+9 & 8+18 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 34 \\ 29 & 26 \end{pmatrix}$$

Therefore, $AB = \begin{pmatrix} 25 & 34 \\ 29 & 26 \end{pmatrix}$.

Example 7.6

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, then find A^2.

Solution

$$A^2 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2\times2+1\times3 & 2\times1+1\times4 \\ 3\times2+4\times3 & 3\times1+4\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 4+3 & 2+4 \\ 6+12 & 3+16 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 18 & 19 \end{pmatrix}$$

Therefore, $A^2 = \begin{pmatrix} 7 & 6 \\ 18 & 19 \end{pmatrix}$.

Example 7.7

Given $A = \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & -1 \\ -2 & 5 \end{pmatrix}$, find: (a) AB and (b) BA.

Solution

(a) $$AB = \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix} \times \begin{pmatrix} 6 & -1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 9\times6+7\times(-2) & 9\times(-1)+7\times5 \\ 8\times6+6\times(-2) & 8\times(-1)+6\times5 \end{pmatrix}$$

$$= \begin{pmatrix} 54-14 & -9+35 \\ 48-12 & -8+30 \end{pmatrix} = \begin{pmatrix} 40 & 26 \\ 36 & 22 \end{pmatrix}$$

Therefore, $AB = \begin{pmatrix} 40 & 26 \\ 36 & 22 \end{pmatrix}$.

(b) $$BA = \begin{pmatrix} 6 & -1 \\ -2 & 5 \end{pmatrix} \times \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} 6\times9+(-1)\times8 & 6\times7+(-1)\times6 \\ (-2)\times9+5\times8 & (-2)\times7+5\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 54-8 & 42-6 \\ -18+40 & -14+30 \end{pmatrix} = \begin{pmatrix} 46 & 36 \\ 22 & 16 \end{pmatrix}$$

Therefore, $BA = \begin{pmatrix} 46 & 36 \\ 22 & 16 \end{pmatrix}$.

Note: $AB \neq BA$. This means that matrix multiplication is not commutative, with exception when a matrix is multiplied by an identity matrix, that is, $AI = IA$ or when a matrix is multiplied by its inverse.

Example 7.8

Given $P = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$.

Find: (a) P(QR) (b) (PQ)R

Solution

(a) $P(QR) = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

$= \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 27 & 37 \\ 11 & 15 \end{pmatrix}$

$= \begin{pmatrix} 130 & 178 \\ 92 & 126 \end{pmatrix}$

Therefore, $P(QR) = \begin{pmatrix} 130 & 178 \\ 92 & 126 \end{pmatrix}$.

(b) $(PQ)R = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

$= \begin{pmatrix} 14 & 10 \\ 10 & 7 \end{pmatrix} \times \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

$= \begin{pmatrix} 130 & 178 \\ 92 & 126 \end{pmatrix}$

Therefore, $(PQ)R = \begin{pmatrix} 130 & 178 \\ 92 & 126 \end{pmatrix}$.

Note: Since $P(QR) = (PQ)R$; hence, matrix multiplication is associative.

Identity matrix for multiplication

The identity property for matrix multiplication is such that for any square matrix A, AI = IA where I is the matrix multiplicative identity.

Example 7.9

Given the matrices $A=\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}$, $B=\begin{pmatrix}a & c\\b & d\end{pmatrix}$, and $I=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Find:

(a) AI and IA (b) BI and IB.

Solution

(a) $AI=\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}\times\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}=A$

$IA=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\times\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}=\begin{pmatrix}2 & 4\\3 & 1\end{pmatrix}=A.$

(b) $BI=\begin{pmatrix}a & c\\b & d\end{pmatrix}\times\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}a & c\\b & d\end{pmatrix}=B$

$IB=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\times\begin{pmatrix}a & c\\b & d\end{pmatrix}=\begin{pmatrix}a & c\\b & d\end{pmatrix}=B.$

Exercise 7.2

Answer the following questions:

1. Given $A=\begin{pmatrix}3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}2\\3\end{pmatrix}$. Find: (a) AB and (b) BA

2. Given $A=\begin{pmatrix}3 & 2\\1 & 5\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$, and $C=\begin{pmatrix}5 & 7\\6 & -8\end{pmatrix}$. Find:
(a) AB (b) BC (c) AC
(d) CA (e) A^2 (f) C^2

3. Given $L = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$, $M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, and $N = \begin{pmatrix} 5 & 7 \\ 6 & -8 \end{pmatrix}$. Find:

(a) $3L - 2M$ (b) $4M + 6N$ (c) $6M - 4N$

(d) $5(L + M)$ (e) $5(L - M)$

4. Using the matrices $R = \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix}$, $S = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, and $T = \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}$; find:

(a) $R(ST)$ (b) $(RS)T$

5. Given $A = \begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} -2 & 4 \\ 4 & 5 \end{pmatrix}$. Find:

(a) $A(B+C)$ (b) $AB+AC$

(c) Is matrix multiplication distributive over matrix addition? Give reasons.

6. Given $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $F = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Find: (a) EF (b) GE (c) EH

7. Find: (a) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & -7 \\ 6 & 8 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} -5 & -7 \\ 6 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} 18 & -12 \\ 13 & -14 \end{pmatrix} \times \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} -36 & -18 \\ -24 & -12 \end{pmatrix} \times \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$

8. Given that $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $K = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{-\sqrt{2}} \end{pmatrix}$. Find:

(a) HK (b) KH

9. (a) Calculate AB in each of the following :

(i) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(iii) $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) Is it possible to calculate BA in each case? Why?

10. (a) Given $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Find: A if $AB = \begin{pmatrix} 6 & 5 \\ 4 & 2 \end{pmatrix}$

(b) Given $P = \begin{pmatrix} -2 & 6 \\ 4 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$. Find: Q if $PQ = \begin{pmatrix} 4 & -4 \\ 6 & -6 \end{pmatrix}$

11. If $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix}$. Find: (a) AB (b) BA (c) A^2

12. If $X = \begin{pmatrix} -5 & 4 \\ 3 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 7 & -3 \\ 8 & 2 \end{pmatrix}$. Find: (a) $3X + 2Y$ (b) $4X - Y$

13. If $A = \begin{pmatrix} 9 & 4 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$. Find: (a) BA (b) $3A - 2B$

14. If $R = \begin{pmatrix} -4 & 7 \\ 5 & -4 \end{pmatrix}$ and $S = \begin{pmatrix} -9 & 5 \\ -2 & 6 \end{pmatrix}$. Find: (a) 4R (b) $S + 2R$ (c) SR

15. If $X = \begin{pmatrix} 12 & -3 \\ 0 & 7 \end{pmatrix}$ and $Y = \begin{pmatrix} -9 & 15 \\ 3 & -7 \end{pmatrix}$. Find:

(a) XY (b) X^2 (c) $X + 3Y$

16. If $J = \begin{pmatrix} 7 & 2 \\ 5 & 3 \end{pmatrix}$ and $K = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$, find JK.

17. If $U = \begin{pmatrix} -6 & 4 \\ 4 & 5 \end{pmatrix}$ and $V = \begin{pmatrix} -8 \\ 3 \end{pmatrix}$, find: (a) UV (b) U^2V

Determinant of a matrix

Determinant is a scalar number that is the function of the entries of the square matrix. Determinant of a matrix is obtained by subtracting the product of the entries of the anti–diagonal (or off – diagonal) from the product of the entries of the main diagonal (or leading diagonal) and is denoted by |A| or det(A) where A any $n \times n$ matrix. Given $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Then, $|A| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$.

Determinants exist for square matrices only.

For example, (2×2), (3×3) ..., $(n \times n)$; where $n \in \mathbb{N}$.

Example 7.10

Given $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, find $|A|$.

Solution

$$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

$$= (1\times4) - (2\times3)$$

$$= 4 - 6$$

$$= -2$$

Therefore, $|A| = -2$.

Example 7.11

Find the determinant of $A = \begin{pmatrix} 21 & 10 \\ -1 & 12 \end{pmatrix}$.

Solution

$$|A| = \begin{vmatrix} 21 & 10 \\ -1 & 12 \end{vmatrix}$$

$$= 21\times12 - (-1\times10)$$

$$= 252 + 10$$

$$= 262$$

Therefore, $|A| = 262$.

Example 7.12

If $T = \begin{pmatrix} 2k & 25 \\ 20 & 4k \end{pmatrix}$, find the values of k given that $|T| = 300$.

Solution

$$|T| = \begin{vmatrix} 2k & 25 \\ 20 & 4k \end{vmatrix}$$

$$= 8k^2 - 500. \text{ But, } |T| = 300$$

$$8k^2 - 500 = 300$$

$$8k^2 = 800$$

$$k^2 = 100$$

$$k = \pm\sqrt{100} = \pm 10$$

Therefore, $k = 10$ or $k = -10$.

Singular and non-singular matrices

If the determinant of a matrix is zero, then the matrix is called singular.

For example, $P = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, then, $|P| = 8 - 8 = 0$.

Therefore, P is a singular matrix because its determinant is zero.

If a matrix has non–zero determinant, then the matrix is called non–singular.

For example, $Q = \begin{pmatrix} 6 & 3 \\ 2 & 4 \end{pmatrix}$ then, $|Q| = 24 - 6 = 18$.

Then, Q is a non–singular matrix because its determinant is non–zero.

Generally, if $R = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $a, b, c,$ and d are any real numbers, then R is a singular matrix if $|R| = ad - bc = 0$ and non–singular matrix when $|R| = ad - bc \neq 0$.

Exercise 7.3

Answer the following questions:

In questions 1 to 10 calculate the determinant of the matrix and state whether the matrix is singular or non – singular.

1. $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$

2. $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

3. $C = \begin{pmatrix} 52 & 58 \\ 18 & 17 \end{pmatrix}$

4. $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

5. $E = \begin{pmatrix} 1 & 6 \\ 5 & 30 \end{pmatrix}$

6. $F = \begin{pmatrix} -29 & -18 \\ 13 & -15 \end{pmatrix}$

7. $G = \begin{pmatrix} 0 & 0 \\ 4 & 5 \end{pmatrix}$

8. $H = \begin{pmatrix} 0 & 11 \\ 0 & 12 \end{pmatrix}$

9. $J = \begin{pmatrix} 125 & -18 \\ 15 & -8 \end{pmatrix}$

10. $J = \begin{pmatrix} a & b \\ ak & bk \end{pmatrix}$, where a and b are any real numbers and k is any non–zero real number.

11. (a) If $K = \begin{pmatrix} a & 10 \\ 14 & 50 \end{pmatrix}$, find a given that $|K| = 10$.

    (b) If $L = \begin{pmatrix} a & 2x \\ x & 0 \end{pmatrix}$, find x given that $|L| = -8$.

    (c) Given that $A = \begin{pmatrix} x+1 & x-1 \\ 2x & x \end{pmatrix}$ is a singular matrix, find the possible values of x.

Inverse of a matrix

The inverse of a 2×2 matrix A is another 2×2 matrix denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity matrix.

Properties of inverse of matrices

Let A and B be matrices with A^{-1} and B^{-1} respectively. The following properties hold:

1. $AA^{-1} = A^{-1}A = I$, where I is an identity matrix.
2. The inverse of matrix A exists if and only if A is a non – singular matrix. *i.e.* $|A| \neq 0$.
3. When $|A| = 0$, then A is a singular matrix, hence, $\frac{1}{|A|}$ is undefined and the A^{-1} does not exist
4. $(A^{-1})^{-1} = A$.
5. $(AB)^{-1} = B^{-1}A^{-1}$.

Consider a 2×2 matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where *a*, *b*, *c*, and *d* are any real numbers.

Let the inverse of A be $A^{-1} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ where *p*, *q*, *r*, and *s* are any real numbers.

Since, $AA^{-1} = I$ then, $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} ap+cq & ar+cs \\ bp+dq & br+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating corresponding elements we obtain:

$$ap + cq = 1 \quad (1)$$

$$bp + dq = 0 \quad (2)$$

$$ar + cs = 0 \quad (3)$$

$$br + ds = 1 \quad (4)$$

We need to solve for p, q, r, and s in terms of a, b, c, and d.
Multiply equation (1) by b and equation (2) by a we get

$$bap + bcq = b \quad (5)$$

$$abp + adq = 0 \quad (6)$$

Solving equations (5) and (6) simultaneously we get

$$bcq - adq = b$$

$$-(ad - bc)q = b, \text{ factorizing } -q$$

Hence, $q = \dfrac{-b}{ad - bc}$, dividing both sides by $-(ad - bc)$

Multiply equation (1) by d and equation (2) by c we get

$$adp + cdq = d \quad (7)$$

$$bcp + cdq = 0 \quad (8)$$

Solving equations (7) and (8) simultaneously we get

$$adp - bcp = d$$

$$(ad - bc)p = d, \text{ factorizing } p$$

Hence, $p = \dfrac{d}{ad - bc}$, dividing both sides by $ad - bc$

Multiply equation (3) by b and equation (4) by a we get

$$abr + bcs = 0 \quad (9)$$

$$abr + ads = a \quad (10)$$

Solving equations (9) and (10) simultaneously we get

$$bcs - ads = -a$$

$$-(ad - bc)s = -a, \text{ factorizing } -s$$

$$(ad - bc)s = a, \text{ multiplying by negative sign both sides}$$

$$\text{Hence, } s = \frac{a}{ad - bc}, \text{ dividing both sides by } ad - bc$$

Multiply equation (3) by d and equation (4) by c we get

$$adr + cds = 0 \qquad (11)$$

$$bcr + cds = c \qquad (12)$$

Solving equations (11) and (12) simultaneously we get

$$adr - bcr = -c$$

$$(ad - bc)r = -c, \text{ factorizing } r$$

$$\text{Hence, } r = \frac{-c}{ad - bc}, \text{ dividing both sides by } ad - bc$$

$$\text{Therefore, } A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Generally, in order to obtain A^{-1} for a 2×2 matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ one needs to use the following steps:

1. Interchange the elements of the leading diagonal.
2. Change the signs of the elements in the anti – diagonal, if the element is positive become negative and vice – versa.
3. Divide each element by the determinant of A.

Inverses do exist for matrices having non – zero determinant. Singular matrices have no inverse because they have zero determinant and division by zero is not defined.

Consider the two matrices $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ and $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$, the product of the matrices is:

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \times \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 10-9 & -6+6 \\ 15-15 & -9+10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product of these two matrices gives the identity matrix.

If $A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, then $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ is known as the inverse of matrix A and is written as A^{-1}.

Therefore, finding the inverse of a matrix can be done in two ways:

(i) By simultaneous equations

(ii) By formula

Example 7.13

Given $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, find: (a) A^{-1} (b) $(A^2)^{-1}$

Solution

(a) $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$|A| = 1\times4 - 2\times3$

$= 4 - 6 = -2$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Therefore, $A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$.

$$A^2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 10 & 22 \end{pmatrix}$$

then, $(A^2)^{-1} = \frac{1}{|A^2|}\begin{pmatrix} 22 & -15 \\ -10 & 7 \end{pmatrix}$

$$= \frac{1}{154-150}\begin{pmatrix} 22 & -15 \\ -10 & 7 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 22 & -15 \\ -10 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{2} & -\frac{15}{4} \\ \frac{-5}{2} & \frac{7}{4} \end{pmatrix}$$

Therefore, $(A^2)^{-1} = \begin{pmatrix} \frac{11}{2} & -\frac{15}{4} \\ -\frac{5}{2} & \frac{7}{4} \end{pmatrix}$.

Example 7.14

Which of the following matrices have inverses?

(a) $B = \begin{pmatrix} -1 & 3 \\ 2 & 6 \end{pmatrix}$ (b) $C = \begin{pmatrix} -85 & 15 \\ 68 & -12 \end{pmatrix}$

Solution

(a) $|B| = \begin{vmatrix} -1 & 3 \\ 2 & 6 \end{vmatrix} = -6 - 6 = -12$, Since $|B| \neq 0$, then B^{-1} exist.

(b) $|C| = \begin{vmatrix} -85 & 15 \\ 68 & -12 \end{vmatrix} = 1\,020 - 1\,020 = 0$, Since $|C| = 0$, then C^{-1} does not exist.

Example 7.15

Find the inverse of matrix A given that $A = \begin{pmatrix} 6 & 8 \\ 2 & 3 \end{pmatrix}$.

Solution

By using simultaneous equations:

Let $\begin{pmatrix} 6 & 8 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} w & y \\ x & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$6w + 8x = 1$ (1)

$2w + 3x = 0$ (2)

Multiplying equation (2) by 3 and eliminating w gives:

$$-\begin{cases} 6w + 8x = 1 \\ 6w + 9x = 0 \end{cases}$$
$$-x = 1$$

Hence, $x = -1$

Substituting $x = -1$ into equation (1) gives:

$6w + 8(-1) = 1$, $6w - 8 = 1$, $6w = 9$

Hence, $w = \frac{3}{2}$.

$6y + 8z = 0$ (3)

$2y + 3z = 1$ (4)

Multiplying equation (4) by 3 and eliminating y gives:

$$-\begin{cases} 6y + 8z = 0 \\ 6y + 9z = 3 \end{cases}$$
$$-z = -3$$

Hence, $z = 3$.

Substituting $z = 3$ into equation (4) gives:

$2y + 9 = 1$

$2y = -8$

Hence, $y = -4$.

Therefore, $A^{-1} = \begin{pmatrix} \frac{3}{2} & -4 \\ -1 & 3 \end{pmatrix}$.

Alternatively: By using the formula:

If $A = \begin{pmatrix} w & y \\ x & z \end{pmatrix}$, $A^{-1} = \frac{1}{wz - xy}\begin{pmatrix} z & -y \\ -x & w \end{pmatrix}$

Hence, if $A = \begin{pmatrix} 6 & 8 \\ 2 & 3 \end{pmatrix}$, then

$$A^{-1} = \frac{1}{18-16}\begin{pmatrix} 3 & -8 \\ -2 & 6 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -8 \\ -2 & 6 \end{pmatrix}$$

Therefore, $A^{-1} = \begin{pmatrix} \frac{3}{2} & -4 \\ -1 & 3 \end{pmatrix}$.

Example 7.16

If $Z = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$ then show that $Z^2 - 6Z + 17I = 0$, and find Z^{-1}.

Solution

$$Z^2 = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix}$$

$$6Z = 6\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 12 & -18 \\ 18 & 24 \end{pmatrix}$$

$$17I = 17\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}$$

Then, $Z^2 - 6Z + 17I = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix} - \begin{pmatrix} 12 & -18 \\ 18 & 24 \end{pmatrix} + \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}$

$$= \begin{pmatrix} -5-12+17 & -18+18+0 \\ 18-18+0 & 7-24+17 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $Z^2 - 6Z + 17I = 0$ shown.

$$|Z| = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 8 - (-9) = 17$$

$$Z^{-1} = \frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{pmatrix}$$

Therefore, $Z^{-1} = \begin{pmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{pmatrix}$.

Exercise 7.4

Answer the following questions:

In questions 1 to 10 determine whether the following matrices are singular or non – singular. Hence, determine their inverses.

1. $A = \begin{pmatrix} 4 & 4 \\ -4 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$
3. $C = \begin{pmatrix} 15 & 12 \\ 30 & 24 \end{pmatrix}$
4. $D = \begin{pmatrix} 0 & 7 \\ 5 & 0 \end{pmatrix}$
5. $E = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}$
6. $F = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$
7. $M = \begin{pmatrix} 16 & -8 \\ 24 & 12 \end{pmatrix}$
8. $H = \begin{pmatrix} 3 & 12 \\ 5 & 20 \end{pmatrix}$
9. $G = \begin{pmatrix} 50 & 2 \\ 150 & 60 \end{pmatrix}$

10. $J = \begin{pmatrix} a & ak \\ b & bk \end{pmatrix}$ where a, b, and k are any real numbers, k not equal to 0.

11. Using the matrix in question 1, find: (a) $(A^2)^{-1}$ (b) $(A^3)^{-1}$

12. Using the matrices in question 5 and 6, find: (a) $(E^{-1})^2$ (b) $(F^{-1})^2$

13. If $N = \begin{pmatrix} 4 & -6 \\ 6 & 8 \end{pmatrix}$, then find the value of $N^2 + 6N - 18I$ where I is an identity matrix.

Activity 7.2: Solving simultaneous equations by using matrices

In a group or individually, perform the following tasks:

Given that each kilogram of food M contains 10 units of calcium and 4 units of iron. Each kilogram of food N contains 6 units of calcium and 4 units of iron, and the mix of M and N contains 92 units of calcium and 44 units of iron.

1. If the variables x and y represent the number of kilograms of food M and N respectively, then use the given information to formulate the system of linear simultaneous equation.
2. Transform the system of linear equations into matrix form.
3. Find the determinant and inverse of the matrix.
4. Use the inverse in task 3 to find the number of kilograms of each food.
5. Share your findings with your neighbours for more inputs.

Use of matrices in solving simultaneous equations

Matrices can be used for solving simultaneous equations. Consider the product of a 2×2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and a 2×1 matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, where a, b, c, and d are any real numbers and x and y are real variables.

Thus, $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$.

The elements of the product matrix are linear expressions.

The general system of linear simultaneous equations has the form:

$$\begin{cases} ax + cy = p \\ bx + dy = q \end{cases} \qquad (13)$$

where p and q are real numbers.

For any system of simultaneous equations a matrix method can be used to solve for x and y provided that the determinant of the coefficient matrix is not zero; that is $ad - bc \neq 0$.

The coefficient matrix of equation (13) is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. The system can be written in matrix form as follows:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

The inverse of the matrix of coefficients of x and y is: $\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Pre – multiply the inverse matrix on each side of the matrix equation:

$$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \times \begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \times \begin{pmatrix} p \\ q \end{pmatrix}$$

Recall that $AA^{-1} = A^{-1}A = I$ for any matrix A. Then:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} dp-cq \\ aq-bp \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dp-cq}{ad-bc} \\ \frac{aq-bp}{ad-bc} \end{pmatrix}$$

Therefore, $x = \left(\frac{dp-cq}{ad-bc}\right)$ and $y = \left(\frac{aq-bp}{ad-bc}\right)$ provided that $ad - bc \neq 0$.

Example 7.17

Use matrix method to solve the following simultaneous equations:

$$5x + 6y = 11$$
$$7x + 8y = 15$$

Solution

In matrix form, the system of equation becomes $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \end{pmatrix}$

Let $A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ then $|A| = 40 - 42 = -2$

Then, $A^{-1} = -\frac{1}{2}\begin{pmatrix} 8 & -6 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix}$

Pre-multiply the inverse matrix to both sides of the matrix equation to get,

$$\begin{pmatrix} -4 & 3 \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 11 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, $x = 1$ and $y = 1$.

Example 7.18

Solve the following simultaneous equations by using matrix method:

$$4x + 2y = 40$$
$$x + 3y = 35$$

Solution

The matrix of coefficients of x and y is $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

$\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = (4\times3)-(1\times2) = 12-2 = 10$

The inverse of $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}$

The matrix equivalent to the equation is $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 35 \end{pmatrix}$

Pre – multiply inverse matrix to both sides of the equivalent equation:

$$\begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix} \times \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix} \times \begin{pmatrix} 40 \\ 35 \end{pmatrix}$$

thus,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

Therefore, $x = 5$ and $y = 10$.

Example 7.19

A business person is running a grocery near a certain school. She is selling pieces of cake and soda. Each piece of cake costs 1 500 shillings and each bottle of soda costs 500 shillings. In one afternoon, the person made a total of 255 500 shillings by selling a total of 225 pieces of cake and soda. Use the given information to:

(a) Formulate the system of simultaneous equations.

(b) Transform the system of linear simultaneous equations in a matrix form.

(c) Find the inverse matrix in part (b).

(d) Use the inverse obtained in (c) to find the number of pieces of cake and bottles of soda sold.

Solution

(a) Let x denotes the number of pieces of cake sold and y denotes the number of bottles of soda sold. The system of linear simultaneous equation becomes:

$$1\,500x + 500y = 255\,500$$
$$x + y = 225$$

(b) In matrix form, the system of linear simultaneous equations becomes

$$\begin{pmatrix} 1\,500 & 500 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 255\,500 \\ 225 \end{pmatrix}$$

(c) Let $J = \begin{pmatrix} 1\,500 & 500 \\ 1 & 1 \end{pmatrix}$, then $|J| = \begin{vmatrix} 1\,500 & 500 \\ 1 & 1 \end{vmatrix} = 1\,500 - 500 = 1\,000$

Therefore, $J^{-1} = \dfrac{1}{1\,000}\begin{pmatrix} 1 & -500 \\ -1 & 1\,500 \end{pmatrix}$.

(d) Pre – multiply J^{-1} to both sides of the equivalent equation in (b)

$$\frac{1}{1\,000}\begin{pmatrix} 1 & -500 \\ -1 & 1\,500 \end{pmatrix}\begin{pmatrix} 1\,500 & 500 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1\,000}\begin{pmatrix} 1 & -500 \\ -1 & 1\,500 \end{pmatrix}\begin{pmatrix} 255\,500 \\ 225 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1\,000}\begin{pmatrix} 143\,000 \\ 82\,000 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 143 \\ 82 \end{pmatrix}$$

Therefore, 143 pieces of cake and 82 bottles of soda were sold.

Exercise 7.5

Answer the following questions:

Solve the following system of simultaneous equations for question 1 to 8 by using matrix method.

1. $7x - 2y = 29$
   $7x + y = 38$

2. $3x + 2y = 12$
   $7x - 4y = 2$

3. $10x - 3y = 36$
   $8x + 3y = 18$

4. $4x - 2y = 7$
   $4x + 5y = 40$

5. $-6x + 5y = 16$
   $7x + 6y = 44$

6. $2x - 7y - 1 = 0$
   $5x + 17y - 37 = 0$

7. $x = 5 - 6y$
   $x = 9y$

8. $8x = -2y + 13$
   $16x + y + 14 = 0$

9. A woman owns 60 domestic animals. Each of her domestic animals is either a cow or a chicken. If the domestic animals have the total of 200 legs, use matrix method to find the number of cows and the number of chickens which the woman owns.

10. The sum of ages of the father and her daughter is 52 years. Eight years ago the age of the father was 5 times that of her daughter. By using the inverse matrix method, find the age of the father and her daughter.

11. Jesca and Ally went to the market to buy tomatoes and onions. Jesca had 160 000 shillings and Ally had 208 000 shillings. Jesca bought 400 tomatoes and 600 onions. Ally bought 480 tomatoes and 800 onions. By using the inverse matrix method, find how much did they spend for each item.

Activity 7.3: Rotating an object through 90°

Resources: white paper, a ruler, and a protractor.

In a group or individually, perform the following tasks:

1. Trace a paper and copy the shapes as shown in Figure 7.1. Use the tracing paper to rotate each shape half a turn about the point marked with a dot.

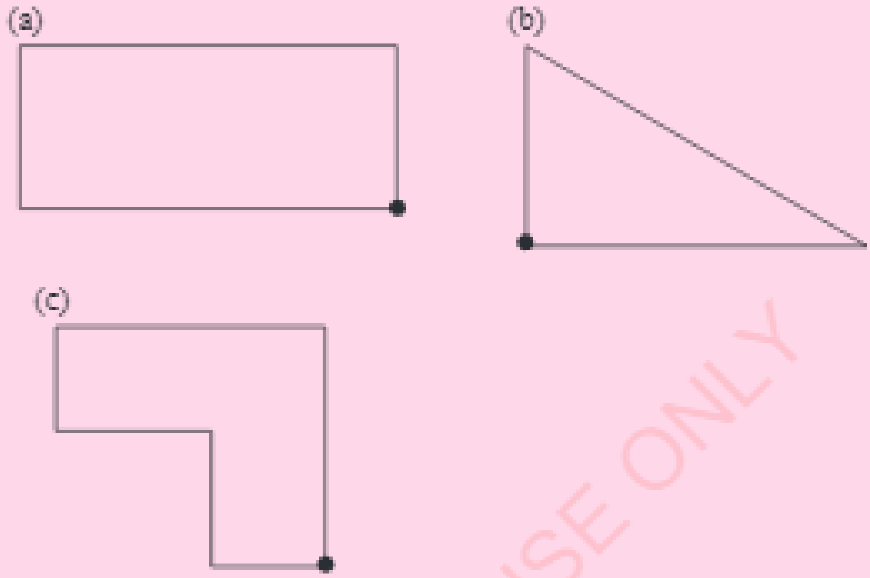


Figure 7.1: *Stationary points*

2. Rotate each of the shapes in Figure 7.1 a quarter of a turn anticlockwise about the point marked with a dot. Draw the x – axis and y – axis in the middle of a page so that x and y take values from -8 to $+8$. What have you observed?
3. Draw the triangle ABC at A(1, 3), B(1, 6), C(3, 6); rotate ABC through 90° clockwise about (0, 0) and mark A′ B′ C′.
4. Draw the triangle DEF at D(3, 3), E(6, 3), and F(6, 1); rotate DEF through 90° clockwise about (0, 0) mark D′ E′ F′.
5. Draw the triangle PQR at P(–4, 7), Q(–4, 5), R(–1, 5); rotate PQR through 90° anticlockwise about (0, 0) mark P′ Q′ R′.
6. Describe what have you observed in tasks 3 to 5.
7. Share your findings with the whole class through presentation and discussion for more inputs.

A transformation in a plane is a mapping which shifts an object from one position to another within the same plane. Examples of transformations in the xy – plane are; **reflections**, **rotations**, **enlargement**, and **translations**.

Consider a transformation T and any point P(x, y) in the xy – plane such that T maps the point P(x, y) into the point P′(x', y') as shown in Figure 7.2. Then point P′(x', y') is called the image of P(x, y) under transformation T. In Basic mathematics this mapping is denoted by T[x, y] = T′(x', y') which reads T of (x, y) is equal to ordered pair (x', y').

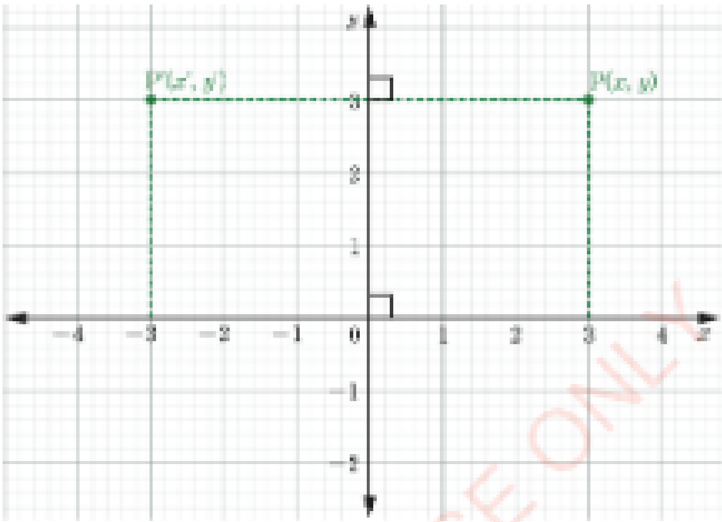


Figure 7.2: *Transformation of a point in the xy – plane*

Reflection

When looking at a mirror the real face of the observer as far as in front of the mirror the image is behind the mirror. In addition to this, the size of the image is equal to the size of the object (See Figure 7.3).

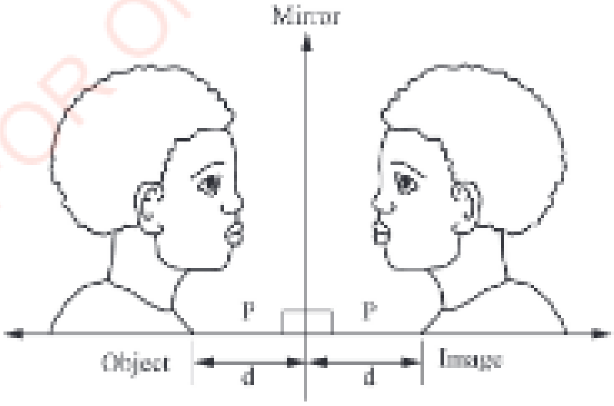


Figure 7.3: *Reflection on the mirror*

In Figure 7.3, the object is shown at a distance d from the mirror. Similarly, the image is at a distance d from the mirror and the line joining corresponding points of the image to the object crosses the mirror is at right angles. Therefore, the mirror is a perpendicular bisector of the lines joining corresponding points from the object to its image.

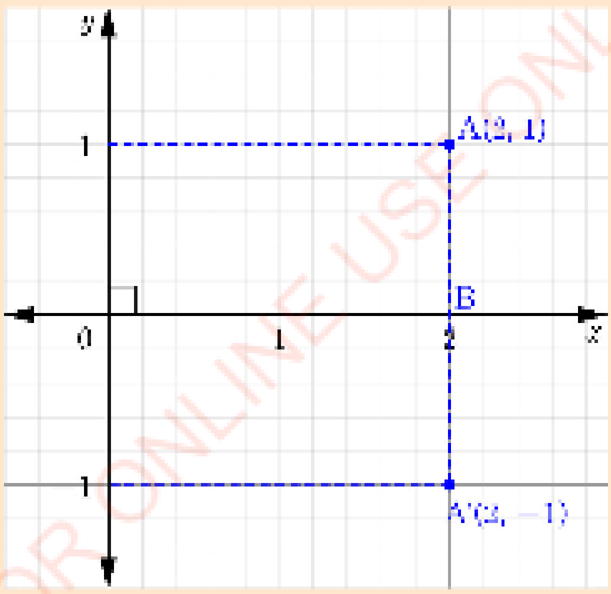
A transformation in which the size of the image is equal to the size of the object is called an isometric mapping. Reflection is an example of an isometric mapping.

Example 7.20

Find the image of the point A(2, 1) after a reflection in the x – axis.

Solution

Plot point A and its image A′ such that AA′ crosses the x – axis at right angles at B where $\overline{AB} = \overline{A'B}$ as shown in the following figure.



The figure shows that the image of A(2, 1) under a reflection in the x – axis is A′(2, –1). The notation used for a reflection is the letter M.

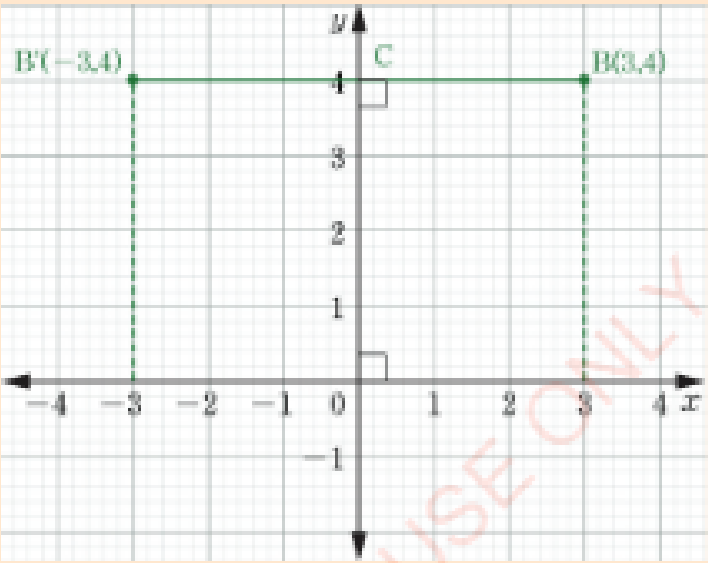
Therefore, $M_x(2, 1) = (2, -1)$ means that the reflection of A(2, 1) in the x – axis is A′(2, –1). Similarly, $M[x, y] = M'(x', y')$ means that the reflection of point (x, y) is point (x', y').

Example 7.21

Find the image of the point B(3, 4) under a reflection in the y – axis.

Solution

Refer to the following figure. Draw BB' at right angles to the y – axis at C, such that $\overline{BC} = \overline{B'C}$.



The given figure shows that the image of $(3, 4)$ under a reflection in the y – axis is $(-3, 4)$.

Therefore, $M_y[3, 4] = (-3, 4)$.

Now consider the image of the point $P(x, y)$ when reflected in the line inclined at an angle α passing through the origin. Then, $y = x\tan\alpha$ where $\tan\alpha$ is the gradient of the line (See Figure 7.4). Draw vector $\overrightarrow{OP}$ inclined at β with the coordinates of P being $P(x, y)$, $\overrightarrow{OP}'$ is the image of $\overrightarrow{OP}$ under reflection on $\overrightarrow{OS}$. By the physical laws of reflection $\overrightarrow{PP}'$ is perpendicular to the line of reflection.

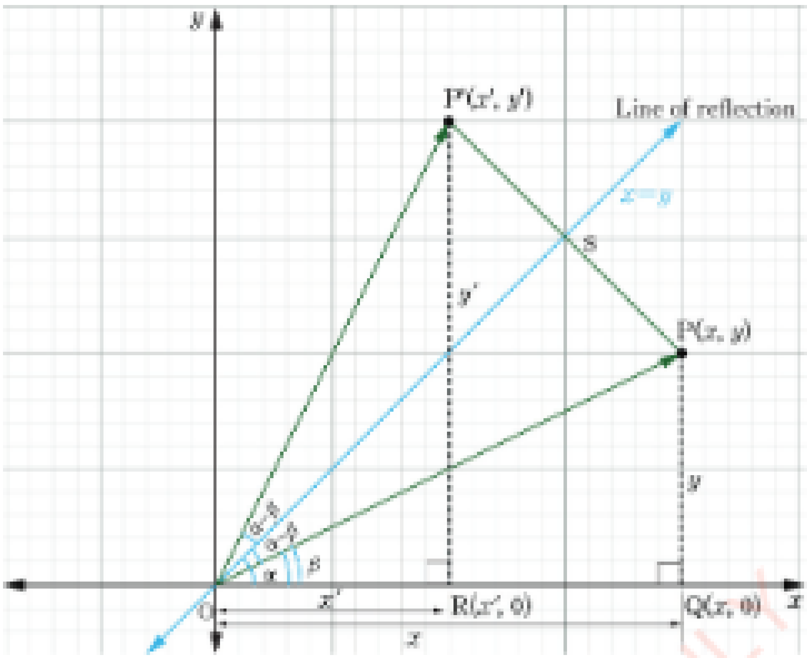


Figure 7.4: *Reflection of a line*

In Figure 7.4, $P'\hat{O}S = P\hat{O}S = \alpha - \beta$ and $\overline{OP}$ is equal to $\overline{OP'}$. Draw $\overline{PQ}$ perpendicular to the x – axis at Q. From coordinate geometry the coordinates of Q are $(x, 0)$. $\overline{OQ} = x$ and $\overline{PQ} = y$. Triangle OPQ is right – angled at Q.

Hence, $x = \overline{OP}\cos\beta$ and $y = \overline{OP}\sin\beta$. Now, draw $\overline{P'R}$ perpendicular to the x – axis at R. Then the coordinates of R are $(x', 0)$. Length of the line segment $OR = x'$ and length of $\overline{RP'} = y'$. $\Delta OP'R$ is right – angled at R.

$P'\hat{O}S = P\hat{O}S = \alpha - \beta$. Since reflection is an isometric mapping, angle

$P'\hat{O}R = \alpha - \beta + \alpha - \beta + \beta = 2\alpha - \beta$

Hence,

$x' = \overline{OP'}\cos(2\alpha - \beta)$

$= \overline{OP}\cos(2\alpha - \beta)$ since, $\overline{OP'} = \overline{OP}$

$= \overline{OP}[\cos 2\alpha\cos\beta + \sin 2\alpha\sin\beta]$

$= \overline{OP}\cos\beta\cos 2\alpha + \overline{OP}\sin\beta\sin 2\alpha$

But, $x = \overline{OP}\cos\beta$ and $y = \overline{OP}\sin\beta$

$$x' = x\cos 2\alpha + y\sin 2\alpha$$

Also, $y' = \overline{OP}\sin(2\alpha - \beta)$

$$= \overline{OP}[\sin 2\alpha\cos\beta - \cos 2\alpha\sin\beta]$$

$$= \overline{OP}\sin 2\alpha\cos\beta - \overline{OP}\cos 2\alpha\sin\beta$$

$$= \overline{OP}\cos\beta\sin 2\alpha - \overline{OP}\sin\beta\cos 2\alpha$$

But, $x = \overline{OP}\cos\beta$ and $y = \overline{OP}\sin\beta$

Therefore, $y' = x\sin 2\alpha - y\cos 2\alpha$.

This information can be summarized as follows:

If M is a reflection in the line inclined at α then $M[x, y] = (x', y')$ where $x' = x\cos 2\alpha + y\sin 2\alpha$ and $y' = x\sin 2\alpha - y\cos 2\alpha$

These two linear equations are called the transformation equations for reflection. In a matrix form these equations are written as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ is called the matrix of reflection in the line inclined at angle α passing through the origin.

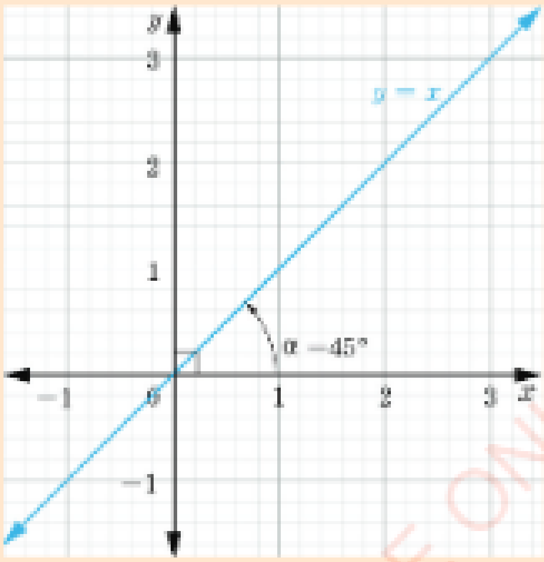
The column matrix $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under the transformation M. The column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ is equal to the position vector (x, y) and sometimes it is called column vector. Similarly, the column matrix $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is equal to the position vector (x', y').

Example 7.22

Find the image of the point $A(1, 2)$ after a reflection in the line $y = x$.

Solution

Gradient of $y = x$ is 1 (See the following figure).



Since the gradient of the line $y = x$ is 1, then $\tan\alpha = 1$, which gives $\alpha = 45°$.

Hence, $M = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

$$M_{y=x} = \begin{pmatrix} \cos 90° & \sin 90° \\ \sin 90° & -\cos 90° \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Given $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0+2 \\ 1+0 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore, the image of the point $A(1, 2)$ is $A'(2, 1)$.

Example 7.23

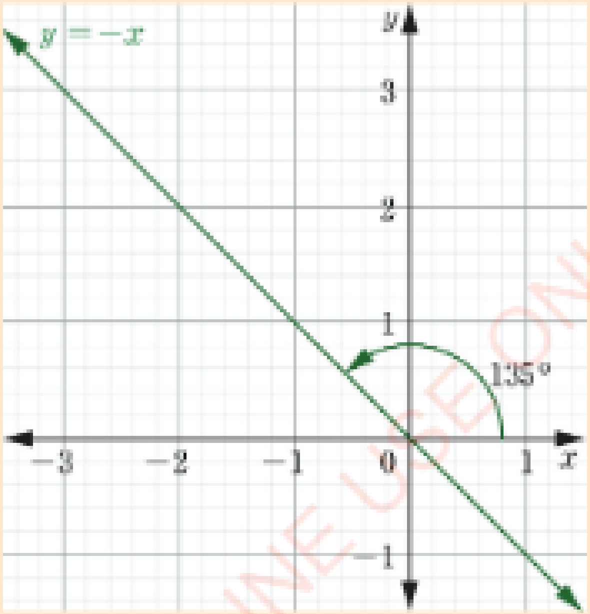
Find the image of the point B(3, 4) after a reflection in the line $y = -x$ followed by another reflection in the line $y = 0$.

Solution

Let $M_{y=-x}$ be the reflection in the line $y = -x$ and M_x be the reflection in the line $y = 0$.

Then, $M_{y=-x} = (x', y')$ and $M_x[x', y'] = (x'', y'')$

The gradient of the line $y = -x$ is -1



Then $\tan\alpha = -1$ which gives $\alpha = 135°$ as shown in the given figure.

$$M_{y=-x} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} \cos 270° & \sin 270° \\ \sin 270° & -\cos 270° \end{pmatrix}$$

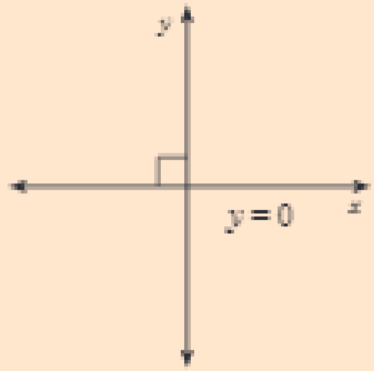
$$M_{y=-x} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

Hence, $(x', y') = (-4, -3)$.

The gradient of the line $y = 0$ is 0.

Then $\tan\alpha = 0$ which gives $\alpha = 0°$ as shown in the following figure.



$$M_x = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} \cos 0° & \sin 0° \\ \sin 0° & -\cos 0° \end{pmatrix}$$

Thus, $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

Hence, $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Therefore, $(x'', y'') = (-4, 3)$.

Alternatively: $(x'', y'') = M_x M_{y=-x}(x, y)$ can be interpreted in terms of matrix multiplication by the equation: $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$

Where $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

In which case the same result is obtained:

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Therefore, $(x'', y'') = (-4, 3)$. Note that the matrix product $M_x M_{y=-x}$ is interpreted as $M_{y=-x}$ followed by M_x and not vice versa. Hence, the reflection M_1 followed by M_2 is equivalent to the matrix product M_2M_1.

Example 7.24

Find the equation of the image of the line $y = 2x + 5$ after a reflection in the line $y = x$.

Solution

Choose any two points on the line $y = 2x + 5$, P(0, 5) and $Q\left(-\frac{5}{2}, 0\right)$ respectively. Reflection in the line $y = x$ is governed by the matrix

$$M = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \quad \text{where} \quad \alpha = 45°$$

$$= \begin{pmatrix} \cos 90° & \sin 90° \\ \sin 90° & -\cos 90° \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let $P'(x', y')$ and $Q'(x'', y'')$ be the image of $P(0, 5)$ and $Q\left(-\frac{5}{2}, 0\right)$ respectively.

Then, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Hence, $(x', y') = (5, 0)$

and $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} -\frac{5}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{5}{2} \end{pmatrix}$

Hence, $(x'', y'') = \left(0, -\frac{5}{2}\right)$

The image of the line passes through the points $(5, 0)$ and $\left(0, -\frac{5}{2}\right)$.

The equation of image of the line is:

$$\frac{y - 0}{x - 5} = \frac{0 - \left(-\frac{5}{2}\right)}{5 - 0}$$

Then, $\frac{y}{x - 5} = \frac{\frac{5}{2}}{5}$

$$\frac{y}{x - 5} = \frac{1}{2}$$

$$y = \frac{1}{2}x - \frac{5}{2}$$

Therefore, the image of the line $y = 2x + 5$ after a reflection in the line $y = x$ is

$y = \frac{1}{2}x - \frac{5}{2}$.

Exercise 7.6

Answer the following questions:

In questions 1 to 5 write the matrix of reflection in the given line:

1. $y = 0$ (the x – axis).
2. $y = x$.
3. $x = 0$ (the y – axis).
4. $y = -x$.
5. l such that l is inclined at 30° with respect to the positive x – axis.
6. Reflect the point (5, 2) in the line $y = 0$.
7. Reflect the point (5, 2) in the line $y = x$.
8. Reflect the point (1, 2) in the y – axis.
9. Reflect the point (1, 2) in the line $y = -x$.
10. Find the image of the point (3, 4) after a reflection in the line inclined at 30°.
11. Find the image of the point (1, 2) after a reflection in the line $y = x$ followed by another reflection in the line $y = -x$.
12. Draw ΔABC and its image $\Delta A'B'C'$ after a reflection in the line $y = x$ if A(0, 3), B(3, 0), and C(3, 2). What is the line of symmetry of the two figures?
13. Find the image of:

    (a) the line $y = -x$ after a reflection in the y – axis.

    (b) the line $3x + 4y + 6 = 0$ after a reflection in the line $y = -x$.

Rotation

The transformation which shifts a vector by turning it through an angle θ about a fixed point O is called a rotation. This is another example of isometric mapping. The rotation by an angle θ about the origin will be denoted by R_θ. When measured in the anticlockwise direction θ is positive and when measured in the clockwise direction θ is negative.

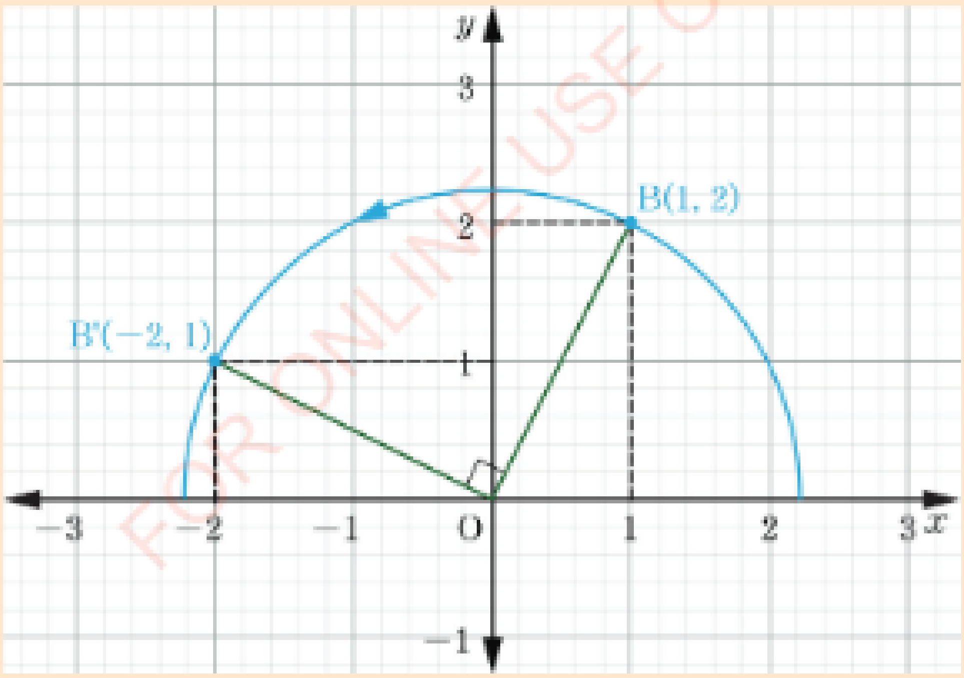
Consider the following example:

Example 7.25

Find the image of the point $B(1, 2)$ after a rotation by 90° about the origin in the anticlockwise direction.

Solution

Use graphical method to locate $B'(x', y')$ by measuring $B\hat{O}B' = 90°$ maintaining $\overline{OB} = \overline{OB'}$ as a radius as shown in the following figure.



Therefore, the image of the point $B(1, 2)$ under $R_{90°}$ is $B'(-2, 1)$.

A more generalized way of obtaining the image (x', y') of a point (x, y) under R_θ is derived as follows:

Suppose $P(x, y)$ be any point in the xy – plane, find $P'(x', y')$ as the image of $P(x, y)$ after a rotation by θ about $O(0, 0)$ as shown in Figure 7.5.

In Figure 7.5, let $\overrightarrow{OP}$ be inclined at angle β.

Note: $|\overrightarrow{OP}| = |\overrightarrow{OP}'|$ because point P and point P′ lie on the arc of a circle having centre O, with radius $\overline{OP}$.

$|\overrightarrow{OP}| = |\overrightarrow{OP}'|$ explains that rotation is an isometric mapping.

Draw $\overline{PA}$ perpendicular to the x – axis at A then the coordinates of A are $(x, 0)$, therefore, $\overline{OA} = x$ and $\overline{AP} = y$.

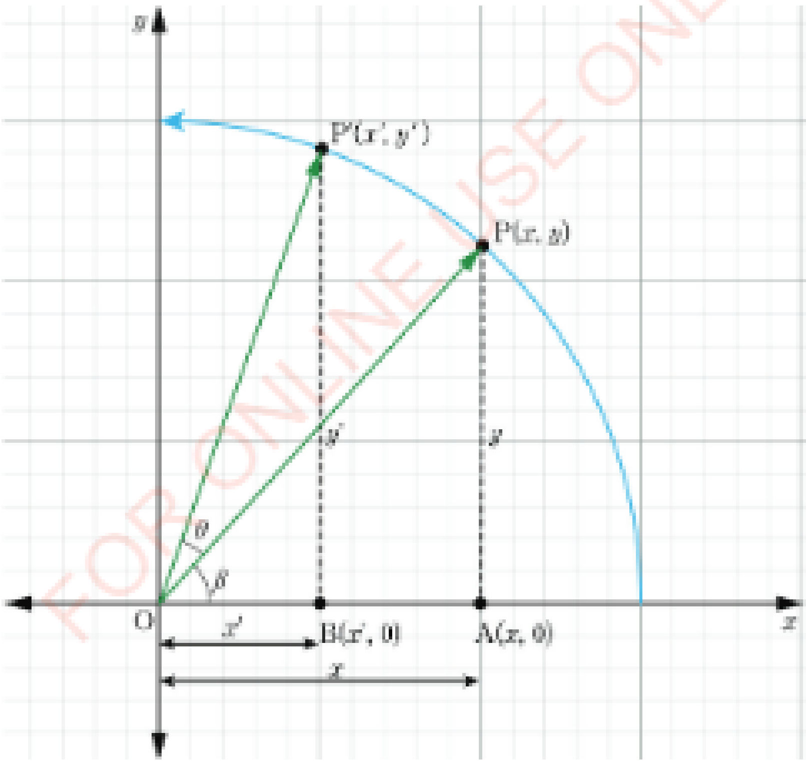


Figure 7.5: *Rotation of a point*

Triangle OAP is a right – angled triangle at A with $P\hat{O}A = \beta$ as shown in Figures 7.5 and 7.6.

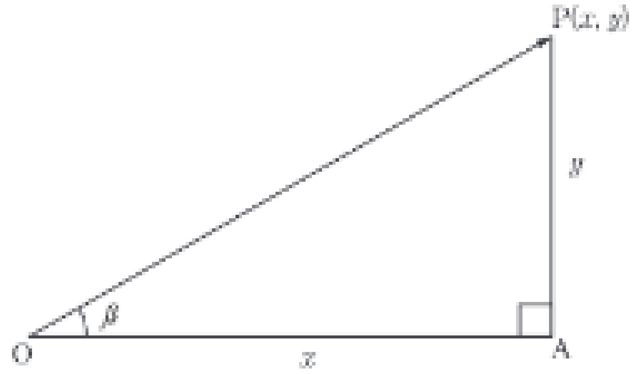


Figure 7.6: *Triangle with point P to be rotated*

Hence, $\frac{x}{|\overrightarrow{OP}|} = \cos\beta$ and $\sin\beta = \frac{y}{|\overrightarrow{OP}|}$

$x = |\overrightarrow{OP}|\cos\beta$ and $y = |\overrightarrow{OP}|\sin\beta$

Again draw $\overline{P'B}$ perpendicular to the x – axis at B. Then, the coordinates of B are $(x', 0)$, hence, $\overline{OB} = x'$ and $\overline{BP'} = y'$.

$\Delta OBP'$ is a right – angled at B with $P'\hat{O}B = \beta + \theta$. (See Figures 7.5 and 7.7).

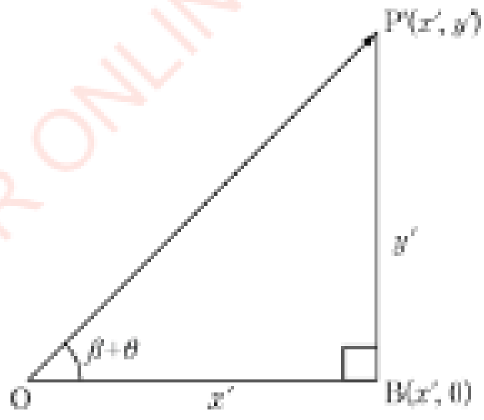


Figure 7.7: *Triangle with a rotated point P*

According to Figure 7.5 and 7.7:

$|\overrightarrow{OP'}| = |\overrightarrow{OP}| = \text{radius}$

$\overrightarrow{OB} = x', \ \overrightarrow{BP'} = y'$

$P'\hat{O}B = \beta + \theta$

Hence, $x = |\overrightarrow{OP'}| \cos(\beta + \theta)$

$= |\overrightarrow{OP}| \cos(\beta + \theta)$

$= |\overrightarrow{OP}| (\cos\beta\cos\theta - \sin\beta\sin\theta)$

$= |\overrightarrow{OP}| \cos\beta\cos\theta - |\overrightarrow{OP}| \sin\beta\sin\theta$

But, $|\overrightarrow{OP}| \cos\beta = x$ and $|\overrightarrow{OP}| \cos\beta = y$

Hence, $x' = x\cos\theta - y\sin\theta$

Similary $y' = |\overrightarrow{OP'}| \sin(\beta + \theta)$

$= |\overrightarrow{OP}| \sin(\beta + \theta)$

$= |\overrightarrow{OP}| (\sin\beta\cos\theta + \cos\beta\sin\theta)$

$= |\overrightarrow{OP}| \sin\beta\cos\theta + |\overrightarrow{OP}| \cos\beta\sin\theta$

But, $|\overrightarrow{OP}| \sin\beta = y$ and $|\overrightarrow{OP}| \cos\beta = x$

Hence, $y' = y\cos\theta + x\sin\theta$

$y' = x\sin\theta + y\cos\theta$

Generally, the rotation of a point (x, y) through angle θ about the origin is given by:

$$R_\theta[x, y] = (x', y') \quad \text{where } x' = x\cos\theta - y\sin\theta, \ y' = x\sin\theta + y\cos\theta.$$

The matrix equivalent equation is:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$$

Where $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, through angle θ about the origin.

Example 7.26

Find the image of the point $(1, 2)$ under a rotation through 180° anticlockwise.

Solution

$R_{180°}[x, y] = (x', y')$ where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Therefore, $(x', y') = (-1, -2)$.

Example 7.27

Find the image of the point (5, 2) under a rotation of 90° followed by another rotation of 180° anticlockwise.

Solution

$R_{180°}R_{90°}(x, y) = (x', y')$ first obtain $(x', y') = R_{90°}[x, y]$.

Then obtain $(x'', y'') = R_{180°}[x', y']$

Now, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

Then $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Therefore, $R_{180°}R_{90°}[(5,2)] = (2, -5)$.

Exercise 7.7

Answer the following questions:

In questions 1 to 6, find the matrix of rotation through.

1. 90° about the origin.
2. –90° about the origin.
3. 270° about the origin.
4. 360° about the origin.
5. 45° about the origin.
6. 180° about the origin.

In questions 7 to 10, find the image of the point (3, 5) after a rotation through:

7. 90° about the origin.
8. –180° about the origin.
9. 270° about the origin.
10. 360° about the origin. What transformation is this?
11. Find the image of the point (1, 2) after a rotation through 90° followed by another rotation by 270° about the origin.
12. Find the image of the point (3, 4) after a rotation about –90° followed by another rotation by –180° about the origin.
13. Find the matrix of rotation in line $y = x$ about the origin.

Translation

Consider a triangle OPQ in Figure 7.8. The transformation T which maps triangle OPQ into triangle O′P′Q′ by moving it 2 units in the positive x direction and 3 units in the positive y direction is called a translation by the vector $(2,3)$. Under this transformation the object size is equal to that of the image.

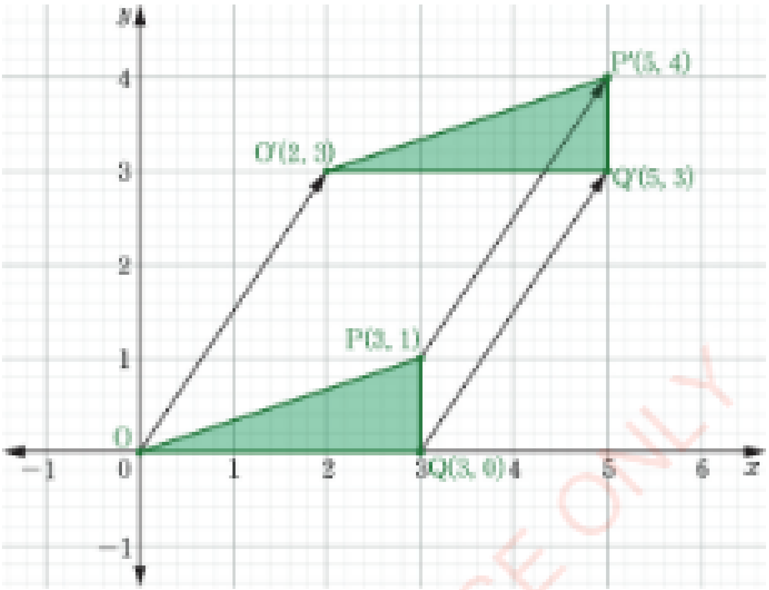


Figure 7.8: *Translation of object by a vector*

The vector $\overrightarrow{OO'} = (2, 3)$ is called the vector of translation.

In this particular example, $T'[(0,0)] = (2,3)$, $T'[(3,0)] = (5,3)$ and $T'[(3,1)] = (5,4)$.

Generally, if T is a translation by a fixed vector (a,b) then T maps every point (x,y) into (x',y') where $(x',y') = (x,y) + (a,b)$.

In a matrix notation this equation is written as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

Example 7.28

If T is a translation by the vector (4, 3), find the image of (1, 2)under this translation.

Solution

$T[(x,y)] = (x',y')$ where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ given, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Therefore, $(x',y') = (5, 5)$.

Example 7.29

A translation T maps the point (–3, 2) into (4, 3). Find where T maps:

(a) the origin (b) the point $(7, 4)$.

Solution

$T[(x,y)] = (x',y')$ then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where $\begin{pmatrix} a \\ b \end{pmatrix}$ is a vector of translation for (a, b) $\in \mathbb{R}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix}$$

Given $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

The fixed vector of translation is $\begin{pmatrix}7\\1\end{pmatrix}$.

Therefore,

(a) $T[(0,0)]=(x',y')$ where $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}7\\1\end{pmatrix}=\begin{pmatrix}7\\1\end{pmatrix}$.

(b) $T[(7,4)]=(x',y')$ where $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}+\begin{pmatrix}7\\1\end{pmatrix}=\begin{pmatrix}14\\5\end{pmatrix}$.

Exercise 7.8

Answer the following questions:

1. A translation takes the origin to the point (–2, 5). Find where it takes the point:
   (a) (–6, 6) (b) (–5, 4)

2. A translation takes every point a distance 1 unit to the left and 2 units downwards. Find where it takes:
   (a) the origin
   (b) the point (1, 1)
   (c) the point (3, 7)

3. A translation T maps the point (3, 2) into the point (–4, –5). Find where T maps:
   (a) (0, 0) (b) (5, 5)
   (c) (9, 4)

4. A translation T takes the origin to the point (3, –2) and a second translation S takes the origin to the point (–2, 1). Find where:
   (a) T followed by S takes the origin.
   (b) S followed by T takes the origin.
   (c) T followed by T takes the origin.

5. A translation T takes the origin to the point (–2, 3). Find a second translation S such that the composition ST leaves the origin fixed.

6. A translation moves the origin a distance 2 units along the line $y=-x$ upwards. Find where it takes:
   (a) the point (0, 0)
   (b) the point (2, –1)
   (c) the point (1, 1).

7. If a translation T takes the origin to the point (8, 7). Given, $U=(-12,12)$, $V=(6,-16)$, find:
   (a) $T[U+V]$
   (b) $T[U]+T[V]$

8. Find T[1, 0], T[0, 1], and T[1, 1] if T[10, 0] = (9, 8).

9. Find the image of the line $3x+4y+6=0$ under a translation by the vector (–6, –1).

Enlargement

The transformation which magnifies an object such that its image is proportionally increased or decreased in size by some factor is called enlargement. The general matrix of enlargement is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ where k is a non – zero real number called the linear scale factor. This transformation is non–isometric.

Generally, the enlargement of the point (x, y) by the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is given by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$.

Example 7.30

Find the image of the square with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$, and $C(0, 1)$ under the enlargement matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

Hence, sketch both the object and the image on the same pair of axes.

Solution

By using the formula $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}$ where $k = 4$.

$$O' : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ hence, } O'(0,0) = O(0,0).$$

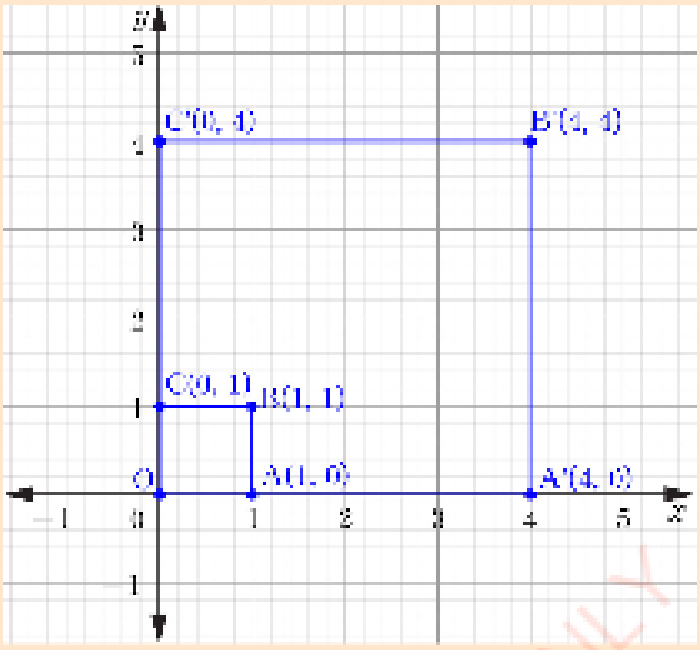
$$A' : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \text{ hence, } A'(4,0).$$

$$B' : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \text{ hence, } B'(4,4).$$

$$C' : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \text{ hence, } C'(0,4).$$

Hence, the image is the square with vertices $O'(0, 0)$, $A'(4, 0)$ $B'(4, 4)$, and $C'(0, 4)$.

Therefore, the following is a sketch of both the object and the image:



Example 7.31

Find the image of the point $(6, 9)$ under enlargement by the matrix $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$.

Solution

By using the formula $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ where $k = \frac{1}{3}$ and $(x, y) = (6, 9)$ gives

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore, $(x', y') = (2, 3)$.

Exercise 7.9

Answer the following questions:

1. Find the image of the point (1, 2) under enlargement by the matrix $T = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

2. Find the image of the point (4, 4) under enlargement by the matrix $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.

3. Find the image of the point $\left(-\frac{1}{2}, -\frac{1}{3}\right)$ under enlargement by the matrix $T = \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix}$.

4. Suppose T is a matrix of enlargement by factor k. Find the matrix S such that $ST[(x, y)] = (x', y')$.

5. Find the enlargement matrix which maps the point (–3, 4) into the point (18, –24).

6. Sketch the image of the triangle having vertices A(1, 2), B(3, 4), and C(3, 8) under the enlargement matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

7. Draw the image of the unit circle of unit radius and centre O(0, 0) under the enlargement matrix $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

8. Find the matrix which enlarges the vector $r = (3, 4)$ to the vector $r' = (15, 20)$.

9. Find the image of each of the following points under enlargement by matrix $T = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$:

   (a) (1, 2)

   (b) (3, 4).

Chapter summary

1. Types of matrices:

(a) Row matrix: This is a matrix having only one row.

For example, $N = (2 \quad 5)$.

(b) Column matrix: This is a matrix having only one column.

For example, $N = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(c) Null matrix: This is a matrix with all zero elements.

For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(d) Square matrix: This is a matrix having the same number of rows and columns. For example, $G = \begin{pmatrix} 4 & 6 \\ 2 & -8 \end{pmatrix}$.

(e) Diagonal matrix: This is a square matrix in which all the elements are zero except the elements in the leading diagonal. For example,

$V = \begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$.

(f) Unit matrix or Identity matrix: This is a diagonal or square matrix in which the diagonal elements in the leading diagonal equal to 1.

For example, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(g) Scalar matrix: This is a square matrix whose elements in the main diagonal are the same and the rest of the elements are zero.

For example, $R = \begin{pmatrix} -21 & 0 \\ 0 & -21 \end{pmatrix}$.

2. Addition of Matrices: Two matrices A and B may be added provided they are of the same order. Generally,

If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} f & h \\ g & i \end{pmatrix}$, then

$$A + B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} f & h \\ g & i \end{pmatrix} = \begin{pmatrix} a+f & c+h \\ b+g & d+i \end{pmatrix}.$$

3. Subtraction of matrices:

   Two matrices A and B may be subtracted provided they are of the same order. Generally;

   If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} f & h \\ g & i \end{pmatrix}$, then $A - B = \begin{pmatrix} a-f & c-h \\ b-g & d-i \end{pmatrix}$.

4. Multiplication of matrices:

   Two matrices A and B may be multiplied if the number of columns of matrix A is equal to the number of rows in matrix B. Generally;

   If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} f & h \\ g & i \end{pmatrix}$; then

   $$AB = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} f & h \\ g & i \end{pmatrix} = \begin{pmatrix} af+cg & ah+ci \\ bf+dg & bh+di \end{pmatrix}.$$

5. Identity matrix for multiplication:

   A non zero matrix A remains unchanged when it is multiply by an identity matrix.

   If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then AI = IA = A, where A is any non zero matrix and I is the identity matrix.

6. Determinant of a matrix, if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $|A| = ad - bc$.

7. (a) Singular matrix: This is the matrix whose determinant is equal to zero.

   (b) Non – singular matrix: This is the matrix whose determinant is not equal to zero.

   (c) The inverse of a matrix A is denoted by A^{-1}.

   Generally, $A^{-1}A = AA^{-1} = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

8. Forms of simultaneous equations in matrix.

   If $\begin{cases} ax + cy = e \\ bx + dy = f \end{cases}$, then the equivalent matrix form is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$.

9. Transformation equation for reflection matrix is given as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}.$$

10. Transformation equation for rotation matrix is given as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix}.$$

11. Transformation equation for translation matrix by a vector (a, b) is given as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

12. Transformation equation for enlargement matrix is given as:

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & o \\ o & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ where k is a non zero real number.

Revision exercise 7 (a)

Answer the following questions:

1. Find the sum of the matrices:

$B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$

2. Find the values of x and y in each of the following:

(a) $\begin{pmatrix} x & 2y \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -8 \\ 0 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} x+3 \\ 2-y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} 2x-y \\ 0+y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3. Given matrices $A = \begin{pmatrix} 5 & 6 \\ 8 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -6 \\ -8 & -7 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$. Find :

   (a) $A + (B + C)$ and $(A + B) + C$

   (b) $A + B$ and $B + A$

   (c) What is the relationship between matrices A and B ?

4. Using the matrices in question 3, find:

   (a) AB and BA (b) A(BC) and (AB)C

5. Given matrices $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   Complete the following multiplication table:

×	A	B	C	D
A			D	A
B			A	
C			B	
D			C	

6. Find the determinant of each of the following matrices:

   (a) $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ (b) $B = \begin{pmatrix} 5 & 2 \\ 5 & 3 \end{pmatrix}$ (c) $C = \begin{pmatrix} 6 & 1 \\ 2 & 2 \end{pmatrix}$

   (d) $D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (e) $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

7. Indicate whether the matrix has an inverse or not for each of the following matrices:

   (a) $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (b) $Q = \begin{pmatrix} 1 & 2 \\ 9 & 8 \end{pmatrix}$ (c) $R = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$

   (d) $S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ (e) $T = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$

8. Find the inverse of the matrices in question 6.

9. Which matrices are singular among the following?

(a) $F=\begin{pmatrix}2 & 3\\ 3 & 4\end{pmatrix}$ (b) $G=\begin{pmatrix}0 & 0\\ 4 & 5\end{pmatrix}$ (c) $H=\begin{pmatrix}1 & 3\\ 2 & 6\end{pmatrix}$

(d) $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (e) $J=\begin{pmatrix}4 & 2\\ 6 & 3\end{pmatrix}$ (f) $K=\begin{pmatrix}3 & 15\\ 4 & 20\end{pmatrix}$

(g) $L=\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix}$

10. Use matrix method to solve each of the following pairs of simultaneous equations:

(a) $3x-4y=2$, $x+2y=3$ (b) $-2y+2x=7$, $4x+5y=2$ (c) $2x+5y=9$, $x+3y=5$

(d) $3x+y=6$, $5x+2y=11$ (e) $3x-2y=12$, $2y+6x=10$ (f) $2x+3y-2=0$, $8x-9y-1=0$

11. Use matrix method to solve each of the following pairs of simultaneous equations:

(a) $2x+y=7$, $4x+3y=17$ (b) $x+2y=0$, $3x+5y=1$ (c) $2x+3y=3$, $x+3y=0$

(d) $4x-y=-5$, $7x-3y=-5$ (e) $10x+5y+3=0$, $5x+10y+9=0$ (f) $2x+3y-2=0$, $8x-9y-1=0$

12. Use matrix method where possible to solve each of the following simultaneous equations:

(a) $3x+4y=8$, $2x+3y=13$ (b) $2x+3y=18$, $3x+5y=29$ (c) $2x-3y=4$, $2x+3y=6$

(d) $27x-24y=9$, $18x-16y=6$

13. In a class, there are ten more boys than girls. If there were one more girl, there would be twice as many boys as girls. Use matrix method to find the number of boys and girls in the class.

14. A student finds that it is possible to buy 12 pencils and 10 rulers for TSh 210. Alternatively, it is possible to buy 20 pencils and 4 rulers for TSh 160 at the same price per each unit item. What are the unit prices per pencil and ruler? (Use matrix method).

15. Use matrix method to solve each of the following pairs of simultaneous equations:

(a) $2x+6y=13$
$8x+y=6$

(b) $x+3y=5$
$7x-6y=44$

16. Use matrix method to solve each of the following pairs of simultaneous equations:

(a) $\frac{2}{5}x+\frac{1}{3}y=1$
$\frac{3}{5}x-\frac{1}{9}y=7$

(b) $-7x+12y=11$
$50x+9y=47$

(c) $6x-5y=3$
$3x+11y=39$

(d) $3x-5y=21$
$x-4y=3$

17. If matrices $A=\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}$, $B=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$, and $C=\begin{pmatrix}8 & 0\\-4 & 2\end{pmatrix}$, find:

(a) $4A+2B-6C$

(b) $-4A-2B+6C$

18. If $A=\begin{pmatrix}7 & -9\\4 & -5\end{pmatrix}$ and $B=\begin{pmatrix}5 & 2\\5 & 4\end{pmatrix}$, find:

(a) $((A^{-1})^{-1})^{-1}$

(b) $(AB)^{-1}$

19. If $A=\begin{pmatrix}6 & 0\\0 & 1\end{pmatrix}$, find:

(a) A^2

(b) A^3

20. Find the unknowns in each of the following:

(a) $\begin{vmatrix}2x & 0\\0 & -4x\end{vmatrix}=-128$

(b) $A=\begin{pmatrix}6 & 2t\\t & 0\end{pmatrix}$ and $|A|=-50$

(c) $B=\begin{pmatrix}3 & y-1\\y+1 & 1\end{pmatrix}$ and B is singular

(d) $C=\begin{pmatrix}t & 1\\t-25 & 1-t\end{pmatrix}$ where C is a singular matrix

Revision exercise 7 (b)

Answer the following questions:

1. Find the images of (a, b) under reflection in the line $y = x$.

2. Using the general matrix of reflection, find the matrices of reflection in:
   (a) the x– axis (b) $y = x$
   (c) the y – axis (d) $y = -x$

3. Find the image of the point (5, 6) under a reflection in the line $y = x$.

4. Find the image of the point (5, 6) under a reflection in the line $y = x$. followed by another reflection in the line $y = -x$.

5. Find $R_{90°}[i]$ and $R_{90°}[j]$ where $R_{90°}$ stands for a rotation of 90° anticlockwise about the origin.

6. Using the general matrix of rotation find the matrices of rotation through:
   (a) 90° (b) –180°
   (c) 270° (d) 360° about the origin

7. Find the image of the point (6, 5) under a rotation through 180° about the origin.

8. Find the image of the point (6, 5) under a rotation through:
   (a) 90° followed by another rotation of 180°.
   (b) 180° followed by another rotation of –90°.

9. Write the general matrix of enlargement.

10. Find the image of the point D(–6, 2) under the enlargement by a matrix
$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

11. What is the image of $(1, 2)$ under the transformation matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ followed by $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$?

12. Find the image of the circle of radius 1 unit having its centre at $(1, 1)$ under an enlargement transformation by factor 5. Hence, draw the circle and its image on the same xy – plane.

13. Find the transformation matrix S which will not change the size of the unit circle after a transformation by the matrix $T = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.

14. A translation T takes the point $(3, 5)$ to the point $(5, 3)$. Find where it takes:
    (a) the point $(5, 5)$
    (b) the point $(3, 3)$

15. A translation by the vector (3, 4) is applied to the coordinates of the vertices O(0, 4), A(4, 0), and B(4, 5) of a right – angled triangle OAB. Find the coordinates of the images of these vertices, hence sketch both $\Delta O'A'B'$ and ΔOAB on the same xy – plane.

16. A translation T shifts every point in the plane by 4 units upwards along the line $y = -x$. Find the image of:
    (a) the origin
    (b) the point (4, 4) under this translation

17. Find the image of the line $2x - 7y + 9 = 0$ under:
    (a) a reflection in the x – axis.
    (b) a rotation by 180° clockwise about the origin.
    (c) a translation by the vector (1, 2).

Project 7

1. Draw a triangle with vertices A(1, 3), B(1, –1), and C(–2, –1) on the graph paper.
   (a) Move all points 4 units in the positive direction parallel to the x – axis followed by 2 units in the positive direction parallel to the y – axis.
   (b) Draw the image of the triangle after translation and name it $A'B'C'$.

2. On the graph paper draw the coordinates on xy – axis:
   (a) plot the points P(–2, 1), Q(0, 1), R(1, 0), S(3, 4), and T(–2, 4).
   Join them in order to form the closed shape PQRST.
   (b) draw the image of PQRST after a reflection in the x – axis.

3. Describe fully the transformation which maps shape A into shape B.

4. Choose any four coordinates ABCD in the xy – plane such that, if you join them the result is a rhombus ABCD; What is the name of a shape formed when the coordinates of the vertices ABDC are reflected in the line $y = -x$.

Chapter Eight

Linear programming

Introduction

Linear programming is a branch of Mathematics which refers to the analysis of mathematical problems in which a linear function (objective function) of several variables is to be optimized when the variables are subject to a number of constraints in the linear inequalities. In this chapter, you will learn how to formulate and solve system of simultaneous equations graphically, formulate and find solution set of simultaneous linear inequalities graphically, formulate an objective function from word problems, locate corner points of the feasible region, find the minimum and maximum values using the objective function, and applying linear programming in solving real life problems. The competencies developed will be applied in various fields such as business, manufacturing, transportation, investment planning, and optimum allocation of resources in manufacturing of goods, and many other applications.

Simultaneous equations

Activity 8.1: Solving simultaneous equations graphically

In a group or individually, perform the following tasks:

Consider two numbers such that five times the first number plus two times the second number is twenty, and the first number minus six times the second number is twelve. Find graphically the numbers by using the following steps:

1. Identify the variables (unknowns) from the word problem by letters.
2. Write system of linear equations with two unknowns.
3. Find x and y–intercepts.
4. Plot intercepts of the first line on the xy – plane and join them using a ruler and do the same for the second line.

5. Determine the coordinates of the point of intersection, if the lines intersect. This offers the solution of simultaneous equations.

6. Share your findings with your friends for more inputs.

Note: If the lines do not intersect, they are parallel. In this case, the corresponding simultaneous equations have no solution.

Example 8.1

Solve the following system of simultaneous equations using graphical method

$2x - y = 1$

$3x + 3y = 6$

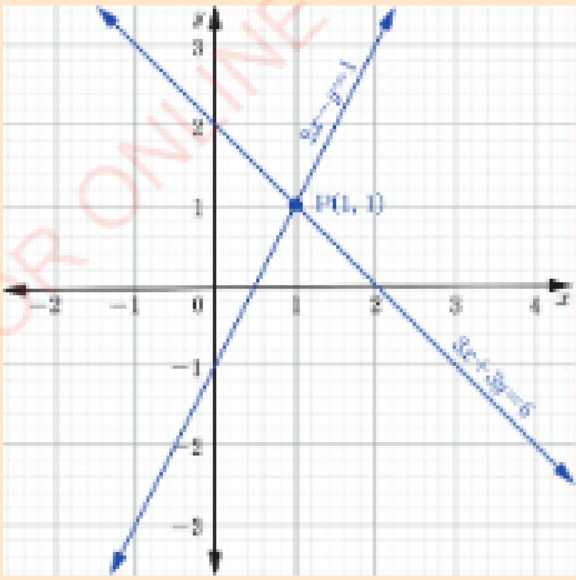
Solution

Determine where the lines intercept the coordinate axes.

$2x - y = 1$ intercepts the coordinate axes at $\left(\frac{1}{2}, 0\right)$ and $(0, -1)$.

$3x + 3y = 6$ intercepts the coordinate axes at $(0, 2)$ and $(2, 0)$.

The following figure shows the solution of the system:



The lines intersect at point P(1, 1).

Therefore, $(x, y) = (1, 1)$ is the solution to the given system of simultaneous equations, which is: $x = 1$ and $y = 1$.

Check this solution by substituting the values in each of the given equations.

Substituting $x = 1$ and $y = 1$ into the equation $2x - y = 1$, gives:

$$2(1) - 1 = 1$$
$$2 - 1 = 1$$
$$1 = 1 \text{ (true)}$$

Substituting $x = 1$ and $y = 1$ into the equation $3x + 3y = 6$, gives:

$$3(1) + 3(1) = 6$$
$$3 + 3 = 6$$
$$6 = 6 \text{ (true)}$$

Another way of checking the solution obtained using the graphical method is by using elimination or substitution method. Then, for the system of simultaneous equations in example 8.1, this can be done by using elimination method as follows:

$$2x - y = 1 \qquad (1)$$
$$3x + 3y = 6 \qquad (2)$$

Multiply equation (1) by 3 and equation (2) by 1 to make the coefficients of y in equation (1) and in equation (2) equal. We have the following equations:

$$6x - 3y = 3 \qquad (3)$$
$$3x + 3y = 6 \qquad (4)$$

To eliminate y, add equation (3) and equation (4) because the signs of the coefficient are different, which gives:

$$9x = 9$$

Hence, $x = 1$.

Similarly, multiply equation (1) by 3 and equation (2) by 2 to make coefficients of x in equation (1) and equation (2) equal.

$$6x - 3y = 3 \qquad (5)$$
$$6x + 6y = 12 \qquad (6)$$

To eliminate x, subtract equation (6) from equation (5) because the signs of the coefficients are the same, the resulting equation will be:

$$-9y = -9$$

Hence, $y = 1$.

Therefore, $(x, y) = (1, 1)$ as it can be read from the graph.

Example 8.2

Solve the following system of simultaneous equations graphically and check your answer using the substitution method.

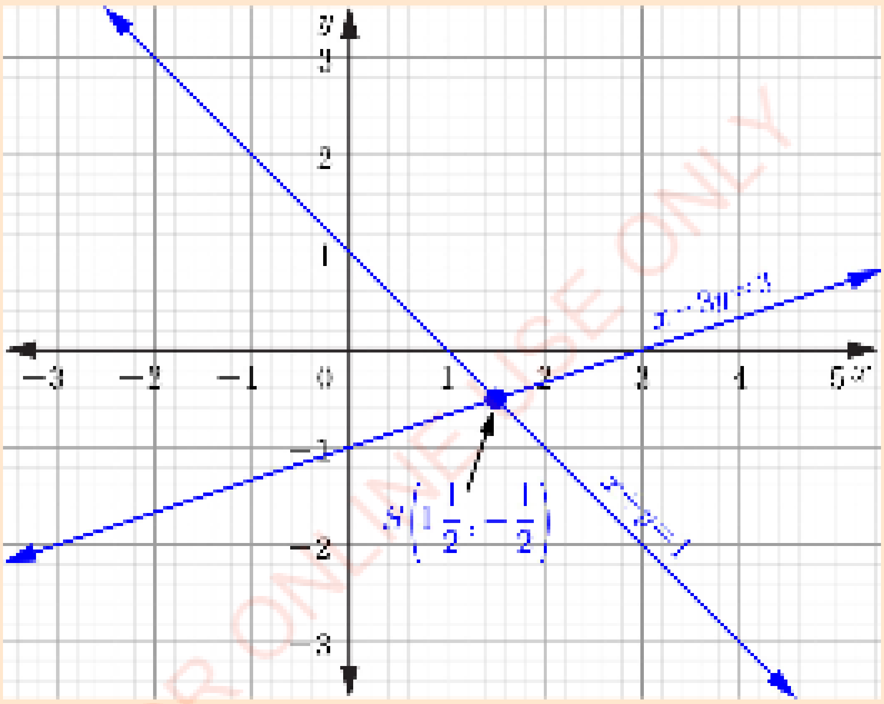
$$x + y = 1$$
$$x - 3y = 3$$

Solution

$x + y = 1$ intercepts the coordinate axes at $(0, 1)$ and $(1, 0)$.

$x - 3y = 3$ intercepts the coordinate axes at $(0, -1)$ and $(3, 0)$.

Draw the lines on xy – plane as shown in the following figure:



The lines intersect at a point $S\left(1\frac{1}{2}, -\frac{1}{2}\right)$.

Therefore, $x = 1\frac{1}{2}$ and $y = -\frac{1}{2}$ is the solution of the given system of simultaneous equations.

Check the solution $(x, y) = \left(1\frac{1}{2}, -\frac{1}{2}\right)$ by using the substitution method.

$$x + y = 1 \qquad (1)$$

$$x - 3y = 3 \qquad (2)$$

From equation (1), making y the subject of the equation, yields

$$y = 1 - x \qquad (3)$$

Substitute equation (3) into equation (2) to obtain $x - 3(1 - x) = 3$ which on rearrangement yields

$$4x = 6$$

Hence, $x = 1\frac{1}{2}$.

Substituting $x = 1\frac{1}{2}$ in equation (3) gives:

$$y = 1 - 1\frac{1}{2} = -\frac{1}{2}.$$

Therefore, $(x, y) = \left(1\frac{1}{2}, -\frac{1}{2}\right)$ as it can be read from the graph.

Example 8.3

The average of two numbers is 7, and three times the difference between them is 18. Find the numbers graphically.

Solution

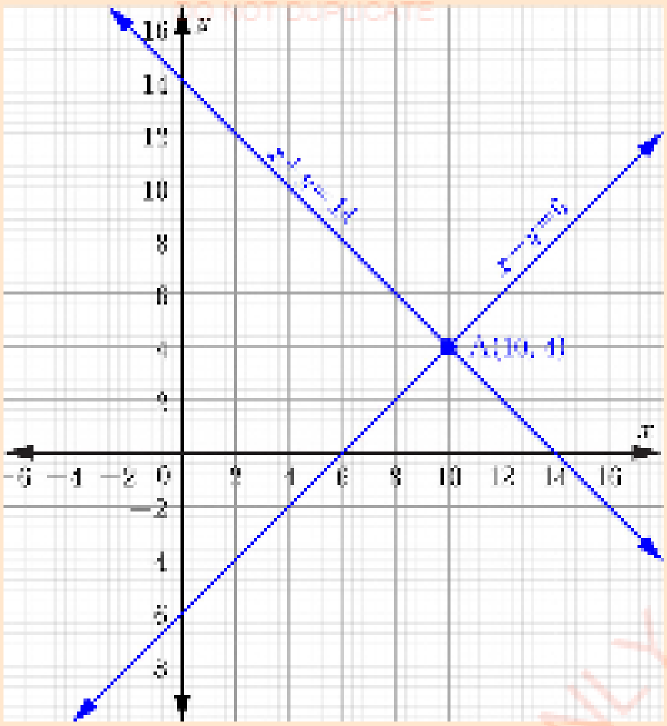
Let the two numbers be x and y, then the system of simultaneous equations are

$\frac{x+y}{2} = 7$ and $3(x - y) = 18$ which simplifies to $x + y = 14$ and $x - y = 6$.

$x + y = 14$ intercepts the coordinate axes at (14, 0) and (0, 14).

$x - y = 6$ intercepts the coordinate axes at (6, 0) and (0, –6).

Then, draw the graph using the identified points as shown in the figure.



In the figure, the two lines intersect at a point $A(10, 4)$. That is $x = 10$ and $y = 4$. Therefore, the numbers are 10 and 4.

Exercise 8.1

Solve the following simultaneous equations graphically and check your answers by a non – graphical method.

1. $x + 2y = 4$
   $3x - 2y = 6$
2. $3y = 2x + 3$
   $x - \frac{2}{3}y = 1$
3. $3p + 4q = 7$
   $5p + 6q = 11$
4. $y - 4x = 9$
   $2y + 3x = 3$
5. $y = 2x + 1$
   $y = 3 - x$
6. $7x + 3y + 12 = 0$
   $5x - 2y + 2 = 0$
7. $4x - 3y = 19$
   $6x + 7y = 17$
8. $3x + 4y = 3$
   $2x - 5y + 5 = 0$
9. $a + 1 = 2b + 2$
   $6a + 3 = 2b + 1$

10. $2x+3y=4$
$6x+9y=10$

11. $7x+4y=11$
$14x+3y=12$

12. $2c+2d=c-d$
$2d+2=c+1$

13. Ali paid 15 000 shillings for 10 oranges and 35 mangoes. Moshi went to the same fruit market and paid 8 800 shillings for 16 oranges and 18 mangoes. Find graphically the price for a mango and for an orange.

14. For his family of 3 adults and 2 children, Abdallah paid 5 800 shillings to travel from town A to town B. Nyake paid a total sum of 6 800 shillings for his 5 children, his wife and himself for the same journey. Using the graphical method, find
(a) how much did a ticket for an adult cost?
(b) how much did Nyake pay for his five children?

15. The sum of the ages of a mother and her daughter is 52 years. Eight years ago, the mother was eight times as old as her daughter. Using the graphical method, find their present ages.

Linear inequalities

Activity 8.2: Solving linear inequalities graphically

In a group or individually, perform the following tasks:

Beatrice got a new job and her monthly income is 526 500 shillings. To qualify to rent an apartment, her monthly income must be at least three times as much as the rent. Find graphically the highest rent she will qualify for by using the following steps:

1. Identify the variables from this word problem.
2. Write down the linear inequalities.
3. Replace the inequality symbol with an equal sign to form the equations of the boundary line.
4. Draw the graph and shade the region satisfying the inequalities.
5. Give the conclusion from graph drawn in task 4.
6. Share your findings with your neighbours for more inputs.

When a linear equation of the form $ax+by+c=0$ where a, b, and c are real numbers and $(a, b) \neq (0, 0)$ is represented by a straight line on xy – coordinate plane, it separates the plane into two disjoint sets called half – planes.

The graph of a linear inequality of one or two variables is the set of all points in the xy – plane which satisfies the inequality. The graph of a linear inequality has a region of the plane whose boundary is a straight line.

Steps for graphing linear inequalities are:

1. Replacing the inequality symbol with equal sign to form the equation of the boundary line.
2. Drawing the graph of the straight line that is the boundary, use solid line if an equal sign ($\leq$ or $\geq$) is included, or dotted line if an equal sign ($<$ or $>$) is not included.
3. Choosing any convenient test point which does not lie on the boundary line and substitute it to the inequality.
4. Shading the region that does not satisfy the test points or region that satisfies the test points.

In Figure 8.1 the straight line represents the equation $y = 4$ and all points on this line have coordinates which satisfy the equation $y = 4$. Points on the half – plane above this straight line satisfy the inequality $y > 4$, while those in the lower half – plane satisfy the inequality $y < 4$.

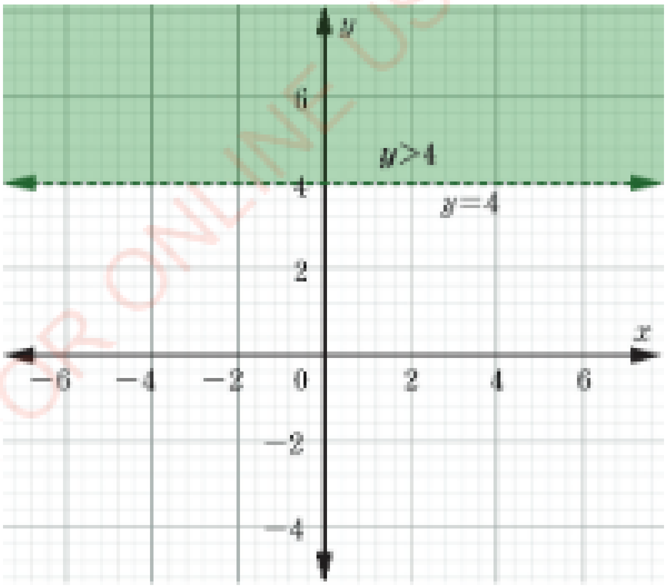


Figure 8.1: *Regions separated by line y = 4*

In Figure 8.2 the straight line represents the equation $x = -1$ and all points on this line have coordinates which satisfy the equation $x = -1$. The half–plane on the left hand side of the straight line $x = -1$ is defined by the inequality $x < -1$, while the one on the right hand side of the line $x = -1$ is represented by the inequality $x > -1$.

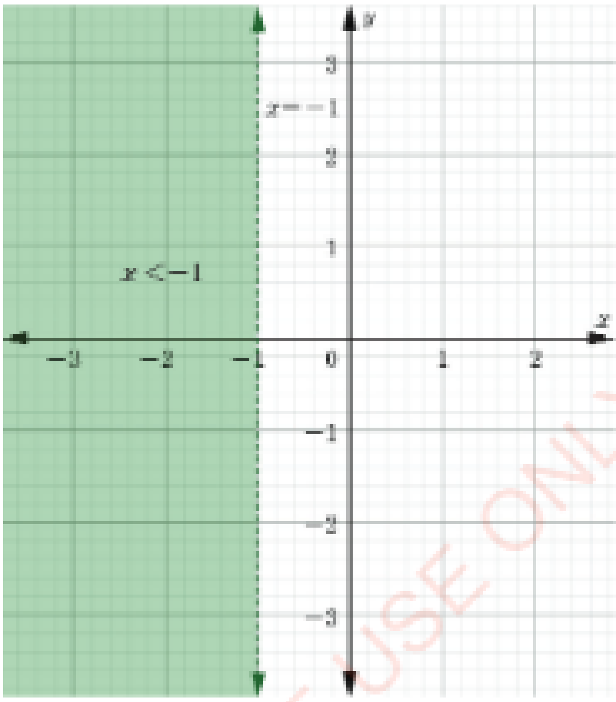


Figure 8.2: *Regions separated by line x = –1*

Shading of regions of linear inequalities

Activity 8.3: Shading of the region of linear inequalities

In a group or individually, perform the following tasks:

Observe Figure 8.3 which shows the boundary lines for the set of inequalities then perform the tasks as follows:

1. Write down all the set of inequalities from Figure 8.3.
2. Explain why the equations are drawn as dotted lines.
3. Share your findings with your neighbours for inputs.

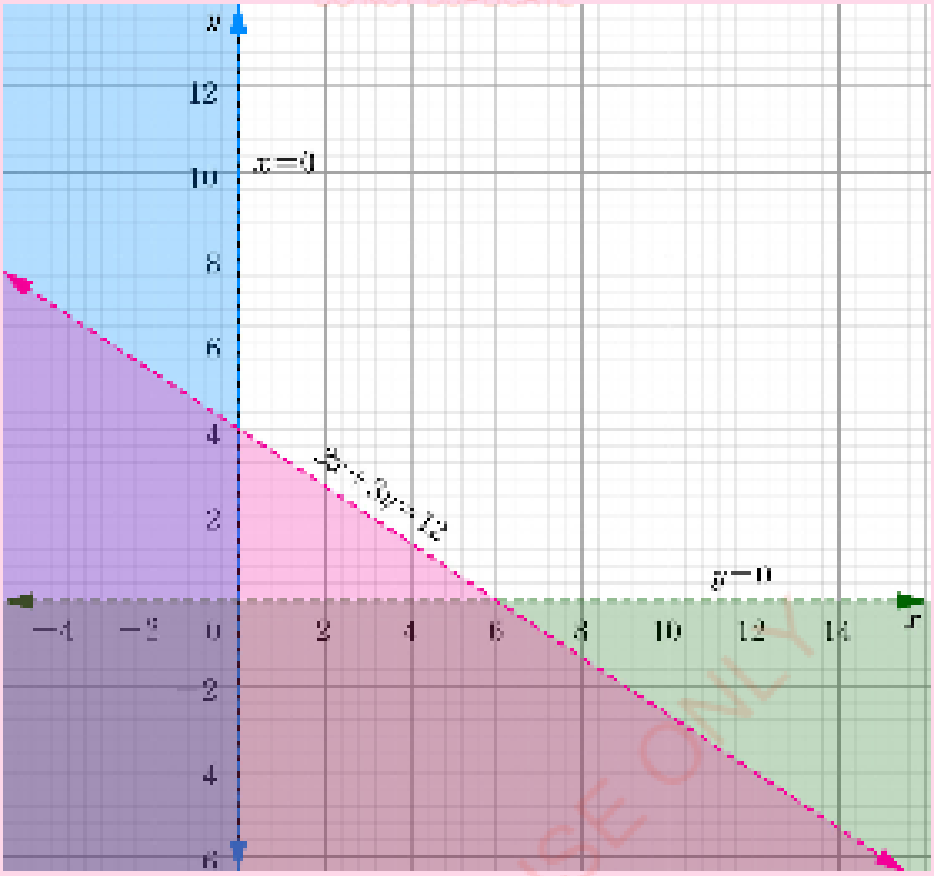


Figure 8.3: *Regions separated by three different lines i.e., $x = 0$, $y = 0$, and $2x + 3y = 12$*

Boundaries of half – planes

Equations are used to describe the boundaries of half – planes. When inequalities are written in the form $x \geq 0$, $y \geq 0$ or $ax + by \leq c$, where a, b, and c are real numbers, the boundary lines are drawn as solid lines to include points on the lines. When points on the boundary line are not to be included the lines are drawn as dotted lines.

Example 8.4

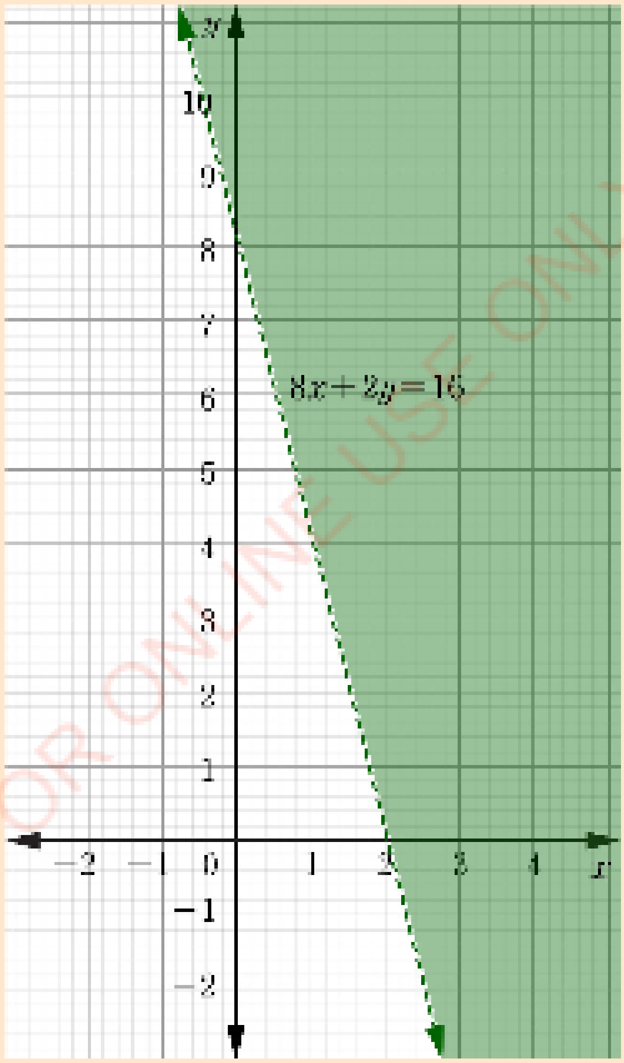
Draw and show the half–planes represented by $8x + 2y < 16$ and shade the unwanted region.

Solution

First, replace the inequality sign with an equal sign to form $8x + 2y = 16$, then determine whether the line intercepts the coordinate axes.

$8x + 2y = 16$ intercepts the coordinate axes at $(0, 8)$ and $(2, 0)$.

Now, draw the line on the xy – coordinate plane as shown in the following figure:



Feasible region

Consider the simultaneous inequalities $2x+3y>12$ and $x-y>2$, whose half–planes are shown in Figure 8.4.

The following steps are useful when drawing the graphs showing feasible region:

1. Replace the inequality sign with an equal sign to form $2x+3y=12$ and $x-y=2$.
2. Determine where the lines intercepts the coordinate axes. i.e.,
   $2x+3y=12$ intercepts the coordinate axes at $(0,4)$ and $(6,0)$.
   $x-y=2$ intercepts the coordinate axes at $(0,-2)$ and $(2,0)$.
3. Use the coordinates to draw the graph as shown in Figure 8.4.
4. Shade the unwanted region as shown in Figure 8.4.
5. Label the feasible region as shown in Figure 8.4.

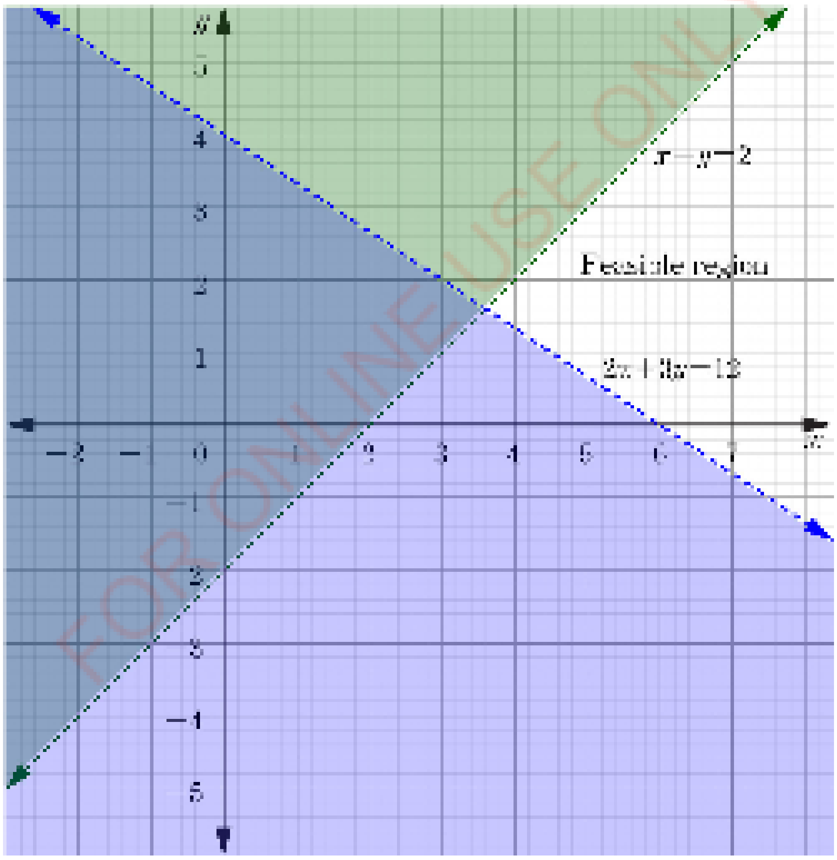


Figure 8.4: *The graphical representation of the feasible region*

Note that the two lines in Figure 8.4 divide the plane into four regions. All points in the unshaded region have coordinates which satisfy both inequalities $2x+3y>12$ and $x-y>2$.

Therefore, all the sets of points in the feasible region are the solutions to the two simultaneous inequalities. This region is called the feasible region.

Example 8.5

Draw the graph and show the feasible region which satisfies the following simultaneous inequalities:

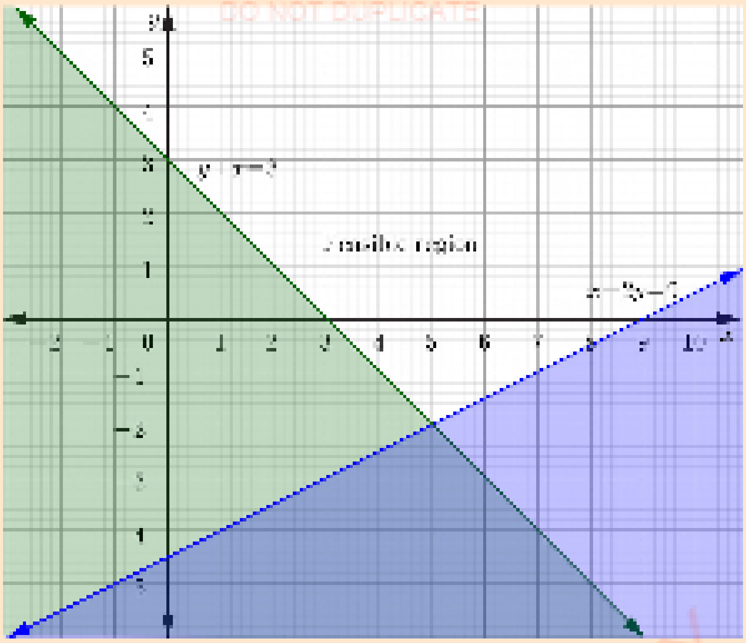
$$y+x\geq 3$$
$$x-2y\leq 9$$

Solution

1. Replace the inequality sign with an equal sign to form $y+x=3$ and $x-2y=9$.
2. Determine where the lines intercepts the coordinate axes. That is,

   $y+x=3$ intercepts the coordinate axes at $(0,3)$ and $(3,0)$.

   $x-2y=9$ intercepts the coordinate axes at $\left(0,-4\frac{1}{2}\right)$ and (9, 0).
3. Use the coordinates in step 2 to draw the graph as shown in the following figure.
4. By checking points above and below the lines, shade the unwanted region for $y+x\geq 3$ and $x-2y\leq 9$ as shown in the following figure.
5. Label the feasible region as shown in the following figure.



Example 8.6

Fatuma was given 250 000 shillings to buy oranges and mangoes. An orange costs 100 shillings while a mango costs 500 shillings. If the number of oranges bought is at least twice the number of mangoes, show graphically the feasible region representing the number of oranges and mangoes she bought, assuming that no fractions of oranges and mangoes are sold at the market.

Solution

Select suitable variables to represent different quantities mentioned in the question. Let x be the number of oranges bought and y be the number of mangoes bought. Form inequalities from the information provided in the question. Since x oranges and y mangoes were bought, the amount of money paid for them was $100x + 500y$ shillings. This amount could not be more than 250 000 shillings because Fatuma had only this amount of money.

Thus: $100x + 500y \leq 250\ 000$ that is, she could have used all the money or less than that. Then, number of oranges bought is at least twice the number of mangoes bought. Thus, $x \geq 2y$ or $x - 2y \geq 0$.

The number of oranges and mangoes bought cannot be negative, but it can be zero or any positive number. Hence, $x \geq 0$ and $y \geq 0$.

Thus, the inequalities obtained from the question are: $100x + 500y \leq 250\ 000$ which simplifies to $x + 5y \leq 2\ 500$

$x - 2y \geq 0$

$x \geq 0,\ y \geq 0$

Determine where the line $x + 5y = 2500$ intercepts the coordinated axes.

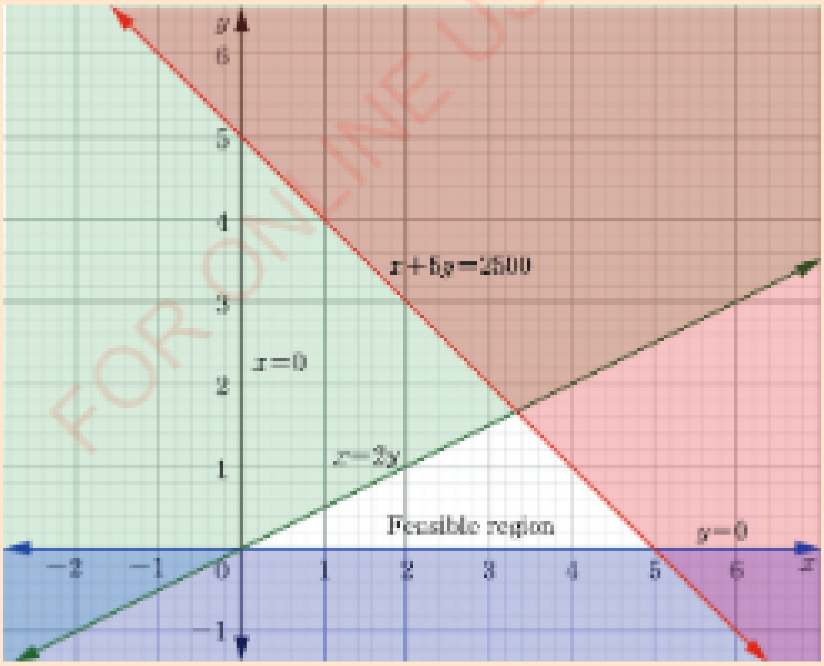
$x + 5y = 2500$ intercepts the coordinate axes at (2500, 0) and (0, 500).

Choose the scale to use when plotting the graph:

Horizontal scale, 1 unit: 500 oranges and

Vertical scale, 1 unit: 100 mangoes

Represent these inequalities on xy – plane (see the figure) and identify the feasible region.



From the figure, all points in the feasible region, including points on the boundary, form the solution set for the problem. But, oranges and mangoes cannot be obtained from the market in fractions, the solution set can only consist of points whose coordinates are whole numbers. All points in the feasible region, including points on the boundaries satisfy the set of inequalities.

Exercise 8.2

Answer the following questions:

1. (a) Draw the graph of the equation $2x - y = 7$. Show which half – plane is represented by $2x - y > 7$ and another one represented by $2x - y < 7$.

   (b) Repeat question 1 (a) using the equation $2x + y = 7$. Which half – plane represents $2x + y > 7$ and which one represents $2x + y < 7$?

   (c) From the graphs drawn in 1 (a) and 1 (b) can it be predicted whether or not the half – plane above the line always corresponds to the inequality '>'? What determines this correspondence?

2. Draw in separate graphs and show the regions representing each of the following inequalities:

   (a) $y < 2x - 1$ (b) $y > 3 - x$

   (c) $5x < 2y + 2$ (d) $7x + 3y + 12 < 0$

3. Shade the unwanted regions for the half – planes representing each of the following simultaneous inequalities:

   (a) $y < 2x - 1$, $y > 3 - x$

   (b) $y > 2x - 1$, $y < -1$

   (c) $5x < 4y + 2$, $4x + 3y + 12 > 0$

   (d) $y < 2x - 1$, $y > 3 - x$, $y > -1$

   (e) $y > 2x - 1$, $y < -1$, $4x + 3y + 12 > 0$

   (f) $-3 < x - y < 2$

4. On the same coordinate axes draw the graphs of the following inequalities: $x + 2y \leq 2$, $y - x \leq 1$, and $y \geq 0$. Which of the following points lie in the graph of the three simultaneous inequalities?

(a) (1, 1) (b) (2, 0) (c) $\left(-\frac{1}{2}, 1\right)$

(d) $\left(1, \frac{1}{2}\right)$ (e) (0, 0) (f) $\left(-\frac{1}{2}, \frac{1}{4}\right)$

5. In each case, draw a graph and show the feasible region which is satisfied by all of the following set of inequalities:

(a) $x > 0$, $2x + y \leq 4$, $2x + 3y \geq 8$

(b) $y > 0$, $x + 3y \leq 9$, $x + y \leq 10$

Which, if any, of the inequalities can be omitted without affecting the answer?

6. Find the points whose coordinates satisfy each of the following simultaneous inequalities:

(a) $x < 4$, $3y - x \leq 6$, $3y + 2x > 6$

(b) $0 \leq x < 4$, $0 \leq 3y - x \leq 6$,

7. Draw a graph to show the set of points whose coordinates satisfy each of the following inequalities:

(a) $2x - y \geq 0$ (b) $y > -2$ (c) $x \leq 3$

8. Draw a graph of the set of points whose coordinates satisfy the following inequalities:

$5x + 6y \leq 60$, $x \leq 6$, $x \geq 0$, and $y \geq 0$

Use the graph to answer the following questions:

(a) Which point in the set has the greatest value of y?

(b) Which points in the set are both x and y integers and $x = 5$?

(c) What is the largest integer value of y so that the point $(3, y)$ belong to the set?

(d) What is the largest integer k such that the point (k, k) belongs to the set?

(e) If k is not an integer, find the largest value of k so that the point (k, k) is in the set.

9. Juma is thinking of two numbers. Two times the first number is at least three and is also not less than three times the second number. Furthermore, three times the first number is not more than the second number plus six. Find the numbers graphically.

In questions 10 – 11, write down the inequalities satisfying the given information.

10. A post office has to transport 15 660 parcels using a lorry, which takes 2 700 parcels at a time, and a van which takes 1 080 parcels at a time. The cost of each journey is 35 000 shillings by lorry and 28 000 shillings by van. The van can make more trips than the lorry. The total cost should not exceed 308 000 shillings.

11. Bulls are scarce in region Y. At market A, each bull costs 420 000 shillings but it is sold with a cow costing 180 000 shillings. At another B market a bull costs 140 000 shillings with two cows each sold at 180 000 shillings. A businessman is prepared to spend up to 5 040 000 shillings on bulls and 4 140 000 shillings on cows.

The objective function

Activity 8.4: Finding the corner points of the feasible region

In a group or individually, perform the following tasks:

1. Draw a graph with equation $x + y = 12$, intersecting the y – axis at A and the x – axis at C and intersecting the line $x = 6$ at B and another line from D(0, 3) passing through point E which is the mid – point of $\overline{AB}$ intersecting the line $x = 6$ at point F.
2. Find:
   (a) the coordinates of point B, E, and F.
   (b) the equation of the line DE.
3. If x and y are subject to the restrictions, $y \geq 0, x \geq 0, x \geq 6$, and $x + y \geq 12$, by shading unwanted region, identify and label the feasible region.
4. Write the coordinates of the corner points of the feasible region.
5. Deduce the minimum and maximum value of $2x + 3y$.
6. Compare the coordinates in task 2(a) and the coordinates of corner points in task 4.
7. Share your findings with your neighbours for inputs.

From activity 8.4, the corner points are the vertices of the feasible region. They can be obtained by either inspecting the coordinates of the corners of a well – drawn feasible region or by solving for points of intersection of the boundary lines.

Maximum and minimum values

The maximum or minimum values of a linear programming problem are obtained by using the **objective function**.

Consider the following examples illustrating two situations which can be solved by using the concept of maximum and minimum values.

The objective function is a linear function which is used to find the solution of a linear programming problem by substituting the corner points of the feasible region.

Example 8.7

A petty businesswoman has 120 000 shillings to spend on exercise books. At the school shop an exercise book costs 800 shillings and at a stationery store it costs 1 200 shillings. The school shop has only 60 exercise books left and a petty businesswoman wants to obtain the greatest number of exercise books possible using the money she has.

(a) Formulate a linear programming problem.

(b) Find the greatest number of exercise books that a petty businesswoman can buy.

Solution

(a) Suppose a petty businesswoman wants to obtain the maximum number of exercise books using the money she has. Then, the first step is to express the information in mathematical form by introducing unknowns or variables to represent the different quantities in the problem.

So, let x be the number of exercise books bought at the school shop and y be the number of exercise books bought at the stationery store. Then, $x \leq 60$, since not more than 60 exercise books can be bought at the school shop.

$800x + 1\,200y \leq 120\,000$, since the price is 800 shillings at the school shop and 1 200 shillings at the stationary store and a petty businesswoman can either spend all the money she has or less than that.

$(x + y)$ gives the greatest number of exercise books bought also $x \geq 0$, $y \geq 0$, since exercise books can either be zero or any positive whole number but certainly not negative.

In this problem, a petty businesswoman has a limited or constrained amount of money. Her objective is to attain the greatest number of exercise books possible $(x + y)$ for her limited amount of money.

The linear equation $f(x, y) = x + y$, where f is any positive whole number referred as the objective function for the problem. The linear inequalities are:

$x \leq 60$, $800x + 1\ 200y \leq 120\ 000$, $x \geq 0$, and $y \geq 0$.

These inequalities are called constraints of the problem. Therefore, the linear programming problems can be formulated as follows:

Objective function is to maximize $f(x, y) = x + y$,

Subject to the constraints:

$800x + 1\ 200y \leq 120\ 000$ which simplifies to $2x + 3y \leq 300$

$x \leq 60$

$x \geq 0$, $y \geq 0$.

(b) Determine where the line $2x + 3y = 300$ intercepts the coordinate axes.

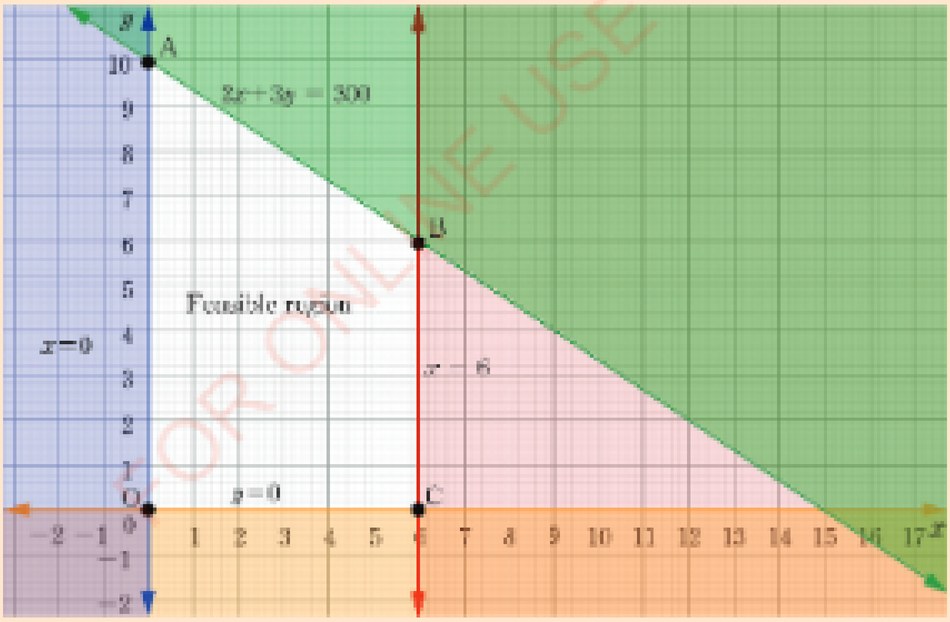
$2x + 3y = 300$ intercepts the coordinate axes at (150, 0) and (0, 100).

Choose the scale to use when plotting the graph:

Horizontal scale, 1 unit: 10 exercise books bought at the school shop and

Vertical scale, 1 unit: 10 exercise books bought at the stationery store

Plot the constraints on a graph as shown in following figure keeping in mind to shade the unwanted region.



Determine the feasible region and from it find a point which will give the greatest value of the objective function. Even though there are other ways of finding this point, it is easier to determine the coordinates of the corner points on the boundary of the feasible region and substitute them in the objective function, $f(x, y) = x + y$. The boundary region OABC is called the feasible region.

From the given figure, the corner points are: $O(0,0)$, $A(0,100)$, $B(60,60)$, and $C(60,0)$.

The following table shows the procedure for testing corner points:

Corner points	**Maximize** $f(x, y) = x + y$
O(0, 0)	0
A(0, 100)	100
B(60, 60)	120
C(60, 0)	60

The point B(60, 60) offers the optimal value to the linear programming problem.

Therefore, the greatest number of exercise books that a petty businesswoman can buy is 120. That is 60 exercise books from the school shop and 60 exercise books from the stationery store.

Note: A point in the feasible region which gives the optimal value of the objective function is called an optimal point. An optimal value is a maximum or minimum value of an objective function. Constraints are linear inequalities defining limitations of decisions. They arise from various sources such as limited resources.

The following are outlined steps for solving linear programming problems by graphical method:

1. Make a summary in a tabular form showing all the given information.
2. Identify the decision variables.
3. Formulate the objective function from the linear programming problem.
4. Formulate the set of constraints according to the given conditions.
5. Draw the graph using the constraints to determine the feasible region.
6. Identify the feasible region and its corner points.

7. By testing the coordinates of the corner points in the objective function, determine the optimal value and optimal point.
8. Make the conclusion based on the requirements of the problem.

Example 8.8

A nutritionist prescribes a special diet for patients containing the following number of units of vitamins A and B per kg of two food types F_1 and F_2. Food type F_1 contains 20 units of vitamin A and 7 units of vitamin B while food type F_2 contains 15 units of vitamin A and 14 units of vitamin B. If the minimum daily intake required is 120 units of vitamin A and 70 units of vitamin B, what is the least total mass of food a patient must have enough of these vitamins?

Solution

Step 1: Make a summary in a tabular form as shown in the following table:

	Nutrients	
Food	**Vitamin A**	**Vitamin B**
F_1	20	7
F_2	15	14
Minimum daily requirement	120	70

Step 2: Let x be number of units of food type F_1 and y be number of units of food type F_2.

Step 3: The objective function is to minimize $f(x, y) = x + y$ (least total mass of food).

Step 4: Write down the constraints:

$20x + 15y \geq 120$ which simplifies to $4x + 3y \geq 24$, since the daily minimum intake of vitamin A is 120 units.

$7x + 14y \geq 70$ which simplifies to $x + 2y \geq 10$, since the daily minimum intake of vitamin B is 70 units.

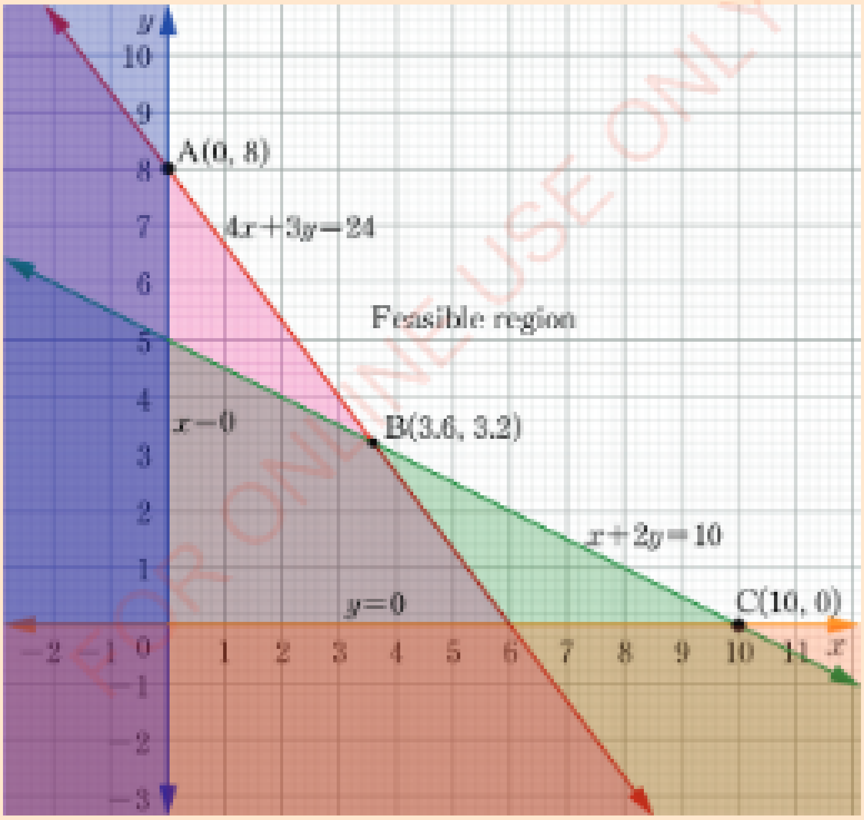
$x \geq 0$, $y \geq 0$ which are non – negativity constraints.

Determine where the lines $4x+3y=24$ and $x+2y=10$ intercepts the coordinate axes as shown in the following tables:

The x and y intercepts for $4x+3y=24$		
x	0	6
y	8	0

The x and y intercepts for $x+2y=10$		
x	0	10
y	5	0

Step 5: Plot the the graph using constraints as shown in the following figure by shading unwanted region:



Step 6: Identify the feasible region and read the coordinates of the corner points on the boundary. From the graph the corner points are: $A(0,8)$, $B(3.6,3.2)$, and $C(10,0)$.

Step 7: The following table shows the procedure for testing the corner points.

Corner points	Maximize $f(x,y)=x+y$
A(0, 8)	8
B(3.6, 3.2)	6.8
C(10, 0)	10

The optimal value is 6.8 and the optimal is B(3.6, 3.2)

Step 8: From the table point B(3.6, 3.2) offers the optimal value to the linear programming problem.

Therefore, the least total mass of food a patient must have is 6.8 units.

Note: In example 8.7, the objective function of the problem was to find the greatest number of exercise books that a petty businesswoman could buy. Consequently, the objective function for the problem was to be maximized. The constraints formed used the "$\leq$" signs, except the non – negative constraints $x \geq 0$ and $y \geq 0$. These are examples of maximization problem. Example 8.8 is a minimization problem where the constraints formed used the "$\geq$" signs.

Exercise 8.3

Answer the following questions:

1. Draw the graph of the equations $2x+3y=12$ and $7x+4y=28$, then show the region satisfied by the following restrictions:
   $2x+3y\leq 12$, $7x+4y\leq 28$, $x\geq 0$, and $y\geq 0$.
   Find the maximum possible value of:
   (a) $f(x,y)=x+y$ subject to given restrictions.
   (b) $f(x,y)=x+5y$ subject to the given restrictions.

2. Construct a polygonal region which contains the points whose coordinates satisfy the following system of inequalities:
$y \leq 2x$, $x \geq 6$, $y \geq 2$, and $2x + 3y \leq 30$.

At which point does:

(a) $y - x$ takes a minimum value?

(b) $y - x$ takes a maximum value?

(c) $5y + 6x$ takes a minimum value?

(d) $4x + 6y$ takes a maximum value?

3. Show on a graph the feasible region under the following constraints:
$x \geq 0$, $y \geq 0$, $2x + 2y \geq 11$, $2x + y \geq 8$, and $2x + 5y \geq 18$.

(a) Under these constraints, minimize the following objective functions:

(i) $f(x, y) = 2x + 3y$

(ii) $f(x, y) = 3x + 2y$

(iii) $f(x, y) = x + 3y$ and for each case state the coordinates of the points and the minimum value.

(b) Repeat (a) when both x and y are integers.

4. A farmer wants to plant coffee and potatoes. Coffee needs 3 men per hectare and potatoes need 3 men per hectare. He has 48 hired labourers available. To maintain one hectare of coffee he needs 250 000 shillings while one hectare of potatoes costs him 100 000 shillings. Find the greatest possible area of land he can sow if he is prepared to use 2 500 000 shillings.

5. A technical school is planning to buy two types of machines. A lather machine needs 3 m^2 of floor space and a drill machine requires 2 m^2 of floor space. The total space available is 30 m^2. The cost of one lather machine is 250 000 shillings, a drill machine costs 300 000 shillings, and the school can spend not more than 3 000 000 shillings. Find the greatest number of machines the school can buy.

6. A farm is to be planted with wheat and maize while observing the following constraints:

	Wheat	Maize	Maximum total
Days labour per hectare	2	1	10
Labour cost per hectare (shs)	60 000	60 000	420 000
Cost of fertilizer per hectare (shs)	40 000	40 000	240 000

(a) Find the greatest area that can be planted.

(b) If wheat yields a profit of 80 000 shillings per hectare while maize yields 60 000 shillings per hectare, how should the area be planted?

7. The manufacturing company has two plants, which produce three types of products namely SMALL, MEDIUM, and LARGE products. The first plant can manufacture 3 units of SMALL product, 12 units of MEDIUM product, and 12 units of LARGE product at the cost of 24 000 shillings per hour. The second plant can manufacture 9 units of SMALL product, 15 units of MEDIUM product, and 3 units of LARGE product at the cost of 18 000 shillings per hour. The company has received an order of 204 units of SMALL product, 690 units of MEDIUM product, and 330 units of LARGE product.

(a) Formulate a linear programming problem to minimize cost.

(b) How many hours should be given to each plant in order to satisfy the order at the least cost?

8. A doctor prescribes that in order to obtain an adequate supply of vitamins A and C, his patient shall have portions of food 1 and food 2. The number of units of vitamin A and vitamin C are given in the following table:

	Nutrients	
Food	**Vitamin A**	**Vitamin C**
F_1	3	2
F_2	1	7

The doctor prescribes a minimum of 14 units of vitamin A and 21 units of vitamin C. What are the least number of portion of food 1 and 2 that will fit the doctor's prescription?

9. A bread dealer proposes to buy up to 110 loaves of bread. White bread costs 2 000 shillings per loaf and brown bread costs 2 500 shillings per loaf and can spend up to 250 000 shillings. The profit on a single loaf of white bread is 1 000 shillings and that of brown bread is 1 200 shillings. How many of each type should he buy so as to get maximum profit?

10. Esther wants to use her farm to plant maize and millet. The farming of maize costs 120 000 shillings per acre and millet costs 200 000 shillings per acre. The funds available in farming the two crops is 2 400 000 shillings and there are only 13 acres available. Maize requires 10 man hours per acre, while millet requires 8 man hours per acre and there are 120 man hours available. The profit per acre of maize is 125 000 shillings and the profit per acre of millet is 110 000 shillings.
    (a) How should she allocate her farm for maximum profit?
    (b) What is the maximum profit?

Chapter summary

1. Steps to be considered when solving simultaneous equations graphically are:
    (a) Draw x and y – axis on the graph paper.
    (b) Prepare the table of values or find the coordinates of intercepts on the coordinate axes. Then, draw the graph of each equation on the xy – plane.
    (c) Read the point of intersection of the two line graphs. This point offers the solution of simultaneous equations.

2. In drawing linear inequality graph.
    (a) Draw a dotted line graph if the inequality is $>$ or $<$ and shade the unwanted region.
    (b) Draw a solid line graph if the inequality is $\geq$ or $\leq$ and shade the unwanted region.

3. When solving the linear programming questions, the following steps should be followed:
    (a) Make a summary in a tabular form showing all the given informations.
    (b) Identify the decision variables.

(c) Formulate the objective function.

(d) Formulate constraints/ inequalities.

(e) Draw the graph using the constraints to determine the feasible region.

(f) Identify the feasible region and its corner points. Here, if the shaded region is the unwanted, then the feasible region is the area which is clear after shading however, if the shading have been done on the wanted region, then the feasible region will be the region with the overlapping shadings.

(g) Use the cornerpoints and objective function to determine the optimal value and optimal point.

(h) Make the conclusion based on the requirements of the problem.

Revision Exercise 8

1. Solve the following simultaneous equations graphically and check your answer by a non – graphical method:

   (a) $\frac{x}{2}+\frac{y}{2}=2$ and $2x+3y=13$

   (b) $2x-y=5$ and $x=1-y$

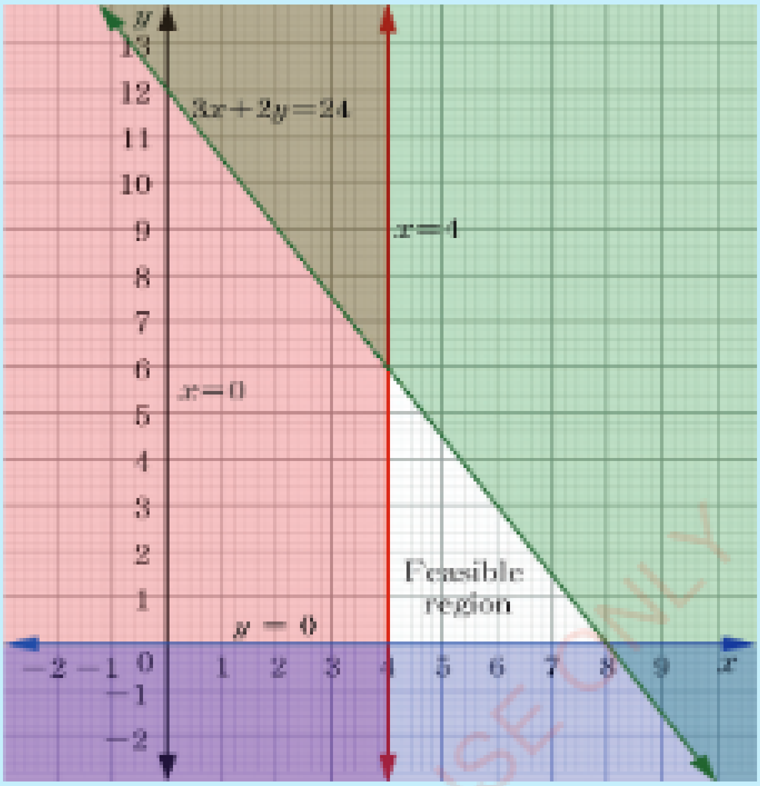
2. Draw and show the half – planes represented by each of the following:

   (a) $y+x\leq 0$ (b) $x+y\leq 1$

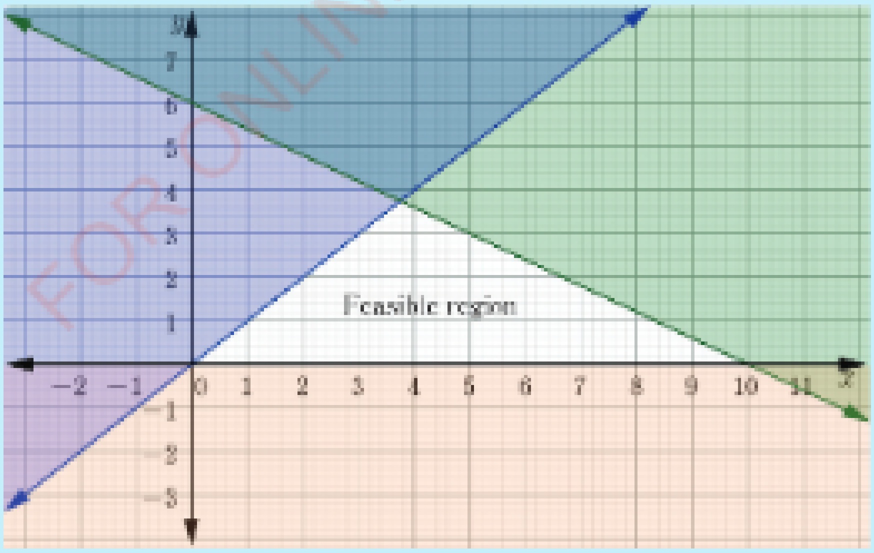
   (c) $2x+3y\leq -6$ (d) $8y\geq 2x+16$

3. Write down four inequalities representing a feasible region in each of the following figures:

(a)



(b)



4. Plot a graph representing the following inequalities: $x \geq 0$, $y \geq 0$, $3x + 2y \leq 18$, and $2x + 4y \leq 16$. List all the points inside the region whose x and y coordinates are both even numbers.
5. A school plans to buy goats. A male goat costs 60 000 shillings and a female one costs 75 000 shillings. The school wants to buy not more than 4 male goats. If it is prepared to spend up to 1 200 000 shillings, find the maximum number of goats it can buy.
6. A choirmaster has 10 acres of land on which he can grow either mangoes or oranges. He has 59 working days available during cultivation season. Mangoes require 5 days per acre of labour while oranges requires 8 days of labour per acre. Net profit from mangoes per acre is 30 000 shillings and that of oranges is 45 000 shillings. How many acres of each crop should he plant to maximize his profit?
7. A manufacturer has 24, 36, and 18 tons of wood, plastics, and steel respectively. To make product A, it requires 1, 3, and 2 tons of wood, plastic and steel respectively while the product B requires 3, 3, and 1 tons of wood, plastic, and steel respectively. Product A sells for 385 500 shillings and product B sells for 475 500 shillings.
   (a) Formulate a linear programming problem to maximize profit.
   (b) How many units of each products should the manufacturer make to obtain the maximum gross income?

Project 8

1. Mr. Juma has x cows and y goats. He has fewer than four cows and more than two goats. He has enough room for maximum of ten animals in total.
2. Express the three conditions in 1 as inequalities.
3. Draw an appropriate pair of axes and label the feasible region which satisfies all the inequalities.
4. State two possible combinations of cows and goats which he can have.
5. On graph paper using scales 1 cm for unit on each axis, mark clearly the region for which $x \geq 0$, $y \geq 0$, $x + y \leq 4$, $3 \leq x \leq 8$, $y \leq 8$, $y \leq 4$, and $2x + 3y \leq 24$. Use graph to find the minimum and maximum values of the function $Q = x + 2y$.

Answers to odd – numbered questions

CHAPTER ONE

Exercise 1.1

1. (a) $\frac{1}{4}$ (b) -1 (c) $-\frac{3}{11}$ (d) $\frac{1}{5}$

   (e) $-\frac{2}{3}$ (f) $-\frac{38}{41}$

3. (a) $y = 3x - 5$ (b) $y = 2x - 1$ (c) $y = \frac{2}{5}x - \frac{23}{5}$

5. (a) $4y - 3x - 21 = 0$ (b) $y + x - 5 = 0$

   (c) $3x + 11y - 35 = 0$ (d) $5y - x + 19 = 0$

   (e) $2x + 3y - 7 = 0$ (f) $82y + 76x - 403 = 0$

7. $k = \frac{10}{3}$

11. x – intercept $= -3\frac{1}{3}$, y – intercept $= 5$

13. $m = \frac{1}{4}$, $c = \frac{17}{4}$ 15. $5y - 3x + 5 = 0$ 17. $y = 5$

19. (a) Gradient $= 3$; y – intercept $= -12$

    (b) Gradient $= 3\frac{1}{2}$; y – intercept $= \frac{3}{2}$

    (c) Gradient $= -\frac{a}{b}$; y – intercept $= -\frac{c}{b}$

    (d) Gradient $= -\frac{b}{a}$; y – intercept $= b$

Exercise 1.2

1. (a) $(5, 3)$ (b) $\left(6, \frac{3}{2}\right)$ (c) $(3, 3)$
   (d) $(-3, 3)$ (e) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (f) $\left(2\frac{7}{8}, 2\frac{1}{4}\right)$
3. $(-5, 7)$
5. $(-7, -2), \left(-1, -\frac{1}{2}\right), \left(-4, -4\frac{1}{2}\right)$
7. $(2, -11)$
9. $x + y - 5 = 0$

Exercise 1.3

1. (a) 13 (b) 5 (c) $\sqrt{(r-p)^2 + (s-q)^2}$
   (d) 4 (e) $\sqrt{52}$ or $2\sqrt{13}$ (f) 13.3
3. $\overline{PQ} = \overline{PR} = \sqrt{137}$ units, the triangle PQR is an isosceles triangle
5. $x = 1$ or -7
7. $(0, -2)$
9. D$(-3, 0)$

Exercise 1.4

1. (a) Not parallel (b) Not parallel
   (c) Not parallel (d) Parallel
3. $x + 3y - 12 = 0$
5. $3x + 4y - 7 = 0$
7. (a) $C = 0$ (b) $C \neq 0,\ A = 0, B \neq 0$
   (c) $C \neq 0,\ B = 0,\ A \neq 0$ (d) $A = B$
9. $\left(\frac{23}{3}, 0\right)$
11. $y = -2$

Exercise 1.5

1. (a) $-\frac{4}{3}$ (b) $\frac{1}{5}$ (c) $-\frac{1}{3}$ (d) $\frac{B}{A}$

3. (a) $\overline{AC}$ and $\overline{BC}$ determine the right angle at C

   (b) $\overline{AB}$ and $\overline{BC}$ determine the right angle at B

   (c) $\overline{AC}$ and $\overline{BC}$ determine the right angle at C

5. $x = 4$

9. $y = -6x + 13$

Revision Exercise 1

1. (a) First (b) Fourth (c) Third (d) Second

3. Have equal distance which is $\sqrt{113}$ units of length.

5. (a) $2x + 5y + 4 = 0$

   (b) $-5x + 2y + 19 = 0$

7. $Q(0, 0)$, $R\left(2, \frac{3}{2}\right)$, $S\left(5, -\frac{3}{2}\right)$

9. $x + 4y + 8 = 0$

11. $\sqrt{5}$

13. $(13, 3)$

15. $\left(\frac{5}{2}, 0\right)$

17. 6 square units

CHAPTER TWO

Exercise 2.1

1. h = 6cm
3. A = 60 dm^2
5. A = 50 cm^2
7. $B\hat{A}C = 90°$
9. A = 167cm^2
11. A = 86.6 cm^2
13. 1 125 trees
15. (a) $x = 2.44$ cm (b) 8.4 cm^2

Exercise 2.2

1. $s = 3.42$ cm, $p = 30.8$ cm
5. 4.62 cm
7. 8 cm
9. $s = 3.71$ cm, $p = 37.1$ cm
11. 6:1

Exercise 2.3

1. 72.31 cm^2
3. $A = 3\sqrt{3}\ dm^2$
5. 152.17 dm^2
7. 5.95 cm
9. 314.1 cm^2
11. 111.25 cm^2

Exercise 2.4

1. (a) 31.4 cm (b) 20.1cm
3. 7.01 dm
5. 62.8 cm^2
7. 56 cm
9. 52 752 shillings
11. 26.86 cm

Exercise 2.5

1. 4:25
3. $\frac{5}{3}$
5. 15 cm; 18 cm
7. Tsh. 57 375
9. 2 square units
11. (a) 4:5 (b) 7.5 cm (c) 64 cm^2

Revision exercise

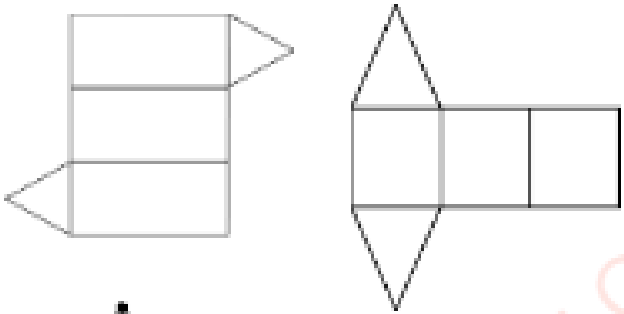
1. $9x^2$ square units
3. 600 cm^2
5. 12 cm
7. 2 400 cm^2
9. 50 cm^2; $4\sqrt{50}$ cm
11. 166.28 cm^2
13. 22 092.02 cm^2
15. 32 cm
17. 28.5 cm^2

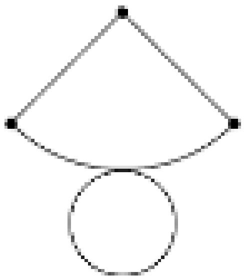
CHAPTER THREE

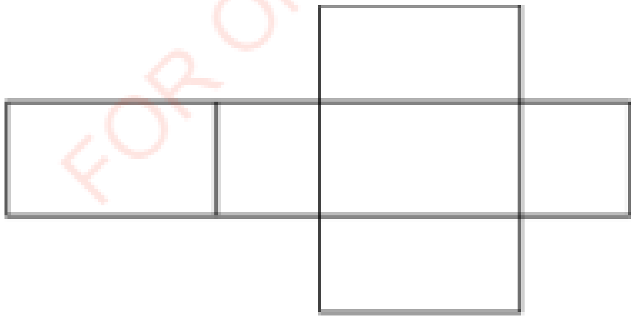
Exercise 3.1

3. 5 faces, 9 edges, 6 vertices
9. Triangular prism, rectangular pyramid
11. (a) Triangular pyramid,
    (b) Rectangular prism,
    (c) Cubes

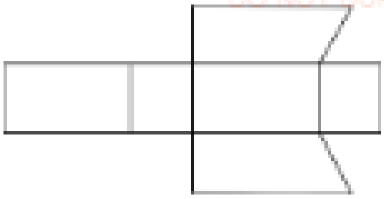
Exercise 3.2

1. 

3. 

5. 

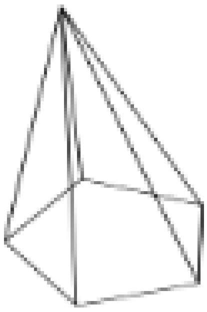
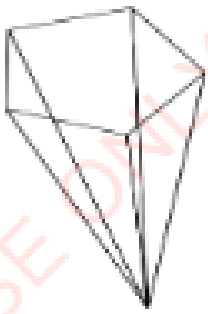
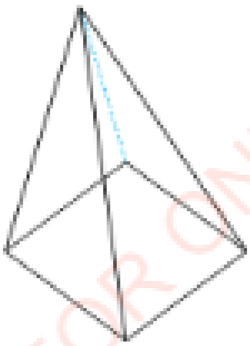
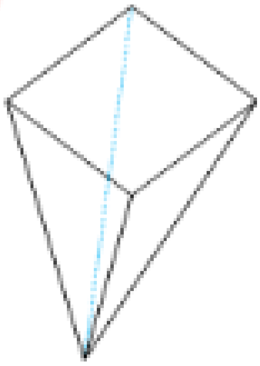
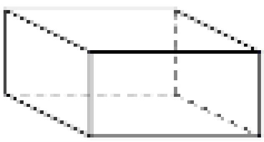
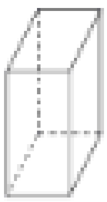
7.



Exercise 3.3

1. (c) and (e)

3. (a) $\overline{VO}$ (b) $U\hat{O}T$ (c) $V\hat{O}S$ or $U\hat{O}V$

5. (a)	 or	
(b)	 or	
(c)	 or	

7. (a) 5 cm (b) 5 cm (c) $2\frac{1}{2}$ cm

9. (a) $T\hat{U}V$ and $P\hat{Q}R$ (b) $S\hat{P}W$ and $V\hat{Q}R$
   (c) $T\hat{V}U$ and $P\hat{R}Q$ (d) $R\hat{S}Q$ and $U\hat{W}V$

Exercise 3.4

1. (a) 3 m (b) 4.38 m
   (c) 33°14′ or 33.23° (d) 36°51′ or 36.87°
3. (a) $\sqrt{116}$ m (b) 48°
5. 27°

Exercise 3.5

1. 52 cm^2 3. 288 cm^2 5. 180 cm^2
7. 62.35 square units 9. 703.36 cm^2 11. 6 cm
13. 301.44 cm^2 15. 96 cm^2 17. 615.44 cm^2
19. 514 457 600 km^2

Exercise 3.6

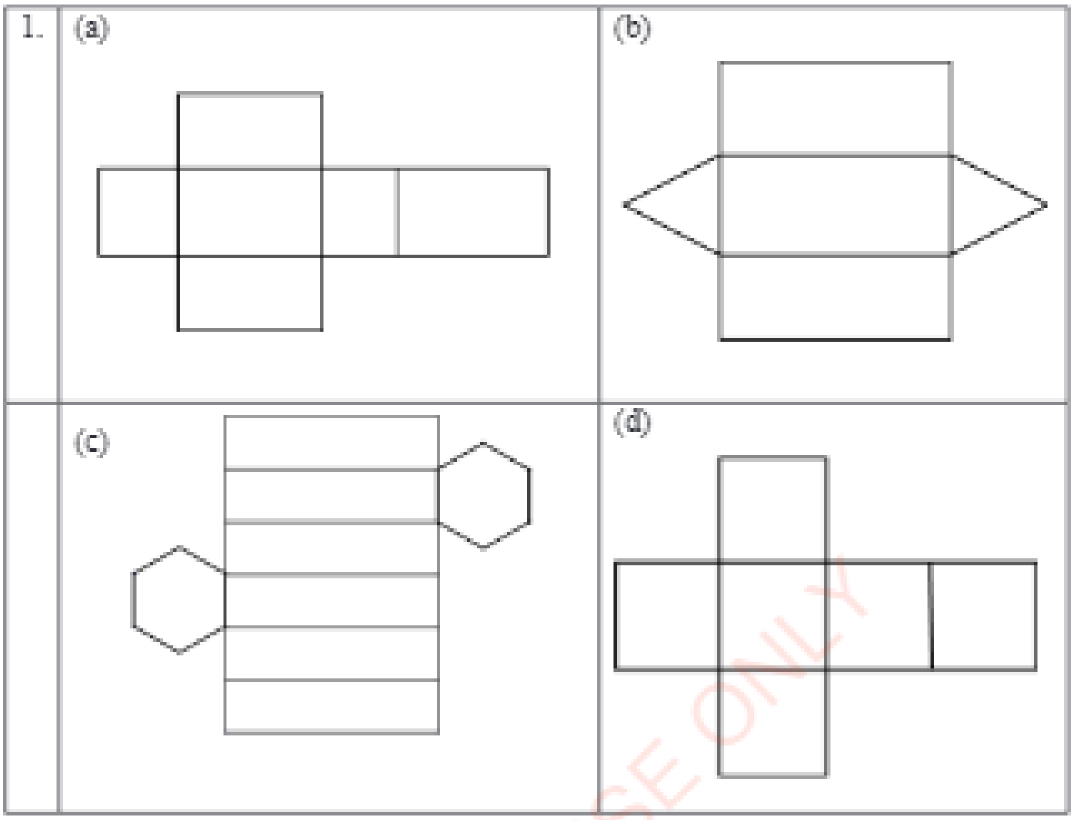
1. (a) 40 cm^2 (b) 36 cm^3 3. 25.5 cm
5. 20 cm 7. 753 600 mm^3 9. $100\sqrt{3}$ cm^2
11. 4.31 cm^3 13. 98.125 dm^3 15. 60 dm^3

Exercise 3.7

1. (a) 150.72 cm^3 (b) 37.68 cm^3
5. 360 cm^3
7. 72 cm^2
9. (a) 60 cm (b) 60.3 cm

Revision Exercise 3(a)

1.



3. (b) 8.25 cm (c) 90°
   (d) Yes (e) 56.30°

5. (a) $\overline{AC} = 10\sqrt{2}$ cm (b) 35.26° or 35°16′
   $\overline{AG} = 10\sqrt{3}$ cm

7. (a) 49.18° (b) 60°15′

9. 10 cm

11. (a) 8.77 m (b) 27°7′ (c) 38°40′

13. (a) 10.58 cm (b) 61°51′ (c) 70°31′ (d) 56°12′

Revision Exercise 3(b)

1. 21 cm
3. (a) 108π cm^3 (b) $6\sqrt{117}$ cm^2
5. 75°10′
7. $\sqrt{\dfrac{154}{\pi}}$ cm
9. 1 017.4 cm^2
11. (a) 528 cm^2 (b) 480 cm^2
13. 992 cm^2
15. 2.1856 dm
17. 4.5 cm^2

CHAPTER FOUR

Exercise 4.1

1. (a) {1, 2, 3, 4, 5, 6}
   (b) {win, draw, loose }
   (c) {January, February, March, April, May, June, July, August, September, October, November, December}
   (d) {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
   (e) {2, 4, 6, 8, 10}

3. (a) {3, 4, 5, 6}
   (b) {February, March, April, May, August, September, October, November, December}
   (c) {5, 7, 9, 11, 13}
   (d) $E' = \{a, c\}$
   (e) $E' = \varnothing$

Exercise 4.2

1. (a) $\frac{3}{20}$ (b) $\frac{1}{5}$ (c) $\frac{51}{100}$ (d) $\frac{13}{25}$

3. $\frac{2}{5}$ 5. $\frac{7}{10}$

7. (a) $\frac{7}{12}$ (b) $\frac{1}{3}$

9. $\frac{1}{3}$

11. (a) $\frac{1}{5}$ (b) $\frac{3}{10}$ (c) $\frac{2}{25}$

13. (a) $\frac{8}{25}$ (b) $\frac{9}{50}$

15. 0.6 17. 1 19. 0.541

Exercise 4.3

1. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$

3. (a) $\frac{5}{12}$ (b) $\frac{11}{12}$ (c) $\frac{1}{6}$ (d) 0

5. $\frac{1}{6}$

7. $\frac{1}{6}$

9. 12 pairs

11. $\frac{5}{6}$

13. (a) 0.75 (b) 0.55

Revision exercise 4

1. (a) $\frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{391}{400}$

3. (a) $\frac{5}{11}$ (b) $\frac{7}{22}$ (c) $\frac{15}{22}$ (d) $\frac{17}{22}$

5. (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) 0

9. $\frac{1}{5}$

11. (a) $\frac{1}{6}$ (b) $\frac{2}{3}$

13. (a) $\frac{51}{105}$ (b) $\frac{54}{105}$ (c) $\frac{6}{7}$

CHAPTER FIVE

Exercise 5.1

1. (a) Negative (b) Negative (c) Positive
   (d) Negative (e) Positive (f) Positive

3. (a) $\sin 50°$ (b) $-\sin 50°$ (c) $-\sin 50°$

5. (a) $-\tan 40°$ (b) $\tan 40°$ (c) $-\tan 40°$

Exercise 5.2

1. $\frac{3}{5}$ 3. $\tan\beta = \frac{\sqrt{6}}{2}$ 5. 0.3746 7. 0.8

11. $\frac{1}{2}$ 13. $\frac{\sqrt{13}}{6}$ 15. $\pm 2\sqrt{2}$ 17. 1.428

19. 0.4663

Exercise 5.3

1. (a) 0.3090, –0.9511, –0.3249 (b) –0.9397, –0.3420, 2.748 (c) –0.6691, 0.7431, –0.9004
   (d) –0.9511, 0.3090, –3.078 (e) –0.3907, –0.9205, 0.4245 (f) 0.9063, –0.4226, –2.1445

3. (a) –25° 53′, –154° 07′ (b) –141° 47′, –321°47′ (c) –18° 50′, –198° 50′

5. $\frac{\sqrt{3}}{4}$ 7. $\sqrt{2}$

9. (a) 27° (b) –168° (c) 156° (d) –95°

Exercise 5.4

1. 135.8 m
3. 13.8 m
5. 33° 46′
   56° 14′
7. 61° 49′
9. (a) 11.3 m
   (b) 5.6 m
11. (a) 41° 48′
    (b) 11.18 m

Exercise 5.5

3. 107° 28′ and 252° 32′

Exercise 5.6

1. (a) b = 2.81cm, c = 3.61 cm, angle A = 76°
   (b) d = 3.88 cm, f = 5.93 cm, angle F = 70°
   (c) $\hat{B}$ = 43° 28′, $\hat{C}$ = 55° 06′, c = 19.08 cm
   (d) $\hat{Z}$ = 33° 55′, $\hat{X}$ = 27° 52′, x = 15.91 cm

3. 66.9 m

5. (a) 5.7 km (b) 6.2 km

7. (a) 29° 42′ (b) 3.8 cm 4.5 m

Exercise 5.7

1. (a) $a = 3.88$ m, $\hat{B} = 56°$, $\hat{C} = 84°$
   (b) $t = 7.5$ cm, $\hat{S} = 29° 8'$, $\hat{U} = 37° 52'$
   (c) $\hat{E} = 37° 58'$, $\hat{F} = 91° 47'$, $\hat{G} = 50° 15'$
   (d) $\hat{J} = 38° 25'$, $\hat{K} = 49° 1'$, $\hat{L} = 92° 34'$
   (e) $\hat{P} = 120° 52'$, $\hat{Q} = 30° 52'$, $\hat{R} = 28° 16'$

3. A = 58° 48′ 5. 18.9 cm 7. 16.65 m

9. 8.7 cm

Exercise 5.8

3. $\frac{\sqrt{6}+\sqrt{2}}{4}$ 9. $\frac{\sqrt{2}-\sqrt{6}}{4}$

Exercise 5.9

3. $\sin s \cos t - \sin t \cos s$ 5. $\frac{\sqrt{6}-\sqrt{2}}{4}$

7. 2 sin A cos A 9. 1 11. $-\frac{\sqrt{2}}{2}$

Exercise 5.10

1. $2+\sqrt{3}$ 5. $\tan B = \frac{3}{4}$ 7. $\sqrt{3}$

9. (a) $\frac{6}{7}$ (b) $\frac{2}{9}$

Revision exercise 5

1. (a) $\frac{\sqrt{6}}{2}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\sqrt{2}$

   (d) $-\left(\frac{1}{2}+\sqrt{2}\right)$ (e) $2\sqrt{3}$ (f) 0

3. (a) −0.2079 (b) −0.7193 (c) −0.8098

   (d) −0.2588 (e) −0.6293 (f) 1.1918

5. (a) −352° 55′, −187° 36′, 7° 6′, 172° 55′

   (b) −235° 24′, −124° 36′, 124° 36′, 235° 24′

   (c) −340° 28′, −160° 28′, 19° 32′, 199° 32′

7. (a) −sin 58° (b) −cos 83° (c) −tan 36°

9. $\hat{A} = 54°$, $\overline{AB} = 20.98$ cm, $\overline{BC} = 17.06$ cm

11. $\hat{A} = 24° 29'$, $\hat{B} = 27° 14'$, $\hat{C} = 128° 17'$

13. 4.42 km

15. 51° 6′

17. 81° 17′

19. (a) − 0.866 (b) undefined (c) − 0.7071 (d) 1

CHAPTER SIX

Exercise 6.1

3. (a) (3, 2) (b) (6, –7) (c) (1, 4) (d) (2, 0) (e) (0, –3)

5. (a) $\overrightarrow{OA} = (3, 4)$ (b) $\overrightarrow{OB} = (5, 3)$

(c) $\overrightarrow{OC} = (7, 8)$ (d) $\overrightarrow{OD} = (u_1, u_2)$

7. $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

Exercise 6.2

1. (a) The magnitude of a vector $\underset{\sim}{r} = (x, y)$ is defined by $|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$

(b) 13 units, $\theta = 202.6°$

3. $\frac{3}{5}$ and $\frac{4}{5}$

5. (a) $\sqrt{2}$ units (b) $\sqrt{2}$ units (c) 3 units

(d) $\sqrt{5}$ units (e) 14 units

Exercise 6.3

1. 3.5

3. $\overline{AB} + \overline{BC} + \overline{CA} = 0$

11. The vector opposite to the resultant is (–6, –9)

13. $6i + 9j$

Exercise 6.4

1. $2\underset{\sim}{a} + 2\underset{\sim}{b} + 4\underset{\sim}{c} = (8, 29)$

3. $3\underset{\sim}{a} - 5\underset{\sim}{b} - 6\underset{\sim}{c} = (5, -20)$

7. $\underset{\sim}{q} = -3\underset{\sim}{a}$

9. (a) $3(\underset{\sim}{p} - \underset{\sim}{q}) = (6, 6)$

   (b) $3(\underset{\sim}{p} + \underset{\sim}{q}) = (48, 36)$

Exercise 6.5

1. $\sqrt{2}$ N, direction is S45°W

3. Magnitude 154.2N, direction is 45.2°E

5. $F_x = -5\sqrt{3}N$, $F_y = 25$ N

7. 643 km/h

Revision exercise 6

1. $3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{i} - 5\underset{\sim}{j}$

3. $4\underset{\sim}{i} + 15\underset{\sim}{j}$

5. $\sqrt{13}$ and $\sqrt{5} + \sqrt{2}$

7. 7.8 N at an angle of 39.8° to the 6 N force

9. $250\left(\sqrt{2} + 2\right)\underset{\sim}{j} - 50\sqrt{2}\underset{\sim}{i}$

11. (a) 13 (b) $t = 1$, $k = -1$

13. 93.4 km/h at an angle of 50.9° upstream

CHAPTER SEVEN

Exercise 7.1

1. (a) $P + Q = \begin{pmatrix} 9 & 5 \\ 6 & 4 \end{pmatrix}$ (b) $Q - P = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

3. (a) $A + B = \begin{pmatrix} 18 & 10 \\ 12 & 8 \end{pmatrix}$

(b) $A - B = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$

5. $A + B = \begin{pmatrix} 7 & 0 \\ 4 & 3 \end{pmatrix}$

7. (a) $\begin{pmatrix} 9 & 8 \\ 7 & 13 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}$

9. (a) $\begin{pmatrix} 2 & 1 \\ 5 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$

11. $w = -4,\ x = 4,\ y = 10,\ z = -4$

Exercise 7.2

1. (a) $AB = 18$ (b) $\begin{pmatrix} 6 & 8 \\ 9 & 12 \end{pmatrix}$

3. (a) $3L - 2M = \begin{pmatrix} 7 & 8 \\ 5 & 13 \end{pmatrix}$ (b) $4M + 6N = \begin{pmatrix} 34 & 38 \\ 32 & -44 \end{pmatrix}$

(c) $6M - 4N = \begin{pmatrix} -14 & -34 \\ -30 & 38 \end{pmatrix}$ (d) $5(L + M) = \begin{pmatrix} 20 & 5 \\ 0 & 30 \end{pmatrix}$

(e) $5(L - M) = \begin{pmatrix} 10 & 15 \\ 10 & 20 \end{pmatrix}$

5. (a) $A(B+C) = \begin{pmatrix} 20 & 48 \\ 6 & 16 \end{pmatrix}$ (b) $AB + AC = \begin{pmatrix} 20 & 48 \\ 6 & 16 \end{pmatrix}$

(c) YES; matrix multiplication under matrix addition is distributive.

7. (a) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} -5 & -7 \\ 6 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} -5 & -7 \\ 6 & 8 \end{pmatrix}$ (e) $\begin{pmatrix} 48 & -102 \\ 40 & -95 \end{pmatrix}$ (f) $\begin{pmatrix} 90 & 144 \\ 60 & 96 \end{pmatrix}$

9. (a) (i) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ (iii) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) No; matrices are not compatible.

11. (a) $\begin{pmatrix} 29 & 14 \\ 21 & 10 \end{pmatrix}$ (b) $\begin{pmatrix} 26 & 19 \\ 18 & 13 \end{pmatrix}$ (c) $\begin{pmatrix} 25 & 18 \\ 18 & 13 \end{pmatrix}$

13. (a) $\begin{pmatrix} 81 & 50 \\ 51 & 32 \end{pmatrix}$ (b) $\begin{pmatrix} 11 & 6 \\ -1 & 14 \end{pmatrix}$

15. (a) $\begin{pmatrix} -117 & 201 \\ 21 & -49 \end{pmatrix}$ (b) $\begin{pmatrix} 144 & -57 \\ 0 & 49 \end{pmatrix}$

(c) $\begin{pmatrix} -15 & 42 \\ 9 & -14 \end{pmatrix}$

17. (a) $\begin{pmatrix} 60 \\ -17 \end{pmatrix}$ (b) $\begin{pmatrix} -428 \\ 155 \end{pmatrix}$

Exercise 7.3

1. $|A| = -10$, A is non singular matrix.
3. $|C| = -160$, C is non singular matrix.
5. $|E| = 0$, E is singular matrix.
7. $|G| = 0$, is singular matrix.
9. $|J| = -730$, J is non singular matrix.
11. (a) $a = 3$ (b) $x = 2$ or $x = -2$ (c) $x = 0$ or $x = 3$

Exercise 7.4

1. $|A| = 32$, A is non singular matrix, $A^{-1} = \begin{pmatrix} \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$

3. $|C| = 0$, C is singular matrix

5. $|E| = 48$, E is non singular matrix, $E^{-1} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$

7. $|M| = 384$, M is non–singular matrix, $M^{-1} = \begin{pmatrix} \frac{1}{32} & \frac{1}{48} \\ -\frac{1}{16} & \frac{1}{24} \end{pmatrix}$

9. $|G| = 2700$, G is non–singular, $G^{-1} = \begin{pmatrix} \frac{1}{45} & -\frac{1}{1350} \\ -\frac{1}{18} & \frac{1}{54} \end{pmatrix}$

11. (a) $\begin{pmatrix} 0 & -\frac{1}{32} \\ \frac{1}{32} & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -\frac{1}{256} & -\frac{1}{256} \\ \frac{1}{256} & -\frac{1}{256} \end{pmatrix}$

13. $\begin{pmatrix} -14 & -108 \\ 108 & 58 \end{pmatrix}$

Exercise 7.5

1. $x = 5,\ y = 3$
3. $x = 3,\ y = -2$
5. $x = \frac{124}{71},\ y = \frac{376}{71}$
7. $x = 3,\ y = \frac{1}{3}$

9. 20 chickens and 40 cows.

11. 100 shillings for a tomato and 200 shillings for an onion.

Exercise 7.6

1. $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3. $M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
5. $M_{30^\circ} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
7. $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$
9. $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$
11. $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$
13. (a) $y = x$ (b) $3y + 4x - 6 = 0$

Exercise 7.7

1. $R_{90^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

3. $R_{270^\circ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

5. $R_{45^\circ} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

7. $(x', y') = (-5, 3)$

9. $(x', y') = (5, -3)$

11. $(x'', y'') = (1, 2)$

13. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

Exercise 7.8

1. (a) $(-8, 11)$ (b) $(-7, 9)$

3. (a) $(-7, -7)$ (b) $(-2, -2)$ (c) $(2, -3)$

5. $(2, -3)$

7. (a) $(2, 3)$ (b) $(10, 10)$

9. $4y + 3x + 28 = 0$

Exercise 7.9

1. $(x', y') = (5, 10)$ 3. $(x', y') = (6, 4)$ 5. $\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$

9. (a) $(7, 14)$ (b) $(21, 28)$

Revision exercise 7(a)

1. $\begin{pmatrix} -1 & 5 \\ 5 & 6 \end{pmatrix}$

3. (a) $A + (B + C) = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ and $(A + B) + C = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$

   (b) $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $(B + A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

   (c) $A = -B$

5.

×	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

7. (a) $|P| = 0$; P has no inverse (b) $|Q| = -10$; Q has an inverse

   (c) $|R| = 0$; R has no inverse (d) $|S| = -2$; S has an inverse

   (d) $|T| = 24$; T has an inverse

9. Singular matrices are $G = \begin{pmatrix} 0 & 0 \\ 4 & 5 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, $J = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$, and

   $K = \begin{pmatrix} 3 & 15 \\ 4 & 20 \end{pmatrix}$